EDM: Notes on Electronic Distance Measurement

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Electronic Distance Measurement (EDM) is a fundamental feature of modern surveying Total Stations. Distance measurements can be made in two modes: [1] with a reflector using an Infrared or visible laser electromagnetic wave and phase measurement or [2] 'reflectorless' using a visible laser beam and pulse time of flight. Reflectorless measurements are limited in length – usually less than 500 m – and are not as accurate as measurements to reflectors; consequently these notes are concerned with distances determined by measuring the phase angle between an emitted and reflected electromagnetic wave.

Electromagnetic waves have frequency *f* and wavelength λ that are related by:

$$\lambda = \frac{c}{f} \tag{1}$$

where *c* is the velocity of electromagnetic waves in a medium. *c* is commonly referred to as the speed of light in the medium.

The Speed of Light, the Metre and the Second

The *speed of light* in vacuum denoted as c_0 is a fundamental physical constant whose value is known to be $299\,792\,458\,\mathrm{m/s}$ and *c* the speed of light in a medium (usually air) is related to c_0 by

$$c = \frac{c_0}{n} \tag{2}$$

where *n* is a dimensionless quantity known as the refractive index of the medium. Now since nothing travels faster than the speed of light in a vacuum then $c < c_0$ and $n > 1$. (The refractive index of air ranges from $n = 1.0001$ to $n = 1.0005$ depending on the temperature and pressure so *c* ranges from $299\,642\,637$ to $299\,762\,482\,\mathrm{m/s}$.)

The *metre* (m) is defined in the International System of Units (SI) as the distance light travels in vacuum in $1/299\,792\,458$ of a second.

The *second* (s) is the duration of $9\,192\,631\,770$ periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom. Atomic clocks (caesium fountain, rubidium laser) on-board GPS satellites are fundamental to position fixing on the Earth (engineerguy 2012).

The Electromagnetic Spectrum

The electromagnetic spectrum orders all electromagnetic waves according to frequency and wavelength. The Sun, Earth and other bodies radiate electromagnetic energy of varying wavelengths that are characterized as sinusoidal waves. The wavelength is the distance from wave crest to wave crest (see Figure 2) and the frequency is the rate at which the sinusoid curve repeats.

Frequency is measured in *hertz* (Hz) in honour of the German physicist Heinrich Hertz (1857–1894) who proved the existence of electromagnetic waves postulated by James Clerk Maxwell's electromagnetic theory of light.

One Hertz = 1 Hz = 1 cycle-per-second

One Kilo Hertz = 1 KHz = 1×10^3 Hz

One Mega Hertz = 1 MHz = 1×10^6 Hz

One Giga Hertz = 1 GHz = 1×10^9 Hz

Wavelength and frequency are related by equation (1) and high frequency corresponds with small wavelength. The visible part of the electromagnetic spectrum has wavelengths between 400 nm and 700 nm where nm is nanometre = 10^{-9} metre. Other useful derived units are

One kilometre	= 1 km	= 1×10^3 m
One millimetre	= 1 mm	= 1×10^{-3} m
One micrometre	= 1 μm	= 1×10^{-6} m
One nanometre	= 1 nm	= 1×10^{-9} m

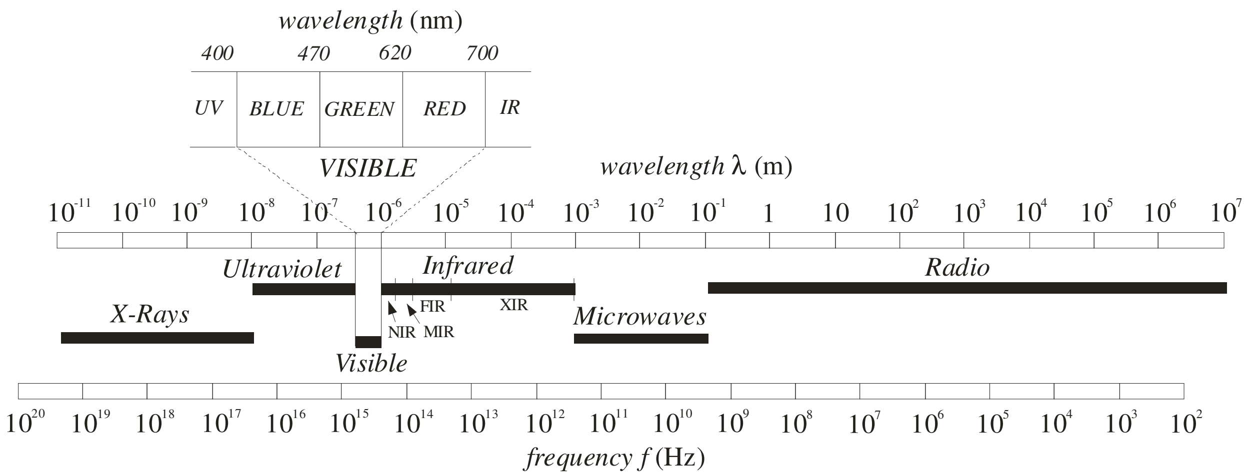

Figure 1. Schematic diagram of the electromagnetic spectrum

The Infrared portion of the spectrum is subdivided by wavelength into Near Infrared (NIR) 700 nm – 3,000 nm, or 0.7 μm – 3.0 μm; Middle Infrared (MIR) 3.0 μm – 6.0 μm; Far Infrared (FIR) 6.0 μm – 15.0 μm and Extreme Infrared (XIR) 15.0 μm – 1 mm.

Most modern Total Stations use electromagnetic waves in the visible RED and NIR region of the electromagnetic spectrum.

Electromagnetic waves and phase angle

An electromagnetic wave can be described by the formula

$$y = A\sin\phi = A\sin(\omega t) \quad (3)$$

$$\phi = \omega t \quad (4)$$

$$\omega = 2\pi f \quad (5)$$

where A is the amplitude, ω is angular velocity (radians/sec), t is time (seconds), ϕ is the phase angle (radians) and f is the frequency (Hertz or cycles/sec)

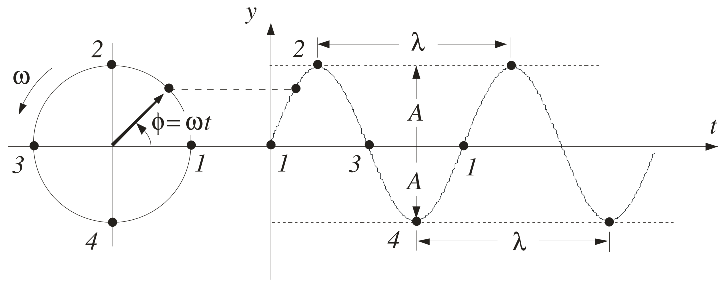

Figure 2.

In Figure 2 the amplitude is the height or depth of the wave above or below the *x*-axis. The wavelength is the distance between peaks (or troughs) and a cycle corresponds to one revolution of the radius vector, i.e., from position 1 to position 1 again.

Considering the phase angle ϕ as a function of time, i.e., the angular velocity ω is unvarying, then $\phi = \phi(t) = \omega t$ and using the Total Increment Theorem we may write

$$\delta\phi = \omega\,\delta t \tag{6}$$

where $\delta\phi$ is a small quantity known as the phase lag and δt is a small time difference or time lag.

Using (6), a wave with a phase lag of $\delta\phi$ can be expressed as

$$\begin{aligned} y &= A\sin(\phi + \delta\phi) \\ &= A\sin(\omega t + \omega\,\delta t) \\ &= A\sin\omega(t + \delta t) \end{aligned} \tag{7}$$

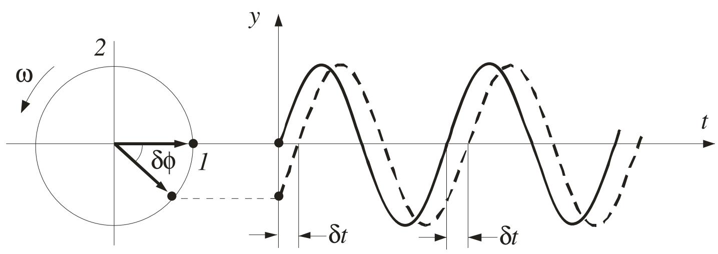

Figure 3. The dashed wave lags the solid wave by $\delta\phi$

Distance from emitted and reflected waves

Consider Figure 4 that shows a schematic diagram of an electromagnetic wave emitted from *A* and at *B* reflected back to *A*. The distance between A and *B* is *d*. [In the case of visible light or near infrared (NIR) the reflectors are corner cubes of glass.]

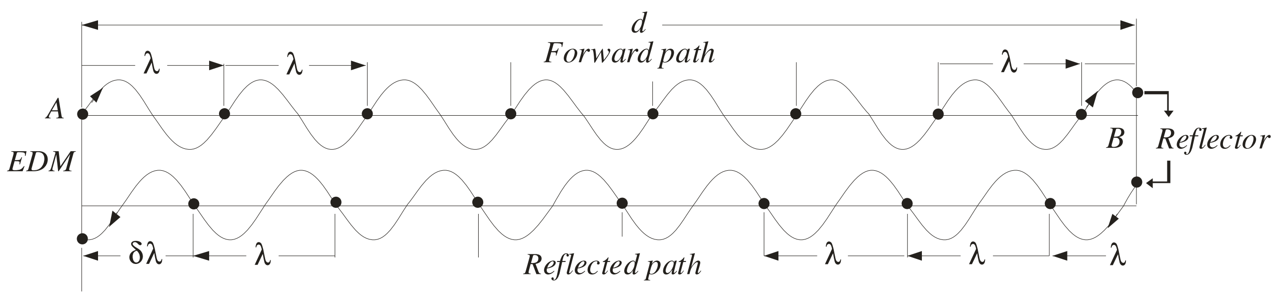

Figure 4. Forward and reflected path of electromagnetic wave.

The total path length (A to B and back to A) is

$$2d = m\lambda + \delta\lambda \tag{8}$$

where m is an integer and $\delta\lambda$ is part of a wavelength.

If t_λ is the time for 1 cycle of the wave, i.e., λ corresponds with t_λ and $\delta\lambda$ corresponds with δt_λ, and c is the speed of light in the medium, then (8) can be written as

$$2d = m\,c\,t_\lambda + c\,\delta t_\lambda \tag{9}$$

Now, using (6), (5) and (1) we may write

$$\delta t_\lambda = \frac{\delta\phi}{\omega} = \frac{\delta\phi}{2\pi f} = \frac{\delta\phi\,\lambda}{2\pi\,c} \tag{10}$$

Substituting (10) into (9) and with $t_\lambda = \lambda/c$ we may write the distance d as

$$d = m\frac{\lambda}{2} + \frac{\delta\phi}{2\pi}\frac{\lambda}{2} \tag{11}$$

With the substitutions

$$U = \frac{\lambda}{2} \tag{12}$$

and

$$L = \frac{\delta\phi}{2\pi}\frac{\lambda}{2} = \frac{\delta\phi}{2\pi}U \tag{13}$$

the distance d can be written as

$$d = mU + L \tag{14}$$

where U is the unit length of the of the distance meter
L is the fraction of the unit length U to be determined by phase measurement
$\delta\phi$ is the measured phase lag (radians)
m is an integral number of unit lengths (and is unknown at this stage)

Equations (11) and (14) are the fundamental equations of electronic distance measurement using measured phase difference.

Digital Phase Measurement

Rueger (1996) outlines a method of digital phase measurement which is followed here.

Digital phase measurement is based on a comparison of two sinusoidal waves (or signals) of equal frequency. One wave is the emitted (or reference signal) and the other is the reflected or return signal, and both signals trigger square waves where the amplitude alternates at the same steady frequency as the sinusoid. The sinusoidal waves, emitted and reflected, are shown as *(1)* and *(2)* in Figure 5. The corresponding square waves are shown as *(3)* and *(4)* respectively.

As the amplitude of the square waves alternate an 'electronic gate' is triggered. The gate opens when the reference signal begins a new cycle and closed when the return signal does the same. When the gate is open, pulses from a high frequency oscillator are accumulated in a counter.

In Figure 5, *(5)* shows the phase counts 'i' between the opening of the gate (GO) and the closing of the gate (GC) and *(6)* shows a full wavelength count 'j'.

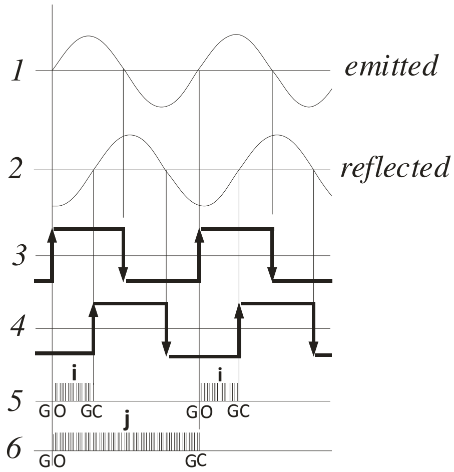

Figure 5. Counting sequences of a digital phase measurement. The curves *1* and *3* represent the reference signal and the curves *2* and *4* the return signal. The counts representing the phase difference between the reference and return signal are depicted on line *5* and the counts for a full cycle of the reference signal on line *6* (Rueger 1996, Fig 4.10)

The ratio $\frac{\delta\phi}{2\pi} = \frac{\text{i counts}}{\text{j counts}}$ and from (13) we have the fractional part of the unit length *U* as

$$L = \frac{\delta\phi}{2\pi} U = \frac{\text{i counts}}{\text{j counts}} \times U \tag{15}$$

For a given unit length *U,* and *L* determined from phase measurements, the 'unknown' in the distances equation (14) is the integer *m.*

The distance *d* can be obtained once *m* is determined in the following way (Rueger 1996).

Suppose that $U_1 = \lambda_1/2$ is the fundamental unit length of the EDM and that by modulation of the electromagnetic wave, two other wavelengths are generated: $\lambda_2 = p \times \lambda_1$ and $\lambda_3 = q \times \lambda_1$ giving rise to unit lengths U_2 and U_3 and fractional parts L_1, L_2, L_3 from phase measurements. These give rise to three equations

$$\begin{aligned} d &= m_1 U_1 + L_1 \\ d &= m_2 U_2 + L_2 \\ d &= m_3 U_3 + L_3 \end{aligned}$$

Since *p* and *q* are known, the equations can be manipulated and a solution for m_1 obtained which yields the distance *d.* Rueger (1996, Appendix C) has an example of this technique used by an early manufacturer of EDM (AGA Geodimeter Model 6A, circa 1970).

Alternatively, the distance *d* may be obtained by an addition of fractional parts *L* of multiple unit lengths.

Suppose that $U_1 = \lambda_1/2 = 10$ m is the fundamental unit length of the EDM and that by modulation of the electromagnetic wave, three other unit lengths are generated: $U_2 = 10 \times U_1 = 100$ m, $U_3 = 100 \times U_1 = 1000$ m and $U_4 = 1000 \times U_1 = 10000$ m. Now suppose that phase measurement is accurate to one part in 10,000 of the unit length and the following phase measurements are obtained

$U_1 = 10$ m phase measurement = $\dfrac{\delta\phi}{2\pi} = 0.8250$ $L_1 = \underline{8.250}$ m

$U_2 = 100$ m phase measurement = $\dfrac{\delta\phi}{2\pi} = 0.3682$ $L_2 = \underline{3}6.82$

$U_3 = 1000$ m phase measurement = $\dfrac{\delta\phi}{2\pi} = 0.5243$ $L_3 = \underline{5}24.3$

$U_4 = 10000$ m phase measurement = $\dfrac{\delta\phi}{2\pi} = 0.2188$ $L_4 = \underline{2}188$

The distance *d* is then the summation of the 'fine measurement' 8.250 m + the 'coarse measurements' 30 m, 500 m and 2000 m = 2538.250 m. This technique of multiples of unit lengths and coarse measurements to identify 10's, 100's, 1000's etc. is a more common approach to resolving the distance in EDM.

The Carrier wave and the Modulated wave

The *carrier* wave in the EDM component of modern Total Stations is in the visible RED or the near infrared (NIR) region of the electromagnetic spectrum. Waves in this part of the spectrum are least affected by atmospheric conditions (temperature, pressure, humidity), have good dispersion properties (i.e., narrow beams can be generated) and good reflecting properties. The carrier wave can be *modulated* by a known waveform so that the modulated wave carries useful information. In EDM the modulating waveform changes the amplitude of the carrier wave so that the modulated wave has a desirable frequency f_{MOD} and wavelength λ_{MOD}. This type of modulation is known as *amplitude modulation*.

Figure 6 shows three waves in a time period of 0.003 sec. The first is a carrier wave with frequency $f_{\text{CAR}} = 20,000\,\text{Hz} = 20\,\text{MHz}$, amplitude $A_{\text{MOD}} = 5$ and wavelength $\lambda_{\text{CAR}} = \dfrac{c_0}{f_{\text{CAR}}} = 14,989.62$ m.

The second is a modulating wave with the same amplitude as the carrier wave and a frequency of 1/10th of the carrier wave $f_{\text{MOD}} = 2,000\,\text{Hz} = 2\,\text{MHz}$. The third wave is the modulated wave.

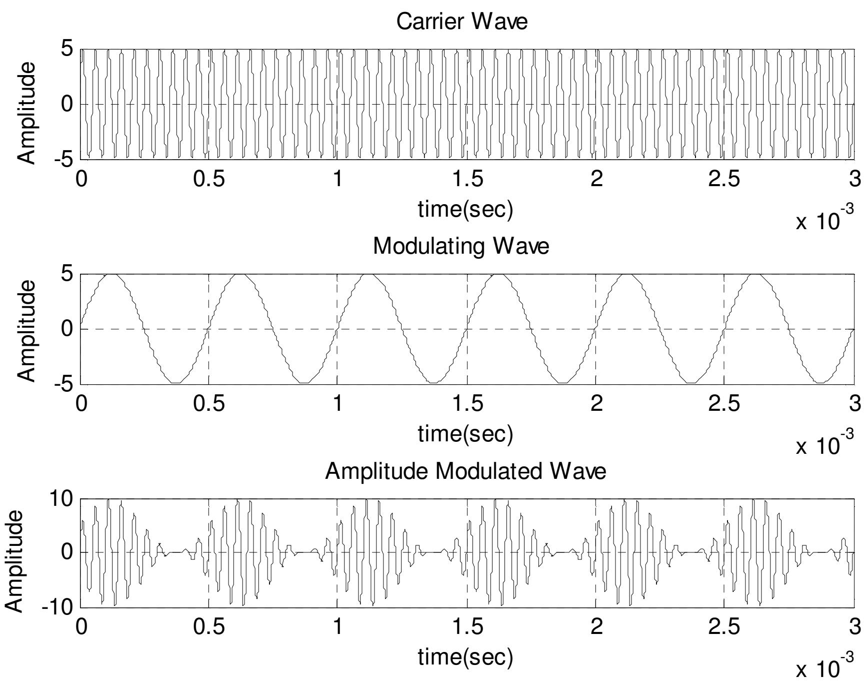

Figure 6. Carrier wave, Modulating wave and Modulated wave.

Propagation of Electromagnetic Waves through the Atmosphere

Atmospheric Transmittance

The transmittance of the atmosphere is the ratio of incident radiant power to transmitted radiant power and is a measure of the attenuation (and extinction) of wave propagation (Rueger 1996). We could imagine atmospheric transmittance of visible white light as the ratio of brightness of a searchlight at 1 m and at 1000 m.

The transmittance is a function of numerous variables: wavelength; distance; temperature; barometric pressure; atmospheric water vapour content; gaseous content of the atmosphere; dust and other particles in the atmosphere. Electromagnetic radiation may be scattered by air molecules and other aerosols in the atmosphere and/or absorbed by the atmosphere. In the infrared part of the electromagnetic spectrum, transmittance is affected by water vapour (H_2O), carbon dioxide (CO_2) and ozone (O_3) in the atmosphere and at certain wavelengths there is almost total absorption and or scattering.

In EDM, the transmittance of the atmosphere affects the range of the instrument and the strength (or power) of the reflected signal.

Figure 7 shows the transmittance of electromagnetic radiation in the visible and near infrared (NIR) portion of the spectrum. In the NIR band water vapour is the main cause of attenuation at 1, 1.5, 2 and 3 μm. The TPS1200 series of LEICA Total Stations use a near infrared (NIR) carrier wave of 0.780 μm and the TPS1200+ series use a visible RED carrier wave of 0.658 μm.

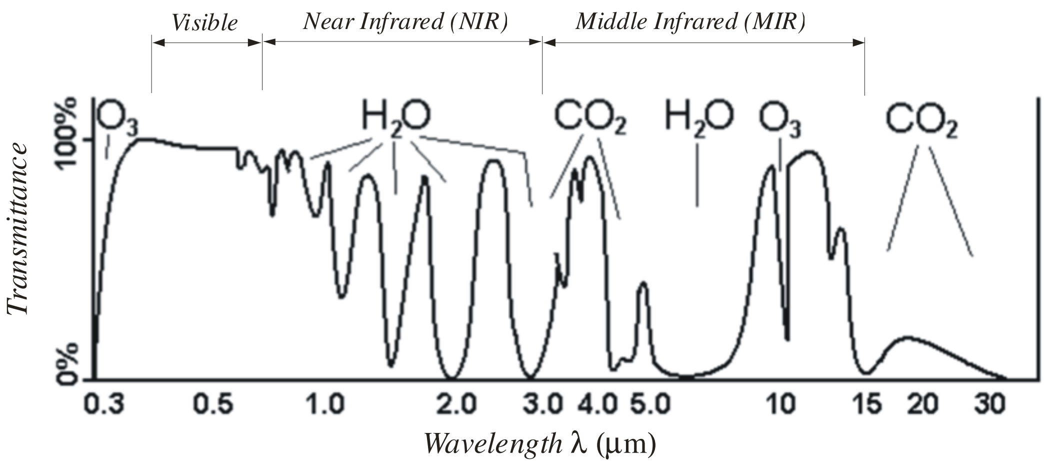

Figure 7. Atmospheric Transmittance for visible and near infrared wavelengths.

Refractive Index *n*, Refractivity *N* and Group Refractive Index n_g

The refractive index *n* is the ratio of the speed of light in vacuum c_0 and the speed of light in the medium (usually the atmosphere) *c*

$$n = \frac{c_0}{c} \tag{16}$$

The refractivity *N* is defined as

$$N = (n - 1) \times 10^6 \tag{17}$$

For example: if $n = 1.000\,299\,264\,637$ then $N = 0.000299264637 \times 10^6 = 299.264637$

n and N are dimensionless quantities.

The refractive index of the atmosphere is a function of the gaseous composition of the air, water vapour pressure, temperature (wet and dry bulb) and the frequency/wavelength of the radiated signal.

Formula for calculating refractive index are often complex. A simple empirical formula, proposed by the French mathematician Augustin-Louis Cauchy (1789 – 1857) has the general form

$$n = B + \frac{C}{\lambda_0^2} + \frac{D}{\lambda_0^4} + \cdots \tag{18}$$

where n is the refractive index, λ_0 is the wavelength in vacuum and $B, C, D,$ etc. are coefficients that can be determined from experiment/measurements. Whilst the theory on which Cauchy based this equation was later found to be incorrect, his mathematically simple equation is a close approximation for more complex relationships in certain bands of the electromagnetic spectrum.

A two-term form of Cauchy's equation is used to model the refractive index of common optical materials used in the manufacture of reflecting corner-cube prisms where λ_0 is in micrometres (μm).

$$n = B + \frac{C}{\lambda_0^2} \tag{19}$$

Material	B	C (μm)
Fused silica	1.4580	0.00354
Borosilicate glass BK7	1.5046	0.00420
Hard crown glass K5	1.5220	0.00459

In electro-optical EDM the refractive index is dependent on the wavelength of the visible or near infrared radiation which is a modulated carrier wave (see the lower wave in Figure 6) where the wave crests of the carrier wave and the modulating wave are moving at different velocities through the atmosphere.

The *group velocity* c_g of the signal which is smaller than the individual velocities is given as the differential equation

$$c_g = c - \lambda \frac{dc}{d\lambda} \tag{20}$$

where c and λ are the speed of light and wavelength in the medium.

Denoting the *group refractive index* as n_g equation (16) can be rearranged as

$$c_0 = c_g n_g \tag{21}$$

Substituting (21) into (20) gives

$$\frac{c_0}{n_g} = \frac{c_0}{n} - \lambda \frac{d}{d\lambda}\left(\frac{c_0}{n}\right) = \frac{c_0}{n} - \lambda \frac{d}{dn}\left(\frac{c_0}{n}\right)\frac{dn}{d\lambda} = \frac{c_0}{n} + \frac{\lambda c_0}{n^2}\frac{dn}{d\lambda} = \frac{c_0}{n}\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)$$

giving

$$n_g = \frac{n}{1 + \dfrac{\lambda}{n}\dfrac{dn}{d\lambda}} \tag{22}$$

If λ_0 is the wavelength in vacuum then $\lambda = \lambda_0 / n$ and the group refractive index becomes

$$n_g = n - \lambda_0 \frac{dn}{d\lambda_0} \tag{23}$$

Differentiating (18) with respect to λ_0 gives

$$\frac{dn}{d\lambda_0} = -\left(2\frac{C}{\lambda_0^3} + 4\frac{D}{\lambda_0^5} + \cdots \right) \tag{24}$$

And substituting (18) and (24) into (23) and simplifying gives

$$n_g = B + 3\frac{C}{\lambda_0^2} + 4\frac{D}{\lambda_0^4} + \cdots \tag{25}$$

A formula of this form was used by Barrel and Sears (1939) to model the group refractive index of air for the visible spectrum.

More recent measurements of the group refractive index by Ciddor (1996) – whose equations are regarded as definitive – have led to a Barrel & Sears type equation (Rueger 1998, eq. 4)

$$N_g = \left(n_g - 1 \right) \times 10^6 = 287.6155 + \frac{4.88660}{\lambda^2} + \frac{0.06800}{\lambda^4} \tag{26}$$

where N_g, n_g are the group refractivity and refractive index respectively and λ is the carrier wavelength of the EDM signal in micrometre (μm). This formula is for air at standard conditions: temperature $t = 0^\circ$ C; pressure $p = 1013.25$ hPa; partial water vapour pressure $e = 0.0$ hPa; 0.0375% $C0_2$ and can be used for wavelengths in the range 650 nm to 850 nm. Results from this closed formula deviate less than 0.25 ppm from the more accurate formula of Ciddor (1996).

Note that pressure is in units of hectopascal (hPa). Pascal[1] is an SI derived unit and hecto (h) = 10^2 so 1 hectopascal = 100 pascal. The conversion between hectopascal and the old pressure unit of millibar (mb or mbar) is simple: 1000 hPa = 1000 mb. Also 1000 hPa = 750 mm of Mercury (Hg)

Equation (26) was adopted by the International Association of Geodesy (IAG) during its 22nd General Assembly, Birmingham, U.K., 19-30 July 1999 and is recommended for computation of n_g for accuracies of 1 ppm with visible and near infrared waves in the atmosphere.

[1] A unit of pressure in the SI system named in honour of Blaise Pascal (1623–1662) a French mathematician, physicist, religious philosopher, and master of French prose. Pascal laid the foundation for the modern theory of probabilities, formulated what came to be known as Pascal's law of pressure, and propagated a religious doctrine that taught the experience of God through the heart rather than through reason. Pascal tested the theories of Galileo and Evangelista Torricelli (an Italian physicist who discovered the principle of the barometer). To do so, he reproduced and amplified experiments on atmospheric pressure by constructing mercury barometers and measuring air pressure, both in Paris and on the top of a mountain overlooking Clermont-Ferrand. These tests paved the way for further studies in hydrodynamics and hydrostatics. Pascal invented the syringe and created the hydraulic press, an instrument based upon the principle that became known as Pascal's Law: pressure applied to a confined liquid is transmitted undiminished through the liquid in all directions regardless of the area to which the pressure is applied. His publications on the problem of the vacuum (1647-48) added to his reputation (Encyclopaedia Britannica 1999).

Example: The LEICA TPS1200+ series use a carrier wave of 0.658 μm and using (26)

$$N_g = (n_g - 1)\times 10^6 = 287.6155 + \frac{4.88660}{0.658^2} + \frac{0.06800}{0.658^4}$$
$$= 287.6155 + 11.2863887 + 0.3627483$$
$$= 299.264637$$

and
$$n_g = 1.000\,299\,264\,637$$

Refractive Index of Light n_L

For calculating the refractive index of light n_L for actual atmospheric conditions the International Association of Geodesy (IAG) recommends the following formula (Rueger 1998)

$$N_L = (n_L - 1)\times 10^6 = \left(\frac{273.15}{1013.25}\times\frac{N_g p}{273.15+t}\right) - \frac{11.27\,e}{273.15+t} \qquad (27)$$

where N_L, n_L are the group refractivity and refractive index of light respectively, N_g is the group refractivity of visible or near infrared waves calculated from (26), *t* is the temperature in degrees Celsius (°C), *p* is the atmospheric pressure in hectopascal (hPa) and *e* is the partial water vapour pressure in hectopascal (hPa).

Pressure *p*, Partial Water Vapour Pressure *e*, Saturation Water Vapour Pressure *E*, and Humidity *h*

Consider an element of area δA of a surface on which a force δF is exerted; the pressure *p* is

$$p = \lim_{A\to 0}\frac{\delta F}{\delta A} = \frac{dF}{dA}$$

For conditions where the force *F* is uniformly distributed over an area *A* then

$$p = \frac{Force}{Area}$$

The units of pressure are newtons per square metre (N/m^2) or pascals (Pa).

Atmospheric pressure is the pressure exerted by the weight of air in a column of the atmosphere of cross-section area *A* above the earth's surface. Atmospheric pressure varies with altitude, and for a fixed location, varies slightly from time to time (high- and low-pressure weather systems). For reference purposes, a standard atmosphere has a value of pressure $p = 1013.25$ hPa.

In a mixture of gases, each gas has a *partial pressure* which is the hypothetical pressure of that gas if it alone occupied the volume of the mixture at the same temperature. The total pressure of a mixture of gases is the sum of the partial pressures of each individual gas in the mixture.

Water vapour is the gaseous phase of water and can be produced from the evaporation or boiling of liquid water or in the process of turning ice into water (the transition from solid to liquid). Water vapour is an invisible constituent of the atmosphere and is continuously generated by evaporation and removed by condensation.

Suppose that in an experiment a closed vessel of volume *V* m^3 contains 1 kg of perfectly dry air at a pressure p_a and temperature *t*. If *x* kg of water vapour at the same temperature is injected into the vessel then the pressure will increase to:

$$p = p_a + e$$

where e is the *partial water vapour pressure*. Both the air and water vapour occupy the same volume V and are at the same temperature t.

If water vapour continues to be injected, the partial water vapour pressure e will continue to rise, but only to a certain limiting value. The partial pressure of the dry air p_a will remain constant.

If more water vapour is injected and the temperature remains constant then the excess will condense and collect as liquid water on the base and sides of the vessel. At this stage, the space containing the air has become *saturated,* and $e = E$ where E is the *saturation water vapour pressure.*

This experiment could be repeated, starting with the vessel evacuated and containing no air, and exactly the same amount of water vapour could be injected before condensation commenced (provided that the temperature remained constant). This leads to the conclusion that the saturation water vapour pressure E depends only on the temperature and not on the presence of other gasses.

For purposes of definition water vapour is said to be saturated when it can exist in stable thermodynamic equilibrium with a plane surface of water or ice. Hence we have E_W and E_{ICE} denoting saturation water vapour pressure over water and ice respectively.

Humidity is the amount of water vapour in the air and *relative humidity h* is a ratio of partial water vapour pressure to saturated water vapour pressure expressed as a percentage

$$h = \frac{e}{E} \times 100 \tag{28}$$

Partial water vapour pressure e can be derived by measurement of *dry* and *wet-bulb* temperatures of an aspiration psychrometer. This is a device that draws air at a certain velocity over two mercury-in-glass thermometers; one having a (normal) bulb containing mercury and the other a bulb that is covered by gauze that is soaked with water. The first is the dry-bulb thermometer recording temperature t and the second is the wet-bulb thermometer recording temperature t'. The partial water vapour pressure e is given by (Rueger 1996)

$$e = E'_W - 0.000662\, p\left(t - t'\right) \tag{29}$$

where E'_W is saturation water vapour pressure over water in hPa for temperature t'
p is atmospheric pressure in hPa
t is (dry-bulb) temperature in °C
t' is the wet-bulb temperature in °C

Saturation water vapour pressure over water for the temperature range of -70 °C to 50 °C can be calculated from (Murray 1967)

$$E'_W = 6.1078\ \mathrm{e}^{\left(\frac{17.269t'}{237.30+t'}\right)} \tag{30}$$

$\mathrm{e} = 2.718281828...$ is the base of natural logarithms.

Accuracy required for *p, t* and *e* in the calculation of Refractive Index n_L

By rearranging (27) n_L can be expressed as a function of the variables p (pressure), t (temperature) and e (partial water vapour pressure)

$$n_L = n_L\left(p, t, e\right) = \frac{Dp}{273.15 + t} \times 10^{-6} - \frac{11.27e}{273.15 + t} \times 10^{-6} + 1 \tag{31}$$

where $D = \frac{273.15}{1013.25} N_g$. Using the Total Increment Theorem, the increment δn_L can be written as

$$\delta n_L = \frac{\partial}{\partial p}(n_L)\delta p + \frac{\partial}{\partial t}(n_L)\delta t + \frac{\partial}{\partial e}(n_L)\delta e \tag{32}$$

where the partial derivatives are:

$$\frac{\partial}{\partial p}(\delta n_L) = \frac{D}{273.15+t}\times 10^{-6}; \quad \frac{\partial}{\partial t}(\delta n_L) = -\left(\frac{Dp - 11.27e}{(273.15+t)^2}\right)\times 10^{-6}; \quad \frac{\partial}{\partial e}(\delta n_L) = -\frac{11.27}{273.15+t}\times 10^{-6}$$

Equation (32) may be written as

$$\delta n_L \times 10^6 = \frac{D}{273.15+t}\delta p - \left(\frac{Dp - 11.27e}{(273.15+t)^2}\right)\delta t - \frac{11.27}{273.15+t}\delta e \tag{33}$$

and the increments $\delta p, \delta t, \delta e$ can be regarded as small systematic errors that induce an error of δn_L in the calculation of the refractive index n_L.

Example Following Rueger (1996): For a temperature $t = 15\,^\circ\mathrm{C}$, a pressure $p = 1007$ hPa, a partial water vapour pressure $e = 13$ hPa and a group refractivity $N_g = (n_g - 1)\times 10^6 = 304.5$ giving $D = \frac{273.15}{1013.25} N_g = 82.086528$, the error in the calculated refractive index is

$$\delta n_L \times 10^6 = 0.285\,\delta p - 0.994\,\delta t - 0.039\,\delta e \tag{34}$$

The significance of (34) can be summarized as follows (Rueger 1996):

1. an error in p of 1 hPa affects the refractive index and distance by 0.3 ppm.
2. an error in t of 1 °C affects the refractive index and distance by 1 ppm.
3. an error in e of 1 hPa affects the refractive index and distance by 0.04 ppm.

We can deduce that temperature is critical for the determination of refractive index if errors in distance are not to exceed 1 ppm. Also, the partial vapour pressure need not be known very accurately. Indeed some manufacturers ignore *e* entirely in their documentation and formulae.

Measured Distance and Velocity Corrections

The distance *d* from phase difference measurement of an emitted and reflected electromagnetic wave is given by (11).

An alternative is to consider the *time of flight* of a small pulse of emitted and reflected electromagnetic radiation and *d* can be calculated from

$$2d = c\,\delta t \tag{35}$$

where *c* is the speed of light in the medium (the atmosphere) and δt is the time of flight (the time difference between the emission of the pulse and its reflected return).

The Measured Distance d' and First Velocity Correction K'

Using (2) or (16) we may write (35) as

$$d = \frac{c_0}{n}\left(\tfrac{1}{2}\delta t\right) = d' + K' = \frac{c_0}{n_{\mathrm{REF}}}\left(\tfrac{1}{2}\delta t\right) + K' \tag{36}$$

where n is the refractive index of the medium (the atmosphere), c_0 is the speed of light in vacuum, $d' = \frac{c_0}{n_{\text{REF}}}\left(\tfrac{1}{2}\delta t\right)$ is the *measured distance* and can be supposed as the uncorrected displayed distance of the EDM and K' is a small correction known as the *first velocity correction*.

n_{REF} is the *reference refractive index* of the EDM and is an instrument constant.

n_{REF} is defined [using (1) and (2)] as

$$n_{\text{REF}} = \frac{c_0}{\lambda_{\text{MOD}}\, f_{\text{MOD}}} = \frac{c_0}{2U\, f_{\text{MOD}}} \tag{37}$$

where λ_{MOD} is the modulation wavelength (for fine readings) of the EDM and is exactly half of the unit length U of the EDM. n_{REF} is fixed by the manufacturer by adopting an instrument unit length U and a suitable modulation frequency f_{MOD} so that n_{REF} corresponds to average atmospheric conditions.

Equation (36) may be rearranged and simplified as

$$K' = \frac{c_0}{n_{\text{REF}}}\left(\tfrac{1}{2}\delta t\right)\left(\frac{n_{\text{REF}} - n}{n}\right) = d'\left(\frac{n_{\text{REF}} - n}{n}\right) \tag{38}$$

n is close to unity and where it appears in the denominator of (38) it may be replaced by unity without incurring errors greater than 0.02 ppm in the correction K'. So to a reasonable accuracy the (corrected) distance d is

$$d = d' + K' \tag{39}$$

where d' is the (uncorrected) displayed distance of the EDM and K' is the first velocity correction

$$K' = d'\left(n_{\text{REF}} - n\right) \tag{40}$$

Using (40) gives

$$\begin{aligned} K' &= d'\left(n_{\text{REF}} - n\right) \\ &= d'\left\{n_{\text{REF}} - 1 + 1 - n - 1 + 1\right\} \\ &= d'\left\{\left(n_{\text{REF}} - 1\right) + 1 - \left(\left(n - 1\right) + 1\right)\right\} \\ &= d'\left\{\left(n_{\text{REF}} - 1\right) - \left(n - 1\right)\right\} \end{aligned} \tag{41}$$

Rearranging (27) gives an equation for $n_L - 1$ as

$$\left(n_L - 1\right) = \left(\frac{Dp}{273.15 + t}\right)\times 10^{-6} - \frac{11.27\, e}{273.15 + t}\times 10^{-6} \tag{42}$$

where $D = \frac{273.15}{1013.25} N_g$

Substituting (42) for $(n-1)$ in (41) and simplifying gives the first velocity correction (Rueger 1996, 1998)

$$K' = d' \times 10^{-6}\left\{C - \frac{Dp}{273.15 + t} + \frac{11.27\, e}{273.15 + t}\right\} \tag{43}$$

where: d' is the (uncorrected) displayed distance of the EDM in metres
p is the atmospheric pressure in hPa
t is the temperature in ^{o}C
e is the partial water vapour pressure in hPa

$$C = (n_{REF} - 1) \times 10^6$$

$$D = \frac{273.15}{1013.25} N_g$$

The reference refractive index n_{REF} and the group refractivity N_g are given by (37) and (26) respectively.

> [Manufacturers often supply values for *C* and *D* in equation (43) and also, often use a representative value for partial water vapour pressure *e*. Tables and charts for first velocity corrections related to 'standard conditions' are generally supplied by the manufacturer in the instrument handbook.]

The Curved Path Length *D*, Chord Length *C* and Second Velocity Correction K''

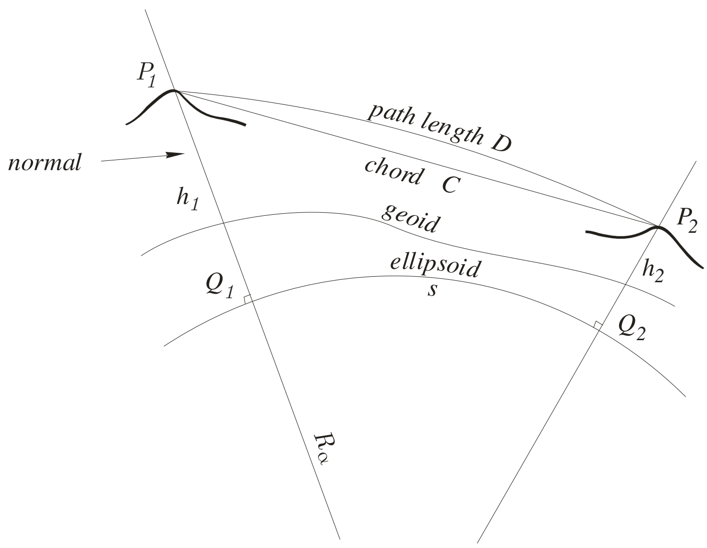

Figure 8. Curved path *D* of electromagnetic wave

Figure 8 shows points P_1 and P_2 some distance apart on the Earth's terrestrial surface. The normals to the ellipsoid[2] passing through P_1 and P_2 intersect the ellipsoid at Q_1 and Q_2, and h_1, h_2 are the ellipsoidal heights (distances Q_1P_1 and Q_2P_2) at P_1 and P_2. s is the *geodesic*[3] distance Q_1Q_2 and in Figure 8 is approximated by the length of a circular arc of radius R_α.

Figure 8 also shows the geoid which is a gently undulating equipotential surface of the Earth's gravity field approximating global mean sea level in a least squares sense. The Australian Height

[2] The ellipsoid is a surface of revolution created by rotating an ellipse, having semi-axes *a, b* ($a > b$), about its minor axis. The minor axis of the ellipsoid is parallel (or nearly parallel) with the Earth's axis of revolution and the equator of the ellipsoid is perpendicular to the axis of revolution.

[3] The geodesic is the shortest path length between two points on a surface. On a plane this is a straight line, on a sphere it is a great circle arc and on an ellipsoid the geodesic is curved path having curvature and torsion (twist).

Datum (AHD) is an approximation of the geoid. Heights derived from GPS observations are ellipsoidal heights but heights related to the Australian Height Datum are connected to the geoid.

R_α is the *radius of curvature* of the ellipsoid at P_1 in the direction α_{12} where α_{12} is the azimuth (clockwise angle from true north) of the geodesic from Q_1 to Q_2.

$$R_\alpha = \frac{\rho_1 \nu_1}{\rho_1 \sin^2 \alpha_{12} + \nu_1 \cos^2 \alpha_{12}} \tag{44}$$

where ρ, ν are the radii of curvature in the meridian and prime vertical respectively at P_1.

The Curved Path Length *D* and Coefficient of Refraction *k*

In Figure 8, the path followed by the electromagnetic wave is *D* and due to the differing layers of the Earth's atmosphere that the wave is moving through (with differing refractive indices *n*) this path will be curved. i.e., the path *D* with have a curvature and a radius of curvature r_α that will generally change over the length of the curve. Curvature and radius of curvature have an inverse relationship and we may define a *coefficient of refraction k* as a ratio of curvatures

$$k = \frac{\text{curvature of path D}}{\text{curvature of ellipsoid}} = \frac{1/r_\alpha}{1/R_\alpha} = \frac{R_\alpha}{r_\alpha} \tag{45}$$

It is not possible to determine r_α and hence *k* directly, but values of *k* can be inferred from measurements and theory and a generally accepted value for electromagnetic waves in the visible or near infrared spectrum is (Rueger 1996)

$$k_L = 0.13 \qquad\qquad (r_\alpha \approx 8R_\alpha) \tag{46}$$

[This value may vary considerably under differing atmospheric conditions during the day.]

Note that there are other definitions of 'coefficient of refraction' and care should be taken using values from other sources or from geodetic reductions such as height differences from vertical circle observations where the coefficient of refraction may have a different definition.

The Chord Length *C*

Suppose that the curved path *D* is approximated by a circular arc of radius equal to the radius of curvature $r = r_\alpha$ and that this circular arc subtends an angle θ at the centre. The *chord distance C* is given by

$$C = 2r \sin \tfrac{1}{2}\theta \tag{47}$$

But $k = R/r$, $2r = 2R/k$, $D = r\theta$ and $\tfrac{1}{2}\theta = (Dk)/(2R)$. So equation (47) can be written as

$$C = \frac{2R}{k} \sin\left(\frac{Dk}{2R}\right) \tag{48}$$

Using the series expansion for $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ equation (48) can be written as

$$C = \frac{2R}{k}\left\{\frac{Dk}{2R} - \frac{1}{6}\left(\frac{Dk}{2R}\right)^3 + \frac{1}{120}\left(\frac{Dk}{2R}\right)^5 - \cdots\right\}$$

$$= D - \frac{D^3k^2}{24R^2} + \cdots \tag{49}$$

Without any loss of accuracy *D* can be replaced with the displayed (uncorrected) EDM distance d' and the chord distance *C* given by

$$C = d' + \text{ path curvature correction} \tag{50}$$

and

$$\text{path curvature correction} = -\frac{\left(d'\right)^3 k^2}{24R^2} \tag{51}$$

where d' is the displayed (uncorrected) EDM distance
k is the refraction (see k_L above)
R is the radius of curvature of the ellipsoid in the direction of the measured line [see (44)]

For a distance 36 km ($d' = 36000$ m) measured by a near infrared EDM ($k_L = 0.13$) with $R = 6378000$ m the path curvature correction = 0.0008 m.

The Second Velocity Correction K''

For precise EDM, atmospheric conditions (temperature, pressure, partial water vapour pressure) should be determined at both ends of a measured line and mean results used to calculate the first velocity corrections. However, a small error is introduced (by adopting the mean values) and a further small correction known as the *second velocity correction,* should be applied.

The second velocity correction may be written as (Rueger 1996)

$$K'' = -\left(k - k^2\right)\frac{d'^3}{12R^2} \tag{52}$$

Equation (52) is based on the work of Saastamoinen (1964) and its derivation is beyond the scope of these notes.

For a distance of 16 km ($d' = 16000$ m) measured by a near infrared EDM ($k_L = 0.13$) with $R = 6378000$ m the second velocity correction $K'' = -0.0009$ m .

The second velocity correction and path curvature corrections are often ignored for short EDM lines $\left(d' < 10 \text{ km}\right)$

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