NP-complete Partitioning Problems

Subset Sum: **Given a list of** t **positive integers** $S = \{x_1, x_2, \ldots, x_t\}$ **and an integer** B **, is there a subset** $S' \subseteq S$ **s.t.** $\sum_{x_i \in S'} x_i = B$ **.**

- **Yes instance:** $S = \{1, 2, 5, 7, 8, 10, 11\}, B = 22$ **.**
- **No instance:** $S = \{4, 10, 11, 12, 15\}, B = 28$ **.**

Note: **It is still NP-complete if** $B = \sum_i x_i / 2$

3-Partition **Given a list of** $3t$ **positive integers** $S = \{x_1, x_2, \ldots, x_{3t}\}$ **with** $\sum_{x_i \in S} x_i = tB$ **, and each** x_i **satisfying** $B/4 < x_i < B/2$ **, can you partition** S **into** t **groups of size 3, such that each group sums to exactly B.**

- **Yes instance:** $S = \{26, 26, 27, 28, 29, 29, 31, 33, 39, 40, 45, 47\}$
- **No instance:** $S = \{26, 26, 27, 28, 29, 29, 31, 33, 38, 40, 45, 48\}$ **(I think)**

4 groups of 100

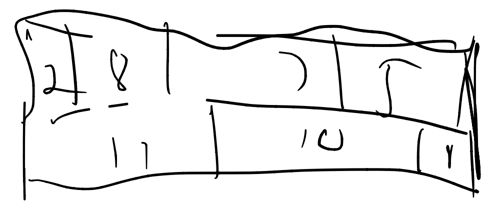
$P||C_{\max}$

Problem: Given n jobs with processing times p_j, schedule them on m machines so as to minimize the makespan.

Decision version: Given n jobs with processing times p_j and a number D, can you schedule them on m machines so as to complete by time D.

Sample inputs:

- **Jobs are** $\{1, 2, 5, 7, 8, 10, 11\}$ **, 2 machines,** $D = 22$ **.** yes
- **Jobs are** $S = \{4, 4, 10, 11, 12, 15\}$ **, 3 machines** $D = 20$ **.** no



known NP-c problem new problem

Reduction: Subset sum reduces to $P||C_{\max}$ **.**

Idea of reduction: Given a subset sum instance, create a 2-machine instance of $P||C_{\max}$ **, with** $p_j = x_j$ **and** $D = B$ **. Now there is a feasible schedule iff there is a subset summing to B.**

Subset Sum $\leq P||C_{max}$

- Given a input to subset sum $S = \{x_1, \dots x_n\}$ B, with $B = \sum x_i / 2$ ← poly time
- Form an input to $P||C_{max}$ with n jobs, $p_i = x_i$, 2 machines $D = B$.
- Solve $P||C_{max}$ output yes/no

<u>Show</u> Subset Sum outputs yes $\iff$ $P||C_{max}$ outputs yes

<u>Pf</u> $\Rightarrow$ If subsetsum = yes, then there are two subsets of jobs S_1, S_2 each summing to B, $\therefore$ the jobs on each machine sum to $B = D$, so the answer is yes

$\Leftarrow$ If $P||C_{max}$ is yes, then $C_{max} \leq D$, but $\sum p_j = 2D$, so $C_{max} = D$, so the schedule gives 2 sets of jobs, each of total size D, therefore it gives a partition of S into 2 sets of size B.

$\boxtimes$

Also, the reduction is just copying the input $\therefore$ polynomial time.

$1|r_j|L_{\max}$

Reduction: **Reduce 3-partition to** $1|r_j|L_{\max}$**.**

3-Partition **Given a list of** $3t$ **positive integers** $S = \{x_1, x_2, \ldots, x_{3t}\}$ **with** $\sum_{x_i \in S} x_i = tB$ **, can you partition** S **into** t **groups of size 3, such that each group sums to exactly B,** each $\frac{B}{4} \leq x_i \leq \frac{B}{2}$.

Given a 3-partition instance, we will creat a $1|r_j|L_{\max}$ **instance in the following way:**

Jobs: $n = 4t - 1$ **jobs,** $t - 1$ **of which are dummy jobs**

Dummy Jobs:

j	r_j	p_j	d_j
1	B	1	B+1
2	2B +1	1	2B+2
3	3B + 2	1	3B+3
⋮	⋮	⋮	⋮
$t-1$	$(t-1)B + (t-2)$	1	$(t-1)B + (t-1)$

Real Jobs:

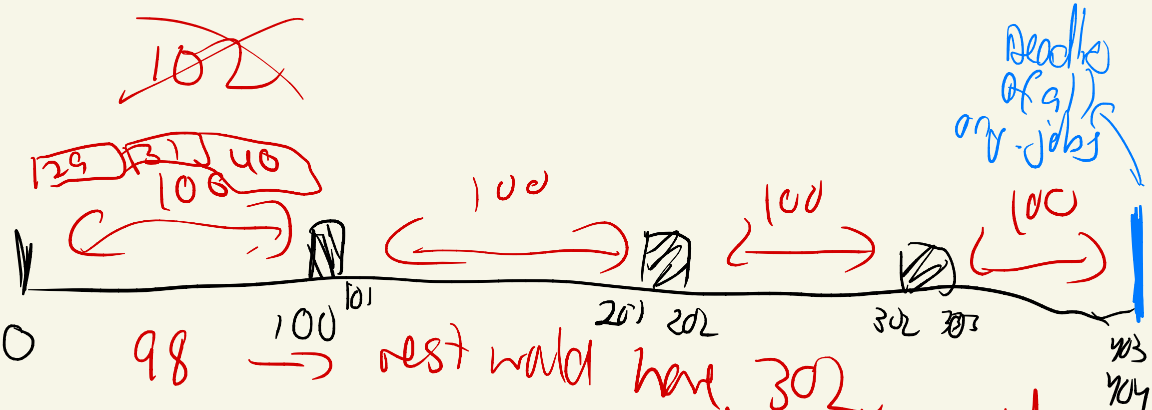
- **indexed** t **through** $4t - 1$ **.**
- **All have** $r_j = 0$

- **All have** $d_j = tb + (t-1)$
- $p_j = x_{j-(t-1)}$

3 partition = yes $\iff$ $L_{max} = 0$

Reduction so that all jobs meet their deadlines ~~iff~~ 3-partition has a solution

(12 jobs 4 groups of 3 each summing to 100)



98 $\rightarrow$ rest would have 302 and any job would miss a deadline

$r_j = 100$ $p_j = 1$ $d_j = 101$

$r_j = 201$ $p_j = 1$ $d_j = 202$

- **All have** $d_j = tb + (t-1)$
- $p_j = x_{j-(t-1)}$

j	r_j	p_j	d_j
1	100	1	10
2	201	1	202
3	302	1	303
4	0	26	403
	0	26	⋮
	⋮	27	
		28	
		⋮	
15	0		407

- **All have** $d_j = tb + (t-1)$
- $p_j = x_{j-(t-1)}$

poly time

Show

3-partition is yes

$\Longleftrightarrow$ $C_{max} = 0$

$\Rightarrow$ if 3-partition is yes, schedule follows the 3-partition sol.

$\Leftarrow$ if 3-partition is no, then one group is $> B$, and that forces a job to miss its deadline.

- **All have** $d_j = tb + (t-1)$
- $p_j = x_{j-(t-1)}$

Proof

Dummy Jobs:

j	r_j	p_j	d_j
1	**B**	**1**	**B+1**
2	**2B +1**	**1**	**2B+2**
3	**3B + 2**	**1**	**3B+3**
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t-1$	$(t-1)B+(t-2)$	**1**	$(t-1)B+(t-1)$

Real Jobs:

- **indexed** t **through** $4t-1$ **.**
- **All have** $r_j = 0$
- **All have** $d_j = tb + (t-1)$
- $p_j = x_{j-(t-1)}$

Idea of Proof: **Argue that there is a schedule with** $L_{\max} = 0$ **iff the partition instance is yes.**