Notes on Elementary Particle Physics

Instructor: MVN Murthy

Teaching Assistant: Dibyakrupa Sahoo

August–December 2012

Chapter 1

Natural Units, Scales, Notations & Conventions

In this chapter we survey the various units, scales, notations and conventions we shall be using in our discussions ahead. More over this is done keeping in mind the current practices followed by the modern literatures in high energy physics.

1.1 Natural units

As is true for any branch of physics, particle physics is based on experiments. And these experiments look for the most elementary constituents of matter. These necessarily involve probing at extremely small length scales (typically of the order of 10^{-15} m or less). And the typical masses involved are 10^{-27} kg. The standard system of units in physics, in general, is the International System of Units (SI). In particle physics we use a system of units known as the '**Natural Units**'.

There are two fundamental constants in relativistic quantum mechanics:

1. Reduced Planck's constant $\hbar$ (dimension ML^2T^{-1}): $\hbar \equiv \dfrac{h}{2\pi} = 1.054\,571\,628(53) \times 10^{-34}$ J s.

2. Speed of light in vacuum c (dimension LT^{-1}): $c = 299\,792\,458$ m s^{-1}.

In natural units, we specify the energy in units of GeV (1 GeV = 10^9 eV)[1]. This choice is motivated by the fact that the rest energy of a nucleon (proton or neutron) is approximately 1 GeV. In the natural units, we put

$$\hbar = c = 1, \tag{1.1}$$

so that we won't have to worry about the pesky $\hbar$ and c that appear in most of the equations in particle physics. Instead of mass, length and time, we use mass, action ($\hbar$) and speed (c). In these units. In

[1] One eV (electron-Volt) is the energy gained by an electron when accelerated by a potential difference of one volt.

these units

$$E = E'c^2, \qquad t = t'\frac{\hbar}{c^2}, \qquad p = p'c, \qquad v = v'c, \qquad l = l'\frac{\hbar}{c}, \qquad e^2 = e'^2\hbar c, \qquad J = J'\hbar$$

where all the primed quantities are either dimensionless or some power of mass. We can always use dimensional analysis to figure out where the $\hbar$'s and c's enter a formula. This deliberate sloppiness in dealings with $\hbar$'s and c's allows us to express:

1. mass (m), momentum (mc) and energy (mc^2) in terms of GeV; and
2. length ($\hbar/mc$) and time ($\hbar/mc^2$) in terms of GeV^{-1}.

The following table summarizes the relations between the SI units and Natural units for mass, length and time.

Quantity	SI unit	Natural unit	Relation (Conversion factor)
Mass (M)	kg	GeV	$1\ \text{GeV} = 1.78 \times 10^{-27}$ kg
			$1\ \text{kg} = 5.61 \times 10^{26}$ GeV
Length (L)	m	GeV^{-1}	$1\ \text{GeV}^{-1} = 0.1975 \times 10^{-15}$ m
			$1\ \text{m} = 5.07 \times 10^{15}\ \text{GeV}^{-1}$
Time (T)	s	GeV^{-1}	$1\ \text{GeV}^{-1} = 6.59 \times 10^{-25}$ s
			$1\ \text{s} = 1.52 \times 10^{24}\ \text{GeV}^{-1}$

In particle physics, we deal with cross-sections often. Cross-section has dimension of area (L^2). In particle physics it is expressed in a unit called **barn** (b): $\boxed{1\ \text{b} = 10^{-28}\ \text{m}^2}$. If we use the conversion factor given in the table above, we get

$$1\ \text{b} = 2570.49\ \text{GeV}^{-2},$$
$$\implies 1\ \text{GeV}^{-2} = 0.389 \times 10^{-3}\ \text{b} = 0.389\ \text{mb}.$$

Another important quantity that comes up when we study interactions between particles is their electric charge. The electric charge of an electron in SI units (coulomb) is given by

$$e = 1.602\,176\,487(40) \times 10^{-19}\ \text{C}, \tag{1.2}$$

and the fine structure constant is given by a dimensionless number

$$\alpha = \frac{\frac{1}{4\pi\epsilon_0}\frac{e^2}{(\hbar/mc)}}{mc^2} = \frac{e^2}{4\pi\epsilon_0\hbar c} \simeq \frac{1}{137}. \tag{1.3}$$

Here ϵ_0 is the permittivity of free space, while its permeability is denoted by μ_0 and satisfies the relation $\epsilon_0\mu_0 = \frac{1}{c^2}$. In the description of various interactions, such units are not generally useful, so physicists have considered another set of units (known as **Heaviside-Lorentz units**) where

$$\epsilon_0 = \mu_0 = \hbar = c = 1, \tag{1.4}$$

keeping the value of α unchanged (as it should be since this is a constant)

$$\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}. \tag{1.5}$$

In the Heaviside-Lorentz units the Maxwell's equations take the following form:

$$\vec{\nabla} \cdot \vec{E} = \rho, \tag{1.6a}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{1.6b}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1.6c}$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{J}, \tag{1.6d}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, ρ is the charge density and $\vec{J}$ is the current density. Electric charge is also a constant in the Heaviside-Lorentz units, the charge of electron being

$$e \simeq 0.3028. \tag{1.7}$$

For sake of completeness we provide the SI prefixes in the following table.

Factor	Prefix	Symbol	Factor	Prefix	Symbol
10^{24}	yotta-	Y	10^{-24}	yocto-	y
10^{21}	zetta-	Z	10^{-21}	zepto-	z
10^{18}	exa-	E	10^{-18}	atto-	a
10^{15}	peta-	P	10^{-15}	femto-	f
10^{12}	tera-	T	10^{-12}	pico-	p
10^{9}	giga-	G	10^{-9}	nano-	n
10^{6}	mega-	M	10^{-6}	micro-	μ
10^{3}	kilo-	k	10^{-3}	milli-	m
10^{2}	hecto-	h	10^{-2}	centi-	c
10^{1}	deka-	da	10^{-1}	deci-	d

Example 1.1. de Broglie wavelength: The de Broglie wavelength associated with a 1 GeV photon is given by:

$$\lambda_\gamma^{dB} = \frac{h}{p} = \frac{2\pi\hbar c}{E} = 2\pi \text{ GeV}^{-1} = 2\pi \times 0.1975 \times 10^{-15} \text{ m} = 0.395\pi \text{ fm}.$$

Example 1.2. Compton wavelength: The compton wavelength of a particle of mass m is defined as

$$\lambda^C = \frac{\hbar c}{mc^2} = \frac{\hbar}{mc}.$$

In natural units the Compton wavelength is

$$\lambda^C = \frac{1}{m}.$$

So the Compton wavelength for a pion (whose mass is approximately 140 MeV) is

$$\lambda^C_\pi = \frac{1}{140 \text{ MeV}} = \frac{10^3}{140 \text{ GeV}} = \frac{0.1975 \times 10^3}{140} \text{ fm} = 1.411 \text{ fm}.$$

Similarly the Compton wavelength for an electron (whose mass is approximately 0.5 MeV) is

$$\lambda^C_e = \frac{1}{0.5 \text{ MeV}} = \frac{10^3}{0.5 \text{ GeV}} = \frac{0.1975 \times 10^3}{0.5} \text{ fm} = 395 \text{ fm}.$$

If we approximate the proton's mass to be 1 GeV, then its Compton wavelength is

$$\lambda^C_p = 1 \text{ GeV}^{-1} = 0.1975 \text{ fm}.$$

Example 1.3. Classical electron radius: The classical electron radius, also known as the Lorentz radius or the Thomson scattering length, is given by

$$r_e = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mc^2}. \tag{1.8}$$

In the '**Natural-Heaviside-Lorentz units**' we have $\epsilon_0 = \mu_0 = \hbar = c = 1$ and $e \simeq 0.3028$. Using these values we have

$$r_e = \frac{e^2}{4\pi m} \simeq \frac{0.3028^2}{4\pi(0.5 \text{ MeV})} = \frac{0.3028^2 \times 0.1975 \times 10^3}{4\pi \times 0.5} \text{ fm} = 2.88 \text{ fm} \tag{1.9}$$

One can compare it with the more accurate value is $r_e = 2.817\ 940\ 289\ 4(58)$ fm.

Example 1.4. The Bohr radius: The Bohr radius is the radius of the lowest energy stable orbit of the atomic electron. The energy of the electron in a Hydrogen atom is given by

$$E \simeq \frac{p^2}{2m} - \frac{\alpha}{r}, \tag{1.10}$$

where the first term is kinetic energy and the second term is the electrostatic energy. The momentum p

scales as $1/r$ and hence

$$E \simeq \frac{1}{2mr^2} - \frac{\alpha}{r}. \tag{1.11}$$

Taking the stability condition $\frac{dE}{dr} = 0$ we get the Bohr radius:

$$r = \frac{1}{\alpha m} \simeq \frac{137}{0.5 \text{ MeV}} = \frac{137 \times 10^3 \times 0.1975 \times 10^{-15}}{0.5} \text{ m} = 0.541 \times 10^{-10} \text{ m} = 0.541 \text{ Å}$$

1.2 Ranges and Strengths of the Four Fundamental Interactions

Every particle (massive or massless[2]) is subject to gravitational interaction. Particles that are electrically charged experience electromagnetic interaction. There are two more interactions responsible for happenings in the domain of elementary particles, namely the strong interaction responsible for binding nucleons inside a nucleus and the weak interaction which figures itself in decay processes. There is no classical analogue for these two short ranged forces unlike the electromagnetic and gravity which are long ranged. All the fundamental interactions are possible by via exchange of some elementary particles, which are variously called as *messenger particles*, *force carriers*, *intermediate bosons* and *gauge bosons.* Many a times when these elementary particles are involved in interactions, they cannot be observed; they act as *virtual particles.*

1.2.1 Virtual Particles and Fundamental Interactions

Particle creations may sometimes appear to violate energy conservation, but only for a limited period of time as allowed by the Heisenberg unicertainty principle:

$$\Delta E \; \Delta t \geqslant \frac{\hbar}{2}. \tag{1.12}$$

Thus a particle with energy E can come into existence for a time Δt which does not exceed $\frac{\hbar}{2E}$. During its brief life, the virtual particle can travel a maximum distance:

$$\Delta l = c\Delta t = \frac{\hbar c}{2E}. \tag{1.13}$$

Now for an interaction to have long range (*wider area of influence*) the corresponding force carrier has to travel large distances. If the force carrier has no mass, then its energy can be arbitrarily small ($E \to 0$) such that the range of that interaction becomes infinity. On the other hand if the force carrier is massive with mass M (say), the range of the interaction is upto the distance $\frac{\hbar}{2Mc}$. Photon being massless, the electromagnetic interaction, which it mediates, is of infinite range. The carrier particle

[2]Remember that photon is massless, but yet it is affected by gravity. Light bends under gravity. This is because space itself is bent due to presence of matter and energy according to the General Theory of Relativity.

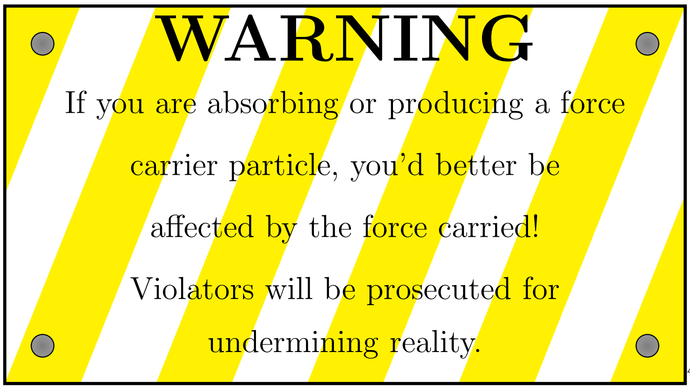
'graviton' (still undiscovered, so only speculated) is also massless (if it exists), and thus leads to the infinite range of the gravitational interaction that it mediates. The $W^{\pm}$ and Z bosons, which mediate the weak interaction are massive (approximately of 90 GeV), so the range of weak interaction is given by:

$$R_{\text{weak}} = \frac{1}{180}\ \text{GeV}^{-1} = \frac{0.1975}{180} \times 10^{-15}\ \text{m} = 1.097 \times 10^{-18}\ \text{m}. \tag{1.14}$$

The strong interaction is mediated by massless gluons, so it might be thought to be of infinite range. However, in fact it is confined to nuclear dimensions only. This is because of the property of 'confinement'. We can however find out the range of strong interaction by analysing the force between two nucleons, which is mediated by pions (having mass around 140 MeV). This is a *residual* strong force[3]. If we consider the mass of pion then the range of strong interaction is given by:

$$R_{\text{strong}} = \frac{1}{280\ \text{MeV}} = \frac{197.5}{280} \times 10^{-15}\ \text{m} = 0.7053\ \text{fm}. \tag{1.15}$$

This is exactly half of the Compton wavelength for pion.



[4]

Thus far we have talked about the ranges of the fundamental interactions. However, there is another aspect to these interactions and that is their relative strengths.

1.2.2 Strengths of Fundamental Interactions

We shall consider the four fundamental interactions one by one.

Gravitational Interaction Let us consider two protons of mass M_p, separated by a distance r. The gravitational potential energy between them is given by Newton's law:

$$V = G_N \frac{M_p^2}{r}, \tag{1.16}$$

[3]Residual means that it is a consequences of gluonic exchanges amongst the consituting quarks of the nucleons. The idea of pion exchange is due to the Japanese physicist Hideki Yukawa.

[4]The content and essential design is taken from one such picture in the beautiful website 'The Particle Adventure': www.particleadventure.org

where G_N is Newton's gravitational constant:

$$G_N = 6.674\,28(67) \times 10^{-11}\ \text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2} = 6.708\,81(67) \times 10^{-39}\ \text{GeV}^{-2}. \tag{1.17}$$

Suppose we consider the two protons to be inside a nucleus, i.e. say $r = 10^{-15}$ m $= 5.07$ GeV^{-1}. Then

$$V \simeq 6.708 \times \frac{1}{5.07} \times 10^{-39}\ \text{GeV} = 1.323 \times 10^{-39}\ \text{GeV}. \tag{1.18}$$

This energy is thus found to be extremely negligible in comparison to the mass of proton itself. However, it must be noted that

$$\begin{aligned}
\sqrt{G_N} =& 0.819 \times 10^{-19}\ \text{GeV}^{-1}, \\
=& 0.819 \times 0.1975 \times 10^{-34}\ \text{m} = 1.617 \times 10^{-35}\ \text{m} = L_{\text{Planck}}, \\
=& 0.819 \times 6.59 \times 10^{-44}\ \text{s} = 5.397 \times 10^{-44}\ \text{s} = T_{\text{Planck}}, \\
\text{and} \qquad \frac{1}{\sqrt{G_N}} =& \frac{1}{0.819} \times 10^{19}\ \text{GeV} = 1.221 \times 10^{19}\ \text{GeV} = M_{\text{Planck}},
\end{aligned}$$

where $L_{\text{Planck}}, T_{\text{Planck}}, M_{\text{Planck}}$ are called as the *Planck length*, the *Planck time* and the *Planck mass* respectively. At these Planck scales gravitation becomes the dominant interaction of the four in the realm of elementary particles. Assuming that at the Planck mass scale the gravitational interaction has as much strength as electromagnetic interaction at the proton mass scale, we obtain the relative strength of gravitational interaction as follows:

$$\alpha_{\text{gr}} = \frac{M_p^2}{M_{\text{Planck}}^2} \alpha_{\text{em}}. \tag{1.19}$$

However, we know that $\alpha_{\text{em}} = 1/137$. Therefore

$$\alpha_{\text{gr}} = \frac{1}{137 \times (1.221)^2 \times 10^{38}} = 0.4896 \times 10^{-40}. \tag{1.20}$$

Looking at these extremely small numbers, it must be clear that gravity has no measure bearings in the realm of elementary particles at our currently available energies.

Weak Interaction Weak interaction is the driving mechanism behind most of radioactive decays, e.g.

$$\begin{aligned}
n &\to p + e^- + \overline{\nu}_e, \\
\text{O}^{14} &\to \text{N}^{14} + e^- + \overline{\nu}_e.
\end{aligned}$$

All these decays are characterised by their half-life. The average life-time of the neutron is about

885 s and the half-life of O^{14} is about 71.4 s. Its possible to estimate the half-life for such decays by folowing the Fermi theory of weak interaction[5]. The Fermi theory contains a constant called as Fermi constant G_F which is a measure of the strength of the weak interaction:

$$\begin{aligned}
G_F =& 1.166\,37(1) \times 10^{-5} \text{ GeV}^{-2}, \\
\sqrt{G_F} =& 3.415 \times 10^{-3} \text{ GeV}^{-1}, \\
=& 3.415 \times 0.1975 \times 10^{-18} \text{ m} = 0.674 \times 10^{-18} \text{ m}, \\
=& 3.415 \times 6.59 \times 10^{-28} \text{ s} = 2.2505 \times 10^{-27} \text{ s}, \\
\text{and} \qquad \frac{1}{\sqrt{G_F}} =& 292.8 \text{ GeV}.
\end{aligned}$$

So at around 300 GeV the weak interaction strength becomes comparable to that of electromagnetic interaction. At the proton mass scale the relative strength of weak interaction is as follows:

$$\alpha_{\text{wk}} = G_F M_p^2 \alpha_{\text{em}} = \frac{1.166}{137} \times 10^{-5} = 0.851 \times 10^{-7} \tag{1.21}$$

Electromagnetic Interaction We are very familiar with the electromagnetic interaction. The strength of electromagnetic interaction is determined by the dimensionless number $\alpha \equiv \alpha_{\text{em}}$, which is

$$\alpha_{\text{em}} = \frac{e^2}{4\pi\epsilon_0 \hbar c} = \frac{1}{137}. \tag{1.22}$$

Strong Interaction This interaction not only binds the quarks and antiquarks into various baryons and mesons, but also leads to the residual strong force that binds the nucleons inside any nucleus. The electromagnetic binding enegy for a proton-antiproton ($p\overline{p}$) system is about 14 keV, however the binding energy for a deuteron (np) is about 2 MeV. So the residual strong force is about 100 times stronger than the electromagnetic force. In the same way the strong force is around 100 to 1000 times stronger than the electromagnetic force. So

$$\alpha_{\text{st}} = \begin{cases} 100\ \alpha_{\text{em}} & \sim 1, \\ 1000\ \alpha_{\text{em}} & \sim 10. \end{cases} \tag{1.23}$$

We conclude that the relative strengths of the four fundamental interactions are in the order:

$$\boxed{\alpha_{\text{st}} : \alpha_{\text{em}} : \alpha_{\text{wk}} : \alpha_{\text{gr}} :: 1\text{-}10 : 10^{-2} : 10^{-7} : 10^{-40}} \tag{1.24}$$

[5]Remember that this is not the accurate description of weak interaction. However at the low energies involved in the radioactive decay modes it is fairly accurate.

Note. *It must be kept in mind that the values given here and those that will be given in the next section are not always true. In some particular particle event deviations from these numbers can be found. Here we are trying to make only a few gross quantitative comparisons amongst the four fundamental interactions. Our estimates are not valid for all processes, but these are typical values (that is, most particle events will testify these numbers to be approximately correct). Since we have found that gravity plays absolutely no role in particle events at the currently achieved (or near future) energy regimes, we shall henceforth drop gravity from our discussion.*

1.3 Typical Cross-sections and Mean Life-times

The cross-section is directly proportional to the square of the coupling constant of the relevant underlying interaction. So we expect that

$$\boxed{\sigma_{\rm st} : \sigma_{\rm em} : \sigma_{\rm wk} :: 1\text{-}10^2 : 10^{-4} : 10^{-14}} \tag{1.25}$$

The mean life-time of a particle decaying via a particular channel (involving mediation by a particular interaction) is inversely proportional to the coupling constant of the mediating interaction. Thus we expect that

$$\boxed{\tau_{\rm st} : \tau_{\rm em} : \tau_{\rm wk} :: 1\text{-}10^{-2} : 10^4 : 10^{14}} \tag{1.26}$$

These are, however, only *approximate ratios.* We have to look at some specific examples to find out the typical values of cross-sections and mean life-times.

1.3.1 Typical Cross-sections

Let us consider the electromagnetic scattering of electron and positron to muon and antimuon:

$$e^-(p_1) + e^+(p_2) \to \mu^+(k_1) + \mu^-(k_2).$$

The cross-section for this process is given by

$$\sigma_{e^-e^+} = \alpha^2 f(s, m_e, m_\mu), \tag{1.27}$$

where f is a function of the center-of-momentum energy ($\sqrt{s}$ where $s = (p_1 + p_2)^2 = (k_1 + k_2)^2$), and the masses: m_e and m_μ. Note that the total cross-section is in general a function of Lorentz invariant variables, which in this case are the square of the sum of the two four momenta in the initial state (or final state) and the masses.

At very high energies we may neglect the masses of the particles, and purely by dimensional considerations the cross section must be given by

$$\sigma_{e^-e^+} \simeq \frac{\alpha^2}{s}. \tag{1.28}$$

The exact expression is in fact

$$\sigma_{e^-e^+} = \frac{4\pi\alpha^2}{3s}. \tag{1.29}$$

At a center-of-momentum energy of 1 GeV, we have

$$\sigma_{e^-e^+} = \frac{4\pi\alpha^2}{3s} = \frac{4\pi}{3\times 137^2}\ \mathrm{GeV}^{-2} = \frac{4\pi}{3\times 137^2} \times 0.1975^2 \times 10^{-30}\ \mathrm{m}^2 = 87.05\ \mathrm{nb}. \tag{1.30}$$

If we consider that the energy of each colliding beam is E_b, then $\sqrt{s} = 2E_b$. Therefore

$$\sigma_{e^-e^+} = \frac{21.7625\ \mathrm{nb}}{E_b^2(\mathrm{in\ GeV}^2)}. \tag{1.31}$$

Another example of electromagnetic scattering is the scattering of low-energy photon on proton. At low-energy (which corresponds to long-wavelength photons) we can use the Thomson formula for the scattering cross-section. In our case

$$\sigma_{\gamma p} = \frac{8\pi}{3}\left(\frac{\alpha}{m_p}\right)^2 \simeq \frac{8\pi}{3}\left(\frac{1}{137}\ \mathrm{GeV}^{-1}\right)^2 = 1.741\times 10^{-35}\ \mathrm{m}^2 = 174.1\ \mathrm{nb}. \tag{1.32}$$

Let us now consider the strong scattering of proton and proton. The charge radius of the proton as measured by experiments (electron- proton scattering) is about 0.81 fm. This is infact larger than the compton wavelength of the proton. Because the strong interaction strength is close to unity, the cross-section for proton-proton scattering is given by

$$\sigma_{pp} = \pi r_p^2 \simeq 3.141 \times 0.81^2 \times 10^{-30}\ \mathrm{m}^2 = 2.061 \times 10^{-30}\ \mathrm{m}^2 = 20.61\ \mathrm{mb}$$

using the classical analogy for the cross-section. Indeed the experimental value is close to this, about 45 mb at close to 1 GeV energy.

Therefore, the ratio of electron-positron and proton-proton scattering cross-sections is given by

$$\frac{\sigma_{e^-e^+}}{\sigma_{pp}} = \frac{87.05\ \mathrm{nb}}{45\ \mathrm{mb}} = 1.934\times 10^{-6}. \tag{1.33}$$

Similarly

$$\frac{\sigma_{\gamma p}}{\sigma_{pp}} = \frac{174.1\ \mathrm{nb}}{45\ \mathrm{mb}} = 3.869\times 10^{-6}. \tag{1.34}$$

These are consistent with our observation

$$\sigma_{\rm st} : \sigma_{\rm em} :: 1\text{-}10^2 : 10^{-4} \text{ which implies } \sigma_{\rm st} : \sigma_{\rm em} :: 1 : 10^{-4}\text{-}10^{-6}.$$

Now let us consider a scattering event that is mediated by weak interaction gauge boson. We shall consider the scattering of an electron-neutrino and a neutron into an electron and a proton:

$$\nu_e + n \rightarrow e^- + p.$$

The total cross-section may be written as

$$\sigma_{\nu_e n} = G_F^2 f(s, m_e). \tag{1.35}$$

Unlike the electromagnetic and strong interactions the coupling strength $G_F = [L^2]$ is not dimensionless. Therefore from dimensional arguments the cross section must go as

$$\sigma_{\nu_e n} = G_F^2\ s. \tag{1.36}$$

If the center-of-momentum energy $\sqrt{s}$ is about 1 GeV, then

$$\sigma_{\nu_e n} = G_F^2\ s = 1.16637^2 \times 10^{-10}\ \text{GeV}^{-2} = 1.360 \times 0.1975^2 \times 10^{-40}\ \text{m}^2 = 53.05\ \text{fb}. \tag{1.37}$$

Then

$$\frac{\sigma_{\nu_e n}}{\sigma_{pp}} = \frac{53.05\ \text{fb}}{45\ \text{mb}} = 1.179 \times 10^{-12}. \tag{1.38}$$

Another example of weak scattering is the scattering of electron-neutrino and electron. The cross-section for this scattering is given by

$$\sigma_{\nu_e e} = \frac{G_F^2\ s}{\pi}. \tag{1.39}$$

If we consider that $\sqrt{s} = 1$ GeV, then

$$\sigma_{\nu_e e} = \frac{G_F^2}{\pi}\ \text{GeV}^2 = \frac{1.16637^2 \times 0.1975^2}{\pi} \times 10^{-40}\ \text{m}^2 = 16.89\ \text{fb}. \tag{1.40}$$

Therefore

$$\frac{\sigma_{\nu_e e}}{\sigma_{pp}} = \frac{16.89\ \text{fb}}{45\ \text{mb}} = 3.75 \times 10^{-13}. \tag{1.41}$$

If we consider the laboratory frame of reference where the electron is at rest before the interaction with neutrino. In such a case we can approximate s by $2m_eE_\nu$, where m_e is the mass of electron and E_ν is the energy of the neutrino in laboratory. If E_ν is given in units of GeV, then

$$\sigma_{\nu_e e} = \frac{2G_F^2 m_e E_\nu}{\pi} = \frac{2 \times (1.16637 \times 10^{-5}\ \text{GeV}^{-2})^2 \times 0.511 \times 10^{-3}\ \text{GeV} \times E_\nu(\text{in GeV})}{\pi}$$

$$=E_\nu(\text{in GeV}) \times 0.44256 \times 0.1975^2 \times 10^{-43}\ \text{m}^2 = E_\nu(\text{in GeV}) \times 1.726 \times 10^{-45}\ \text{m}^2$$

$$=E_\nu(\text{in GeV}) \times 1.726 \times 10^{-17}\ \text{b} = E_\nu(\text{in GeV}) \times 17.26\ \text{atto-barn (ab)}.$$

So if $E_\nu = 1$ GeV, then $\sigma_{\nu_e e} = 17.26$ ab. Therefore, for this case

$$\frac{\sigma_{\nu_e e}}{\sigma_{pp}} = \frac{17.26\ \text{ab}}{45\ \text{mb}} = 3.83 \times 10^{-16}. \tag{1.42}$$

This result is consistent with our observation

$$\sigma_{\text{st}} : \sigma_{\text{wk}} :: 1\text{-}10^2 : 10^{-14} \text{ which implies } \sigma_{\text{st}} : \sigma_{\text{wk}} :: 1 : 10^{-14}\text{-}10^{-16}.$$

So we conclude that our observations for the relative cross-sections are approximately correct.

1.3.2 Typical Mean life-times

Let us consider a particle decay that proceeds via strong interaction. One such decay is: $\Delta^{++} \to \pi^+ + p$. Experimentally, the peak corresponding to Δ^{++} has a full width at half maximum of $\Gamma_{\Delta^{++}} \simeq 100$ MeV. Now the life-time of Δ^{++} is given by

$$\tau_{\Delta^{++}} = \frac{1}{\Gamma_{\Delta^{++}}} = \frac{1}{100}\ \text{MeV}^{-1} = 10\ \text{GeV}^{-1} = 6.59 \times 10^{-24}\ \text{s}. \tag{1.43}$$

Let us now consider a decay that proceeds via electromagnetic interaction, e.g. $\pi^0 \to \gamma + \gamma$. The life-time of π^0 is found to be about 10^{-16} s. So

$$\frac{\tau_{\pi^0}}{\tau_{\Delta^{++}}} \simeq \frac{10^{-16}}{6.59 \times 10^{-24}} = 1.52 \times 10^7. \tag{1.44}$$

Now let us look at one decay that proceeds via weak interaction: $\Sigma \to n + \pi$. The life-time of Σ is about 10^{-10} s. So

$$\frac{\tau_{\Sigma}}{\tau_{\Delta^{++}}} \simeq \frac{10^{-10}}{10^{-24}} = 10^{14}. \tag{1.45}$$

As we can see these ratios confirm that our observation for mean life-times is approximately corrct:

$$\tau_{\text{st}} : \tau_{\text{em}} : \tau_{\text{wk}} :: 1\text{-}10^{-2} : 10^4 : 10^{14} \text{ which implies } \tau_{\text{st}} : \tau_{\text{em}} : \tau_{\text{wk}} :: 1 : 10^4\text{-}10^6 : 10^{14}\text{-}10^{16}.$$

We would like to stress that all these ratios about cross-sections and life-times are given here only to make out a gross quantitative comparisons amongst the various fundamental interactions.

1.4 Notation and Conventions

We shall follow the notation and conventions as given below. If there is any deviation from these, then they would be specified at their place of usage. Also not all the conventions and notations are listed below. The most general ones have been listed, to make the presentation of the latter concepts clear and smoother.

Lorentz indices	μ, ν, ρ, σ, etc.
Three-vector indices	i, j, k, l, etc.
Three-vector	$\vec{x}$
Unit vector	$\hat{x} = \frac{\vec{x}}{\lvert\vec{x}\rvert}$
Kronecker delta	$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$
Levi-Civita tensor	ϵ_{ijk}: totally antisymmetric. $\epsilon_{123} = 1$.
Contractions	$\epsilon_{ijk}\epsilon_{ijk} = 6$, $\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}$, $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.
Dot product	$\vec{A} \cdot \vec{B} = \sum_i A_i B_i$
Cross product	$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$
metric	$g_{\mu\nu} = g^{\mu\nu} =$ diag$(1, -1, -1, -1)$
	$g^{\nu}_{\mu} = g^{\nu\sigma} g_{\mu\sigma} = \delta^{\nu}_{\mu} = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases}.$
Contravariant four-vector	$A^{\mu} = (A^0, \vec{A})$, $x^{\mu} = (t, \vec{x})$
Covariant four-vector	$A_{\mu} = g_{\mu\nu} A^{\nu} = (A^0, -\vec{A})$, $x_{\mu} = (t, -\vec{x})$
Lorentz invariant	$A \cdot B \equiv A_{\mu} B^{\mu} = g_{\mu\nu} A^{\mu} B^{\nu} = A^0 B^0 - \vec{A} \cdot \vec{B}$
antisymmetric tensor	$\epsilon_{\mu\nu\rho\sigma}$. $\epsilon_{0123} = +1$ and $\epsilon^{0123} = -1$
contractions	$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = -24$, $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\tau} = -6 g^{\sigma}_{\tau}$, $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\tau\omega} = -2(g^{\rho}_{\tau} g^{\sigma}_{\omega} - g^{\rho}_{\omega} g^{\sigma}_{\tau})$
derivatives	$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$, $\partial^{\mu} \equiv \frac{\partial}{\partial x_{\mu}} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$
	$\partial_{\mu}\partial^{\mu} = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$
	$\partial \cdot A = \partial_{\mu} A^{\mu} = \frac{\partial A^0}{\partial t} + \vec{\nabla} \cdot \vec{A}$
	$A \overleftrightarrow{\partial^{\mu}} B = A(\partial^{\mu} B) - (\partial^{\mu} A) B$

1.5 Particle Nomenclature

With discovery of large number of particles, it has been really troublesome to name and denote them. During the early days of particle physics, both Greek and Latin characters were used. They are stil used, though there has been some modification of names for hadrons. The only particle which uses both Greek and Latin characters is the particle J/Ψ.

There is one convention we shall follow in these notes. An anti-particle can be denoted by putting a horizontal line above the particle name, e.g. anti-proton $\equiv \overline{\text{proton}}$, positron $\equiv \overline{\text{electron}}$, anti-neutrino $\equiv \overline{\text{neutrino}}$ etc. The practice of denoting the anti-particle by putting a horizontal line over the particle symbol is very ancient in particle physics.

1.5.1 Particles denoted by Greek Characters

Particle	Symbol	Name
alpha particle	α	alpha
beta particle (electron)	β	beta
gamma (photon)	γ	gamma
delta particle	Δ	Delta
eta particle	η	eta
lambda particle	Λ	Lambda
muon	μ	mu
neutrino	ν	nu
xi particle (cascade)	Ξ	Xi
pion	π	pi
rho particle	ρ	rho
sigma particle	Σ	Sigma
tauon or tau	τ	tau
upsilon or bottomium	Υ	Upsilon
chi	χ	chi
psi or charmonium	ψ	psi
omega minus	Ω	Omega

1.5.2 Particles denoted by Latin Characters

This is a very long list of particles. We know about the folowing: b (bottom quark), c (charm quark), d (down quark), e (electron), g (gluon), h or H (Higgs), K (kaons), n (neutron), p (proton), s (strange quark), t (top quark), u (up quark), W (charged weak gauge boson), Z (neutral weak gauge boson).

1.5.3 Naming Scheme for Hadrons

Hadrons being the most numerous particles to populate the detectors in high energy collisions, it is emphatic to systematically name them. The Particle Data Group has been revising its naming scheme for hadrons since 1986. The ones which are given below are according to the 2012 Particle Data Book. For a concise and original account on the naming scheme for hadrons, have a look at J. Beringer *et al.* (Particle Data Group), Phys. Rev. D86, 010001 (2012).

"Flavor neutral" Mesons: The 'flavor neutral' mesons are those mesons that have all the heavy flavor quantum numbers zero, i.e. $S = C = B = T = 0$. In the following table we list the naming scheme for these mesons by specifying their quark-antiquark content and by the value of ${}^{2S+1}L_J$, where S[6], L, J stand for spin, orbital and total angular momenta of the $q\overline{q}$ system.

$q\overline{q}$ content	${}^{2S+1}L_J$			
	${}^1(L \text{ even})_J$	${}^1(L \text{ odd})_J$	${}^3(L \text{ even})_J$	${}^3(L \text{ odd})_J$
$u\overline{d}, u\overline{u} - d\overline{d}, d\overline{u}$ $(I = 1)$	π	b^a	ρ	a
$d\overline{d} + u\overline{u}$ and/or $s\overline{s}$ $(I = 0)$	η, η'	h, h'	ω, ϕ	f, f'
$c\overline{c}$	η_c	h_c	ψ^b	χ_c
$b\overline{b}$	η_b	h_b	Υ	χ_b
$t\overline{t}$	η_t	h_t	θ	χ_t

[a]Do not confuse this with bottom quark.
[b]This is the same J/ψ particle.

Note. *Although there are names for the $t\overline{t}$ mesons, such bound states are unlikely to be formed and found. Top quark is evidently so heavy that even before it can pair with $\overline{\text{top}}$, it decays in laboratory. We have not shown any electric charges of the mesons in the above table, but they are usually placed on the top-right corner of the symbol.*

"Flavored" Mesons: By 'flavored' mesons we mean those mesons which have nonzero heavy flavor quantum numbers, i.e. either $S \neq 0$, or $C \neq 0$, or $B \neq 0$, or $T \neq 0$. The main symbol for such a meson is an upper-case italic letter that indicates the heavier quark (or $\overline{\text{quark}}$) as follows:

$$s \to \overline{K},\ \overline{s} \to K, \qquad c \to D,\ \overline{c} \to \overline{D}, \qquad b \to \overline{B},\ \overline{b} \to B, \qquad t \to T,\ \overline{t} \to \overline{T}.$$

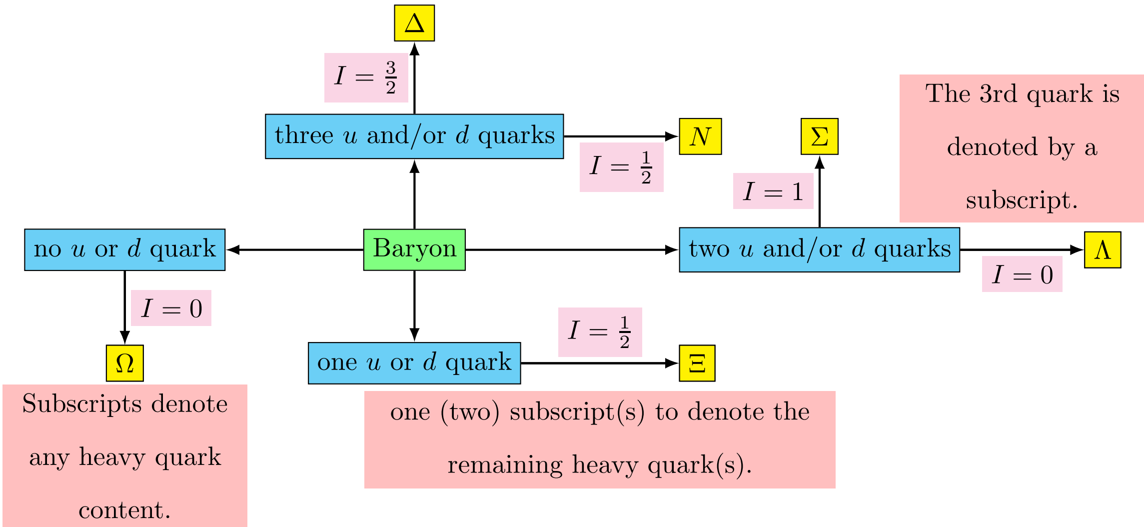
If the lighter quark is not an up (u) or down (d) quark, its identity is specified by keeping its symbol as a subscript. In the literature, it is also found that physicists use the subscripts u (d) also to specify that the lighter quark is u (d).

Example 1.5. Following are some examples illustrating the above scheme of nomenclature.

K mesons		D mesons		B mesons	
$q\overline{q}$ content	Symbol	$q\overline{q}$ content	Symbol	$q\overline{q}$ content	Symbol
$u\overline{s}$	K^+	$u\overline{c}$	$\overline{D}^0$	$u\overline{b}$	B^+
$d\overline{s}$	K^0	$d\overline{c}$	D^-	$d\overline{b}$	B^0
$\overline{u}s$	K^-	$\overline{u}c$	D^0	$\overline{u}b$	B^-
$\overline{d}s$	$\overline{K}^0$	$\overline{d}c$	D^+	$\overline{d}b$	$\overline{B}^0$
		$s\overline{c}$	D_s^-	$s\overline{b}$	B_s^0
		$\overline{s}c$	D_s^+	$\overline{s}b$	$\overline{B}_s^0$

[6]Do not confuse this S with strangeness.

Baryons: For baryons the old symbols N (nucleons), Δ, Λ, Σ, Ξ, and Ω are extensively and exclusively used. The following diagram succinctly explains the current nomenclature for baryons.



Example 1.6. The four light quarks u, d, c and s can be combined in 20 different ways to create baryons and in 16 different ways to create mesons[7]. The sixteen mesons are grouped into a 15-plet and a singlet. Figures 1.1 and 1.2 show the ground-state pseudoscalar and vector mesons respectively. Figures 1.3 and 1.4 show the ground state baryon multiplets with spins $\frac{1}{2}$ and $\frac{3}{2}$ respectively. The scheme of nomenclature as described above is used in all these diagrams and the quark contents are also explicitly shown.

EXERCISE

Question 1.1. What is the energy of an electron that has a de Broglie wavelength of 10^{-16} m?

Question 1.2. In units of the electron Bohr radius, what would be the Bohr radius for a muonic atom and pionic atom.

Question 1.3. The size of the proton (charge radius) is approximately 1 fm. Typically one needs a probe whose wave length is much less than this size to probe the structure of the proton. Suppose we assume that a photon probe has a wavelength less than 1/10 fm, calculate the energy of the photon required to probe the internal structure of the photon.

Question 1.4. The pions are unstable particles. Investigate the decay modes of charged and neutral pions. Assuming an equal number $\pi^{\pm}$ are enter the earths atmosphere (approximately correct), what particles are left in what ratios after all the pions and even their decay products have decayed.

Question 1.5. Suppose the proton could decay with a life time of 10^{30} years, how many cubic meters of water would have to be observed if one wanted to have about 100 events in a year.

Question 1.6. Low energy neutrinos pass through a piece of solid iron- if the neutrino-nucleon cross section is about $\sigma \approx 10^{-47}$ m^2, estimate the mean free path of the neutrinos in iron (density of iron is 8

[7]How do we arrive at these numbers? The answer will be given in Chapter 2.

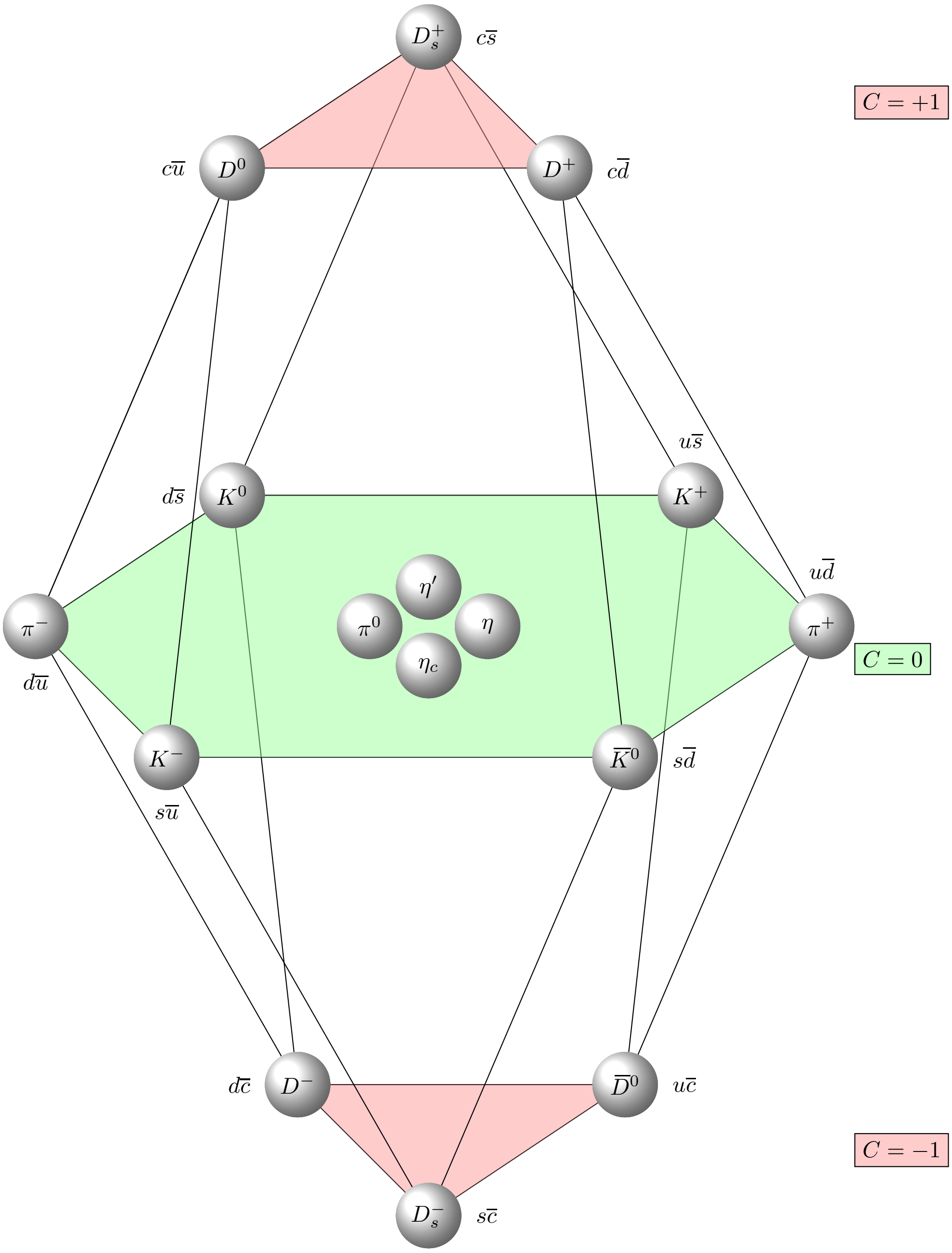


Figure 1.1: Pseudo-scalar meson super-multiplet with charm quark included.

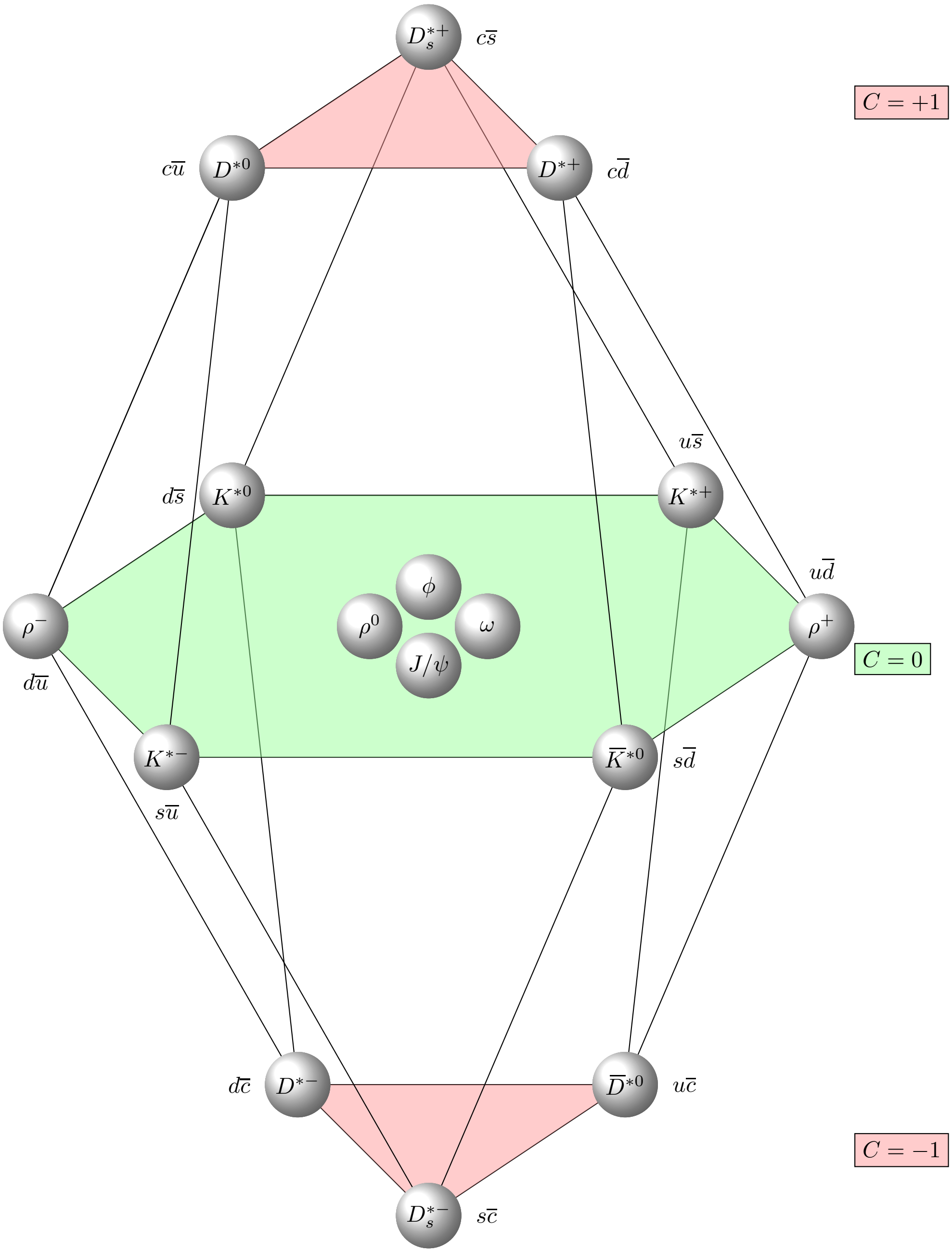


Figure 1.2: Vector meson super-multiplet with charm quark included.

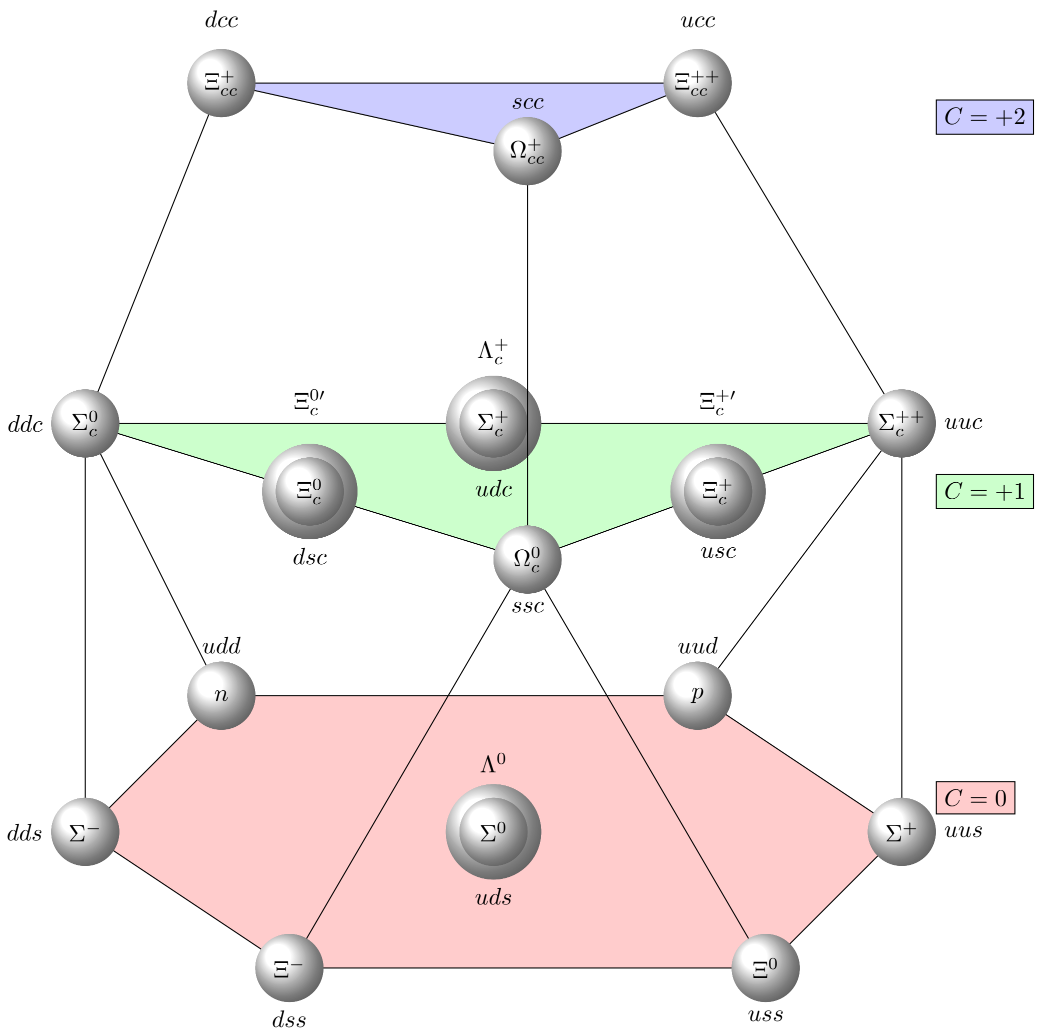


Figure 1.3: Spin-$\frac{1}{2}$ baryon super-multiplet with charm quark included.

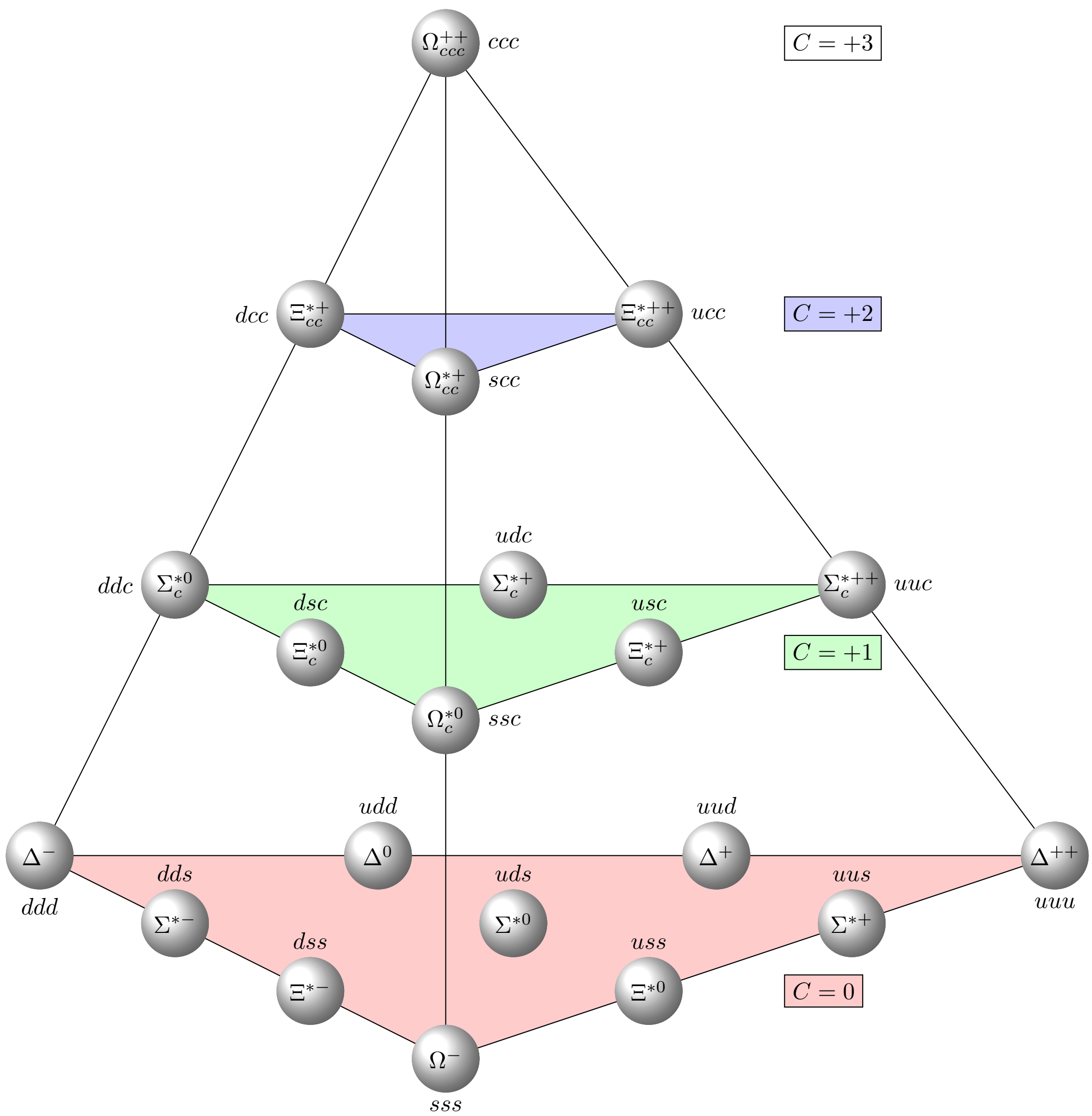


Figure 1.4: Spin-$\frac{3}{2}$ baryon super-multiplet with charm quark included.

times the density of water).

Question 1.7. Verify that the spin of the neutral pion can be deduced from the fact that it decays into two photons. Photons have spin-1 and are massless.

Question 1.8. Free neutron is an unstable particle with a life time of about 13 minutes. Investigate the decay mode of the neutron. Is it possible to have more than one decay mode for the neutron?

Question 1.9. Neutrons bound in nucleus like He^4 or O^{16} remain stable. Why? Apply the same reasons to understand why neutrons in some heavier nuclei are allowed to decay.

Question 1.10. Consider a world in which the masses of neutrons and protons are equal. What would be the consequences, how would this world look like?

Question 1.11. Construct a baryon multiplet like the one shown in Fig. 1.3, by taking the bottom quark instead of the charm quark as the heavier quark.

Chapter 2

Symmetries and Conservation Laws

When learning about the laws of physics you find that there is a large number of complicated and detailed laws, laws of gravitation, of electricity and magnetism, nuclear interactions and so on. But across the variety of these laws there sweep great general principles which all the laws seem to follow. Examples of these are the principles of conservation, some qualities of symmetry. . . .

Richard P. Feynman

Symmetry considerations are a powerful tool to explore and understand the behaviour of elementary particles. They provide the backbone of our theoretical formulations. Even when some of the apparent symmetries are not exact they provide a basis for classification of states assuming exact symmetry and allow us to look at possible sources and pattern of symmetry breaking.

We know that every elementary particle is characterised by a set of quantum numbers. These quantum numbers summarize the intrinsic properties of the particle, and therefore are called as *internal* quantum numbers[1]. The existence of these quantum numbers implies that there are some underlying symmetries in the realm of elementary particles. In this chapter we will discuss some such symmetries that are relevant in particle physics.

Symmetries can be classified into two broad categories:

- **Global Symmetry:** A global symmetry is one which is valid at all spacetime points. The existence of quantum numbers in a system always arise from the invariance of the system under a *global* geometrical transformation.

- **Local Symmetry:** A local symmetry is one which has different symmetry transformations at differents points in spacetime. Such symmetries play a pivotal role in physics, as they form the basis of what are known as *gauge theories*.

[1] By saying that some property of a system is internal we want to stress that the particular property under consideration has nothing to do with the dynamical state of the system, which is described by other conserved quantities such as energy, momentum or angular momentum.

Symmetries that are relevant in particle physics may be classified as follows:

- **Permutation symmetry:** This is also called as the exchange symmetry, and deals with the symmetry of the system under permutation (or exchange) of identical particles. It results in Bose-Einstein statistics (for bosons) and Fermi-Dirac Statistics (for fermions).

- **Continuous symmetry:** This type of transformation deals with the symmetry of the system under infinitesimally small (therefore continuous) transformations. Translation in space and time, rotation in space, Lorentz transformation are examples of such symmetry. The corresponding quantum numbers are *additive*, i.e. the quantum number associated with a given symmetry of a composite system is obtained by adding together (algebraically or vectorially) the corresponding quantum numbers of all the components of the system.

- **Discrete symmetry:** This kind of symmetry is evidently not continuous. The system exhibits symmetries which are steps apart from each other. Space inversion, time reversal, charge-conjugation are examples of discrete symmetry. The quantum numbers are *multiplicative* in this case, i.e. such a quantum number of a composite system is given by the product of the quantum numbers of all the constituents.

- **Unitary symmetry:** Such symmetries arise from phase transformations of fields, or from generaised rotations in the internal space of the system. They are related to conservation of many generalised "charges", for example, U(1) symmetries are associated with conservation of electric charge, baryon number, lepton number; SU(2) symmetry is associated with conservation of isospin, SU(3)(flavour) symmetry is associated with conservation of flavors and SU(3) (colour) symmetry is associated with conservation of color. The associated quantum numbers are *additive.*

We shall briefly discuss the first three symmetries. We will not discuss about unitary symmetry in this chapter. It is most conveniently discussed in the context of quark models. It suffices to say that there are many conservation laws which arise from invariance of the Hamiltonian under the so-called U(1) (or phase) transformations, like the conservation of lepton number, Baryon number, etc in a manner that is analogous to the charge conservation.

2.1 Permutation Symmetry

All physically identical, indistinguishable, many particle states must have definite symmetry under permutation symmetry. It is an observed fact that all particles are either bosons or fermions depending on their behaviour with respect to another particle of the same kind. The state of a system of identical bosons is symmetric under permutations where as a system of identical fermions is anti-symmetric under permutations. While one may be able to define systems with mixed symmetry (symmetric under some exchanges and antisymmetric under others) they are not realised in nature. Indeed this symmetry played fundamental a role in the formulation of quarks with colour as the basic entities in the theory of strong interactions. More about this later.

2.2 Continuous Symmetries

A symmetry under space translation implies that the interaction energy between two particles is independent of their positions but depends only on their relative distance. Classically the Lagrangian $\mathscr{L}$, which is a function of generalised coordinates q_i and generalised velocities $\dot{q}_i$: $\mathscr{L} \equiv \mathscr{L}(q_i, \dot{q}_i)$, remains unchanged under the displacement $q_i \to q_i + \delta q_i$. That is

$$\frac{\partial \mathscr{L}}{\partial q_i} = 0$$

Then by virtue of the equations of motion[2], we have

$$\frac{dp_i}{dt} = 0$$

which is a statement of the conservation of momentum. Similarly invariance under translation in time leads to the conservation of energy.

In quantum mechanics if there is a continuous operation like rotation or translation, say G , it may be generated from transformations which differ infinitesimally from the identity transformation

$$G = 1 - i\epsilon g,$$

where g is the Hermitian generator of the symmetry operator in question. (There may be more than one generator.) For example for rotations about z-axis, it is the z-component of the angular momentum. By definition G is a unitary transformation. Suppose the Hamiltonian $\mathscr{H}$ is invariant under G, then we have

$$G^\dagger \mathscr{H} G = \mathscr{H}$$

This is equivalent to

$$[g, \mathscr{H}] = 0$$

[2]Given a lagrangian $\mathscr{L} \equiv \mathscr{L}(q_i, \dot{q}_i)$ the *Euler-Lagrange equations of motion* are given by $\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathscr{L}}{\partial q_i} = 0$. The canonically conjugate momentum p_i to the coordinate q_i is defined as $p_i \equiv \frac{\partial \mathscr{L}}{\partial \dot{q}_i}$.

and by virtue of Heisenberg equation of motion, we have

$$\frac{dg}{dt} = -i[g, \mathscr{H}] = 0$$

and hence g, or more precisely its quantum expectation value, is a constant of motion. For example if $\mathscr{H}$ is invariant under rotations, then the angular momentum about the axis of rotation is a constant of motion.

Furthermore, when two operators commute, they can be simultaneously diagonalised. The set of eigenfunctions will be labelled by the eigenvalues, quantum numbers, of both operators. If the Hamiltonian for a transition is invariant under the transformation, then the quantum numbers labelling the initial state will also be conserved. This is a very powerful result which results in selection rules for reactions to occur.

2.3 Discrete symmetries

All symmetry operations in quantum mechanics are not necessarily continuous. The Hamiltonian may also be invariant under discrete transformations, for example space-time inversion. We consider three important symmetries here, namely, Parity, Charge Conjugation and Time Reversal.

2.3.1 Parity

We first consider parity or space inversion. Classically under a parity transformation $\vec{r} \to -\vec{r}$ and $\vec{p} \to -\vec{p}$. That is a right-handed coordinate system is changed to a left-handed coordinate system. This can not be achieved by rotation which is a continuous transformation in three-space dimensions. Hence it is a discrete symmetry. Infact it is easy to verify that the determinant of the transformation matrix is positive for rotation matrices where as for Parity it is negative.

If $|\alpha\rangle$ is a quantum mechanical state then we require under space inversion,

$$\langle\alpha|P^\dagger \vec{r} P|\alpha\rangle = -\langle\alpha|\vec{r}|\alpha\rangle$$

We accomplish this by stating that under parity transformation,

$$P^\dagger \vec{r} P = -\vec{r}$$

or

$$\vec{r} P = -P\vec{r}$$

where we have used the fact that P is unitary. Thus the position and parity anticommute. Further, since two inversions cancel the effect of each other, we have,

$$P^2 = 1$$

or equivalently,

$$P^{-1} = P^{\dagger} = P$$

The Parity operator is not only unitary but also hermitian with eigenvalues +1 or -1.

By definition the angular momentum is $\vec{L} = \vec{r} \times \vec{p}$. Clearly,

$$[L, P] = 0$$

Since L is the generator of rotations, parity commutes with rotations,

$$[R, P] = 0$$

.

If the Hamiltonian is invariant under parity transformation, then the states are eigenstates of the parity. Consider the wavefunction of a rotationally invariant system in three dimensions:

$$\psi_{nlm} = R_{nl}(r) Y_{lm}(\theta, \phi)$$

for example hydrogen atom. Under parity transformationwe have, $r \to r, \theta \to \pi - \theta, \phi \to \pi + \phi$ in spherical coordinates. Thus

$$P\psi_{nlm} = (-1)^l \psi_{nlm}$$

using the property of the spherical harmonics.

"Intrinsic Parity" is a notion that is applied to all the elementary particles. The word intrinsic is used in the same sense in which spin is referred to as intrinsic. To clarify consider for example the orbital angular momentum operator $L_i = (\vec{r} \times \vec{p})_i$. In quantum mechanics the operator L_i is defined as,

$$L_i = -i(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j})$$

where the indices are taken around cyclically. Further L_i satisfy the angular momentum algebra,

$$[L_i, L_j] = i\epsilon_{ijk} L_k$$

The commutation relation is very general and applies to spin-angular momentum also,

$$[S_i, S_j] = i\epsilon_{ijk}S_k$$

and to the total angular momentum

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

where $J_i = L_i + S_i$. However there is no spacial representation for S analogous to L. In this sense the spin has no classical analogue and is an intrinsic property of quantum mechanical objects. Consequently the parity of a state described by the eigenfunction of orbital angular momentum is given by,

$$P\psi_{nlm} = \eta_\psi(-1)^l\psi_{nlm}$$

where η_ψ denotes the intrinsic parity of the quantum particle. Further as in the other case,

$$\eta_\psi^2 = 1$$

It is in this sense we refer to parity as an intrinsic property of the state when it is an eigenstate of parity. There is no classical analogue.

Intrinsic parity of the Photon : As an example consider the intrinsic parity of photon. The electromagnetic interaction conserves parity. The current j_μ of a charged particle couples to the electromagnetic field (photon) through

$$j_\mu A^\mu$$

where

$$j_\mu = (\vec{j}, \rho)$$

and

$$A^\mu = (\vec{A}, A_0)$$

in the four-vector notation. Under parity,

$$(\vec{j}, \rho) \rightarrow (-\vec{j}, \rho)$$

since $\vec{j} = \rho\vec{v}$, where $\vec{v}$ is the velocity. The electromagnetic interaction is invariant under parity only if

$$(\vec{A}, A_0) \rightarrow (-\vec{A}, A_0)$$

Thus the intrinsic parity of the photon has to be negative just like any position vector.

Intrinsic parity of the pion : When parity is conserved the intrinsic parity of a particle may be determined relative to others whose intrinsic parity is known: For example consider a reaction

$$A \to B + C$$

Conservation of parity implies

$$\eta_A = \eta_B \eta_C (-1)^L$$

where L is the relative angular momentum of the final state particles.

Thus the intrinsic parity of pion may be determined using the scattering process

$$\pi^- + d \to n + n$$

Using the relation

$$(\text{parity } \pi)(\text{parity } d) = (\text{parity } nn)$$

it is easy to show that the intrinsic parity of the pion should be negative. One needs to assume that the intrinsic parity of proton and neutron to be the same. Infact we **define** the intrinsic parity of the proton to be +1 and define the parity of various other particles relative to that of the proton. While the pion has spin zero, deuteron has spin 1. Since the pion is absorbed almost at rest by the deuteron the relative angular momentum in the initial state is zero. Thus the total angular momentum in the initial state is $J = 1$ entirely due to the spin of the deuteron. Since the neutron is a $S = 1/2$ particle, using angular momentum conservation we have the following options in the final state:

$$\begin{aligned}
|\psi_{nn}^{(1)}\rangle &= |J = 1, S = 1, L = 0, 2\rangle \\
|\psi_{nn}^{(2)}\rangle &= |J = 1, S = 1, L = 1\rangle \\
|\psi_{nn}^{(3)}\rangle &= |J = 1, S = 0, L = 1\rangle
\end{aligned}$$

Antisymmetry excludes all but the second wave function. Hence the parity of the pion is given by,

$$\eta_\pi \eta_d = \eta_n \eta_n (-1)^L$$

$$\eta_\pi = -1$$

Hence the parity of the pion with respect to proton or neutron is negative.

A system whose dynamics is given by Schrödinger or Klein-Gordon equation, the wave function in the inverted system describes a particle with opposite momentum.

2.3.2 Charge Conjugation

Charge conjugation operator C is in many ways similar to Parity. By definition it inverts all internal charges (electric, baryon number, lepton number etc) of a particle thus relating it to its anti-particle and vice versa. The space-time coordinates are unchanged. For example electric charge

$$Q \xrightarrow{C} -Q$$

that is

$$|\psi(Q, \vec{p}, \vec{s}\rangle \xrightarrow{C} |\psi(-Q, \vec{p}, \vec{s})\rangle$$

Thus the quantum mechanical state of a proton, say, under charge conjugation is transformed into the state of an anti-proton.

$$|p\rangle \xrightarrow{C} |\bar{p}\rangle$$

Therefore as in the case of parity we have,

$$C^2 = 1$$

or equivalently,

$$C^{-1} = C$$

Thus C is not only unitary but also hermitian with eigenvalues +1 or -1.

Since C reverses the charges, it also reverses the electric and magnetic fields. As a result the photon has negative eigenvalue under C. However Maxwell equations are invariant under charge conjugation. From the decay

$$\pi^0 \to \gamma + \gamma$$

we conclude that the pion is even under charge conjugation. While some charge neutral states like photon, neutral pion are eigenstates under C, it is not always so- for example

$$|n\rangle \xrightarrow{C} |\bar{n}\rangle$$

Invariance under P or C would then mean that the transitions would occur to only states with the same eigenvalue in the initial and final states. Note that the eigenvalues of these operators are multiplicative. Strong and electromagnetic interactions respect these symmetries, where as in weak interactions these are violated. However, the combination of P and C is still a symmetry to a good approximation though it is violated in some systems.

2.3.3 Time Reversal

The discussion of time reversal symmetry is some what more complicated. Classically both Newton's equations and Maxwell's equations are invariant under time reversal. We briefly discuss the situation in quantum mechanics where at the outset it appears not to be so since the Schroedinger equation is first order in time.

Suppose $\psi(x,t)$ is a solution of the Schroedinger equation,

$$i\frac{\partial \psi}{\partial t}(x,t) = (-\frac{1}{2m}\nabla^2 + V)\psi(x,t)$$

then it is easy to see that the time reversed state $\psi(x,-t)$ is not a solution because of the first order time derivative. However, it is easy to check that $\psi^*(x,-t)$ is a solution by complex conjugation:

$$i\frac{\partial \psi^*}{\partial t}(x,-t) = (-\frac{1}{2m}\nabla^2 + V)\psi^*(x,-t)$$

Thus we can conjecture that the time reversal has some thing to do with complex conjugation.

Another way of looking at this is to preserve the probability invariant under time reversal. Following Wigner we may then require

$$\langle\psi|\psi\rangle = \langle T\psi|T\psi\rangle$$

There are two ways of achieving this which is obvious if we look at two different quantum states. We may have

$$\langle\phi|\psi\rangle = \langle T\phi|T\psi\rangle$$

as in ordinary transformations or

$$\langle\phi|\psi\rangle^* = \langle T\phi|T\psi\rangle$$

Since the first choice leads to the trouble mentioned above with respect to the dynamical equation, we may choose,

$$T\psi(x,t) = \psi^*(x,-t)$$

Therefore for any Hermitian operator O,

$$\langle\psi|O|\phi\rangle = \langle T\phi|TOT^{-1}|T\psi\rangle$$

Taking the absolute square gets rid of the complex conjugate problem and the probability remains invariant.

How do we choose TOT^{-1}? Here are some examples,

$$TxT^{-1} = x$$

$$TpT^{-1} = -p$$

etc.

Thus for any process $i \to f$

$$|M_{i \to f}| = |M_{f \to i}|$$

where M denotes the matrix element for a given transition. Thus the probability is the same if the initial and final states are reversed as it happens in any time reversal transformation. This is known as the **principle of detailed balance**. The physical cross-section however is not necessarily the same since the flux and finals state phase space are different. Using Fermi's golden rule the transition rates are given by

$$W_{i \to f} = \frac{2\pi}{\hbar} |M_{i \to f}|^2 \rho_f,$$

$$W_{f \to i} = \frac{2\pi}{\hbar} |M_{i \to f}|^2 \rho_i.$$

While the probabilities are the same the rates may be different since the density of states of the end products $\rho_{i,f}$ are not necessarily the same. These can be quite different depending on the masses and number of particles. This is how one reconciles the time reversal invariance with the law of entropy increase.

2.3.4 CPT theorem

While the discrete symmetries C,P and T appear to violated, the combined operation CPT is an exact symmetry. Any theory that is invariant under Lorentz transformations must have CPT symmetry- CPT theorem. There is no known violation of the CPT symmetry and is consistent with all known experimental observations. The theorem has many consequences:

1. Spin-Statistics theorem: The connection between the spin of the particle and its statistics- for example the spin half particles obey Fermi statistics where as the integer spin particles obey Bose-Einstein statistics.

2. Particles and anti-particles have identical masses and life times.

3. All internal quantum numbers of anti-particles are opposite to those of the particles.

2.4 Problems:

1. Find reasons that could forbid $\gamma \to \gamma\gamma$. What would happen if the photon had mass?

2. Can an electron and a positron annhilite to a single photon?

3. Consider the decay of the particle $\Delta \to \pi + N$, where the spin of the Δ particle is 3/2. Determine the parity of the Δ.

4. Consider the process $\mathrm{Co}^{60} \to \mathrm{Ni}^{60} + e^- + \bar{\nu}_e$. Show that

$$\langle \cos\theta \rangle = \left\langle \frac{\vec{S}.\vec{p}}{|\vec{S}||\vec{p}|} \right\rangle$$

is non-zero if parity is violated. Here S is the spin of the nucleus and p is the momentum of the electron.

Chapter 3

Hadrons and the Quark Model

During the 50's and 60's hundreds of hadrons, or strongly interacting particles, were discovered. The concept of "elementary" particle took a beating. The picture was dramatically simplified when it was realised that they could be organised in multiplets, which in turn could be understood in terms of combinations of elementary constituents called quarks. The quark model proposed by Gell-Mann accounts qualitatively for the masses of light hadrons in the region of 1-2 GeV mass range.

The hadrons are divided into two broad categories called mesons (integer spin) and baryons (half-odd integral spin with an additional quantum number called the baryon number). The following table summarises the low lying hadrons classified according to their spin and parity. We have already alluded to the isospin and strangeness before. As far as strong interactions are concerned both isospin and strangeness are conserved exactly. The table 3.1 also shows the assignment of isospin and strangeness quantum numbers. By inspection it is easy to see that there exists a relation between the charges of the particles and other quantum numbers:

$$Q = I_z + \frac{Y}{2} = I_3 + \frac{B+S}{2}, \tag{3.1}$$

where I_3 is the isospin projection, Y is the hypercharge which is the sum of baryon number and strangeness quantum number. This is the well known Gell-Mann-Nishijima relation. Infact the original assignments of quantum numbers were made using this relation as well[1].

The deliberate arrangement of mesons into groups of (8+1) and baryons into groups of (8) and (10) is suggestive of a classification scheme about which we will say more.

In quantum mechanics the degeneracy of eigenvalues is an indication of an underlying symmetry. From the table the following facts emerge:

- Isospin multiplets of the same J^P are almost exactly degenerate- for example (p,n), $\pi^{\pm,0}$. Thus

[1] With the discovery of new flavours or quantum numbers Gell-Mann Nishijima's original relation has been generalised to include an expanded list of particles

Particle	Mass(MeV)	J^P	Isospin	Strangeness
pseudoscalar Mesons: 8 + 1				
$\pi^{\pm,0}$	140	0^-	1	0
K^+, K^0	495	0^-	1/2	1
$\bar{K}^0, K^-$	495	0^-	1/2	-1
η^0	550	0^-	0	0
η'^0	960	0^-	0	0
vector Mesons: 8 + 1				
$\rho^{\pm,0}$	770	1^-	1	0
K^{*+}, K^{*0}	890	1^-	1/2	1
$\bar{K}^{*0}, K^{*-}$	890	1^-	1/2	-1
ω^0	780	1^-	0	0
ϕ^0	1020	1^-	0	0
spin 1/2 Baryons: 8				
p,n	940	$1/2^+$	1/2	0
Λ^0	1115	$1/2^+$	0	-1
$\Sigma^{\pm,0}$	1190	$1/2^+$	1	-1
$\Xi^{0,-}$	1315	$1/2^+$	1/2	-2
spin 3/2 Baryons: 10				
$\Delta^{++,+,0,-}$	1232	$3/2^+$	3/2	0
$\Sigma^{*\pm,0}$	1385	$3/2^+$	1	-1
$\Xi^{*0,-}$	1523	$3/2^+$	1/2	-2
Ω^-	1672	$3/2^+$	0	-3

Table 3.1: Hadrons and their properties

isospin symmetry is exact in strong interactions. The generators of isospin transformations commute with the Hamiltonian. Small mass differences among the multiplets may then be attributed to isospin breaking effects due to other interactions.

- The hadrons within each J^P group are approximately "degenerate" to varying degrees. Baryons are degenerate to within 30 percent, where as with mesons it would be questionable. In the following analysis we concentrate more on Baryons and discuss mesons only in passing.

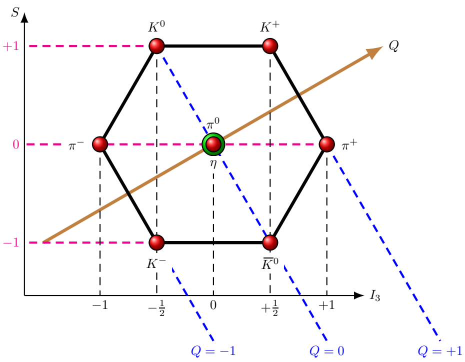
One can construct from the list given above sets of $I_3 - Y$ plots which will be identified with the weight diagrams of the SU(3) group later.

The SU(3) scheme outlined by Gell-Mann had dramatic prediction that Ω^- particle, which was then not yet discovered, should be there to complete the decuplet $J^P = 3/2^+$. Indeed it was found.

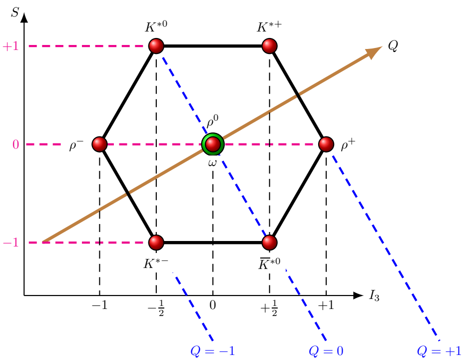
We note a couple of important points without details here:

(1) From the known experimental data on Baryon excited states only states with $I = 1/2, 3/2$ have been seen. This fact, as we shall see later, is crucial for the quark model where only the minimal three quarks are require to construct baryons. If $I = 5/2$ state is observed it would require minimum five quarks.

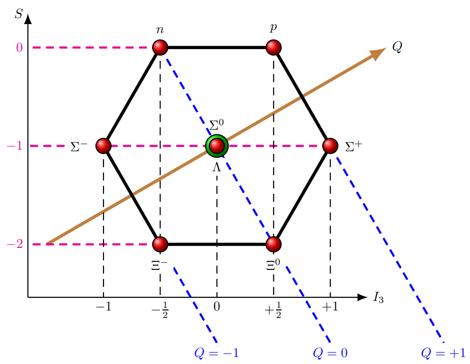
(2) The excitation spectra of N, Δ, Λ are approximately similar. Even though their constituents may be different combinations of various quarks, the approximate similarity indicates a certain universality of the confining potential- namely flavour independence.



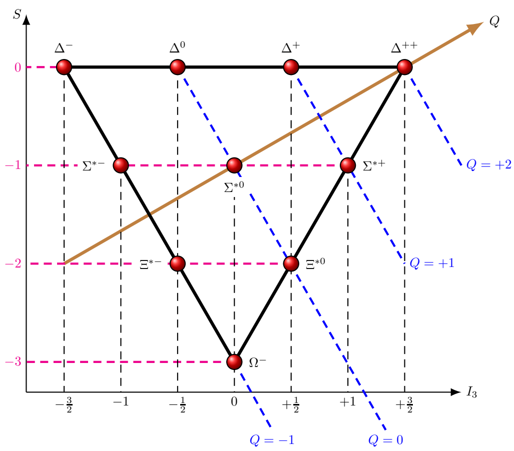
(a) The pseudo-scalar meson octet ($J^P = 0^-$).



(b) The vector meson octet ($J^P = 1^-$).



(c) The spin-$\frac{1}{2}$ baryon octet ($J^P = \frac{1}{2}^+$).



(d) The spin-$\frac{3}{2}$ baryon decuplet ($J^P = \frac{3}{2}^+$).

Figure 3.1: $SU(3)$ weight diagrams.

3.1 The quark model

With the list of "fundamental particles" increasing following the discovery of more excited states of particles in table 3.1, and new particles of even higher masses discovered, the question that whether all of them could be regarded as elementary or fundamental was looming large. The anomalous magnetic moments of nucleons also pointed to the existence of a substructure.

One feature we have noticed of the hadrons when arranged according to their J^P is that they come neatly arranged in various multiplets.

1. Baryons: $8(1/2^+) \oplus 10(3/2^+)$
2. Mesons: $9(0^-) \oplus 9(1^-)$

Gell-Mann and Zweig(1964) proposed that such a multiplet structure naturally arises when hadrons are thought of as composites of more fundamental objects- quarks which are again fermions with spin 1/2.

The minimal non-trivial configuration for generating Baryons, which are also fermions, is to bind three quarks (qqq). Each quark is assigned a baryon number 1/3 which ensures the fundamental baryons have unit baryon number. Note that the baryon number is additive like charge.

Mesons are composites of quark(B=1/3) and anti-quark(B=-1/3) pairs so that the baryon number of mesons is zero. Since the quarks have spin 1/2. mesons will necessarily have integral spin.

The next hypothesis introduced by Gell-Mann is that these quarks span the fundamental representation of the group SU(3) which has dimension 3 and the anti-quarks span the conjugate representation of dimension $\bar{3}$. These assumptions are sufficient to see the emergence of the hadron multiplet structure:

1. Mesons $(q\bar{q})$: $3 \otimes \bar{3} = 1 \oplus 8$
2. Baryons (qqq) : $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus \bar{8} \oplus 10$

where the right hand side shows the dimensionality of higher dimensional representations obtained as a direct sum of the irreducible representations (by taking the Kronecker product of the fundamental representation). The notation will be clarified later but the resemblence to the observed multiplet structure is clear.

While the above classification scheme is shown to work, the fundamental representation is never realised in nature leading to the notion of **quark confinement**. At this stage therefore the quarks merely serve as mnemonics for the classification of hadrons in which they have been permanantly bound. The recent evidence of the decay of the **top** quark in the D0 experiment in Fermilab has however provided the first solid evidence for the reality of quarks.

In the next few sections we consider simple examples using the spin analogy to clarify many group theoretical notions that are used here.

3.2 SU(2) - Spin and Isospin

To simplify the analysis some what we start with the non-strange hadrons. The only symmetry we have to use here is the isospin which is conserved. Hence the states have definite isospin labels. The non-strange baryons are arranged as follows:

- I=1/2 : p, n
- I=3/2 : $\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}$

Suppose $\psi_i^\alpha, \psi_j^\beta$ are basis vectors corresponding to two unitary irreducible representations of a compact Lie group, where α, β label the representation and i, j label vectors in each representations, the basis vectors (tensors) of the Kronecker product represenation are given by the product $\psi_i^\alpha \psi_j \beta$.

In general these need not form the basis of an irreducible representation. However, the basis of any irreducible representation contained in the product can be expanded interms of the product tensors. The coefficients of such an expansion are called Clebsch-Gordon coefficients generalising from the example of the rotation group where they were formulated first. For example,

$$\psi_k^\gamma = \sum_{i,j} C(\alpha, \beta, \gamma; i, j, k) \psi_i^\alpha \psi_j^\beta$$

where ψ_k^γ form the basis of an irreducible representation contained in the product.

Consider for example the D^j representation of the rotation group $R(3)$. The product is written as,

$$D^{j_1} \otimes D^{j_2} = D^{|j_1+j_2|} \oplus ... \oplus D^{|j_1-j_2|}$$

where each irreducible representation is characterised by well defined permutation symmetry. For example, the group of transformations on a spin 1/2 system is given by the representation $D^{1/2}$. For a system of two spin-half objects, we have

$$D^{1/2} \otimes D^{1/2} = D^1 \oplus D^0$$

which is simply a statement of the fact that the two spin half particles may be combined into a spin-1 or spin-0 system. In terms of dimensionalities this may also be written as,

$$2 \otimes 2 = 3 \oplus 1$$

We note that the representation $D^{1/2}$ defines the unitary irreducible representation of lowest dimension of the group SU(2). The above group theoretical statements may be illustrated easily by the following example. Consder explicitly the states of a spin half particle.

Let,

$$
\begin{aligned}
|\uparrow\rangle &= \left|S = \frac{1}{2}, S_z = +\frac{1}{2}\right\rangle \\
|\downarrow\rangle &= \left|S = \frac{1}{2}, S_z = -\frac{1}{2}\right\rangle
\end{aligned}
$$

be the basis vectors of the fundamental representation of SU(2) which is a group of Unitary-Unimodular 2×2 matrices. The product states are four in number,

$$
|\uparrow\uparrow\rangle, \quad |\uparrow\downarrow\rangle, \quad |\downarrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle .
$$

Except the first and the last others do not have definite symmetry under permutation. One may project these into states with definite permutation symmetry:

$$
\begin{aligned}
|1, +1\rangle &= |\uparrow\uparrow\rangle \\
|1, \quad 0\rangle &= \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \\
|1, -1\rangle &= |\downarrow\downarrow\rangle
\end{aligned}
$$

which is equivalent to the statement

$$
|1, m\rangle = \sum_{m_1, m_2} C\left(\frac{1}{2}, \frac{1}{2}, 1; m_1, m_2, m\right) \left|\frac{1}{2}, m_1\right\rangle \left|\frac{1}{2}, m_2\right\rangle
$$

which span the spin-1 representation of a combination of two spin-1/2 particles. Note that the representation is completely symmetric under the exchange of the two spins.

The other combination is antisymmetric and leads to the spin-0 representation of the two particle system.

$$
|0, 0\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}
$$

which is equivalent to the statement

$$
|0, 0\rangle = \sum_{m_1, m_2} C\left(\frac{1}{2}, \frac{1}{2}, 0; m_1, m_2, 0\right) \left|\frac{1}{2}, m_1\right\rangle \left|\frac{1}{2}, m_2\right\rangle
$$

Note that

$$
\begin{aligned}
J^2 |j, m\rangle &= j(j+1) |j, m\rangle \\
J_z |j, m\rangle &= m |j, m\rangle
\end{aligned}
$$

While combining two spin-1/2 objects it is sufficient to look at the symmetry properties of CG coefficinets to get the symmetry property of the state

$$C(j_1, j_2, j; m_1, m_2, m) = (-1)^{j_1+j_2-j} C(j_2, j_1, j; m_1, m_2, m)$$

Example of a physical system for two spin-1/2 objects is the deuteron.

3.2.1 A system of three spin-1/2 objects

Applying the CG theorem,

$$D^{1/2} \otimes D^{1/2} \otimes D^{1/2} = [D^1 \oplus D^0] \otimes D^{1/2} = D^{3/2} \oplus D^{1/2} \oplus D^{1/2}$$

or interms of multiplicities we have

$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus \bar{2}$$

Thus there are two spin 1/2 representations (distinguished by their permutation symmetry and one spin 3/2 representation.

The states that span these representations may be constructed explicitly:

- $$\begin{aligned} |3/2, m\rangle &= \sum_{m_1,m_2} C(1, 1/2, 3/2; m_1, m_2, m) \, |1, m_1\rangle \, |1/2, m_2\rangle \\ |3/2, 3/2\rangle &= |\uparrow\uparrow\uparrow\rangle \\ |3/2, 1/2\rangle &= \frac{|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle}{\sqrt{3}} \\ |3/2, -1/2\rangle &= \frac{|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle}{\sqrt{3}} \\ |3/2, -3/2\rangle &= |\downarrow\downarrow\downarrow\rangle \end{aligned}$$

  Collectively we refer to these states as χ_s and are explicitly symmetric.

- $$\begin{aligned} |1/2, m\rangle &= \sum_{m_1,m_2} C(1, 1/2, 1/2; m_1, m_2, m) \, |1, m_1\rangle \, |1/2, m_2\rangle \\ |1/2, 1/2\rangle &= \frac{2\,|\uparrow\uparrow\downarrow\rangle - (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\,|\uparrow\rangle}{\sqrt{6}} \\ |1/2, -1/2\rangle &= \frac{2\,|\downarrow\downarrow\uparrow\rangle - (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)\,|\downarrow\rangle}{\sqrt{6}} \end{aligned}$$

Collectively we call these states χ_λ. Note that these states are not symmetric or antisymmetric under exchange of spins. These are called Mixed-symmetry states- symmetric in 1-2 with no particular symmetry with respect the third spin.

- $$\begin{aligned} |1/2, m\rangle &= \sum_{m_1,m_2} C(0,1/2,1/2;0,m,m)\,|0,0\rangle\,|1/2,m\rangle \\ |1/2,1/2\rangle &= \frac{(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\,|\uparrow\rangle}{\sqrt{2}} \\ |1/2,-1/2\rangle &= \frac{(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)\,|\downarrow\rangle}{\sqrt{2}} \end{aligned}$$

  Collectively we call these states χ_ρ. These are again called Mixed-symmetry states- antisymmetric in 1-2 with no particular symmetry with respect the third spin.

The precise number of states in each representation correspond to the multiplicities obtained from the CG theorem.

3.2.2 Combining Isospin states

We may carry out the same excercise in the isospin space. The rotations in isospin space are analogous to the rotations in the spin space. The fundamental group is again SU(2) and is spanned by two vectors **u** and **d** referring to the up and down quark states. Analogy with spin is clear once we identify $|\uparrow\rangle \rightarrow |u\rangle$ and $|\downarrow\rangle \rightarrow |d\rangle$. The construction of states in the isospin space then proceeds the same way as in the spin space.

By analogy with spin the u-quark has $I = 1/2, I_3 = 1/2$ and the d-quark has $I = 1/2, I_3 = -1/2$. All the quarks carry spin-1/2 and are fermions under permutation symmetry.

Using the Gell-Mann - Nishijima formula the charges of the quarks may be obtained as follows:

$$Q_u = I_3 + (B+S)/2 = 2/3$$

$$Q_d = I_3 + (B+S)/2 = -1/3$$

since strangeness S=0 and Baryon number B=1/3 for u and d quarks by definition. Thus the quarks carry fractional charges.

Following table summarises the states of two isospin 1/2 particles: Obviously no such di-quark systems

I, I_3	State	Charge Q
I=1 Triplet:		
1,1	uu	4/3
1,0	$\frac{ud+du}{\sqrt{2}}$	1/3
1,-1	dd	-2/3
I=0 Singlet:		
0,0	$\frac{ud-du}{\sqrt{2}}$	1/3

I, I_3	State	Charge Q	Baryon
I=3/2 Δ ϕ_s			
3/2,3/2	uuu	2	Δ^{++}
3/2,1/2	$\frac{uud+duu+udu}{\sqrt{3}}$	1	Δ^{+}
3/2,-1/2	$\frac{udd+dud+ddu}{\sqrt{3}}$	0	Δ^{0}
3/2,-3/2	ddd	-1	Δ^{-}
I=1/2 Nucleon doublet:ϕ_λ			
1/2,1/2	$\frac{2uud-(ud+du)u}{\sqrt{6}}$	1	p
1/2,-1/2	$\frac{2ddu-(ud+du)d}{\sqrt{6}}$	1	n
I=1/2 Nucleon doublet:ϕ_ρ			
1/2,1/2	$\frac{(ud-du)u}{\sqrt{2}}$	1	p
1/2,-1/2	$\frac{(ud-du)d}{\sqrt{2}}$	1	n

with non-integral charges appearr in nature. However we need the above construction to construct systems of three I=1/2 particles. Using the spin analogy the following table summarises the system of three quarks (qqq) which will be identified with the baryon states.

3.2.3 Spin-Isospin States of definite symmetry

The spin-isospin state of the Δ particle with S=I=3/2 is given by

$$|\Delta\rangle = \chi_s \phi_s$$

However the nucleon states with S=I=1/2 have many possible combinations which have the same quantum numbers as the proton and the neutron, infact too many for comfort since there are exactly two members of the doublet that we should extract.

$$\chi_\rho\phi_\rho, \chi_\rho\phi_\lambda, \chi_\lambda\phi_\lambda, \chi_\lambda\phi_\rho$$

So we have four instead two states. But none of these states has a well defined symmetry or antisymmetry under permutations, while the Δ is completely symmetric under spin as well as isospin indices.

If we demand a completely symmetry under exchange as in the case of Delta states then one gets the following combination:

$$|N\rangle = \frac{\chi_\rho\phi_\rho + \chi_\lambda\phi_\lambda}{\sqrt{2}}$$

On the otherhand a completely antisymmetric state would have the combination

$$|N\rangle = \frac{\chi_\rho \phi_\lambda - \chi_\lambda \phi_\rho}{\sqrt{2}}$$

where $N = p, n$ depending upon the isospin projection of ϕ state.

In nuclear three body problem the nuclei $He^3(ppn)$ and $H^3(pnn)$ play the roles analogous to that of proton(uud) and neutron(ddu). The choice of the particular combination of the spin-isospin state is dictated by the fact that the state of a system of fermions must be antisymmetric in all indices. Since in the ground state wave function of these two nuclei (L=0) is completely symmetric, one choses the antisymmetric wave function given above. The ground state static properties are well reproduced by such a combination. Thus it might seem that there is an unambigious choice for the Nucleon from the above two choices. However, the delta states given above are completely symmetric under spin-isospin indices. *The question therefore hangs on the fate of the Spin-Statistics Theorem.* We will address this issue next.

3.2.4 Spin-Statistics Problem: Origin of colour

Consider the state of Δ particle. As remarked before the spin-isospin state of this particle is completely symmetric under permutations. Its $J^P = 3/2^+$ and hence it is even under parity. It is also the ground state of the $I = 3/2$ state. Quantum mechanics tells us that the ground state of any system with even parity must be spacially symmetric under permutations. For example the ground states of the hydrogen molecule, Helium and Oxygen nuclei, etc. Thus we find ourselves in the piquant situation where the Δ state is a completely symmetric in $space \otimes spin \otimes isospin$ coordinates.

The spin-statistics theorem tells us that a state of a system of ferrmions has to be completely anti-symmetric. Thus we encounter a paradoxical situation that spin-statistics theorem may not hold for the Δ states in particular[2].

A way out of this dilemma is to introduce a new quantum number called **Colour**. Thus each quark (u or d) comes in three colours and the wave function of the baryons is completely antisymmetric in the colour space. Thus all baryons have

$$B_{colour} = \epsilon_{ijk} q_i q_j q_k$$

where $i, j, k = red, green, blue$, the three colours (you may take 1,2,3 for the indices). The full wave function of the Delta state is then given by,

$$|\Delta\rangle = \epsilon_{ijk} q_i q_j q_k [\psi_{space} \chi_s \phi_s]$$

[2] Historically many solutions wer proposed- Parastatistics by Greenberg and coloured quarks with integral charge called the Han-Nambu model. But the experimental evidence is firmly against these proposals

which is on the whole an antisymmetric state. One may wonder if the above decomposition smells of non-relativistic quantum mechanics which may not be wholly valid for quarks since their masses are not very large. Indeed the situation with nucleons will clarify this issue further.

We may now extend the arguement given above for the nucleon states also. As we have seen there are two combinations available for nucleons:

$$|N\rangle = \frac{\chi_\rho \phi_\rho + \chi_\lambda \phi_\lambda}{\sqrt{2}}$$

$$|N\rangle = \frac{\chi_\rho \phi_\lambda - \chi_\lambda \phi_\rho}{\sqrt{2}}$$

combined from the mixed symmetry states of spin and isospin. Once again we assume the spacial part is symmetric since both nucleon form the ground state of the $J^P = 1/2^+$ spectrum of baryons. Since the second combination is completely antisymmetric, it may seem as though we do not have the spin-statistics problem. However, since the quarks have to be coloured in order to preserve the antisymmetry of the Δ states, it is natural to choose the symmetric states and impose antisymmetry condition by invoking colour. Thus we choose the nucleon states to be,

$$|N\rangle = \epsilon_{ijk} q_i q_j q_k \frac{\chi_\rho \phi_\rho + \chi_\lambda \phi_\lambda}{\sqrt{2}}$$

which is now completely antisymmetric.

An even stronger evidence of the choice of the combinations given above for nucleons, hence for colour, actually comes from the experimental measurement of the static magnetic moment of the nucleons. We discuss this below.

The experimental data on the neutron and proton magnetic moments gives,

$$\frac{\mu_n = -1.91}{\mu_p = 2.79} = -0.685$$

The corresponding magnetic moment operator in terms of the basic quark operators is given by,

$$M_z = \sum_{i=1}^{3} \mu \sigma_{iz} e_i$$

where μ is the unit of quark magnetic moment which we keep arbitrary since we do not know this. e_i is the charge of the i-th quark and σ_{iz} is the z-component of the Pauli spin vector $\vec{\sigma}$. We are therefore interested in evaluating

$$\mu_{n,p} =< N = n, p|M_z|N = n, p >$$

Note that the operator involves only the spin and isospin operators. We concentrate only this part of the wave-function. Because these states of the nucleon are either fully symmetric or antisymmetric we have the identity,

$$\mu_{n,p} = 3\mu\langle N = n, p|e_3\sigma_{3z}|N = n, p\rangle$$

The matrix elements in the spin space are given by,

$$\begin{aligned}
\langle\chi_\rho|\sigma_{3z}|\chi_\rho\rangle &= 1 \\
\langle\chi_\rho|\sigma_{3z}|\chi_\lambda\rangle &= 0 \\
\langle\chi_\lambda|\sigma_{3z}|\chi_\lambda\rangle &= -1/3
\end{aligned}$$

Similarly in the isospin space we have for protons

$$\begin{aligned}
\langle\phi^p_\rho|e_3|\phi^p_\rho\rangle &= 2/3 \\
\langle\phi^p_\rho|e_3|\phi^p_\lambda\rangle &= 0 \\
\langle\phi^p_\lambda|e_3|\phi^p_\lambda\rangle &= 0
\end{aligned}$$

and for neutrons

$$\begin{aligned}
\langle\phi^n_\rho|e_3|\phi^n_\rho\rangle &= -1/3 \\
\langle\phi^n_\rho|e_3|\phi^n_\lambda\rangle &= 0 \\
\langle\phi^n_\lambda|e_3|\phi^n_\lambda\rangle &= 1/3
\end{aligned}$$

Substituting these in the spin-isospin wave functions of the neutron and proton we have,

$$\begin{aligned}
\mu_n &= -2\mu/3 \\
\mu_p &= \mu
\end{aligned}$$

and therefore the ratio is given by,

$$\frac{\mu_n}{\mu_p} = -2/3$$

whereas the experimental value is given by -0.685 which is in excellant agreement considering the crude assumptions made. On the other hand if we had chosen the antisymmetric combination in the spin-isospin space disregarding the colour hypothesis, we would have obtained,

$$\frac{\mu_n}{\mu_p} = -2$$

in contradiction with experiment. Thus we have now evidence for colour from two independent approachesthe spin-statistics theorem and the experimental data on the static magnetic moments of the neutron and proton. Note that we did not need to fix μ the basic unit of magnetic moment of the quarks- it just cancelled out in the ratios.

3.2.5 Constituent Quarks

The ratio of the magnetic moments as calculated before does not fix the unit of the quark magnetic moment. As in the case of the electron if we assume that the Dirac magnetic moment of the quarks to be given by the expressions:

$$\mu_u = \frac{e_u}{2m_u} = \frac{2\mu}{3} \quad \mu_d = \frac{e_d}{2m_d} = \frac{-\mu}{3}$$

Assuming $m = m_u = m_d$ we have for the proton magnetic moment

$$\mu_p = 2.79\frac{e}{2M_P} = \frac{e}{2m}$$

where m is the quark mass, we immediately get,

$$m = \frac{M_p}{2.79} = 336MeV$$

This mass is often referred to as the constituent quark mass. Unlike the mass of the electron which enters the QED Lagrangian as a fundamental quantity, the constituent quark mass has no firm theoretical basis except to define a scale for discussing the low energy and static properties of the nucleon.

3.2.6 Other evidences for colour

We conclude this discussion with few more remarks on the colour quantum number: Some of the strongest evidence for colour comes from experiments. Consider the following ratio which is now experimentally measured:

$$\frac{\sigma(e^+e^- \to hadrons(q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

which is the ratio of the total cross-sections for electron-positron annihilation to either quarks or muons. Typically such a total cross-section is obtained by summing over all the final states. Thus in the numerator one sums over all the spin-isospin (around 1 GeV. At higher energies one has to sum over other quarks as well) states and in the denominator we sum over the spin states of the muons. If quarks come in three colours, one needs to sum over these as well. As it turns out merely summing over spin and flavours underestimates the ratio by a factor close to three suggesting the existence of an extra degree of freedom. Imposing the requirement that the quarks come in three colours solves this puzzle as well.

The strongest evidence to date comes from the following decay:

$$\pi^0 \to \gamma\gamma$$

It is some what complicated to discuss this case without a background in quantum field theory. It suffices to say that the π decay to two photons proceeds through the mediation of quarks. Once the amplitude is obtained by summing over all quark states. Without imposing the colour degree of freedom, the decay amplitude is underestimated by a factor of 3, and hence the rate by a factor of 9. Including colour the calculated decay rate agrees with experiments within errors.

3.3 SU(3) Flavour States

We have constructed states of non-strange baryons using the SU(2) isospin doublet of quarks (u,d). Extending these arguements to construct hadrons using the triplet of quarks (u,d,s) is straight-forward if more cumbersome. We shall mention briefly how the hadron octets and decuplets mentioned in the beginning of this section are obtained using three basic quark **flavours**

Regarding the triplet (u,d,s) as the basis spanning the fundamental representation of SU(3), we can combine any two of them first. There are nine such combinations which may be arranged as

$$3 \otimes 3 = 6 \oplus 3$$

using the expansion of Kronecker product. Explicitly these di-quark states can be written as

$$uu, dd, ss, \frac{ud + du}{\sqrt{2}}, \frac{us + su}{\sqrt{2}}, \frac{sd + ds}{\sqrt{2}}$$

which are 6 completely symmetric states and

$$\frac{ud - du}{\sqrt{2}}, \frac{us - su}{\sqrt{2}}, \frac{sd - ds}{\sqrt{2}}$$

which are 3 completely antisymmetric states.

Similarly combining three quarks we obtain,

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

where the representation with dimension 10 is completely symmetric given by,

$$uuu, ddd, sss, (uud)_{sym}, (uus)_{sym}, (udd)_{sym}, (sdd)_{sym}, (ssu)_{sym}, (ssd)_{sym}, (uds)_{sym}$$

where $(uud)_{sym}$ means a completely symmetric arrangement of (uud) etc. These quark states correspond to the spin 3/2 decuplet representation of the baryons.

The singlet under SU(3) with dimensionality 1 is the completely antisymmetric combination of (uds) quarks. The two octets are mixed symmetry representations.

Thus we could generate the weight diagrams of SU(3) analogous to the Gell-Mann's scheme for hadrons interms of their quark contents.

Combining these states with states of definite spin proceeds as in the case of combining isospin and spin states.

Appendix: Introduction to SU(2) and SU(3)

In general SU(N) is a group of $N \times N$ unitary unimodular matrices.

$$UU^\dagger = 1, \quad det(U) = 1$$

In general we may therefore write,

$$U = \exp{(i\theta_a T_a)}, \quad a = 1, ..., N^2 - 1$$

where θ_a are the parameters of the group and T_a are the hermitian(because the elements are unitary) generators of the group.

The generators obey the following properties:

$$\begin{aligned} \text{Trace}(T_a) &= 0 \\ \text{Trace}(T_a T_b) &= \delta_{ab} \end{aligned}$$

and

$$[T_a, T_b] = i f_{abc} T_c$$

which defines the algebra of the generators completely.

SU(2) is the group of 2×2 unitary unimodular matrices. It is also the lowest dimensional nontrivial representation of the rotation group. The generators may be chosen to be

$$T_a = \frac{1}{2}\sigma_a; \quad a = 1, 2, 3$$

where σ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.2}$$

The basis for this representation is conventionally chosen to be the eigenvectors of σ_3 that is the column vectors,

$$|1/2, 1/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1/2, -1/2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3}$$

which describe a spin-1/2 particle with the projection $m = 1/2, -1/2$ respectively. As we have seen this fundamental representation of SU(2) may be combined to build higher dimensional representation corresponding to the spins $J = 1, 3/2, 2, ...$ etc. Note that there is only one diagonal generator. In general for SU(N) there can atmost be $N - 1$ diagonal generators which is known as the rank of the group. The rank of the group is also equal to the number of Casimir operators- the states that span the representation are eigenstates of this operator. For example the Casimir operator of the SU(2) is J^2. The states are simultaneous eigenstates of J^2 and J_z.

The group SU(3) is the group of 3×3 unitary unimodular matrices. The generators may be chosen to be

$$T_a = \frac{1}{2}\lambda_a; \quad a = 1, ...8$$

where λ are given by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.4}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \tag{3.5}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} /\sqrt{3} \tag{3.6}$$

We note a few points here:

- The generators T_1, T_2, T_3 generate an SU(2) subgroup of SU(3) and the algebra of these generators closes among themselves.

- The diagonal generators commute among themselves.

$$[H_i, H_j] = 0$$

hence the algebra is closed. The diagonal generators define a subalgebra called the Cartan subalgebra. The elements of this subalgebra are $m = N - 1$ in number where m is the rank of the group. All states in a representation D are labelled by the eigenvalues of H_i such that

$$\{H_i\} |\ \rangle = \{\mu_i\} |\ \rangle$$

and $\vec{\mu}_i = \{\mu_i\}$ is called the weight vector.

For the group SU(3) we have chosen $H_1 = \lambda_3/2, H_2 = \lambda_8/\sqrt{3}$. The eigenvectors may be chosen to be,

$$|1/2, 1/3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |{-1/2}, 1/3\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |0, -2/3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.7}$$

We may easily identify the quantum numbers of these states with isospin and hypercharge of u(1/2,1/3), d(-1/2, 1/3) and s(0,-2/3) quarks. Thus the three quarks u,d and s form the basis of the fundamental representation of SU(3).

3.3.1 Conjugate representation

Suppose T_a are generators of some representation D of the group, then

$$[T_a, T_b] = i f_{abc} T_c$$

and $-T_a^*$ also satisfy the same algebra

$$[T_a^*, T_b^*] = i f_{abc} T_c^*$$

Therefore $-T_a^*$ also generate a representation $\bar{D}$ of the same dimension. The states are again eigenstates of the diagonal generators of the group. Thus we have, for example,

$$D \to \bar{D}$$

$$H_1 \to -H_1, \quad H_2 \to -H_2$$

Under this change,

$$u = |1/2, 1/3\rangle \to \bar{u} = |-1/2, -1/3\rangle$$

$$d = |-1/2, 1/3\rangle \to \bar{d} = |1/2, -1/3\rangle$$

$$s = |0, -2/3\rangle \to \bar{s} = |0, 2/3\rangle$$

interms of flavour states of SU(3). Note that in the conjugate representation all the charges (hyper) are reversed.

Thus if we choose the vectors that span the fundamental representation of SU(3) as quarks, the vectors that span the conjugate representation are anti-quarks. Indeed while there were many choices for the fundamental group for three quarks like O(3), SO(3), SU(3) became a natural choice since its representations are not real unlike SO(3).

3.4 Problems:

1. Explicitly construct the wavefunction of the Δ^{++} state which is completely antisymmetric.

2. Using isospin symmetry show that the transition rates for $\Delta \to \pi + N$ are in the following ratio:

$$\Delta^{++} \to p\pi^{+} : \Delta^{+} \to p\pi^{0} : \Delta^{++} \to p\pi^{-} = 3 : 2 : 1$$

3. Using isospin analysis show that $\rho^0 \to \pi^0\pi^0$ is forbidden.

4. Use isospin invariance to show that the reaction cross-section for $pp \to \pi^+ d$ is twice that of

$$np \to \pi^0 d$$