NOTES ON GENERAL RELATIVITY (GR) AND GRAVITY

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Abstract. These are notes on General Relativity (GR) and Gravity.

As of March 23, 2015, I find that the Central Lectures given by Dr. Frederic P. Schuller for the WE Heraeus International Winter School to be, unequivocally, the best, most lucid, and well-constructed lecture series on General Relativity and Gravity. Instead of reinventing the wheel, I write these notes to build upon and supplement the video lectures and tutorials already created by them. This includes my corrections, comments, relations to other aspects of theoretical physics, and code implementing calculations in GR.

It should be noted that for symbolic computation, I heavily use the SageManifolds v.0.7 package for Sage Math. My goal in this area is this: we see a concept or idea from GR and we go from the equation on the blackboard or textbook and into (Python/Sage Math) code that immediately computes a calculation.

I keep these notes available online, openly accessible, and free for anyone, anytime (with your (financial) help and contribution at Tilt/Open or Patreon, which is a subscription service). I want to keep these notes openly accessible because I want to encourage anyone to freely edit, copy, and make their own notes in the spirit of open-source software.

I continuously update these notes and post them here ernestyalumni.wordpress.com

The stated goal of the WE Heraeus International Winter School on Gravity and Light is to take the student from an introduction to the research frontier (cf. http://www.gravity-and-light.org/lectures). I want to get myself and other students or ambitious non-academic (maybe he or she is a working professional who had studied physics before in college, went to work in another field, maybe even, gasp, investment banking or mobile app developer, but still is curious and passionate about physics and want to contribute) equipped with all the tools available to do research, do calculations, to design experiments or collect data. Again, we're not here to reinvent the wheel. I'm not trying to make a General Relativity appreciation class, but this is a serious attempt towards training to do research.

Date: 23 mars 2015.

1991 *Mathematics Subject Classification.* General Relativity.

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Part 1. WE Heraeus International Winter School on Gravity and Light

Introduction (from EY)

The International Winter School on Gravity and Light held *central lectures* given by Dr. Frederic P. Schuller. These lectures on General Relativity and Gravity are unequivocally and undeniably, the best and most lucid and well-constructed lecture series on General Relativity and Gravity. The mathematical foundation from topology and differential geometry from which General Relativity arises from is solid, well-selected in rigor. The lectures themselves are well-thought out and clearly explained.

Even more so, the International Winter School provided accompanying Tutorial Sessions for each of the lectures. I had given up hopes in seeing this component of the learning process ever be put online so that anyone and everyone in the world could learn through the Tutorial process as well. I was afraid that nobody would understand how the Tutorial or "Office Hours" session was important for students to digest and comprehend and work out-doing exercises-the material presented in the lectures. This International Winter School gets it and shows how online education has to be done, to do it in an excellent manner, moving forward.

For anyone who is serious about learning General Relativity and Gravity, I would simply point to these video lectures and tutorials.

What I want to do is to build upon the material presented in this International Winter School. Why it's important to me, and to the students and practicing researchers out there, is that the material presented takes the student from an introduction to the research frontier. That is the stated goal of the International Winter School. I want to dig into and help contribute to the cutting edge in research and this entire program with lectures and tutorials appears to be the most direct and sensible route directly to being able to do research in General Relativity and Gravity. -EY 20150323

1. Lecture 1: Topology

1.1. Lecture 1: Topological Spaces.

Definition 1. Let M be a set.
A **topology** $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$, $\mathcal{P}(M)$ power set of M: set of all subsets of M. satisfying

(i) $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$

(ii) $U \in \mathcal{O}, \quad V \in \mathcal{O} \Longrightarrow U \bigcap V \in \mathcal{O}$

(iii) $U_\alpha \in \mathcal{O}, \quad \alpha \in \mathcal{A} \quad \Longrightarrow \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha \right) \in \mathcal{O}$

$\mathcal{O}\}$ utterly useless

Definition 2. $\mathcal{O}_{\text{standard}} \subseteq \mathcal{P}(\mathbb{R}^d)$

EY : 20150524

I'll fill in the proof that $\mathcal{O}_{\text{standard}}$ is a topology.

Proof. $\emptyset \in \mathcal{O}_{\text{standard}}$
since $\forall\, p \in \emptyset$, $\exists\, r \in \mathbb{R}^+$: $\mathcal{B}_r(p) \subseteq \emptyset$ (i.e. satisfied "vacuously")

Suppose $U, V \in \mathcal{O}_{\text{standard}}$.

Let $p \in U \bigcap V$. Then $\exists\, r_1, r_2 \in \mathbb{R}^+$ s.t. $\mathcal{B}_{r_1}(p) \subseteq U$
$\mathcal{B}_{r_2}(p) \subseteq V$

Let $r = \min\{r_1, r_2\}$.
Clearly $\mathcal{B}_r(p) \subseteq U$ and $\mathcal{B}_r(p) \subseteq V$. Then $\mathcal{B}_r(p) \subseteq U \bigcap V$. So $U \bigcap V \in \mathcal{O}_{\text{standard}}$.

Suppose, $U_\alpha \in \mathcal{O}_{\text{standard}}$, $\forall\, \alpha \in \mathcal{A}$.
Let $p \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. Then $p \in U_\alpha$ for at least 1 $\alpha \in \mathcal{A}$.
$\exists\, r_\alpha \in \mathbb{R}^+$ s.t. $\mathcal{B}_{r_\alpha}(p) \subseteq U_\alpha \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. So $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{O}_{\text{standard}}$ □

1.2. **2. Continuous maps.**

1.3. **3. Composition of continuous maps.**

1.4. **4. Inheriting a topology.** EY : 20150524

I'll fill in the proof that given f continuous (cont.), then the restriction of f onto a subspace S is cont. If you want a reference, check out Klaus Jänich [2, pp. 13, Ch. 1 Fundamental Concepts, Sec. Continuous Maps]

If cont. $f : M \to N$, $S \subseteq M$, then $f|_S$ cont.

Proof. Let open $V \subseteq N$, i.e. $V \in \mathcal{O}_N$ i.e. V in the topology $\mathcal{O}_N$ of N.

$$f|_S^{-1}(V) = \{m \in M | \; f|_S(m) \in V\}$$

Now $f^{-1}(V) = \{m \in M | f(m) \in V\}$.
So $f^{-1}(V) \bigcap S = f|_S^{-1}(V)$

Now f cont. So $f^{-1}(V) \in \mathcal{O}_N$.
and recall $\mathcal{O}_S| := \{U \bigcap S | U \in \mathcal{O}_M\}$.

so $f^{-1}(V) \bigcap S = f|_S^{-1}(V) \in \mathcal{O}_S$ i.e. $f|_S^{-1}(V)$ open.
$\Longrightarrow f|_S$ cont. □

Topology Tutorial Sheet

filename : `main.pdf`
The WE-Heraeus International Winter School on Gravity and Light: Topology

EY : 20150524

What I won't do here is retype up the solutions presented in the Tutorial (cf. `https://youtu.be/_XkhZQ-hNLs`): the presenter did a very good job. If someone wants to type up the solutions and copy and paste it onto this LaTeX file, in the spirit of open-source collaboration, I would encourage this effort.

Instead, what I want to encourage is the use of as much CAS (Computer Algebra System) and symbolic and numerical computation because, first, we're in the 21st century, second, to set the stage for further applications in research. I use Python and Sage Math alot, mostly because they are open-source software (OSS) and fun to use. Also note that the structure of Sage Math modules matches closely to Category Theory.

In checking whether a set is a topology, I found it strange that there wasn't already a function in Sage Math to check each of the axioms. So I wrote my own; see my code snippet, which you can copy, paste, edit freely in the spirit of OSS here, titled `topology.sage`:

gist github ernestyalumni topology.sage
Download topology.sage

Loading `topology.sage`, after changing into (with the usual Linux terminal commands, cd, ls) by

```
sage: load(''topology.sage'')
```

Exercise 2: Topologies on a simple set.

Question Does $\mathcal{O}_1 := \dots$ constitute a topology . . . ?.

Solution: Yes, since we check by typing in the following commands in Sage Math:

```
emptyset in O_1
Axiom2check(O_1) # True
Axiom3check(O_1) # True
```

Question What about $\mathcal{O}_2$ . . . ?.

Solution: No since the 3rd. axiom fails, as can be checked by typing in the following commands in Sage Math:

```
emptyset in O_2
Axiom2check(O_2) # True
Axiom3check(O_2) # False
```

2.

3.

4.

5.

6.

7. Lecture 7: Connections

$$\begin{aligned}
&\nabla_X f = Xf = (df)(X) \text{ but (not quite)} \\
&X : C^\infty(M) \to C^\infty(M) \\
&df : \Gamma(TM) \to C^\infty(M) \\
&\nabla_X : C^\infty(M) \to C^\infty(M)
\end{aligned}$$

$$
\begin{array}{ccc}
\nabla_X : C^\infty(M) & \longrightarrow & C^\infty(M) \\
\vdots\downarrow & & \vdots\downarrow
\end{array}
$$

$$
\nabla_X : \begin{array}{c} TM^p \otimes T^*M^q \text{ i.e.} \\ \binom{p}{q} \text{ tensor field} \end{array} \longrightarrow \begin{array}{c} TM^p \otimes T^*M^q \text{ i.e.} \\ \binom{p}{q} \text{ tensor field} \end{array}
$$

7.1. **Directional derivatives of tensor fields.** manifold with connection is quadruple $(M, \mathcal{O}, \mathcal{A}, \nabla)$

topology $\mathcal{O}$

atlas $\mathcal{A}$

Consider chart $(U, x) \in \mathcal{A}$

Definition 3. $\forall$ pair $(X, (p,q) - \text{tensor field}) \equiv (X, (p,q) - TF)$,

connection ∇ on smooth manifold $(M, \mathcal{O}, \mathcal{A})$

$\nabla : (X, (p,q) - TF) \to (p,q) - TF$ s.t.

(i) $\nabla_X f = Xf$

(ii) $\nabla_X (T + S) = \nabla_X T + \nabla_X S$

(iii)

$$\nabla_X (T(\omega, Y)) = (\nabla_X T)(\omega, T) + T(\nabla_X \omega, Y) + T(\omega, \nabla_X Y)$$

"Leibnitz" rule.

As

$$T \otimes S(\omega_{(1)} \dots \omega_{(p+r)}, Y_{(1)} \dots Y_{(q+s)}) = T(\omega_{(1)} \dots \omega_{(p)}, Y_{(1)} \dots Y_{(q)}) \cdot S(\omega_{(p+1)} \dots \omega_{(p+r)}, Y_{(q+1)} \dots Y_{(q+s)})$$

so

$$\nabla_X (T \otimes S) = (\nabla_X T) \otimes S + T \otimes \nabla_X S$$

(iv) $\nabla_{fX+Z} T = f \nabla_X T + \nabla_Z T$ C^∞-linear

7.2. **New structure on** $(M, \mathcal{O}, \mathcal{A})$ **required to fix** ∇**.** There are $(\dim M)^3$ many $\Gamma^i{}_{jk}$

$$
\begin{aligned}
&\Gamma^i{}_{jk} : U \to \mathbb{R} \\
&p \mapsto \left(dx^i (\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x^j}) \right)(p)
\end{aligned}
$$

Now $\nabla_{\frac{\partial}{\partial x^m}} (dx^i) = ?$

$$\underbrace{\nabla_{\frac{\partial}{\partial x^m}}(\underbrace{dx^i\left(\frac{\partial}{\partial x^j}\right)}_{\delta^i{}_j})}_{\| \quad \text{(iii)}} = \frac{\partial}{\partial x^m}(\delta^i{}_j) = 0$$

$$= \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i\right)\left(\frac{\partial}{\partial x^j}\right) + dx^i(\underbrace{\nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^j}}_{\Gamma^q{}_{jm}\frac{\partial}{\partial x^q}}) = 0$$

$$\Longrightarrow \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i\right)\left(\frac{\partial}{\partial x^j}\right) = -\Gamma^i_{jm}$$

$$\nabla_{\frac{\partial}{\partial x^m}} dx^i = -\Gamma^i_{jm} dx^j$$

Hence

$$(\nabla_X Y)^i = X(Y^i) + \Gamma^i_j \underbrace{m}_{\text{last entry goes in direction of } X} Y^j X^m$$

$$(\nabla_X \omega)_i = X(\omega_i) + -\Gamma^j_{im}\omega_j X^m$$

Note that for the immediately above expression for $(\nabla_X Y)^i$, in the second term on the right hand side, Γ^i_{jm} has the last entry at the bottom, m going in the direction of X, so that it matches up with X^m. This is a good mnemonic to memorize the index positions of Γ.

<u>summary so far</u>:

$$(\nabla_X Y)^i = X(Y^i) + \Gamma^i_{jm} Y^j X^m$$

$$(\nabla_X \omega)_i = X(\omega_i) + -\Gamma^j_{im}\omega_j X^m$$

similarly, by further application of Leibnitz

T a $(1,2)$-TF (tensor field)

$$(\nabla_X T)^i{}_{jk} = X(T^i{}_{jk}) + \Gamma^i{}_{sm} T^s{}_{jk} X^m - \Gamma^s{}_{jm} T^i{}_{sk} X^m - \Gamma^s{}_{km} T^i{}_{js} X^m$$

What is a Euclidean space:
$(M = \mathbb{R}^n, \mathcal{O}_{\text{st}}, \mathcal{A})$ smooth manifold.
Assume $(\mathbb{R}^n, \text{id}_{\mathbb{R}^n}) \in \mathcal{A}$ and

$$(\Gamma^i_{(x)})_{jk} = dx^i\left((\nabla_{\underline{\text{E}}})_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\right) \overset{!}{=} 0$$

7.3. **Change of Γ's under change of chart.** (U, x), $(V, y) \in \mathcal{A}$ and $U \cap V \neq \emptyset$

$$\Gamma^i_{jk}(y) := dy^i\left(\nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j}\right) = \frac{\partial y^i}{\partial x^q} dx^q \left(\nabla_{\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}} \frac{\partial x^s}{\partial y^j}\frac{\partial}{\partial x^s}\right)$$

Note ∇_{fX} is C^∞-linear for fX

covector dy^i is C^∞-linear in its argument

$$\Longrightarrow \Gamma^i_{jk}(y) = \frac{\partial y^i}{\partial x^q} dx^q \left(\frac{\partial x^p}{\partial y^k}\left[\left(\nabla_{\frac{\partial}{\partial x^p}} \frac{\partial x^s}{\partial y^j}\right)\frac{\partial}{\partial x^s} + \frac{\partial x^s}{\partial y^j}\left(\nabla_{\frac{\partial}{\partial x^p}} \frac{\partial}{\partial x^s}\right)\right]\right) =$$

$$= \frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}\frac{\partial x^s}{\partial y^j}\delta^q{}_s + \frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\frac{\partial x^s}{\partial y^j}\Gamma^q_{sp}(x)$$

$$\Gamma^i_{jk}(y) = \frac{\partial y^i}{\partial x^q}\frac{\partial^2 x^q}{\partial y^j \partial y^k} + \frac{\partial y^i}{\partial x^q}\frac{\partial x^s}{\partial y^j}\frac{\partial x^p}{\partial y^k}\Gamma^q_{sp}(x) \tag{7.1}$$

Eq. (7.1) is the change of connection coefficient function under the change of chart $(U \cap V, x) \to (U \cap V, y)$

7.4. **Normal Coordinates.**

Tutorial 7 Connections. Exercise 1. : True or false?

(a)

- $\nabla_{fX} Y = f \nabla_X Y$ by definition so $\nabla_{fX} = f \nabla_X$ i.e. ∇_X is $C^\infty(M)$-linear in X
- $f \in C^\infty(M)$ is a $(0,0)$-tensor field. $\nabla_X f = Xf \equiv X(f)$ by definition.
- If the manifold is flat, I'm assuming that means that the manifold is globally a Euclidean space, and by definition, $\Gamma = 0$.

$$\nabla_X Y = X^j \frac{\partial}{\partial x^j}(Y^i)\frac{\partial}{\partial x^i} + \Gamma^i_{\ jk} Y^k X^k \frac{\partial}{\partial x^i} = X^j \frac{\partial Y^i}{\partial x^j}\frac{\partial}{\partial x^i} + 0$$

and similarly for any (p,q)-tensor field, i.e.

$$\nabla_X T = X^j \frac{\partial T^{i_1 \dots i_p}_{j_1 \dots j_q}}{\partial x^j}$$

- $$\nabla_X f = X^j \frac{\partial f}{\partial x^j} = X \cdot \mathrm{grad}(f)$$

- $\forall\, (U,x) \in \mathcal{A}$, locally (after working out the first few cases, and doing induction, one can look up the expression for the local form; I found it in Nakahara's **Geometry, Topology and Physics**, Eq. 7.26, and it needs to be modified for the convention of order of bottom indices for Γ:

$$\nabla_\nu t^{\lambda_1 \dots \lambda_p}_{\mu_1 \dots \mu_q} = \partial_\nu t^{\lambda_1 \dots \lambda_p}_{\mu_1 \dots \mu_q} + \Gamma^{\lambda_1}_{\kappa\nu} t^{\kappa \lambda_2 \dots \lambda_p}_{\mu_1 \dots \mu_q} + \dots + \Gamma^{\lambda_p}_{\kappa\nu} t^{\lambda_1 \dots \lambda_{p-1}\kappa}_{\mu_1 \dots \mu_q} - \Gamma^{\kappa}_{\mu_1 \nu} t^{\lambda_1 \dots \lambda_p}_{\kappa \mu_2 \dots \mu_q} - \dots - \Gamma^{\kappa}_{\mu_q \nu} t^{\lambda_1 \dots \lambda_p}_{\mu_1 \dots \mu_{q-1}\kappa}$$

Clearly, ∇_X is uniquely fixed $\forall\, p \in M$ by choosing each of the $(\dim M)^3$ many connection coefficient functions Γ.

(b)

- $\nabla : \mathfrak{X}(M) \to \mathfrak{X}(M)$

  $\nabla : (p,q)$-tensor field $\mapsto (p,q)$-tensor field
- By definition, ∇ satisfies the Leibniz rule.
-
-
-

Exercise 2. : **Practical rules for how ∇ acts** Torsion-free covariant derivative boils down to a connection coefficient function Γ that is symmetric in the bottom indices.

- $$\nabla_X f = X(f) = X^i \frac{\partial f}{\partial x^i}$$
- $$(\nabla_X Y)^a = X^i \frac{\partial Y^a}{\partial x^i} + \Gamma^a_{\ jk} Y^j X^k$$
- $$(\nabla_X \omega)_a = X^i \frac{\partial \omega_a}{\partial x^j} - \Gamma^i_{\ ak} \omega_i X^k$$
- $$(\nabla_m T)^a_{\ bc} = \frac{\partial}{\partial x^m}(T^a_{\ bc}) + \Gamma^a_{\ im} T^i_{bc} - \Gamma^i_{\ bm} T^a_{ic} - \Gamma^j_{\ cm} T^a_{bj}$$
- $$(\nabla_{[m} A)_{n]} = (\nabla_m A)_n - (\nabla_n A)_m = \frac{\partial A_n}{\partial x^m} - \Gamma^i_{\ nm} A_i - \left(\frac{\partial A_m}{\partial x^n} - \Gamma^i_{\ mn} A_i \right) = \frac{\partial A_m}{\partial x^m} - \frac{\partial A_m}{\partial x^n}$$

- $$(\nabla_m \omega)_{nr} = \frac{\partial \omega_{nr}}{\partial x^m} - \Gamma^i_{nm}\omega_{ir} - \Gamma^i_{rm}\omega_{ni}$$

Exercise 3. : Connection coefficients

Question .

The connection coefficient functions Γ in chart $(U \cap V, y)$ is given, in terms of chart $(U \cap V, x)$ as follows:

Recall Eq. (7.1)

$$\Gamma^i_{jk}(y) = \frac{\partial y^i}{\partial x^q}\frac{\partial^2 x^q}{\partial y^j \partial y^k} + \frac{\partial y^i}{\partial x^q}\frac{\partial x^s}{\partial y^j}\frac{\partial x^p}{\partial y^k}\Gamma^q_{sp}(x)$$

8. Lecture 8: Parallel Transport & Curvature (International Winter School on Gravity and Light 2015)

8.1. Parallelity of vector fields.

Definition 4. (1) **parallely transported** along smooth curve $\gamma : \mathbb{R} \to M$

if

(8.1) $$\boxed{\nabla_{v_\gamma} X = 0}$$

(2) A slightly weaker condition

is "parallel"

$$(\nabla_{v_{\gamma,\gamma(\lambda)}} X)_{\gamma(\lambda)} = \mu(\lambda) X_{\gamma(\lambda)}$$

8.2. Autoparallely transported curves.

Definition 5. curve $\gamma : \mathbb{R} \to M$ is called
autoparallely transported if

(8.2) $$\nabla_{v_\gamma} v_\gamma \overset{!}{=} 0$$

8.3. Autoparallel equation.

$$\nabla_{v_\gamma} v_\gamma = 0$$

in summary:

(8.3) $$\ddot{\gamma}^m_{(x)}(\lambda) + (\Gamma^m_{(x)})_{ab}(\gamma(\lambda))\dot{\gamma}^a_{(x)}(\lambda)\dot{\gamma}^b_{(x)}(\lambda) = 0$$

8.4. Torsion.

Definition 6. torsion of a connection ∇ is the $(1,2)$-**tensor field**

(8.4) $$T(\omega, X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X, Y])$$

(Inside a cloud)

$[X, Y]$ vector field defined by

$$[X, Y]f := X(Yf) - Y(Xf)$$

Proof. check T is C^∞-linear in each entry

$$T(\omega, fX, Y) = \omega(\nabla_{fX} Y - \nabla_Y (fX) - [fX, Y])$$

□

Definition 7. A $(M, \mathcal{O}, \mathcal{A}, \nabla)$ is called torsion-free if $T = 0$

In a chart

$$\begin{aligned} T^i{}_{ab} := T\left(dx^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) &= dx^i(\dots) \\ &= \Gamma^i{}_{ab} - \Gamma^i{}_{ba} = 2\Gamma^i{}_{[ab]} \end{aligned}$$

From now on, in these lectures, we only use torsion-free connections.

8.5. **Curvature.**

Definition 8. **Riemann curvature** of a connection ∇ is the $(1,3)$-tensor field

$$\mathrm{Riem}(\omega, Z, X, Y) := \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \tag{8.5}$$

Proof. do it: C^∞-linear in each slot. □

Tutorials $\mathrm{Riem}^i{}_{jab} = \dots$

TUTORIAL 8 PARALLEL TRANSPORT & CURVATURE

Exercise 1.

Exercise 2. : Where connection coefficients appear

It was suggested in the tutorial sheets and hinted in the lecture that the following should be committed to memory.

Question : Recall the autoparallel equation for a curve γ.

(a)

$$\nabla_{v_\gamma} v_\gamma = 0$$

(b)

$$\nabla_{v_\gamma} v_\gamma = \nabla_{\dot{\gamma}\frac{\partial}{\partial x^\mu}} v_\gamma = \dot{\gamma}^\nu \nabla_{\partial_\nu} v_\gamma = \dot{\gamma}^\nu \left[\frac{\partial v^\mu_\gamma}{\partial x^\nu} + \Gamma^\rho_{\mu\nu} v^\mu_\gamma\right] \frac{\partial}{\partial x^\rho} = \dot{\gamma}^\nu \left[\frac{\partial \dot{\gamma}^\rho}{\partial x^\nu} + \Gamma^\rho_{\mu\nu}\dot{\gamma}^\mu\right]\frac{\partial}{\partial x^\rho} = 0$$

$$\Longrightarrow \boxed{\ddot{\gamma}^\rho + \Gamma^\rho_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu}$$

as, for example, for $F(x(t))$,

$$\frac{dF(x(t))}{dt} = \dot{x}\frac{\partial F}{\partial x} = \frac{d}{dt}F$$

so that

$$\dot{\gamma}^\nu \frac{\partial v^\mu_\gamma}{\partial x^\nu} = \frac{d}{d\lambda} v^\mu_\gamma = \frac{d^2}{d\lambda^2}\gamma^\mu$$

Question : Determine the coefficients of the Riemann tensor with respect to a chart (U, x).

Recall this manifestly covariant definition

$$\mathrm{Riem}(\omega, Z, X, Y) = \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)$$

We want $R^i{}_{jab}$.

now

$$\nabla_X \nabla_Y Z = \nabla_X((Y^\mu \frac{\partial}{\partial x^\mu} Z^\rho + \Gamma^\rho_{\mu\nu} Z^\mu Y^\nu)\frac{\partial}{\partial x^\rho}) = (X^\alpha \frac{\partial}{\partial x^\alpha}(Y^\mu \frac{\partial}{\partial x^\mu} Z^\rho + \Gamma^\rho_{\mu\nu} Z^\mu Y^\nu) + \Gamma^\rho_{\alpha\beta}(Y^\mu \frac{\partial}{\partial x^\mu} Z^\alpha + \Gamma^\alpha_{\mu\nu} Z^\mu Y^\nu) X^\beta)\frac{\partial}{\partial x^\rho}$$

For $X = \partial_a$, $Y = \partial_b$, $Z = \partial_j$, then the partial derivatives of the coefficients of the input vectors become zero.

$$\Longrightarrow \nabla_{\partial_a} \nabla_{\partial_b} \partial_j = \frac{\partial}{\partial x^a}(\Gamma^i_{jb}) + \Gamma^i_{\alpha a}\Gamma^\alpha_{jb}$$

Now

$$[X,Y]^i = X^j \frac{\partial}{\partial x^j} Y^i - Y^j \frac{\partial X^i}{\partial x^j}$$

For coordinate vectors, $[\partial_i, \partial_j] = 0 \; \forall \, i, j = 0, 1 \dots d$.

Thus

$$\boxed{R^i_{\ jab} = \frac{\partial}{\partial x^a}\Gamma^i_{jb} - \frac{\partial}{\partial x^b}\Gamma^i_{ja} + \Gamma^i_{\alpha a}\Gamma^\alpha_{jb} - \Gamma^i_{\alpha b}\Gamma^\alpha_{ja}}$$

Question :Ric$(X,Y) :=$ **Riem**$^m_{\ amb} X^a Y^b$ **define** $(0,2)$**-tensor?.**

Yes, transforms as such:

EY developments. I roughly follow the spirit in Theodore Frankel's **The Geometry of Physics: An Introduction** Second Ed. 2003, Chapter 9 Covariant Differentiation and Curvature, Section 9.3b. The Covariant Differential of a Vector Field. P.S. EY : 20150320 I would like a copy of the Third Edition but I don't have the funds right now to purchase the third edition: go to my tilt crowdfunding campaign, http://ernestyalumni.tilt.com, and help with your financial support if you can or send me a message on my various channels and ernestyalumni gmail email address if you could help me get a hold of a digital or hard copy as a pro bono gift from the publisher or author.

The spirit of the development is the following:

> "How can we express connections and curvatures in terms of forms?" -Theodore Frankel.

From Lecture 7, connection ∇ on vector field Y, in the "direction" X,

$$\nabla_{\frac{\partial}{\partial x^k}} Y = \left(\frac{\partial Y^i}{\partial x^k} + \Gamma^i_{jk} Y^j \right) \frac{\partial}{\partial x^i}$$

Make the ansatz (approche, impostazione) that the connection ∇ acts on Y, the vector field, first:

$$\nabla Y(X) = \left(X^k \frac{\partial Y^i}{\partial x^k} + \Gamma^i_{jk} Y^j X^k \right) \frac{\partial}{\partial x^i} = X^k \left(\nabla_{\frac{\partial}{\partial x^k}} Y \right)^i \frac{\partial}{\partial x^i} = (\nabla_X Y)^i \frac{\partial}{\partial x^i} = \nabla_X Y$$

Now from Lecture 7, Definition for Γ,

$$dx^i \left(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \right) = \Gamma^i_{jk}$$

Make this ansatz (approche, impostazine)

$$\nabla \frac{\partial}{\partial x^j} = \left(\Gamma^i_{jk} dx^k \right) \otimes \frac{\partial}{\partial x^i} \in \Omega^1(M, TM) = T^*M \otimes TM$$

where $\Omega^1(M,TM) = T^*M \otimes TM$ is the set of all TM or vector-valued 1-forms on M, with the 1-form being the following:

$$\Gamma^i_{jk} dx^k = \Gamma^i_{\ j} \in \Omega^1(M) \qquad \begin{aligned} i &= 1 \dots \dim(M) \\ j &= 1 \dots \dim(M) \end{aligned}$$

So $\Gamma^i_{\ j}$ is a $\dim M \times \dim M$ matrix of 1-forms (EY !!!).

Thus

$$\nabla Y = (d(Y^i) + \Gamma^i_j Y^j) \otimes \frac{\partial}{\partial x^i}$$

So the connection is a (smooth) map from TM to the set of all vector-valued 1-forms on M, $\Omega^1(M,TM)$, and then, after "eating" a vector Y, yields the "covariant derivative":

$$\begin{aligned} &\nabla : TM \to \Omega^1(M,TM) = T^*M \otimes TM \\ &\nabla : Y \mapsto \nabla Y \\ &\nabla Y : TM \to TM \\ &\nabla Y(X) \mapsto \nabla Y(X) = \nabla_X(Y) \end{aligned}$$

Now

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] f = \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial x^j} \right) - \frac{\partial}{\partial x^j} \left(\frac{\partial}{\partial x^i} \right) = 0$$

(this is okay as on $p \in (U,x)$; x-coordinates on same chart (U,x))

EY : 20150320 My question is when is this nontrivial or nonvanishing (i.e. not equal to 0).

$$[e_a, e_b] = ?$$

for a frame (e_c) and would this be the difference between a tangent bundle TM vs. a (general) vector bundle?

Wikipedia helps here. cf. wikipedia, "Connection (vector bundle)"

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) = \Omega^1(M,E)$$

$$\nabla e_a = \omega^c_{ab} f^b \otimes e_c$$

$f^b \in T^*M$ (this is the dual basis for TM and, note, this is for the manifold, M

$$\nabla_{f_b} e_a = \omega^c_{ab} e_c \in E$$

$$\omega^c_a = \omega^c_{ab} f^b \in \Omega^1(M)$$

is the connection 1-form, with $a, c = 1 \dots \dim V$. EY : 20150320 This V is a vector space living on each of the fibers of E. I know that $\Gamma(T^*M \otimes E)$ looks like it should take values in E, but it's meaning that it takes vector values of V. Correct me if I'm wrong: ernestyalumni at gmail and various social media.

Let $\sigma \in \Gamma(E)$, $\sigma = \sigma^a e_a$

$$\nabla \sigma = (d\sigma^c + \omega^c_{ab} \sigma^a f^b) \otimes e_c \text{ with}$$

$$d\sigma^c = \frac{\partial \sigma^c}{\partial x^b} f^b$$

$$\Longrightarrow \nabla_X \sigma = \left(X^b \frac{\partial \sigma^c}{\partial x^b} + \omega^c_{ab} \sigma^a X^b \right) e_c = X^b \left(\frac{\partial \sigma^c}{\partial x^b} + \omega^c_{ab} \sigma^a \right) e_c$$

9. Lecture 9: Newtonian spacetime is curved!

Axiom 1 (Newton I:)**.** *A body on which* no *force acts moves uniformly along a straight line*

Axiom 2 (Newton II:)**.** *Deviation of a body's motion from such uniform straight motion is effected by a force, reduced by a factor of the body's reciprocal mass.*

Remark:

(1) 1st axiom - in order to be relevant - must be read as a measurement prescription for the geometry of space ...

(2) Since gravity universally acts on every particle, in a universe with at least two particles, gravity must not be considered a force if Newton I is supposed to remain applicable.

9.1. **Laplace's questions.** Laplace $*1749$

$\dagger 1827$

Q: "Can gravity be encoded in a curvature of space, such that its effects show if particles under the influence of (no other) force we postulated to more along straight lines in this curved space?"

Answer: No!

Proof. gravity is a force point of view

$$m\ddot{x}^\alpha(t) = F^\alpha(x(t))$$
$$m\ddot{x}^\alpha(t) = \underbrace{mf^\alpha}_{F^\alpha}(x(t))$$

$-\partial_\alpha f^\alpha = 4\pi G\rho$ (Poisson)
ρ mass density of matter

(EY : 20150330) You know this, $F = Gm_1m_2/r^2$

$$\ddot{x}^\alpha(t) - f^\alpha(x(t)) = 0$$

Laplace asks: Is this $(\ddot{x}(t))$ of the form

$$\ddot{x}^\alpha(t) + \Gamma^\alpha_{\ \beta\gamma}(x(t))\dot{x}^\beta(t)\dot{x}^\gamma(t) = 0$$

Conclusion: One cannot find Γ s such that Newton's equation takes the form of an autoparallel.

$\square$

9.2. **The full wisdom of Newton I.** use also the information from Newton's first law that particles (no force) move uniformly

introduce the appropriate setting to talk about the difference easily

insight: in spacetime $\boxed{\text{uniform \& straight motion}}$ is simply straight motion

So let's try in spacetime:

let $x : \mathbb{R} \to \mathbb{R}^3$

be a particle's trajectory in space $\longleftrightarrow$ worldline (history) of the particle
$$\begin{aligned} &X : \mathbb{R} \to \mathbb{R}^4 \\ &t \mapsto (t, x^1(t), x^2(t), x^3(t)) := \\ &:= (X^0(t), X^1(t), X^2(t), X^3(t)) \end{aligned}$$

That's all it takes:

Trivial rewritings:

$$\dot{X}^0 = 1$$

$$\Longrightarrow \boxed{\begin{aligned} &\ddot{X}^0 &&= 0 \\ &\ddot{X}^\alpha - f^\alpha(X(t)) \cdot \dot{X}^0 \cdot \dot{X}^0 &&= 0 \end{aligned}} \quad (\alpha = 1,2,3) \Longrightarrow \underset{a=0,1,2,3}{\boxed{\ddot{X}^a + \Gamma^a_{\ bc}\dot{X}^b\dot{X}^c = 0}}$$

antoparallel eqn in spacetime

Yes, choosing $\Gamma^0{}_{ab} = 0$

$$\Gamma^\alpha{}_{\beta\gamma} = 0 = Gamma^\alpha{}_{0\beta} = \Gamma^\alpha{}_{\beta 0}$$

only: $\boxed{\Gamma^\alpha{}_{00} \stackrel{!}{=} -f^\alpha}$

Question: Is this a coordinate-choice artifact?

No, since $R^\alpha{}_{0\beta 0} = -\frac{\partial}{\partial x^\beta} f^\alpha$ (only non-vanishing components) (tidal force tensor, $-$ the Hessian of the force component)

Ricci tensor $\Longrightarrow R_{00} = R^m{}_{0m0} = -\partial_\alpha f^\alpha = 4\pi G\rho$

Poisson: $-\partial_\alpha f^\alpha = 4\pi G \cdot \rho$

writing: $T_{00} = \frac{1}{2} s$

$$\Longrightarrow \boxed{R_{00} = 8\pi G T_{00}}$$

Einstein in 1912 $\boxed{\cancel{R_{ab} = 8\pi G T_{ab}}}$

Conclusion: Laplace's idea works in spacetime

Remark

$$\Gamma^\alpha{}_{00} = -f^\alpha$$

$$R^\alpha{}_{\beta\gamma\delta} = 0 \qquad \alpha, \beta, \gamma, \delta = 1, 2, 3$$

$$\boxed{R_{00} = 4\pi G\rho}$$

Q: What about transformation behavior of LHS of

$$\underbrace{\ddot{x}^a + \Gamma^a{}_{bc} \dot{X}^b \dot{X}^c}_{\underbrace{(\nabla_{v_X} v_X)^a}_{:= a^a \text{ ``acceleration vector''}}} = 0$$

9.3. **The foundations of the geometric formulation of Newton's axiom.** new start

Definition 9. A **Newtonian spacetime** is a quintuple

$$(M, \mathcal{O}, \mathcal{A}, \nabla, t)$$

where $(M, \mathcal{O}, \mathcal{A})$ 4-dim. smooth manifold

$$t : M \to \mathbb{R} \text{ smooth function}$$

(i) "There is an absolute space"

$$(dt)_p \neq 0 \qquad \forall\, p \in M$$

(ii) "absolute time flows uniformly"

$$\nabla dt \underbrace{=}_{\text{space of } (0,2)\text{-tensor fields}} 0 \qquad \text{everywhere}$$

∇dt is a $(0, 2)$-tensor field

(iii) add to axioms of Newtonian spacetime $\nabla = 0$ torsion free

Definition 10. absolute space at time τ

$$S_\tau := \{p \in M | t(p) = \tau\}$$

$$\xrightarrow{dt \neq 0} M = \coprod S_\tau$$

Definition 11. A vector $X \in T_p M$ is called

(a) future-directed if

$$dt(X) > 0$$

(b) spatial if

$$dt(X) = 0$$

(c) past-directed if

$$dt(X) < 0$$

picture

Newton I: The worldline of a particle under the influence of no force (gravity isn't one, anyway) is a future-directed autoparallel i.e.

$$\nabla_{v_X} v_X = 0$$
$$dt(v_X) > 0$$

Newton II:

$$\nabla_{v_X} v_X = \frac{F}{m} \Longleftrightarrow m \cdot a = F$$

where F is a spatial vector field:

$$dt(F) = 0$$

Convention: restrict attention to atlases $\mathcal{A}_{\text{stratefied}}$ whose charts $(\mathcal{U}, x)$ have the property

$$\begin{array}{l} x^0 : \mathcal{U} \to \mathbb{R} \\ x^1 : \mathcal{U} \to \mathbb{R} \\ \vdots \quad \vdots \\ x^3 \end{array} \qquad x^0 = t|_{\mathcal{U}} \qquad \Longrightarrow \qquad \begin{array}{l} 0 \overset{\text{"absolute time flows uniformly"}}{=} \nabla dt \\ 0 = \nabla_{\frac{\partial}{\partial x^a}} dx^0 = -\Gamma^0{}_{ba} \qquad a = 0,1,2,3 \end{array}$$

Let's evaluate in a chart $(\mathcal{U}, x)$ of a stratified atlas $\mathcal{A}_{\text{sheet}}$: Newton II:

$$\nabla_{v_X} v_X = \frac{F}{m}$$

in a chart.

$$(X^0)'' + \cancel{\Gamma^0{}_{cd}(X^a)'(X^b)'}^{\text{stratified atlas}} = 0$$
$$(X^\alpha)'' + \Gamma^\alpha_{\gamma\delta} X^{\gamma'} X^{\delta'} + \Gamma^\alpha_{00} X^{0'} X^{0'} + 2\Gamma^\alpha_{\gamma 0} X^{\gamma'} X^{0'} = \frac{F^\alpha}{m} \qquad \alpha = 1,2,3$$

$$\Longrightarrow (X^0)''(\lambda) = 0 \Longrightarrow X^0(\lambda) = a\lambda + b \qquad \text{constants } a, b \text{ with}$$
$$X^0(\lambda) = (x^0 \circ X)(\lambda) \overset{\text{stratified}}{=} (t \circ X)(\lambda)$$

convention parametrize worldline by absolute time

$$\frac{d}{d\lambda} = a \frac{d}{dt}$$
$$a^2 \ddot{X}^\alpha + a^2 \Gamma^\alpha_{\gamma\delta} \dot{X}^\gamma \dot{X}^\delta + a^2 \Gamma^\alpha_{00} \dot{X}^0 \dot{X}^0 + 2\Gamma^\alpha_{\gamma 0} \dot{X}^\gamma \dot{X}^0 = \frac{F^\alpha}{m}$$
$$\Longrightarrow \underbrace{\ddot{X}^\alpha + \Gamma^\alpha_{\gamma\delta} \dot{X}^\gamma \dot{X}^\delta + \Gamma^\alpha_{00} \dot{X}^0 \dot{X}^0 + 2\Gamma^\alpha_{\gamma 0} \dot{X}^\gamma \dot{X}^0}_{a^\alpha} = \frac{1}{a^2} \frac{F^\alpha}{m}$$

10. Lecture 10: Metric Manifolds

We establish a structure on a smooth manifold that allows one to assign vectors in each tangent space a length (and an angle between vectors in the same tangent space).

From this structure, one can then define a notion of length of a curve.
Then we can look at shortest curves.

Requiring the that the shortest curves coincide with the straightest curves (wrt ∇) will result in ∇ being determined by the metric structure.

$$\text{straight} = \text{short}$$

$$g \quad \overset{T=0}{\rightsquigarrow} \quad \nabla \rightsquigarrow \text{Riem}$$

10.1. Metrics.

Definition 12. A metric g on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ is a $(0,2)$-tensor field satisfying

(i) symmetry $g(X,Y) = g(Y,X) \quad \forall\, X, Y$ vector fields

(ii) non-degeneracy: the musical map

$$\begin{aligned} \flat : \Gamma(TM) &\to \Gamma(T^*M) \\ X &\mapsto \flat(X) \end{aligned}$$

... is a C^∞-isomorphism in other words, it is invertible.

11. Symmetry

EY : 20150321 This lecture tremendously and lucidly clarified, for me at least, what a symmetry of the Lie algebra is, and in comparing structures $(M, \mathcal{O}, \mathcal{A})$ vs. $(M, \mathcal{O}, \mathcal{A}, \nabla)$, clarified differences, and asking about differences is a good way to learn, the difference between $\mathcal{L}$ and ∇, respectively.

11.1.

11.2. Important

11.3. Flow of a complete vector field.

Let $(M, \mathcal{O}, \mathcal{A})$ smooth X vector field on M

Definition 13. A curve $\gamma : I \subseteq \mathbb{R} \to M$ is called an integral curve of X
if

$$v_{\gamma, \gamma(\lambda)} = X_{\gamma(\lambda)}$$

Definition 14. A vector filed X is **complete** if all integral curves have $I = \mathbb{R}$ EY: 20150321 (i.e. domain is all of $\mathbb{R}$)

Ex. minute 48:30 EY : reall good explanation by F.P.Schuller; take a pt. out for an incomplete vector field.

Theorem 1. *compactly supported smooth vector field is complete.*

Definition 15. The flow of a complete vector field X is a 1-parameter family

$$h^X = \mathbb{R} \times M \to M$$

where $\gamma_p : \mathbb{R} \to M$ is the integral curve of X with

$\gamma(0) = p$

Then for fixed $\lambda \in \mathbb{R}$

$$h_\lambda^X : M \to M \text{ smooth}$$

picture $h_\lambda^X(S) \neq S($ if $X \neq 0)$

11.4. **Lie subalgebras of the Lie algebra** $(\Gamma(TM), [\cdot,\cdot])$ **of vector fields.**

(a) $\Gamma(TM) = \{$ set of all vector fields $\}$ $\quad C^\infty(M)$-module = $\mathbb{R}$-vector space

$$\implies [X,Y] \in \Gamma(TM) \qquad [X,Y]f := X(Yf) - Y(Xf)$$

(i) $[X,Y] = -[Y,X]$

(ii) $[\lambda X + Z, Y] = \lambda[X,Y] + [Z,Y]$

(iii) $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$

$(\Gamma(TM), [\cdot,\cdot])$ Lie algebra

(b) Let $X_1 \dots X_s$ for s (many) vector fields on M, such that

Tutorial 11 Symmetry. Exercise 1. : True or false?

(a)
-
- $\phi^* : T^*N \to T*M$ i.e. $\phi^*\nu(X) = \nu(\phi_* X)$ for smooth $\phi : M \to N$, so the pullback of a covector $\nu \in T^*N$ maps to a covector in $T*M$.
-
-
-
-

(b)

(c)

Exercise 2. : Pull-back and push-forward

Question . Let's check this locally

$$\phi^*(df)(X) = (df)(\phi_* X) = (df)(X^i \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}) = X^i \frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j} \text{ where } \quad \phi_* X = X^i \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$$

$$d(\phi^* f)(X) = d(f(\phi))(X) = \frac{\partial f}{\partial y^j}\frac{\partial y^j}{\partial x^i} dx^i(X) = X^i \frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j}$$

So

$$\boxed{\phi^*(df) = d(\phi^* f)} \qquad \forall\, p \in M,\ \forall\, X \in \mathfrak{X}(M)$$

The big idea is that this is a showing of the **naturality** of the pullback ϕ^* with d, i.e. that this commutes:

$$\begin{tikzcd} \Omega^1(M) & \Omega^1(N) \arrow[l, "\phi^*"'] \\ C^\infty(M) \arrow[u, "d"] & C^\infty(N) \arrow[l, "\phi^*"] \arrow[u, "d"'] \end{tikzcd}$$

Question .

$$(\phi_*)^a{}_b := (dy^a)(\phi_*(\frac{\partial}{\partial x^b}))$$

$$\text{Let } g \in C^\infty(N)$$

$$\phi_* \left(\frac{\partial}{\partial x^b} \right) g = \frac{\partial x^b}{g} \phi(p) = \frac{\partial}{\partial x^b} g \phi x^{-1} x(p) = \frac{\partial}{\partial x^b} (g y y^{-1} \phi x^{-1})(x) =$$

$$= \frac{\partial}{\partial x^b} (g y^{-1} (y \phi x^{-1} (x(p)))) = \left. \frac{\partial g}{\partial y}^b \right|_y \left. \frac{\partial y^a}{\partial x^b} \right|_x = \frac{\partial y^a}{\partial x^b} \frac{\partial g}{\partial y^a}$$

Then

$$\phi_* \left(\frac{\partial}{\partial x^b} \right) = \frac{\partial y^a}{\partial x^b} \frac{\partial}{\partial y^a}$$

and so

$$(\phi_*)^a_{\ b} = \frac{\partial y^a}{\partial x^b}$$

Question .

Exercise 3. :Lie derivative-the pedestrian way

Question . While it is true that $\forall\, p \in S^2$, for $x(p) = (\theta, \varphi)$, and $(yix^{-1})(\theta, \varphi) = (y^1, y^2, y^3) \in \mathbb{R}^3$ and that, at this point p, $(y^1)^2/a^2 + (y^2)^2/b^2 + (y^3)^2/c^3 = 1$, this doesn't imply (EY: 20150321 I think) that, globally, it's an ellipsoid (yet). In the familiar charts given,
spherical chart $(U, x) \in \mathcal{A}$ and
$(\mathbb{R}^3, y = \mathrm{id}_{\mathbb{R}^3}) \in \mathcal{B}$
it looks like an ellipsoid, but change to another choice of charts, and it could look something very different.

Question .

Equip $(\mathbb{R}^3, \mathcal{O}_{\text{st}}, \mathcal{B})$ with the Euclidean metric g, and pullback g.

Note that the pullback of the inclusion from $\mathbb{R}^3$ onto S^2 for the Euclidean metric is the following:

$$i^* g \left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j} \right) = g \left(i_* \frac{\partial}{\partial \theta^i}, i_* \frac{\partial}{\partial \theta^j} \right) = g \left(\frac{\partial x^a}{\partial \theta^i} \frac{\partial}{\partial x^a}, \frac{\partial x^b}{\partial \theta^j} \frac{\partial}{\partial x^b} \right) = g_{ab} \frac{\partial x^a}{\partial \theta^i} \frac{\partial x^b}{\partial \theta^j}$$

With $g_{ab} = \delta_{ab}$, the usual Euclidean metric, this becomes the following:

$$g_{ij}^{\text{ellipsoid}} = \frac{\partial x^a}{\partial \theta^i} \frac{\partial x^a}{\partial \theta^j}$$

At this point, one should get smart (we are in the 21st century) and use some sort of CAS (Computer Algebra System). I like Sage Math (version 6.4 as of 20150322). I also like the Sage Manifolds package for Sage Math.

I like Sage Math for the following reasons:

- Open source, so its open and freely available to anyone, which fits into my principle of making online education open and freely available to anyone, anytime
- Sage Math structures everything in terms of Category Theory and Categories and Morphisms naturally correspond to Classes and Class methods or functions in Object-Oriented Programming in Python and theyve written it that way

and I like Sage Manifolds for roughly the same reasons, as manifolds are fit into a category theory framework thats written into the Python code. e.g.

```
sage: S2 = Manifold(2, 'S^2', r'\mathbb{S}^2', start_index=1) ; print S2
sage: print S2
2-dimensional manifold 'S^2'
sage: type(S2)
<class 'sage.geometry.manifolds.manifold.Manifold_with_category'>
```

With code (Ive provided for convenience; you can make your own as I wrote it based upon to example of S^2 on the sagemanifolds documentation website page), load it and do the following:

cf. https://github.com/ernestyalumni/diffgeo-by-sagemnfd/blob/master/S2.sage
http://sagemanifolds.obspm.fr/examples.html

```
sage: load("S2.sage")
sage: U_ep = S2.open_subset('U_{ep}')
sage: eps.<the,phi> = U_ep.chart()
sage: a = var(a)
sage: b = var(b)
sage: c = var("c")
sage: inclus = S2.diff_mapping(R3, {(eps, cart): [ a*cos(phi)*sin(the), b*sin(phi)*sin(the),c*cos(the) ]} , name="inc",latex_name=r'\mathcal{i}')
sage: inclus.pullback(h).display()
inc_*(h) = (c^2*sin(the)^2 + (a^2*cos(phi)^2 + b^2*sin(phi)^2)*cos(the)^2) dthe*dthe - (a^2 - b^2)*cos(phi)*cos(the)*sin(phi)*sin(the) dthe*dphi
- (a^2 - b^2)*cos(phi)*cos(the)*sin(phi)*sin(the) dphi*dthe + (b^2*cos(phi)^2 + a^2*sin(phi)^2)*sin(the)^2 dphi*dphi
sage: inclus.pullback(h)[2,2].expr()
(b^2*cos(phi)^2 + a^2*sin(phi)^2)*sin(the)^2
```

A new open subset U_{ep} was declared in S^2, a new chart $(U_{\text{ep}}, (\theta, \phi))$ was declared, the constants, a, b, c, were declared, and the inclusion map given in the problem

$$y \circ \mathfrak{i} \circ x^{-1} : (\theta, \phi) \mapsto (a \cos\phi \sin\theta, b \sin\phi \sin\theta, c \cos\theta)$$

Then the pullback of the inclusion map $\rangle$ was done on the Euclidean metric h, defined earlier in the file

`S2.sage`

. Then one can access the components of this metric and do, for example,

`simplify_full(),full_simplify(), reduce_trig()`

on the expression.

In Python, I could easily do this, and give an answer quick in LaTeX:

```
sage: for i in range(1,3):
....:     for j in range(1,3):
....:         print inclus.pullback(h)[i,j].expr()
....:         latex(inclus.pullback(h)[i,j].expr() )
....:
c^2*sin(the)^2 + (a^2*cos(phi)^2 + b^2*sin(phi)^2)*cos(the)^2
```

(EY: I'll suppress the LaTeX output but this sage math function gives you LaTeX code)

and so

$$\boxed{\begin{gathered} i^*g = c^2 \sin(the)^2 + \left(a^2 \cos(\phi)^2 + b^2 \sin(\phi)^2\right) \cos(the)^2 \, d\theta \otimes d\theta + \\ -2\left(a^2 - b^2\right) \cos(\phi) \cos(the) \sin(\phi) \sin(the) \, d\theta \otimes d\phi + \\ +\left(b^2 \cos(\phi)^2 + a^2 \sin(\phi)^2\right) \sin(the)^2 \, d\phi \otimes d\phi \end{gathered}}$$

Question .

```
sage: polar_vees = eps.frame()
sage: X_1 = - sin(phi) * polar_vees[1] - cot( the ) * cos(phi) * polar_vees[2]
sage: X_2 = cos( phi ) * polar_vees[1] - cot( the ) * sin( phi) * polar_vees[2]
sage: X_3 = polar_vees[2]
sage: X_2.lie_der(X_1).display()
(cos(the)^2 - 1)/sin(the)^2 d/dphi
sage: X_3.lie_der(X_1).display()
cos(phi) d/dthe - cos(the)*sin(phi)/sin(the) d/dphi
```

```
sage: X_3.lie_der(X_2).display()
sin(phi) d/dthe + cos(phi)*cos(the)/sin(the) d/dphi
```

Indeed, one can check on a scalar field $f_{\text{eps}} \in C^\infty(S^2)$:

```
sage: f_eps = S2.scalar_field({eps: function('f', the, phi ) }, name='f' )
sage: (X_1( X_2(f_eps)) - X_2(X_1(f_eps) ) ).display()
U_{ep} --> R
(the, phi) |--> -D[1](f)(the, phi)
sage: X_2.lie_der(X_1) == -X_3
True
sage: X_3.lie_der(X_1) == X_2
True
sage: X_3.lie_der(X_2) == -X_1
True
```

$$\Longrightarrow \boxed{[X_i, X_j] = -\epsilon_{ijk} X_k}$$

So $\text{span}_{\mathbb{R}}\{X_1, X_2, X_3\}$ equipped with $[\,,\,]$ constitute a Lie subalgebra on S^2 (It's closed under $[\,,\,]$

12. Integration

12.1.

12.2.

12.3. **Volume forms.**

Definition 16. On a smooth manifold $(M, \mathcal{O}, \mathcal{A})$
a $(0, \text{dim}M)$-tensor field Ω is called a <u>volume form</u> if

(a) Ω vanishes nowhere (i.e. $\Omega \neq 0 \ \forall \, p \in M$)

(b) totally antisymmetric

$$\Omega(\dots, \underbrace{X}_{i\text{th}}, \dots, \underbrace{Y}_{j\text{th}} \dots) = -\Omega(\dots, \underbrace{Y}_{i\text{th}}, \dots, \underbrace{X}_{j\text{th}} \dots)$$

In a chart:

$$\Omega_{i_1 \dots i_d} = \Omega_{[i_1 \dots i_d]}$$

<u>Example</u> $(M, \mathcal{O}, \mathcal{A}, g)$ metric manifold

construct volume form Ω from g

In <u>any</u> chart: (U, x)

$$\Omega_{i_1 \dots i_d} := \sqrt{\det(g_{ij}(x))} \epsilon_{i_1 \dots i_d}$$

where **Levi-Civita symbol** $\epsilon_{i_1 \dots i_d}$ is <u>defined</u> as $\epsilon_{123 \dots d} = +1$

$$\epsilon_{1 \dots d} = \epsilon_{[i_1 \dots i_d]}$$

Proof. (well-defined) Check: What happens under a change of charts

$$\begin{aligned}\Omega(y)_{i_1\dots i_d} &= \sqrt{\det(g(y)_{ij})}\epsilon_{i_1\dots i_d} = \\ &= \sqrt{\det(g_{mn}(x)\frac{\partial x^m}{\partial y^i}\frac{\partial x^n}{\partial y^j})}\frac{\partial y^{m_1}}{\partial x^{i_1}}\dots\frac{\partial y^{m_d}}{\partial x^{i_d}}\epsilon_{[m_1\dots m_d]} = \\ &= \sqrt{|\det g_{ij}(x)|}\left|\det\left(\frac{\partial x}{\partial y}\right)\right|\det\left(\frac{\partial y}{\partial x}\right)\epsilon_{i_1\dots i_d} = \sqrt{\det g_{ij}(x)}\epsilon_{i_1\dots i_d}\mathrm{sgn}\left(\det\left(\frac{\partial x}{\partial y}\right)\right)\end{aligned}$$

□

EY : 20150323

Consider the following:

$$\begin{aligned}\Omega(y)(Y_{(1)}\dots Y_{(d)}) &= \Omega(y)_{i_1\dots i_d}Y^{i_1}_{(1)}\dots Y^{i_d}_{(d)} = \\ &= \sqrt{\det(g_{ij}(y))}\epsilon_{i_1\dots i_d}Y^{i_1}_{(1)}\dots Y^{i_d}_{(d)} = \\ &= \sqrt{\det(g_{mn}(x))\frac{\partial x^m}{\partial y^i}\frac{\partial x^n}{\partial y^j}}\epsilon_{i_1\dots i_d}\frac{\partial y^{i_1}}{\partial x^{m_1}}\dots\frac{\partial y^{i_d}}{\partial x^{m_d}}X^{m_1}\dots X^{m_d} = \\ &= \sqrt{\det(g_{mn}(x))\frac{\partial x^m}{\partial y^i}\frac{\partial x^n}{\partial y^j}}\det\left(\frac{\partial y}{\partial x}\right)\epsilon_{m_1\dots m_d}X^{m_1}\dots X^{m_d} = \\ &= \sqrt{\det(g_{mn}(x))}\left|\det\left(\frac{\partial x}{\partial y}\right)\right|\det\left(\frac{\partial y}{\partial x}\right)\epsilon_{m_1\dots m_d}X^{m_1}\dots X^{m_d} = \\ &= \sqrt{\det(g_{mn}(x))}\epsilon_{m_1\dots m_d}\mathrm{sgn}\left(\det\left(\frac{\partial x}{\partial y}\right)\right)X^{m_1}\dots X^{m_d} = \mathrm{sgn}(\det\left(\frac{\partial x}{\partial y}\right))\Omega_{m_1\dots m_d}(x)X^{m_1}\dots X^{m_d}\end{aligned}$$

If $\det\left(\frac{\partial y}{\partial x}\right) > 0$,

$$\Omega(y)(Y_{(1)}\dots Y_{(d)}) = \Omega(x)(X_{(1)}\dots X_{(d)})$$

This works also if Levi-Civita symbol $\epsilon_{i_1\dots i_d}$ doesn't change at all under a change of charts. (around 42:43 https://youtu.be/2XpnbvPy-Zg)

Alright, let's require,

restrict the smooth atlas $\mathcal{A}$

to a subatlas ($\mathcal{A}^\uparrow$ still an atlas)

$$\mathcal{A}^\uparrow \subseteq \mathcal{A}$$

s.t. $\forall\,(U,x),(V,y)$ have chart transition maps $y\circ x^{-1}$

$x\circ y^{-1}$

s.t. $\det\left(\frac{\partial y}{\partial x}\right) > 0$

such $\mathcal{A}^\uparrow$ called an **oriented** atlas

$$(M,\mathcal{O},\mathcal{A},g) \Longrightarrow (M,\mathcal{O},\mathcal{A}^\uparrow,g)$$

Note: associated bundles.

Note also: $\det\left(\frac{\partial y^b}{\partial x^a}\right) = \det(\partial_a(y^b x^{-1}))$ $\qquad$ $\frac{\partial y^b}{\partial x^a}$ is an endomorphism on vector space V. $\varphi : V\to V$

$\det\varphi$ $\quad$ independent of choice of basis

g is a $(0,2)$ tensor field, not endomorphism (not independent of choice of basis) $\sqrt{|\det(g_{ij}(y))|}$

Definition 17. Ω be a volume form on $(M, \mathcal{O}, \mathcal{A}^{\uparrow})$ and consider chart (U, x)

Definition 18. $\omega_{(X)} := \Omega_{i_1 \dots i_d} \epsilon^{i_1 \dots i_d}$ same way $\epsilon^{12 \dots d} = +1$

$\epsilon^{[\dots]}$

one can show

$$\boxed{\omega_{(y)} = \det\left(\frac{\partial x}{\partial y}\right)\omega_{(x)}} \qquad \text{scalar density}$$

12.4. **Integration on <u>one</u> chart domain U.**

Definition 19.

$$\boxed{\int_U f :\overset{(U,y)}{=} \int_{y(U)} d^d\beta \omega_{(y)}(y^{-1}(\beta)) f_{(y)}(\beta)} \tag{12.1}$$

Proof. : Check that it's (well-defined), how it changes under change of charts

$$\int_U f :\overset{(U,y)}{=} \int_{y(U)} d^d\beta \omega_{(y)}(y^{-1}(\beta)) f_{(y)}(\beta) = \underset{(U,y)}{=} \int_{x(U)} \int d^d\alpha \left|\det\left(\frac{\partial y}{\partial x}\right)\right| f_{(x)}(\alpha)\omega_{(x)}(x^{-1}(\alpha) \det\left(\frac{\partial x}{\partial y}\right) =$$
$$= \int_{x(U)} d^d\alpha \omega_{(x)}(x^{-1}(x)) f_{(x)}(\alpha)$$

□

On an oriented metric manifold $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, g)$

$$\int_U f := \int_{x(U)} d^d\alpha \underbrace{\sqrt{\det(g_{ij}(x))(x^{-1}(\alpha))}}_{\sqrt{g}} f_{(x)}(\alpha)$$

12.5. **Integration on the entire manifold.**

13. Lecture 13: Relativistic spacetime

Recall, from Lecture 9, the definition of Newtonian spacetime

$$(M, \mathcal{O}, \mathcal{A}, \nabla, t) \qquad \begin{aligned} &\nabla \text{ torsion free} \\ &t \in C^{\infty}(M) \\ &dt \neq 0 \\ &\nabla dt = 0 \qquad \text{(uniform time)} \end{aligned}$$

and the definition of relativistic spacetime (before Lecture)

$$(M, \mathcal{O}, \mathcal{A}^{\uparrow}, \nabla, g, T) \qquad \begin{aligned} &\nabla \text{ torsion-free} \\ &g \text{ Lorentzian metric}(+---) \\ &T \text{ time-orientation} \end{aligned}$$

13.1. **Time orientation.**

Definition 20. $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, g)$ a Lorentzian manifold. Then a time-orientation is given by a vector field T that

(i) does **not** vanish anywhere

(ii) $g(T, T) > 0$

Newtonian vs. relativistic
Newtonian
X was called future-directed if

$$dt(X) > 0$$

$\forall\, p \in M$, take half plane, half space of $T_p M$
also stratified atlas so make planes of constant t straight
relativistic
half cone $\forall\, p, q \in M$, half-cone $\subseteq T_p M$

This definition of <u>spacetime</u>

Question
I see how the cone structure arises from the new metric. I don't understand however, how the T, the time orientation, comes in

Answer
$(M, \mathcal{O}, \mathcal{A}, g)$ $g \overset{(}{\leftarrow} + - --)$

requiring $g(X, X) > 0$, select cones
T chooses which cone

This definition of <u>spacetime</u> has been made to enable the following physical postulates:

(P1) The worldline γ of a <u>massive</u> particle satisfies

(i) $g_{\gamma(\lambda)}(v_{\gamma,\gamma(lambda)}, v_{\gamma,\gamma(\lambda)}) > 0$

(ii) $g_{\gamma(\lambda)}(T, v_{\gamma,\gamma(\lambda)}) > 0$

(P2) Worldlines of <u>massless</u> particles satisfy

(i) $g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) = 0$

(ii) $g_{\gamma(\lambda)}(T, v_{\gamma,\gamma(\lambda)}) > 0$

<u>picture</u>: spacetime:

Answer (to a question) T is a smooth vector field, T determines future vs. past, "general relativity: we have such a time orientation; smoothness makes it less arbitrary than it seems" -FSchuller,

<u>Claim</u>: 9/10 of a metric are determined by the cone

spacetime determined by distribution, only one-tenth error

13.2. **Observers.** $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, \nabla, g, T)$

Definition 21. An <u>observer</u> is a worldline γ with

$$g(v_\gamma, v_\gamma) > 0$$
$$g(T, v_\gamma) > 0$$

together with a choice of basis

$$v_{\gamma,\gamma(\lambda)} \equiv e_0(\lambda), e_1(\lambda), e_2(\lambda), e_3(\lambda)$$

of each $T_{\gamma(\lambda)}M$ where the observer worldline passes, if $g(e_a(\lambda), e_b(\lambda)) = \eta_{ab} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}_{ab}$

precise: observer = smooth curve in the frame bundle LM over M

13.2.1. *Two physical postulates.*

(P3) A **clock** carried by a specific observer (γ, e) will measure a **time**

$$\tau := \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})}$$

between the two "events"

$$\gamma(\lambda_0) \qquad \text{"start the clock"}$$

and

$$\gamma(\lambda_1) \qquad \text{"stop the clock"}$$

Compare with Newtonian spacetime:

$$t(p) = 7$$

Thought bubble: proper time/eigentime τ

Application/Example.
$$M = \mathbb{R}^4$$
$$\mathcal{O} = \mathcal{O}_{\text{st}}$$
$$\mathcal{A} \ni (\mathbb{R}^4, \text{id}_{\mathbb{R}^4})$$
$$g : g_{(x)ij} = \eta_{ij} \quad ; \quad T^i_{(x)} = (1,0,0,0)^i$$
$$\implies \Gamma^i_{(x)\ jk} = 0 \text{ everywhere}$$

$\implies (M, \mathcal{O}, \mathcal{A}^\uparrow, g, T, \nabla) \qquad \text{Riemm} = 0$
$\implies$ spacetime is flat

This situation is called special relativity.

Consider two observers:

$$\gamma : (0,1) \to M$$
$$\gamma^i_{(x)} = (\lambda, 0, 0, 0)^i$$
$$\delta : (0,1) \to M$$
$$\alpha \in (0,1) : \delta^i_{(x)} = \begin{cases} (\lambda, \alpha\lambda, 0, 0)^i & \lambda \leq \frac{1}{2} \\ (\lambda, (1-\lambda)\alpha, 0, 0)^i & \lambda > \frac{1}{2} \end{cases}$$

let's calculate:

$$\tau_\gamma := \int_0^1 \sqrt{g_{(x)ij}\dot{\gamma}^i_{(x)}\dot{\gamma}^j_{(x)}} = \int_0^1 d\lambda 1 = 1$$
$$\tau_\delta := \int_0^{1/2} d\lambda\sqrt{1-\alpha^2} + \int_{1/2}^1 \sqrt{1^2 - (-\alpha)^2} = \int_0^1 \sqrt{1-\alpha^2} = \sqrt{1-\alpha^2}$$

Note: piecewise integration

Taking the clock postulate (P3) seriously, one better come up with a realistic clock design that supports the postulate. idea.

2 little mirrors

(P4) Postulate

Let (γ, e) be an observer, and
δ be a *massive* particle worldline that is parametrized s.t. $g(v_\gamma, v_\gamma) = 1$ (for parametrization/normalization convenience)

Suppose the observer and the particle *meet* somewhere (in spacetime)

$$\delta(\tau_2) = p = \gamma(\tau_1)$$

This observer measures the 3-velocity (spatial velocity) of this particle as

$$v_\delta : \epsilon^\alpha(v_{\delta,\delta(\tau_2)})e_\alpha \qquad \alpha = 1,2,3 \tag{13.1}$$

where $\epsilon^0, \boxed{\epsilon^1, \epsilon^2, \epsilon^3}$ is the unique dual basis of $e_0, \boxed{e_1, e_2, e_3}$

EY:20150407

There might be a major correction to Eq. (13.1) from the Tutorial 14 : Relativistic spacetime, matter, and Gravitation, see the second exercise, Exercise 2, third question:

$$v := \frac{\epsilon^\alpha(v_\delta)}{\epsilon^0(v_\delta)} e_\alpha \tag{13.2}$$

Consequence: An observer (γ, e) will extract quantities measurable in his laboratory from objective spacetime quantities always like that.

Ex: F Faraday $(0,2)$-tensor of electromagnetism:

$$F(e_a, e_b) = F_{ab} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

observer frame e_a, e_b

$E_\alpha := F(e_0, e_\alpha)$
$B^\gamma := F(e_\alpha, e_\rho)\epsilon^{\alpha\beta\gamma}$ where $\epsilon^{123} = +1$ totally antisymmetric

13.3. **Role of the Lorentz transformations.** Lorentz transformations emerge as follows:
Let (γ, e) and $(\widetilde{\gamma}, \widetilde{e})$ be observers with $\gamma(\tau_1) = \widetilde{\gamma}(\tau_2)$

(for simplicity $\gamma(0) = \widetilde{\gamma}(0)$)

Now

$$e_0, \dots, e_1 \qquad \text{at } \tau = 0$$
$$\text{and } \widetilde{e}_0, \dots, \widetilde{e}_1 \qquad \text{at } \tau = 0$$

both bases for the same $T_{\gamma(0)}M$

Thus: $\widetilde{e}_a = \Lambda^b{}_a e_b \qquad \Lambda \in GL(4)$

Now:

$$\begin{aligned} \eta_{ab} = g(\widetilde{e}_a, \widetilde{e}_b) &= g(\Lambda^m{}_a e_m, \Lambda^n{}_b e_n) = \\ &= \Lambda^m{}_a \Lambda^n{}_b \underbrace{g(e_m, e_n)}_{\eta_{mn}} \end{aligned}$$

i.e. $\Lambda \in O(1,3)$

Result: Lorentz transformations relate the *frames* of *any two observers* at the same point.

"$\widetilde{x}^\mu - \Lambda^\mu{}_\nu x^\nu$" is utter nonsense

Tutorial. I didn't see a tutorial video for this lecture, but I saw that the Tutorial sheet number 14 had the relevant topics. Go there.

14. Lecture 14: Matter

two types of matter

point matter

field matter

point matter

massive point particle

more of a phenomenological importance

field matter

electromagnetic field

more fundamental from the GR point of view

both classical matter types

14.1. **Point matter.** Our postulates (P1) and (P2) already constrain the possible particle worldlines.

But what is their precise law of motion, possibly in the presence of "forces",

(a) without external forces

$$S_{\text{massive}}[\gamma] := m \int d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})}$$

with:

$$g_{\gamma(\lambda)}(T_{\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) > 0$$

dynamical law Euler-Lagrange equation

similarly

$$S_{\text{massless}}[\gamma, \mu] = \int d\lambda \mu g(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})$$

$$\begin{aligned} \delta_\mu \qquad & g(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) = 0 \\ \delta_\gamma \qquad & \text{e.o.m.} \end{aligned}$$

Reason for describing equations of motion by actions is that composite systems have an action that is the sum of the actions of the parts of that system, possibly including "interaction terms."

Example.

$$S[\gamma] + S[\delta] + S_{\text{int}}[\gamma, \delta]$$

(b) presence of external forces
or rather presence of fields to which a particle "couples"

Example

$$S[\gamma; A] = \int d\lambda m \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})} + qA(v_{\gamma,\gamma(\lambda)})$$

where A is a **covector field** on M. A fixed (e.g. the electromagnetic potential)

Consider Euler-Lagrange eqns. $L_{\text{int}} = qA_{(x)}\dot{\gamma}^m_{(x)}$

$$m(\nabla_{v_\gamma} v_\gamma)_a + \underbrace{\left(\frac{\partial L_{\text{int}}}{\partial \cdot^{\,m}_{(x)}}\right)^{\!\cdot}\!\gamma - \frac{\partial L_{\text{int}}}{\partial \gamma^m_{(x)}}}_{*} = 0 \Longrightarrow \boxed{m(\nabla_{v_\gamma} v_\gamma)^a = \underbrace{-qF^a{}_m\dot{\gamma}^m}_{\text{Lorentz force on a charged particle in an electromagnetic field}}}$$

$$\frac{\partial L}{\partial \dot{\gamma}^a} = qA_{(x)a}, \qquad \left(\frac{\partial L}{\partial \cdot^{\,m}}\right)^{\!\cdot}\!\gamma = q \cdot \frac{\partial}{\partial x^m}(A_{(x)m}) \cdot \dot{\gamma}^m_{(x)}$$

$$\frac{\partial L}{\partial \gamma^a} = q \cdot \frac{\partial}{\partial x^a}(A_{(x)m})\dot{\gamma}^m$$

$$* = q\left(\frac{\partial A_a}{\partial x^m} - \frac{\partial A_m}{\partial x^a}\right)\dot{\gamma}^m_{(x)} \quad = q \cdot F_{(x)am}\dot{\gamma}^m_{(x)}$$

$F \leftarrow$ Faraday

$$S[\gamma] = \int (m\sqrt{g(v_\gamma, v_\gamma)} + qA(v_\gamma))d\lambda$$

14.2. **Field matter.**

Definition 22. Classical (non-quantum) field matter is any tensor field on spacetime where equations of motion derive from an action.

Example:

$$S_{\text{Maxwell}}[A] = \frac{1}{4}\int_M d^4x\sqrt{-g}F_{ab}F_{cd}g^{ac}g^{bd}$$

A $(0,1)$-tensor field
= thought cloud: for simplicity one chart covers all of M
$-$ for $\sqrt{-g}$ $(+---)$

$F_{ab} := 2\partial_{[a}A_{b]} = 2(\nabla_{[a}A)_{b]}$

Euler-Lagrange equations for fields

$$0 = \frac{\partial \mathcal{L}}{\partial A_m} - \frac{\partial}{\partial x^s}\left(\frac{\partial \mathcal{L}}{\partial \partial_s A_m}\right) + \frac{\partial}{\partial x^s}\frac{\partial}{\partial x^t}\frac{\partial^2 \mathcal{L}}{\partial \partial_t \partial_s A_m}$$

Example ...

$$(\nabla_{\frac{\partial}{\partial x^m}} F)^{ma} = j^a$$

inhomogeneous Maxwell

thought bubble $j = qv_\gamma$

$$\partial_{[a}F_{b]} - ()$$

homogeneous Maxwell

Other example well-liked by textbooks

$$S_{\text{Klein-Gordon}}[\phi] := \int_M d^4x\sqrt{-g}[g^{ab}(\partial_a\phi)(\partial_b\phi) - m^2\phi^2]$$

ϕ $(0,0)$-tensor field

14.3. **Energy-momentum tensor of matter fields.** At some point, we want to write down an action for the metric tensor field itself.

But then, this action $S_{\text{grav}}[g]$ will be added to any $S_{\text{matter}}[A, \phi, \dots]$ in order to describe the total system.

$$S_{\text{total}}[g, A] = S_{\text{grav}}[g] + S_{\text{Maxwell}}[A, g]$$

$$\begin{aligned} \delta A \quad & :\Longrightarrow \text{ Maxwell's equations} \\ \delta g_{ab} \quad & : \boxed{\frac{1}{16\pi G} G^{ab}} + (-2T^{ab}) = 0 \end{aligned}$$

G Newton's constant

$$G^{ab} = 8\pi G_N T^{ab}$$

Definition 23. $S_{\text{matter}}[\Phi, g]$ is a matter action, the **so-called energy-momentum tensor** is

$$T^{ab} := \frac{-2}{\sqrt{-g}} \left(\frac{\partial \mathcal{L}_{\text{matter}}}{\partial g_{ab}} - \partial_s \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \partial_s g_{ab}} + \dots \right)$$

$-$ of $\frac{-2}{\sqrt{g}}$ is Schrödinger minus (EY : 20150408 F.Schuller's joke? but wise)

choose all sign conventions s.t.

$$T(\epsilon^0, \epsilon^0) > 0$$

Example: For S_{Maxwell}:

$$T_{ab} = F_{am} F_{bn} g^{mn} - \frac{1}{4} F_{mn} F^{mn} g_{ab}$$

$T_{ab} \equiv T_{\text{Maxwell}ab}$

$$\begin{aligned} T(e_0, e_0) &= \underline{E}^2 + \underline{B}^2 \\ T(e_0, e_\alpha) &= (E \times B)_\alpha \end{aligned}$$

Fact: One often does not specify the fundamental action for some matter, but one is rather satisfied to assume certain properties / forms of

$$T_{ab}$$

Example Cosmology: (homogeneous & isotropic)

perfect fluid

of pressure p and density ρ modelled by

$$T^{ab} = (\rho + p) u^a u^b - p g^{ab}$$

radiative fluid

What is a fluid of photons:

$$T^{ab}_{\text{Maxwell}} g_{ab} = 0$$

observe: $T^{ab}_{\text{p.f.}} g_{ab} \overset{!}{=} 0$

$$= (\rho + p) u^a u^b g_{ab} - p \underbrace{g^{ab} g_{ab}}_{4}$$

$$\leftrightarrow \rho_p 0 4p = 0$$

$$\rho = 3p$$

$p = \frac{1}{3}\rho$

Reconvene at 3 pm? (EY : 20150409 I sent a Facebook (FB) message to the International Winter School on Gravity and Light: there was no missing video; it continues on Lecture 15 immediately)

Tutorial 14: Relativistic Spacetime, Matter and Gravitation. Exercise 2: Lorentz force law.

Question electromagnetic potential.

15. Lecture 15: Einstein gravity

Recall that in Newtonian spacetime, we were able to reformulate the Poisson law $\Delta\phi = 4\pi G_N \rho$ in terms of the Newtonian spacetime curvature as

$$R_{00} = 4\pi G_N \rho$$

R_{00} with respect to ∇_{Newton}

$G_N =$ Newtonian gravitational constant

This prompted Einstein to postulate < 1915 that the relativistic field equations for the Lorentzian metric g of (relativistic) spacetime

$$R_{ab} = 8\pi G_N T_{ab}$$

However, this equation suffers from a problem

LHS: $(\nabla_a R)^{ab} \neq 0$
generically

RHS:

$$(\nabla_a T)^{ab} = 0$$

thought bubble: = formulated from an action

Einstein tried to argue this problem away.

Nevertheless, the equations cannot be upheld.

15.1. Hilbert.

Hilbert was a specialist for variational principles.

To find the appropriate left hand side of the gravitational field equations, Hibert suggested to start from an action

$$S_{\text{Hilbert}}[g] = \int_M \sqrt{-g} R_{ab} g^{ab}$$

thought bubble = "simplest action"

aim: varying this w.r.t. metric g_{ab} will result in some tensor

$$G^{ab} = 0$$

15.2. **Variation of $S_{\textbf{Hilbert}}$.**

$$0 \overset{!}{=} \underbrace{\delta}_{g_i} S_{\text{Hilbert}}[g] = \int_M [\underbrace{\delta\sqrt{-g}g^{ab}R_{ab}}_{1} + \underbrace{\sqrt{-g}\delta g^{ab}R_{ab}}_{2} + \underbrace{\sqrt{-g}g^{ab}\delta R_{ab}}_{3}]$$

$$\text{and } 1: \; \delta\sqrt{-g} = \frac{-(\det g)g^{mn}\delta g_{mn}}{2\sqrt{-g}} = \frac{1}{2}\sqrt{-g}g^{mn}\delta g_{mn}$$

thought bubble

$$\delta\det(g) = \det(g)g^{mn}\delta g_{mn}$$

e.g. from

$$\det(g) = \exp \mathrm{tr} \ln g$$

ad 2: $g^{ab}g_{bc} = \delta^a_c$

$$\begin{aligned} &\Longrightarrow (\delta g^{ab})g_{bc} + g^{ab}(\delta g_{bc}) = 0 \\ &\Longrightarrow \delta g^{ab} = -g^{am}g^{bn}\delta g_{mn} \end{aligned}$$

ad 3:

$$\begin{aligned} \Delta R_{ab} \underbrace{=}_{\text{normal coords at point}} & \delta\partial_b\Gamma^m_{\ am} - \delta\partial_m\Gamma^m_{\ ab} + \Gamma\Gamma - \Gamma\Gamma = \\ &= \partial_b\delta\Gamma^m_{\ am} - \partial_m\delta\Gamma^m_{\ ab} = \\ &= \nabla_b(\delta\Gamma)^m_{\ am} - \nabla_m(\delta\Gamma)^m_{\ ab} \\ &\Longrightarrow \sqrt{-g}g^{ab}\delta R_{ab} = \sqrt{-g} \end{aligned}$$

"if you formulate the variation properly, you'll see the variation δ commute with ∂_b" EY : 20150408 I think one uses the integration at the bounds, integration by parts trick

$\Gamma^i_{(x)\ jk} - \widetilde{\Gamma}^i_{(x)\ jk}$ are the components of a $(1,2)$-tensor.

Notation: $(\nabla_b A)^i_{\ g} =: A^i_{\ j;b}$

$$\Longrightarrow \sqrt{-g}g^{ab}\delta R_{ab}$$

$$\underbrace{=}_{\nabla g=0} \sqrt{-g}(g^{ab}\delta\Gamma^m_{\ am})_{;b} - \sqrt{-g}(g^{ab}\delta\Gamma^m_{\ ab})_{;m} = \sqrt{-g}A^b_{\ ;b} - \sqrt{-g}B^m_{\ ,m}$$

Question: Why is the difference of coefficients a tensor?

Answer:

$$\Gamma^i_{(y)\ jk} = \frac{\partial y^i}{\partial x^m}\frac{\partial x^m}{\partial y^j}\frac{\partial x^q}{\partial y^k}\Gamma^m_{(x)\ \ ,nq} + \frac{\partial y^i}{\partial x^m}\frac{\partial^2 x^m}{\partial y^j\partial y^k}$$

Collecting terms, one obtains

$$0 \overset{!}{=} \delta S_{\text{Hilbert}} = \int_M [\frac{1}{2}\sqrt{-g}g^{mn}\delta g_{mn}g^{ab}R_{ab} - \sqrt{-g}g^{am}g^{bn}\delta g_{mn}R_{ab} + \underbrace{(\sqrt{-g}A^a)_{,a}}_{\text{surface}} - \underbrace{(\sqrt{-g}B^b)_{,b}}_{\text{surface term}}]$$

$$= \int_M \sqrt{-g}\delta \underbrace{g_{mn}}_{\text{arbitrary variation}} [\frac{1}{2}g^{mn}R - R^{mn}] \Longrightarrow G^{mn} = R^{mn} - \frac{1}{2}g^{mn}R$$

Hence Hilbert, from this "mathematical" argument, concluded that one may take

$$\boxed{R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G_N T_{ab}}$$

Einstein equations

$$S_{E-H}[g] = \int_M \sqrt{-g}R$$

15.3. **3. Solution of the $\nabla_a T^{ab} = 0$ issue.** One can show ($\to$ Tutorials) that the Einstein curvature

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

satisfy the so-called contracted differential Bianchi identity

$$(\nabla_a G)^{ab} = 0$$

15.4. **Variants of the field equations.**

(a) a simple rewriting:

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G_N T_{ab} = T_{ab}$$

$G_N = \frac{1}{8\pi}$

Contract on both sides g^{ab}

$$R_{ab} - \frac{1}{2} g_{ab} R = T_{ab} || g^{ab}$$
$$R - 2R = T := T_{ab} g^{ab}$$
$$\Longrightarrow R = -T$$
$$\Longrightarrow R_{ab} + \frac{1}{2} g_{ab} T = T_{ab}$$
$$\Longleftrightarrow R_{ab} = (T_{ab} - \frac{1}{2} T g_{ab}) =: \widehat{T}_{ab}$$
$$\boxed{R_{ab} = \widehat{T}_{ab}}$$

(b)

$$S_{E-H}[g] := \int_M \sqrt{-g}(R + 2\Lambda)$$

thought bubble: Λ cosmological constant

History:

1915: $\Lambda < 0$ (Einstein) in order to get a non-expanding universe

>1915: $\Lambda = 0$ Hubble

today $\Lambda > 0$ to account for an accelerated expansion

$\Lambda \neq 0$ can be interpreted as a contribution

$-\frac{1}{2}\Lambda g$ to the energy-momentum

"dark energy"

Question: surface terms scalar?

Answer: for a careful treatment of the surface terms which we discarded, see, e.g. E. Poisson, "A relativist's toolkit" C.U.P. "excellent book"

Question: What is a constant on a manifold?

Answer: $\int \sqrt{-g}\Lambda = \Lambda \int \sqrt{-g} 1$

[back to dark energy]

[Weinberg, QCD, calculated]
idea: 1 could arise as the vacuum energy of the standard model fields

$\Lambda_{\text{calculated}} = 10^{120} \times \Lambda_{\text{obs}}$

"worst prediction of physics"

Tutorials: check that

- Schwarzscheld metric (1916)
- FRW metric
- pp-wave metric
- Reisner-Nordstrom

$\Longrightarrow$ are solutions to Einstein's equations

in high school

$m\ddot{x} + m\omega^2 x^2 = 0$

$x(t) = \cos{(\omega t)}$

ET: [elementary tutorials]

study motion of particles & observers in Schwarzscheld S.T.

Satellite: Marcus C. Werner

Gravitational lensing

odd number of pictures Morse theory (EY:20150408 Morse Theory !!!)

Domenico Giulini

Hamiltonian form Canonical Formulations

Key to Quantum Gravity

Tutorial 13 Schwarzschild Spacetime

EY : 20150408 I'm not sure which tutorial follows which lecture at this point.

The tutorial video is excellent itself. Here, I want to encourage the use of CAS to do calculations. There are many out there. Again, I'm partial to the Sage Manifolds package for Sage Math which are both open-source and based on Python. I'll use that here.

Exercise 1. Geodesics in a Schwarzschild spacetime

Question Write down the Lagrangian.

Load "Schwarzschild.sage" in Sage Math, which will always be available freely here https://github.com/ernestyalumni/diffgeo-by-sagemnfd/blob/master/Schwarzschild.sage:

```
sage: load("Schwarzschild.sage")
4-dimensional manifold 'M'
open subset 'U_sph' of the 4-dimensional manifold 'M'
Levi-Civita connection 'nabla_g' associated with the Lorentzian metric 'g' on the 4-dimensional manifold 'M'
```

and so on.

Look at the code and I had defined the Lagrangian to be

`L`

. To get out the coefficients of L of the components of the tangent vectors to the curve, i.e. t', r', θ', ϕ', denoted

`tp,rp,thp,php`

in my .sage file, do the following:

```
sage: L.expr().coefficients(tp)[1][0].factor().full_simplify()
(2*G_N*M_0 - r)/r
sage: L.expr().coefficients(rp)[1][0].factor().full_simplify()
-r/(2*G_N*M_0 - r)
sage: L.expr().coefficients(php)[1][0].factor().full_simplify()
r^2
sage: L.expr().coefficients(thp)[1][0].factor().full_simplify()
r^2*sin(th)^2
```

Question There are 4 Euler-Lagrange equations for this Lagrangian. Derive the one with respect to the function $t(\lambda)$!.

```
sage: L.expr().diff(t)
0
```

This confirms that $\frac{\partial L}{\partial t} = 0$

For $\frac{d}{d\lambda} \frac{\partial L}{\partial t'}$, then one needs to consider this particular workaround for Sage Math (computer technicality). One takes derivatives with respect to declared variables (declared with var) and then substitute in functions that are dependent upon λ, and then take the derivative with respect to the parameter λ. This does that:

```
sage: L.expr().diff( thp ).factor().subs( r == gamma1 ).subs( thp == gamma3.diff( tau ) ).subs( th == gamma3 ).diff(tau)\
....: .factor()
2*(2*cos(gamma3(tau))*gamma1(tau)*D[0](gamma3)(tau)^2 + 2*sin(gamma3(tau))*D[0](gamma1)(tau)*D[0](gamma3)(tau)
+ gamma1(tau)*sin(gamma3(tau))*D[0, 0](gamma3)(tau))*gamma1(tau)*sin(gamma3(tau))
```

Question Show that the Lie derivative of g with respect to the vector fields $K_t := \frac{\partial}{\partial t}$.

The first line defines the vector field by accessing the frame defined on a chart with spherical coordinates and getting the time vector. The second line is the Lie derivative of g with respect to this vector field.

```
sage: K_t = espher[0]
sage: g.lie_der(K_t).display() # 0, as desired
0
```

EY : 20150410 My question is this: $\forall X \in \Gamma(TM)$ i.e. X is a vector field on M, or, specifically, a section of the tangent bundle, then does

$$\mathcal{L}_X g = 0$$

instantly mean that X is a symmetry for (M, g)? $\mathcal{L}_X g$ is interpreted geometrically as how g changes along the flow generated by X, and if it equals 0, then g doesn't change.

16.

17.

18.

19.

20.

21.

22. Lecture 22: Black Holes

Only depends on Lectures 1-15, so does lecture on "Wednesday"

Schwarzschild solution also vacuum solution (from tutorial EY : oh no, must do tutorial)

Study the Schwarzschild as a vacuum solution of the Einstein equation:

$m = G_N M$ where M is the "mass"

$$g = \left(1 - \frac{2m}{r}\right) dt \otimes dt - \frac{1}{1 - \frac{2m}{r}} dr \otimes dr - r^2 (d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi$$

in the so-called <u>Schwarzschild coordinates</u>

$$\begin{array}{cccc} t & r & \theta & \varphi \\ (-\infty, \infty) & (0, \infty) & (0, \pi) & (0, 2\pi) \end{array}$$

What staring at this metric for a while, two questions naturally pose themselves:

(i) What exactly happens $r = 2m$?

$$\begin{array}{cccc} t & r & \theta & \varphi \\ (-\infty, \infty) & (0, 2m) \cup (2m, \infty) & (0, \pi) & (0, 2\pi) \end{array}$$

(ii) Is there anything (in the real world) beyond $t \to -\infty$?

$t \to +\infty$

<u>idea</u>: Map of Linz, blown up

Insight into these two issues is afforded by stopping to stare.

Look at *geodesic* of g, instead.

22.1. **Radial null geodesics.** null - $g(v_\gamma, v_\gamma) = 0$

Consider null geodesic in "<u>Schd</u>"

$$S[\gamma] = \int d\lambda \left[\left(1 - \frac{2m}{r}\right) \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) \right]$$

with $[\dots] = 0$

and one has, in particular, the t-eqn. of motion:

$$\left(\left(1 - \frac{2m}{r}\right) \dot{t}\right)^{\cdot} = 0$$

$\Longrightarrow$

$$\boxed{\left(1 - \frac{2m}{r}\right) \dot{t} = k} = \text{ const.}$$

Consider <u>radial</u> null geodesics
$\theta \overset{!}{=}$ const. $\qquad \varphi =$ const.

From □ and □

$$\Longrightarrow \dot{r}^2 = k^2 \leftrightarrow \dot{r} = \pm k$$

$$\Longrightarrow r(\lambda) = \pm k \cdot \lambda$$

Hence, we may consider

$$\widetilde{t}(r) := t(\pm k \lambda)$$

<u>Case A:</u> $\oplus$

$$\frac{d\widetilde{t}}{dr} = \frac{\dot{\widetilde{t}}}{\dot{r}} = \frac{k}{\left(1 - \frac{2m}{r}\right) k} = \frac{r}{r - 2m}$$
$$\Longrightarrow \widetilde{t}_+(r) = r + 2m \ln |r - 2m|$$

(**outgoing** null geodesics)

<u>Case b.</u> $\pm$ (Circle around $-$, consider $-$):

$$\widetilde{t}_-(r) = -r - 2m \ln |r - 2m|$$

(**ingoing** null geodesics)

Picture

22.2. **Eddington-Finkelstein.** Brilliantly simple idea:

change (on the domain of the Schwarzschild coordinates) to different coordinates, s.t.
in those new coordinates,
ingoing null geodesics appear as straight lines, of slope -1

This is achieved by

$$\bar{t}(t, r, \theta, \varphi) := t + 2m \ln |r - 2m|$$

<u>Recall</u>: ingoing null geodesic has

$$\widetilde{t}(r) = -(r + 2m \ln |r - 2m|) \qquad (Schdcoords)$$

$$\Longleftrightarrow \bar{t} - 2m \ln |r - 2m| = -r - 2m \ln |r - 2m| + \text{ const.}$$
$$\therefore \bar{t} = -r + \text{ const.}$$

(Picture)

outgoing null geodesics

$$\bar{t} = r + 4m \ln |r - 2m| + \text{ const.}$$

Consider the new chart (V, g) while (U, x) was the Schd chart.

$$\underbrace{U}_{\text{Schd}} \bigcup \{ \text{ horizon } \} = V$$

"chart image of the horizon"

Now calculate the *Schd metric* g w.r.t. Eddington-Finkelstein coords.

$$\begin{aligned} \bar{t}(t, r, \theta, \varphi) &= t + 2m \ln |r - 2m| \\ \bar{r}(t, r, \theta, \varphi) &= r \\ \bar{\theta}(t, r, \theta, \varphi) &= \theta \\ \bar{\varphi}(t, r, \theta, \varphi) &= \varphi \end{aligned}$$

EY : 20150422 I would suggest that after seeing this, one would calculate the metric by your favorite CAS. I like the Sage Manifolds package for Sage Math.

Schwarzschild_BH.sage on github

Schwarzschild_BH.sage on Patreon

Schwarzschild_BH.sage on Google Drive

```
sage: load(``Schwarzschild_BH.sage'')
4-dimensional manifold 'M'
/Applications/Sage-6.6.app/Contents/Resources/sage/local/lib/python2.7/site-packages/sage/geometry/manifolds/utilities.py:28
See http://trac.sagemath.org/11912 for details.
  expr = expr.simplify_radical()
Levi-Civita connection 'nabla_g' associated with the Lorentzian metric 'g' on the 4-dimensional manifold 'M'
Launched png viewer for Graphics object consisting of 4 graphics primitives
```

Then calculate the Schwarzschild metric g but in Eddington-Finkelstein coordinates. Keep in mind to calculate the set of coordinates that uses $\bar{t}$, not $\widetilde{t}$:

```
sage: gI.display()
gI = (2*m - r)/r dt*dt - r/(2*m - r) dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: gI.display( X_EF_I_null.frame())
gI = (2*m - r)/r dtbar*dtbar + 2*m/r dtbar*dr + 2*m/r dr*dtbar + (2*m + r)/r dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
```

References

[1] Eric Poisson, **A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics**, Cambridge University Press, 2004. ISBN 0 521 83091 5

EY's references:

[2] Klaus Jnich (Author), S. Levy (Translator), **Topology** (Undergraduate Texts in Mathematics), Springer; 1st ed. 1984. 2nd Corr. printing 1994 edition (January 1, 1995).
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