4 NPV and the Time Value of Money

LEARNING OBJECTIVES

- Construct a cash flow timeline as the first step in solving problems
- Calculate the present and future value of a single cash flow for any time span given
- Value a series of many cash flows
- Understand how to compute the net present value of any set of cash flows
- Apply shortcuts to value special sets of regular cash flows called *perpetuities* and *annuities*
- Compute the number of periods, cash flow, or rate of return in a loan or investment

notation

C	cash flow	NPV	net present value
C_t	cash flow at date t	P	initial principal or deposit, or equivalent present value
FV	future value	PV	present value
FV_t	future value on date t	PV_t	present value on date t
g	growth rate	r	interest rate
IRR	internal rate of return		
n	number of periods for time value calculations		

INTERVIEW WITH Amanda Wittick, Investors Group

Amanda Wittick is a senior business analyst for Investors Group, one of Canada's largest providers of personal financial planning services. She graduated from the University of Manitoba in 2008 with a Bachelor of Commerce degree.

Amanda works out of the company's Winnipeg office, providing liaison support between the IT and business units. "I help define business needs and facilitate the implementation of projects," she says. "While the project manager defines the project needs and assigns resources, I go through the business requirements and identify the potential risks and their impact on the project."

Her job often requires her to apply some of the tools learned in finance class. "When we're looking at implementing a new project, we need to go through a cost-benefit analysis to see if the project is worthwhile. We calculate the *NPV* of the project when making our decision. For example, since many projects we undertake involve the hiring of new employees, when calculating the *PV* of the employees' salaries we make the assumption that these new employees will be around forever, and we use the perpetuity formula taught in finance class. Our finance department will provide us with an appropriate discount rate."

Amanda credits her finance classes with providing her with the necessary background to excel at her occupation. "My studies in finance have provided me with a crucial understanding of the products we offer our clients, without which I would not be capable of effectively performing my job."

University of Manitoba, 2008

"We calculate the *NPV* of the project when making our decision."

As we discussed in Chapter 3, to evaluate a project a financial manager must compare its costs and benefits. In most cases, the cash flows in financial investments involve more than one future period. Thus, the financial manager is faced with the task of trading off a known upfront cost against a series of uncertain future benefits. As we learned, calculating the net present value does just that, such that if the *NPV* of an investment is positive, we should take it.

Calculating the *NPV* requires tools to evaluate cash flows lasting several periods. We develop these tools in this chapter. The first tool is a visual method for representing a series of cash flows: the *timeline.* After constructing a timeline, we establish three important rules for moving cash flows to different points in time. Using these rules, we show how to compute the present and future values of the costs and benefits of a general stream of cash flows, and how to compute the *NPV*. Although we can use these techniques to value any type of asset, certain types of assets have cash flows that follow a regular pattern. We develop shortcuts for *annuities*, *perpetuities*, and other special cases of assets with cash flows that follow regular patterns.

In Chapter 5, we will learn how interest rates are quoted and determined. Once we understand how interest rates are quoted, it will be straightforward to extend the tools of this chapter to cash flows that occur more frequently than once per year.

4.1 The Timeline

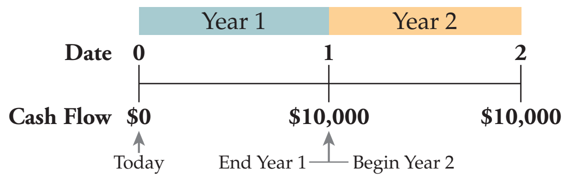
stream of cash flows A series of cash flows lasting several periods.

timeline A linear representation of the timing of (potential) cash flows.

We begin our discussion of valuing cash flows lasting several periods with some basic vocabulary and tools. We refer to a series of cash flows lasting several periods as a **stream of cash flows**. We can represent a stream of cash flows on a **timeline**, a linear representation of the timing of the expected cash flows. Timelines are an important first step in organizing and then solving a financial problem. We use them throughout this text.

Constructing a Timeline

To illustrate how to construct a timeline, assume that a friend owes you money. He has agreed to repay the loan by making two payments of $10,000 at the end of each of the next two years. We represent this information on a timeline as follows:



Date 0 represents the present. Date 1 is one year later and represents the end of the first year. The $10,000 cash flow below date 1 is the payment you will receive at the end of the first year. Date 2 is two years from now; it represents the end of the second year. The $10,000 cash flow below date 2 is the payment you will receive at the end of the second year.

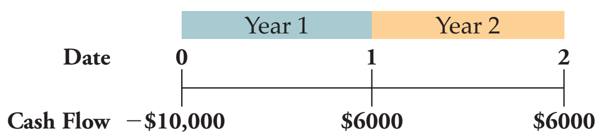
Identifying Dates on a Timeline

To track cash flows, we interpret each point on the timeline as a specific date. The space between date 0 and date 1 then represents the time period between these dates—in this case, the first year of the loan. Date 0 is the beginning of the first year, and date 1 is the end of the first year. Similarly, date 1 is the beginning of the second year, and date 2 is the end of the second year. By denoting time in this way, date 1 signifies *both* the end of year 1 and the beginning of year 2, which makes sense since those dates are effectively the same point in time.[1]

Distinguishing Cash Inflows from Outflows

In this example, both cash flows are inflows. In many cases, however, a financial decision will involve both inflows and outflows. To differentiate between the two types of cash flows, we assign a different sign to each: Inflows (cash flows received) are positive cash flows, whereas outflows (cash flows paid out) are negative cash flows.

To illustrate, suppose you have agreed to lend your brother $10,000 today. Your brother has agreed to repay this loan in two installments of $6000 at the end of each of the next two years. The timeline is



[1]That is, there is no real time difference between a cash flow paid at 11:59 P.M. on December 31 and one paid at 12:01 A.M. on January 1, although there may be some other differences such as taxation that we will overlook for now.

Notice that the first cash flow at date 0 (today) is represented as –$10,000 because it is an outflow. The subsequent cash flows of $6000 are positive because they are inflows.

Representing Various Time Periods

So far, we have used timelines to show the cash flows that occur at the end of each year. Actually, timelines can represent cash flows that take place at any point in time. For example, if you pay rent each month, you could use a timeline such as the one in our first example to represent two rental payments, but you would replace the "year" label with "month."

Many of the timelines included in this chapter are very simple. Consequently, you may feel that it is not worth the time or trouble to construct them. As you progress to more difficult problems, however, you will find that timelines identify events in a transaction or investment that are easy to overlook. If you fail to recognize these cash flows, you will make flawed financial decisions. Therefore, approach *every* problem by drawing the timeline as we do in this chapter.

Concept Check

1. What are the key elements of a timeline?
2. How can you distinguish cash inflows from outflows on a timeline?

4.2 Valuing Cash Flows at Different Points in Time

Financial decisions often require comparing or combining cash flows that occur at different points in time. In this section, we introduce three important rules central to financial decision making that allow us to compare or combine values.

Rule 1: Comparing and Combining Values

Our first rule is that it is only possible to compare or combine values at the same point in time. This rule restates a conclusion introduced in Chapter 3: Only cash flows in the same units can be compared or combined. A dollar today and a dollar in one year are not equivalent. Having money now is more valuable than having money in the future; if you have the money today you can earn interest on it.

Common Mistake Summing Cash Flows Across Time

Once you understand the time value of money, our first rule may seem straightforward. However, it is very common, especially for those who have not studied finance, to violate this rule, simply treating all cash flows as comparable, regardless of when they are received. One example of this is in sports contracts. In 2007, Alex Rodriguez and the New York Yankees were negotiating what was repeatedly referred to as a "$275 million" contract. The $275 million comes from simply adding up all of the payments that he would receive over the ten years of the contract and an additional ten years of deferred payments—treating dollars received in 20 years the same as dollars received today. The same thing occurred when David Beckham signed a "$250 million" contract with the LA Galaxy soccer team.

To compare or combine cash flows that occur at different points in time, you first need to convert the cash flows into the same units by moving them to the same point in time. The next two rules show how to move the cash flows on the timeline.

Rule 2: Compounding

Suppose we have \$1000 today, and we wish to determine the equivalent amount in one year's time. As we saw in Chapter 3, if the current market interest rate is $r = 10\%$, we can use the interest rate factor $(1 + r) = 1.10$ as an exchange rate through time—meaning the rate at which we exchange a currency amount today for the same currency but in one year. That is:

$$(\$1000 \text{ today}) \times (1.10 \text{ \$ in one year / \$ today}) = \$1100 \text{ in one year}$$

In general, if the market interest rate for the year is r, then we multiply by the interest rate factor $(1 + r)$ to move the cash flow from the beginning to the end of the year. We multiply by $(1 + r)$ because at the end of the year you will have $(1 \times$ your original investment) plus interest in the amount of $(r \times$ your original investment). This process of moving forward along the timeline to determine a cash flow's value in the future (its **future value**) is known as **compounding**. *Our second rule stipulates that to calculate a cash flow's future value, you must compound it.*

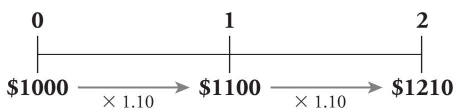
future value The value of a cash flow that is moved forward in time.

compounding Computing the future value of a cash flow over a long horizon by multiplying by the interest rate factors associated with each intervening period.

We can apply this rule repeatedly. Suppose we want to know how much the \$1000 is worth in two years' time. If the interest rate for year 2 is also 10%, then we convert as we just did:

$$(\$1100 \text{ in one year}) \times (1.10 \text{ \$ in two years / \$ in one year}) = \$1210 \text{ in two years}$$

Let's represent this calculation on a timeline:



Given a 10% interest rate, all of the cash flows—\$1000 at date 0, \$1100 at date 1, and \$1210 at date 2—are equivalent. They have the same value to us but are expressed in different units (dollars at different points in time). An arrow that points to the right indicates that the value is being moved forward in time—that is, it is being compounded.

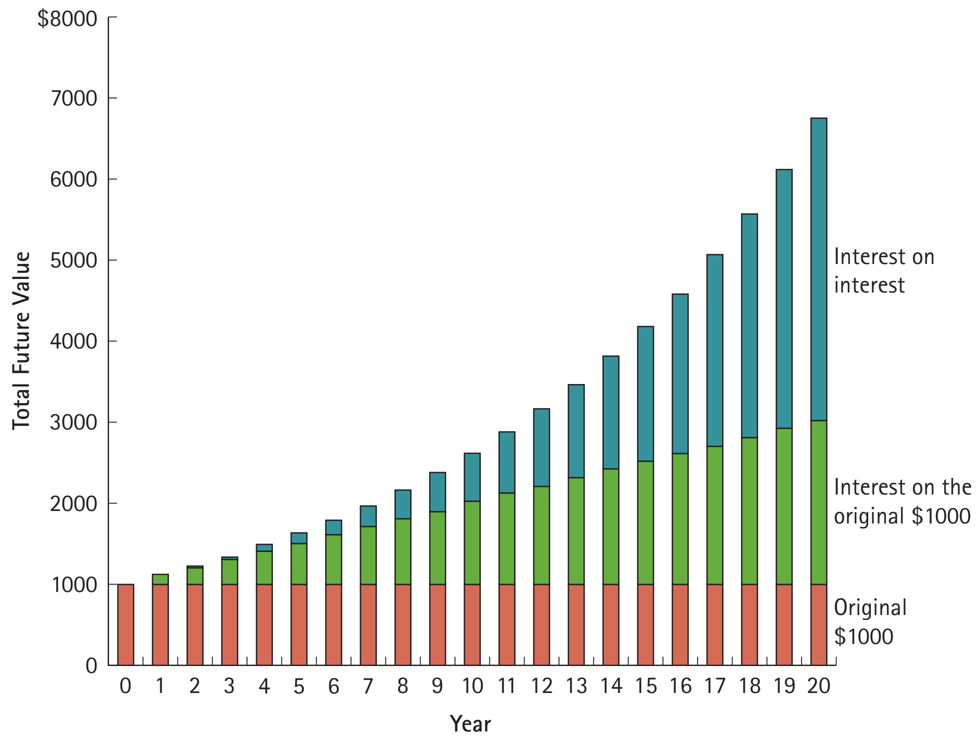
In the preceding example, \$1210 is the future value of \$1000 two years from today. Note that the value grows as we move the cash flow further in the future. In Chapter 3, we defined the time value of money as the difference in value between money today and money in the future. Here, we can say that \$1210 in two years is the equivalent amount to \$1000 today. The reason money is more valuable to you today is that you have opportunities to invest it. As in this example, by having money sooner, you can invest it (here at a 10% return) so that it will grow to a larger amount of money in the future. Note also that the equivalent amount grows by \$100 the first year, but by \$110 the second year. In the second year, we earn interest on our original \$1000, plus we earn interest on the \$100 interest we received in the first year. This effect of earning interest both on the original principal and on the accrued interest, is known as **compound interest**. Figure 4.1 shows how over time the amount of money you earn from interest on interest grows so that it will eventually exceed the amount of money that you earn as interest on your original deposit.

compound interest The combination of earning interest on the original principal and earning interest on accrued interest.

FIGURE 4.1

The Composition of Interest over Time

This bar graph shows how the account balance and the composition of the interest changes over time when an investor starts with an original deposit of $1000, represented by the red area, in an account earning 10% interest over a 20-year period. Note that the turquoise area representing interest on interest grows, and by year 15 has become larger than the interest on the original deposit, shown in green. Over the 20 years of the investment, the interest on interest the investor earned is $3727.50, while the interest earned on the original $1000 principal is $2000. The total compound interest over the 20 years is $5727.50 (the sum of the interest on interest and the interest on principal). Combining the original principal of $1000 with the total compound interest gives the future value after 20 years of $6727.50.



How does the future value change in the third year? Continuing to use the same approach, we compound the cash flow a third time. Assuming the competitive market interest rate is fixed at 10%, we get

$$\$1000 \times (1.10) \times (1.10) \times (1.10) = \$1000 \times (1.10)^3 = \$1331$$

In general, if we have a cash flow now, C_0, to compute its value n periods into the future, we must compound it by the n intervening interest rate factors. If the interest rate r is constant, this calculation yields

Future Value of a Cash Flow

$$FV_n = C_0 \times \underbrace{(1 + r) \times (1 + r) \times \cdots \times (1 + r)}_{n \text{ times}} = C_0 \times (1 + r)^n \qquad (4.1)$$

Rule of 72

Another way to think about the effect of compounding is to consider how long it will take your money to double, given different interest rates. Suppose you want to know how many years it will take for \$1 to grow to a future value of \$2. You want the number of years, *n*, to solve

$$FV_n = \$1 \times (1 + r)^n = \$2$$

If you solve this formula for different interest rates, you will find the following approximation:

Years to double $\approx$ 72 $\div$ (interest rate in percent)

This simple "Rule of 72" is fairly accurate (i.e., within one year of the exact doubling time) for interest rates higher than 2%. For example, if the interest rate is 9%, the doubling time should be about 72 $\div$ 9 = 8 years. Indeed, $1.09^8 = 1.99$! So, given a 9% interest rate, your money will approximately double every eight years.

Rule 3: Discounting

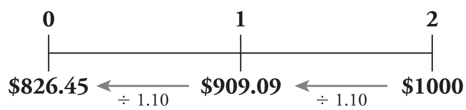
The third rule describes how to put a value today on a cash flow that comes in the future. Suppose you would like to compute the value today of \$1000 that you anticipate receiving in one year. If the current market interest rate is 10%, you can compute this value by converting units as we did in Chapter 3:

$$(\$1000 \text{ in one year}) \div (1.10 \text{ \$ in one year / \$ today}) = \$909.09 \text{ today}$$

That is, to move the cash flow back along the timeline, we divide it by the interest rate factor $(1 + r)$ where r is the interest rate—this is the same as multiplying by the discount factor, $\frac{1}{(1 + r)}$. This process of finding the equivalent value today of a future cash flow is known as **discounting**. *Our third rule stipulates that to calculate the value of a future cash flow at an earlier point in time, we must discount it.*

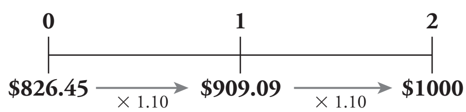
discounting Finding the equivalent value today of a future cash flow by multiplying by a discount factor, or equivalently, dividing by 1 plus the discount rate.

Suppose that you anticipate receiving the \$1000 two years from today rather than in one year. If the interest rate for both years is 10%, you can prepare the following timeline:

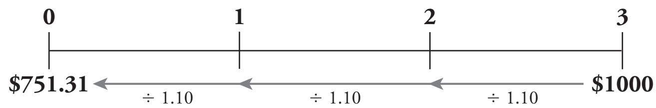


When the interest rate is 10%, all of the cash flows—\$826.45 at date 0, \$909.09 at date 1, and \$1000 at date 2—are equivalent. They represent the same value to us but in different units (different points in time). The arrow points to the left to indicate that the value is being moved backward in time or discounted. Note that the value decreases the further in the future is the original cash flow.

The value of a future cash flow at an earlier point on the timeline is its present value at the earlier point in time. That is, \$826.45 is the present value at date 0 of \$1000 in two years. Recall from Chapter 3 that the present value is the "do-it-yourself" price to produce a future cash flow. Thus, if we invested \$826.45 today for two years at 10% interest, we would have a future value of \$1000, using the second rule of valuing cash flows:



Suppose the \$1000 were three years away and you wanted to compute the present value. Again, if the interest rate is 10%, we have:



That is, the present value today of a cash flow of \$1000 in three years is given by:

$$\$1000 \div (1.10) \div (1.10) \div (1.10) = \$1000 \div (1.10)^3 = \$751.31$$

In general, to compute the present value today (date 0) of a cash flow C_n that comes n periods from now, we must discount it by the n intervening interest rate factors. If the interest rate r is constant, this yields:

Present Value of a Cash Flow

$$PV_0 = C_n \div (1 + r)^n = \frac{C_n}{(1 + r)^n} \tag{4.2}$$

EXAMPLE 4.1

Present Value of a Single Future Cash Flow

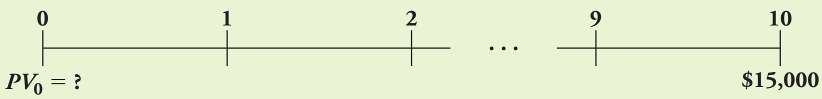
Problem

You are considering investing in a Government of Canada bond that will pay \$15,000 in ten years. If the competitive market interest rate is fixed at 6% per year, what is the bond worth today?

Solution

Plan

First setup your timeline. The cash flows for this bond are represented by the following timeline:



Thus, the bond is worth \$15,000 in ten years. To determine the value today, PV_0, we compute the present value using Equation 4.2 with our interest rate of 6%.

Execute

$$PV_0 = \frac{15{,}000}{1.06^{10}} = \$8375.92 \text{ today}$$

Using a financial calculator or Excel (see the appendix for step-by-step instructions):

	N	I/Y	PV	PMT	FV
Given:	10	6		0	15,000
Solve for:			−8375.92		

Excel Formula: $=PV(RATE,NPER,PMT,FV)=PV(0.06,10,0,15000)$

Evaluate

The bond is worth much less today than its final payoff because of the time value of money.

Using a Financial Calculator: Solving for Present and Future Values

So far, we have used formulas to compute present values and future values. Both financial calculators and spreadsheets have these formulas pre-programmed to quicken the process. In this box, we focus on financial calculators, but spreadsheets such as Excel have very similar shortcut functions.

Financial calculators have a set of functions that perform the calculations that finance professionals do most often. The functions are all based on the following timeline, which among other things can handle most types of loans:

There are a total of five variables: *N*, *PV*, *PMT*, *FV*, and the interest rate, denoted *I/Y*. Each function takes four of these variables as inputs and returns the value of the fifth one that ensures that the *NPV* of the cash flows is zero.

By setting the intermediate payments equal to 0, you could compute present and future values of single cash flows such as we have done above using Equations 4.1 and 4.2. In the examples in Section 4.5, we will calculate cash flows using the PMT button. The best way to learn to use a financial calculator is by practising. We present one example below. We will also show the calculator buttons for any additional examples in this chapter that can be solved with financial calculator functions. Finally, the appendix to this chapter contains step-by-step instructions for using the two most popular financial calculators.

Example

Suppose you plan to invest $20,000 in an account at the Canadian Western Bank paying 8% interest. How much will you have in the account in 15 years? We represent this problem with the following timeline:

To compute the solution, we enter the four variables we know, $N = 15$, $I/Y = 8$, $PV = -20{,}000$, PMT $= 0$, and solve for the one we want to determine: *FV*. Specifically, for the HP-10BII or TI-BAII Plus calculators:

1. Enter 15 and press the N key.
2. Enter 8 and press the I/Y key (I/YR for the HP calculator).
3. Enter −20,000 and press the PV key.
4. Enter 0 and press the PMT key.
5. Press the FV key (for the Texas Instruments calculator, press CPT and then FV).

	N	I/Y	PV	PMT	FV
Given:	15	8	−20,000	0	
Solve for:					63,443

Excel Formula: $= FV(0.08,15,0,-20000)$

The calculator then shows a future value of $63,443.

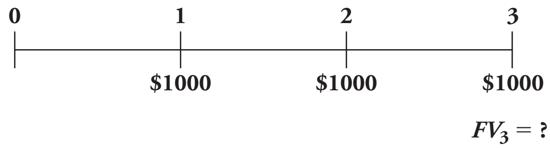
Note that we entered *PV* as a negative number (the amount we are putting *into* the bank), and *FV* is shown as a positive number (the amount we can take *out* of the bank). It is important to use signs correctly to indicate the direction in which the money is flowing when using the calculator functions. You will see more examples of getting the sign of the cash flows correct throughout the chapter.

Excel has the same functions, but it calls “N,” “NPER” and “I/Y,” “RATE”. ***Also, it is important to note that you enter an interest rate of 8% as “8” in a financial calculator, but as 0.08 in Excel.***

Applying the Rules of Valuing Cash Flows

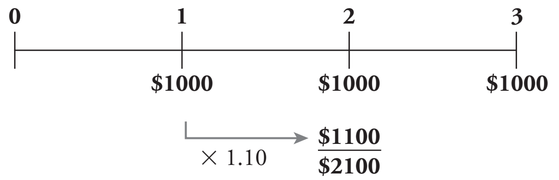
The rules of cash flow valuation allow us to compare and combine cash flows that occur at different points in time. Suppose we plan to deposit into our President's Choice Financial bank account $1000 at the end of each of the next three years. If we earn a fixed 10% interest rate on our savings, how much will we have three years from today (just after the last deposit)?

Again, we start with a timeline:

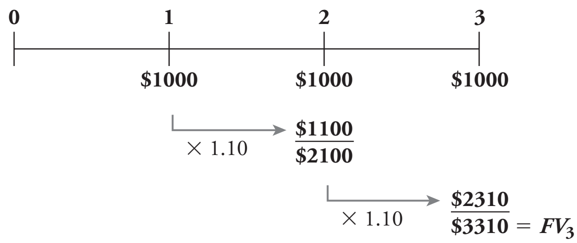


The timeline shows the three deposits we plan to make. We need to compute their value at the end of three years.

We can use the cash flow valuation rules in a number of ways to solve this problem. First, we can take the deposit at date 1 and move it forward to date 2. Because it is then in the same time period as the date 2 deposit, we can combine the two amounts to find out the total in the bank on date 2:

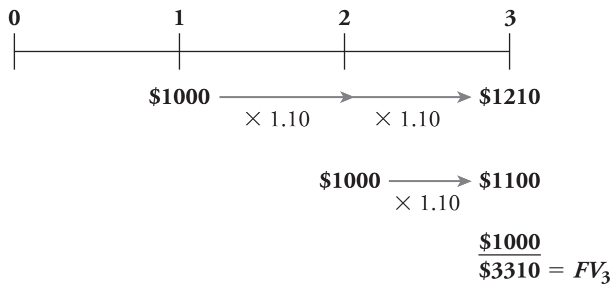


Using the first two rules, we find that our total savings on date 2 will be $2100. Continuing in this fashion, we can solve the problem as follows:



The total amount we will have in the President's Choice Financial bank account at the end of three years is $3310. This amount is the future value of our three $1000 savings deposits.

Another approach to the problem is to compute the future value in year 3 of each cash flow separately. Once all three amounts are in year 3 dollars, we can then combine them.



Both calculations give the same future value of $3310. As long as we follow the rules, we get the same result. The order in which we apply the rules does not matter. The calculation we choose depends on which is more convenient for the problem at hand. Table 4.1 summarizes the three rules of valuing cash flows and their associated formulas.

TABLE 4.1 The Three Rules of Valuing Cash Flows

Rule	Formula
1: Only values at the same point in time can be compared or combined.	None
2: To calculate a cash flow's future value, we must compound it.	Future value of a cash flow: $FV_n = C_0 \times (1 + r)^n$
3: To calculate the present value of a future cash flow, we must discount it.	Present value of a cash flow: $PV_0 = C_n \div (1 + r)^n = \dfrac{C_n}{(1 + r)^n}$

EXAMPLE 4.2

Personal Finance
Computing the Future Value

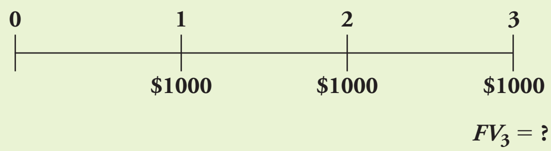
Problem

Let's revisit the savings plan we considered earlier: We plan to save $1000 at the end of each of the next three years. At a fixed 10% interest rate, how much will we have in the President's Choice Financial bank account three years from today?

Solution

Plan

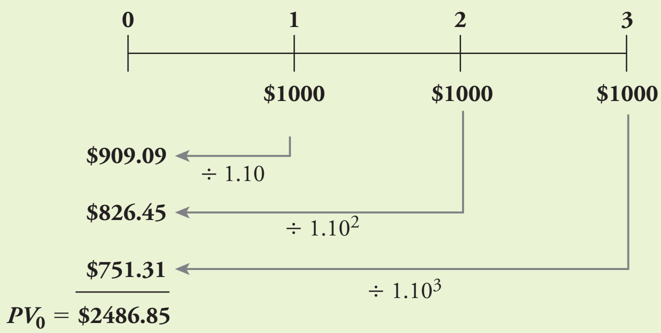
We'll start with the timeline for this savings plan:



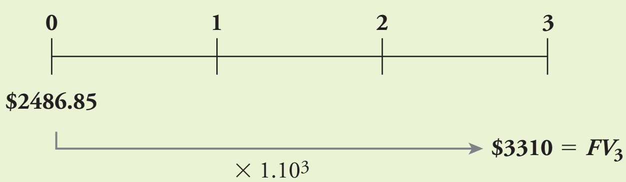
Let's solve this problem in a different way than we did in the text, while still following the rules we established. First we'll compute the present value of the cash flows. Then we'll compute its value three years later (its future value).

Execute

There are several ways to calculate the present value of the cash flows. Here, we treat each cash flow separately and then combine the present values.



Saving $2486.85 today is equivalent to saving $1000 per year for three years. Now let's compute the future value in year 3 of that $2486.85:



Evaluate
This answer of $FV_3 = \$3310$ is precisely the same result we found earlier. As long as we apply the three rules of valuing cash flows, we will always get the correct answer.

Concept Check

3. Can you compare or combine cash flows at different times?

4. What do you need to know to compute a cash flow's present or future value?

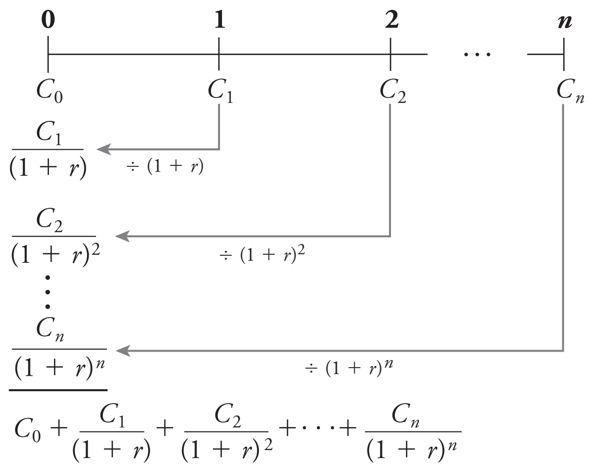
4.3 Valuing a Stream of Cash Flows

Most investment opportunities have multiple cash flows that occur at different points in time. In Section 4.2, we learned the rules to value such cash flows. Now we formalize this approach by deriving a general formula for valuing a stream of cash flows.

Consider a stream of cash flows: C_0 at date 0, C_1 at date 1, and so on, up to C_n at date n. We represent this cash flow stream on a timeline as follows:

Using the rules of cash flow valuation, we compute the present value of this cash flow stream in two steps. First, we compute the present value of each individual cash flow. Then, once the cash flows are in common units of dollars today, we can combine them.

For a given interest rate r, we represent this process on the timeline as follows:



This equation provides the general formula for the present value of a cash flow stream:

$$PV_0 = C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_n}{(1+r)^n} \qquad (4.3)$$

That is, the present value of the cash flow stream is the sum of the present values of each cash flow. Recall from Chapter 3 that we defined the present value as the dollar amount you would need to invest today to produce the single cash flow in the future. The same idea holds in this context. The present value is the amount you need to invest today to generate the cash flows stream $C_0, C_1, \ldots, C_n$. That is, receiving those cash flows is equivalent to having their present value in the bank today.

EXAMPLE 4.3

Personal Finance
Present Value of a Stream of Cash Flows

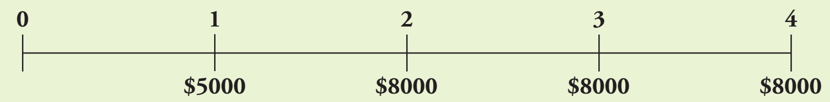
Problem

You have just graduated and need money to buy a new car. Your rich Uncle Henry will lend you the money so long as you agree to pay him back within four years, and you offer to pay him the rate of interest that he would otherwise get by putting his money in a savings account. Based on your earnings and living expenses, you think you will be able to pay him \$5000 in one year, and then \$8000 each year for the next three years. If Uncle Henry would otherwise earn 6% per year on his savings, how much can you borrow from him?

Solution

Plan

The cash flows you can promise Uncle Henry are as follows:



How much money should Uncle Henry be willing to give you today in return for your promise of these payments? He should be willing to give you an amount that is equivalent to these payments in present value terms. This is the amount of money that it would take him to produce these same cash flows. We will (1) solve the problem using Equation 4.3 and then (2) verify our answer by calculating the future value of this amount.

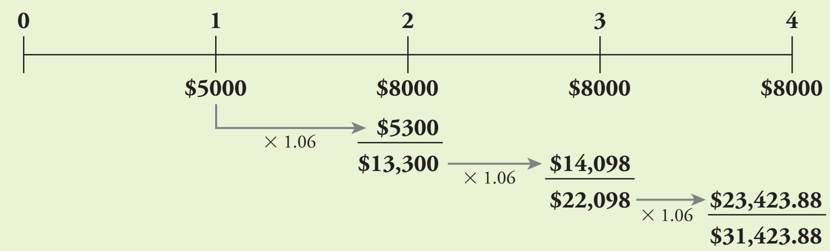
Execute

1. We can calculate the *PV* as follows:

$$PV = \frac{5000}{1.06} + \frac{8000}{1.06^2} + \frac{8000}{1.06^3} + \frac{8000}{1.06^4}$$
$$= 4716.98 + 7119.97 + 6716.95 + 6336.75$$
$$= \$24{,}890.65$$

Now suppose that Uncle Henry gives you the money, and then deposits your payments to him in the bank each year. How much will he have four years from now?

We need to compute the future value of the annual deposits. One way to do so is to compute the bank balance each year:



2. To verify our answer, suppose your uncle kept his \$24,890.65 in the bank today earning 6% interest. In four years he would have:

$$FV_4 = \$24{,}890.65 \times (1.06)^4 = \$31{,}423.87 \text{ in 4 years}$$

We get the same answer both ways (within a penny, which is because of rounding).

Evaluate

Thus, Uncle Henry should be willing to lend you $24,890.65 in exchange for your promised payments. This amount is less than the total you will pay him ($5000 + $8000 + $8000 + $8000 = $29,000) due to the time value of money.

Example 4.3 illustrates that if you want to compute the future value of a stream of cash flows, you can do it directly (the first approach used in Example 4.3), or you can first compute the present value and then move it to the future (the second approach). As always, we use Eq. 4.1 to calculate the future value of any present value. Because we obey the rules of valuing cash flows in both cases, we get the same result.

Concept Check

5. How do you calculate the present value of a cash flow stream?

6. How do you calculate the future value of a cash flow stream?

4.4 The Net Present Value of a Stream of Cash Flows

Now that we have established how to compute present and future values, we are ready to address our central goal: calculating the *NPV* of future cash flows to evaluate an investment decision. The Valuation Principle tells us that the value of a decision is the value of its benefits minus the value of its costs. *NPV* values those benefits and costs in today's dollars. Recall from Chapter 3 that we defined the net present value (*NPV*) of an investment decision as follows:

$$NPV = PV(\text{benefits}) - PV(\text{costs})$$

In this context, the benefits are the cash inflows, and the costs are the cash outflows. We can represent any investment decision on a timeline as a cash flow stream, where the cash outflows (investments) are negative cash flows and the inflows are positive

EXAMPLE 4.4

Personal Finance
Net Present Value of an Investment Opportunity

Problem

You have been offered the following investment opportunity: If you invest $1000 today, you will receive $500 at the end of each of the next two years, followed by $550 at the end of the third year. If you could otherwise earn 10% per year on your money, should you undertake the investment opportunity?

Solution

Plan

As always, start with a timeline. We denote the upfront investment as a negative cash flow (because it is money you need to spend) and the money you receive as a positive cash flow.

0	1	2	3
−$1000	$500	$500	$550

To decide whether you should accept this opportunity, you'll need to compute the *NPV* by computing the present value of the stream.

Execute

The *NPV* is:

$$NPV = -1000 + \frac{500}{1.10} + \frac{500}{1.10^2} + \frac{550}{1.10^3} = \$280.99$$

Evaluate

Because the *NPV* is positive, the benefits exceed the costs and you should make the investment. Indeed, the *NPV* tells us that taking this opportunity is equivalent to getting an extra $280.99 that you can spend today. To illustrate, suppose you borrow $1000 to invest in the opportunity and an extra $280.99 to spend today. How much would you owe on the $1280.99 loan in three years? At 10% interest, the amount you would owe would be:

$$FV_3 = (\$1000 + \$280.99) \times (1.10)^3 = \$1705 \text{ in three years}$$

At the same time, the investment opportunity generates cash flows. If you put these cash flows into a bank account, how much will you have saved three years from now? The future value of the savings is

$$FV_3 = (\$500 \times 1.10^2) + (\$500 \times 1.10) + \$550 = \$1705 \text{ in three years}$$

As you see, you can use your bank savings to repay the loan. Taking the opportunity therefore allows you to spend $280.99 today at no extra cost.

cash flows. Thus, the *NPV* of an investment opportunity is also the *present value* of the stream of cash flows of the opportunity:

$$NPV = PV(\text{benefits}) - PV(\text{costs}) = PV(\text{benefits} - \text{costs})$$

In principle, we have met the goal we set at the beginning of the chapter: How financial managers should evaluate a project. We have developed the tools to evaluate the cash flows of a project. We have shown how to compute the *NPV* of an investment opportunity that lasts more than one period. In practice, when the number of cash flows exceeds four or five (as it most likely will), the calculations can become tedious. Fortunately, a number of special cases do not require us to discount each cash flow separately. We derive these shortcuts in the next section.

Concept Check

7. What benefit does a firm receive when it accepts a project with a positive *NPV*?

8. How do you calculate the net present value of a cash flow stream?

4.5 Perpetuities, Annuities, and Other Special Cases

The formulas we have developed so far allow us to compute the present or future value of any cash flow stream. In this section, we consider two types of cash flow streams, *perpetuities* and *annuities,* and learn shortcuts for valuing them. These shortcuts are possible because the cash flows follow a regular pattern.

perpetuity A stream of equal cash flows that occurs at regular intervals and lasts forever.

consol A bond that promises its owner a fixed cash flow every year, forever.

Perpetuities

A **perpetuity** is a stream of equal cash flows that occur at regular intervals and last forever. One example is the British government bond called a **consol** (or perpetual bond). Consol bonds promise the owner a fixed cash flow every year, forever.

Here is the timeline for a perpetuity:

Note from the timeline that the first cash flow does not occur immediately; *it arrives at the end of the first period.* This timing is sometimes referred to as payment *in arrears* and is a standard convention in loan payment calculations and elsewhere, so we adopt it throughout this text.

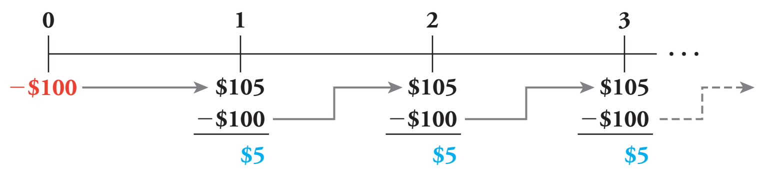
Using the formula for the present value, the present value of a perpetuity with payment C and interest rate r is given by:

$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots$$

Notice that all of the cash flows (C in the formula) are the same because the cash flow for a perpetuity is constant: thus we do not need a time subscript on each one. Also, because the first cash flow is in one period, there is no cash flow at time 0 ($C_0 = 0$).

To find the value of a perpetuity by discounting one cash flow at a time would take forever—literally! You might wonder how, even with a shortcut, the sum of an infinite number of positive terms could be finite. The answer is that the cash flows in the future are discounted for an ever increasing number of periods, so their contribution to the sum eventually becomes negligible.

To derive the shortcut, we calculate the value of a perpetuity by creating our own perpetuity. The Valuation Principle tells us that the value of a perpetuity must be the same as the cost we incurred to create our own identical perpetuity. To illustrate, suppose you could invest \$100 in a bank account paying 5% interest per year forever. At the end of one year, you will have \$105 in the bank—your original \$100 plus \$5 in interest. Suppose you withdraw the \$5 interest and reinvest the \$100 for a second year. Again, you will have \$105 after one year, and you can withdraw \$5 and reinvest \$100 for another year. By doing this year after year, you can withdraw \$5 every year in perpetuity:



By investing \$100 in the bank today, you can, in effect, create a perpetuity paying \$5 per year. Recall from Chapter 3 that the Law of One Price tells us that equivalent cash flows must have the same price in every market. Because the bank will "sell" us (allow us to create) the perpetuity for \$100, the present value of the \$5 per year in perpetuity is this "do-it-yourself" cost of \$100.

Now let's generalize this argument. Suppose we invest an amount P in a bank account with an interest rate r. Every year we can withdraw the interest we have earned, $C = r \times P$, leaving the principal, P, in the bank. Because our cost for creating the perpetuity is only the initial investment of principal (P), the value of receiving C in perpetuity is therefore the upfront cost P. Rearranging $C = r \times P$ to solve for P we have $P = C/r$. Therefore

Present Value of a Perpetuity with Constant Cash Flows, *C*, and Discount Rate, *r*

$$PV_0 = \frac{C}{r} \tag{4.4}$$

Historical Examples of Perpetuities

Companies sometimes issue bonds that they call perpetuities, but in fact are not really perpetuities. For example, according to *Dow Jones International News* (February 26, 2004), in 2004 Korea First Bank sold $300 million of debt in "the form of a so-called 'perpetual bond' that has no fixed maturity date." Although the bond has no fixed maturity date, Korea First Bank has the right to pay it back after 10 years, in 2014. Korea First Bank also has the right to extend the maturity of the bond for another 30 years after 2014. Thus, although the bond does not have a fixed maturity date, it will eventually mature—in either 10 or 40 years. The bond is not really a perpetuity because it does not pay interest forever.

Perpetual bonds were some of the first bonds ever issued. The oldest perpetuities that are still making interest payments were issued by the *Hoogheemraadschap Lekdijk Bovendams,* a seventeenth-century Dutch water board responsible for upkeep of the local dikes. The oldest bond dates from 1624. Two finance professors at Yale University, William Goetzmann and Geert Rouwenhorst, personally verified that these bonds continue to pay interest. On behalf of Yale, they purchased one of these bonds on July 1, 2003, and collected 26 years of back interest. On its issue date in 1648, this bond originally paid interest in Carolus guilders. Over the next 355 years, the currency of payment changed to Flemish pounds, Dutch guilders, and most recently euros. Currently, the bond pays interest of €11.34 annually.

Although the Dutch bonds are the oldest perpetuities still in existence, the first perpetuities date from much earlier times. For example, *census agreements* and *rentes,* which were forms of perpetuities and annuities, were issued in the twelfth century in Italy, France, and Spain. They were initially designed to circumvent the usury laws of the Catholic Church: Because they did not require the repayment of principal, in the eyes of the church they were not considered loans.

By depositing the amount $\frac{C}{r}$ today, we can withdraw interest of $\frac{C}{r} \times r = C$ each period in perpetuity. Note, the first cash flow of C is received one period after the deposit is made. Thus it is important to remember that the PV calculated is one period before the first cash flow.

Our methodology can be summarized as follows. To determine the present value of a cash flow stream, we computed the "do-it-yourself" cost of creating those same cash flows at the bank. This is an extremely useful and powerful approach—and is much simpler and faster than summing those infinite terms!

EXAMPLE 4.5

Personal Finance
Endowing a Perpetuity

Problem

You want to endow an annual graduation party at your university. You want the event to be a memorable one, so you budget $30,000 per year forever for the party. If the university earns 8% per year on its investments, and if the first party is in one year's time, how much will you need to donate to endow the party?

Solution

Plan

The timeline of the cash flows you want to provide is:

This is a standard perpetuity of $30,000 per year. The funding you would need to give the university in perpetuity is the present value of this cash flow stream.

Execute

Use the formula for a perpetuity:

$$PV_0 = C/r = \$30{,}000/0.08 = \$375{,}000 \text{ today}$$

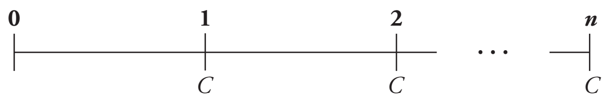
Evaluate

If you donate $375,000 today, and if the university invests it at 8% per year forever, then the graduates will have $30,000 every year for their graduation party.

Annuities

annuity A stream of equal cash flows arriving at a regular interval over a specified time period.

An **annuity** is a stream of n equal cash flows paid at regular intervals. The difference between an annuity and a perpetuity is that an annuity ends after some fixed number of payments. Most car loans, mortgages, and some bonds are annuities. We represent the cash flows of an annuity on a timeline as follows:

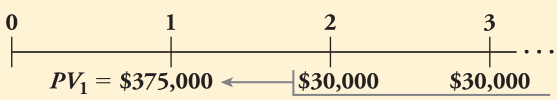


Common Mistake: Discounting One Too Many Times

The perpetuity formula assumes that the first payment occurs at the end of the first period (at date 1). Sometimes perpetuities have cash flows that start later in the future. In this case, we can adapt the perpetuity formula to compute the present value, but we need to do so carefully to avoid a common mistake.

To illustrate, consider the graduation party described in Example 4.6. Rather than starting in one year, suppose that the first party will be held two years from today. How would this delay change the amount of the donation required?

Now the timeline looks like this:



We need to determine the present value of these cash flows, as it tells us the amount of money in the bank needed today to finance the future parties. We cannot apply the perpetuity formula directly, however, because these cash flows are not *exactly* a perpetuity as we defined it. Specifically, the cash flow in the first period is "missing." But consider the situation on date 1—at that point, the first party is one period away and then the cash flows occur regularly. From the perspective of date 1, this *is* a perpetuity, and we can apply the formula (effectively getting PV_1). From the preceding calculation, we know we need $375,000 on date 1 to have enough to start the parties on date 2. We rewrite the timeline as follows:

Our goal can now be restated more simply: How much do we need to invest today to have $PV_1 = \$375{,}000$ in one year? This is a simple present value calculation:

$$PV_0 = \frac{PV_1}{(1 + r)}$$

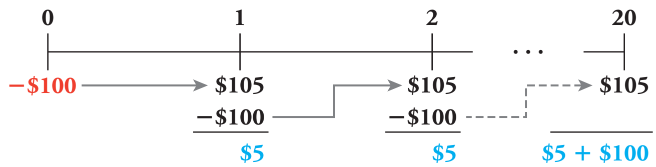
$$PV_0 = \$375{,}000/1.08 = \$347{,}222 \text{ today}$$

A common mistake is to discount the $375,000 twice because the first party is in two periods. *Remember—the present value formula for the perpetuity already discounts the cash flows to one period prior to the first cash flow.* Keep in mind that this common mistake may be made with perpetuities, annuities, and all of the other special cases discussed in this section. All of these formulas discount the cash flows to one period prior to the first cash flow.

Note that just as with the perpetuity, we adopt the convention that the first payment takes place at date 1, one period from today. The present value of an n-period annuity with payment C and interest rate r is:

$$PV_0 = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^n}$$

Present Value of an Annuity. To find a simpler formula, we use the same approach we followed with the perpetuity: find a way to create your own annuity. To illustrate, suppose you invest \$100 in a bank account paying 5% interest. At the end of one year, you will have \$105 in the bank—your original \$100 plus \$5 in interest. Using the same strategy as you did for calculating the value of a perpetuity, suppose you withdraw the \$5 interest and reinvest the \$100 for a second year. Once again you will have \$105 after one year. You can repeat the process, withdrawing \$5 and reinvesting \$100, every year. For a perpetuity, you left the principal in the bank forever. Alternatively, you might decide after 20 years to close the account and withdraw the principal. In that case, your cash flows will look like this:



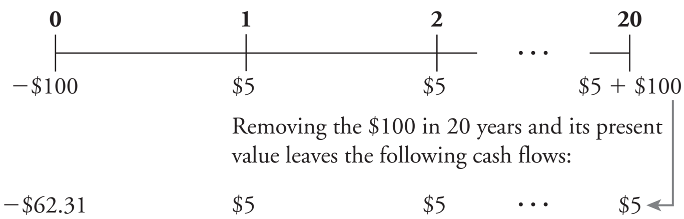
With your initial \$100 investment, you have created a 20-year annuity of \$5 per year, plus you will receive an extra \$100 at the end of 20 years. Again, the Valuation Principle's Law of One Price tells us that because it only took an initial investment of \$100 to create the cash flows on the timeline, the present value of these cash flows is \$100, or:

$$\$100 = PV(\text{20-year annuity of \$5 per year}) + PV(\$100 \text{ in 20 years})$$

So if we invest \$100 now, we can receive \$5 per year for 20 years as well as \$100 in the 20th year, representing the following cash flows:

Rearranging the equation above shows that the cost of a 20-year annuity of \$5 per year is \$100 minus the present value of \$100 in 20 years.

$$\begin{aligned} PV(\text{20-year annuity of \$5 per year}) &= \$100 - PV(\$100 \text{ in 20 years}) \\ &= \$100 - \frac{\$100}{(1.05)^{20}} = \$100 - \$37.69 = \$62.31 \end{aligned}$$



So the present value of \$5 for 20 years is \$62.31. Intuitively, the value of the annuity is the initial investment in the bank account minus the present value of the principal that will be left in the account after 20 years.

The \$5 we receive every year is the interest on the \$100 and can be written as \$100(.05) = \$5. Rearranging, we have \$100 = \$5/.05. If we substitute \$5/.05 into our formula above, we can represent the *PV* of the annuity as a function of its cash flow (\$5), the discount rate (5%), and the number of years (20):

$$PV(\text{20-year annuity of \$5 per year}) = \frac{\$5}{.05} - \frac{\frac{\$5}{.05}}{(1.05)^{20}} = \frac{\overset{C}{\$5}}{.05}\left(1 - \frac{1}{(1.05)^{\overset{n}{20}}}\right)^{*}$$

$$= \$5 \times \frac{1}{\underset{r}{.05}}\left(1 - \frac{1}{(1\underset{r}{.05})^{20}}\right)$$

This method is very useful because we will most often want to know the *PV* of the annuity given its cash flow, discount rate, and number of years. We can write this as a general formula for the present value of an annuity of C for n periods:

Present Value of an *n*-Period Annuity with Constant Cash Flows, *C*, and Discount Rate, *r*

$$PV_0 = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right) \tag{4.5}$$

Let's revisit the cash flows presented in Example 4.2 (\$1000 invested at the end of each of the next three years). Using equation 4.5, we get the following:

$$PV_0 = \$1000 \times \frac{1}{.10}\left(1 - \frac{1}{(1 + .10)^3}\right) = \$2486.85$$

As you can see, this is the same result we obtained when discounting the cash flows individually. Equation 4.5, though, does the calculation in one step instead of having to do calculations for each cash flow as we did in Example 4.2. Using the present value of an annuity formula can save a lot of time when analyzing annuities that have many cash flows.

EXAMPLE 4.6

Personal Finance
Present Value of a Lottery Prize Annuity

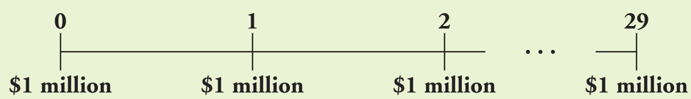
Problem

While vacationing in the US, you purchased a state lottery ticket and now you are the lucky winner of the \$30 million prize. Upon reading the fine print, though, you find that you can take your prize money either as (a) 30 payments of \$1 million per year (starting today), or (b) \$15 million paid today. If the interest rate is 8%, which option should you take?

Solution

Plan

Option (a) provides \$30 million in prize money but paid over time. To evaluate it correctly, we must convert it to a present value. Here is the timeline:



*An early derivation of this formula is attributed to the astonomer Edmond Halley in *Of Compound Interest*, published after Halley's death by Henry Sherwin, Sherwin's Mathematical Tables (London: W. and J. Mount, T. Page and Son, 1761).

Because the first payment starts today, the last payment will occur in 29 years (for a total of 30 payments).[2] The $1 million at date 0 is already stated in present value terms, but we need to compute the present value of the remaining payments. Fortunately, this case looks like a 29-year annuity of $1 million per year, so we can use the annuity formula.

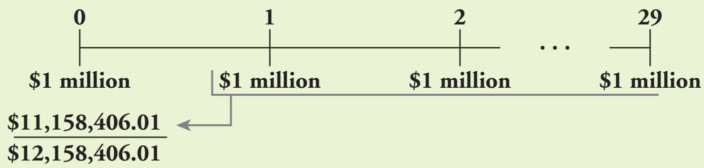
Execute

We use the annuity formula:

$$PV(\text{29-year annuity of \$1 million}) = \$1 \text{ million} \times \frac{1}{0.08}\left(1 - \frac{1}{1.08^{29}}\right)$$

$$= \$11{,}158{,}406.01 \text{ today}$$

Thus, the total present value of the cash flows is $1 million + $11,158,406.01 = $12,158,406.01. In timeline form:



Option (b), $15 million upfront, is more valuable—even though the total amount of money paid is half that of option (a).

Financial calculators or Excel can handle annuities easily—just enter the cash flow in the annuity as the *PMT:*

	N	I/Y	PV	PMT	FV
Given:	29	8		1,000,000	0
Solve for:			−11,158,406.01		
	Excel Formula: =*PV*(*RATE,NPER,PMT,FV*)=*PV*(0.08,29,1000000,0)				

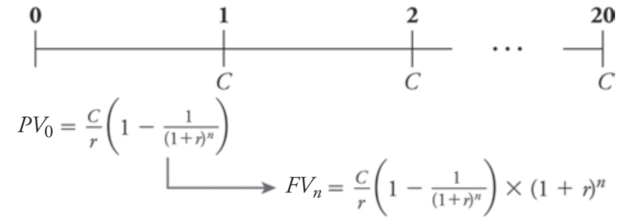
Both the financial calculator and Excel will give you the *PV* of the 29 payments ($11.16 million) to which you must add the first payment of $1 million just as above.

Evaluate

The reason for the difference is the time value of money. If you have the $15 million today, you can use $1 million immediately and invest the remaining $14 million at an 8% interest rate. This strategy will give you $14 million × 8% = $1.12 million per year in perpetuity! Alternatively, you can spend $15 million − $11.16 million = $3.84 million today, and invest the remaining $11.16 million, which will still allow you to withdraw $1 million each year for the next 29 years before your account is depleted.

[2]An annuity in which the first payment occurs immediately is sometimes called an *annuity due*. Throughout this text, we always use the term "annuity" to mean one that is paid in arrears (i.e., at the end of each period).

Future Value of an Annuity. Now that we have derived a simple formula for the present value of an annuity, it is easy to find a simple formula for the future value. If we want to know the value n years in the future, we move the present value n periods forward on the timeline.



As the timeline shows, we compound the present value for n periods at interest rate r:

Future Value of an n-Period Annuity with Constant Cash Flows, C, and Interest Rate, r

$$
\begin{aligned}
FV_n &= PV_0 \times (1+r)^n \\
&= \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right) \times (1+r)^n \\
&= C \times \frac{1}{r}\left((1+r)^n - 1\right) \qquad (4.6)
\end{aligned}
$$

This formula is useful if we want to know how a savings account will grow over time and the investor deposits the same amount every period. We can reevaluate the cash flows in Example 4.2 ($1000 invested at the end of each of three years) to get their future value using Equation 4.6.

$$FV_3 = \$1000 \times \frac{1}{0.10}\left((1+0.10)^3 - 1\right) = \$3310$$

As you can see, this gives the same result, in one step, as the three calculations used in Example 4.2.

EXAMPLE 4.7

Personal Finance
Retirement Savings Plan Annuity

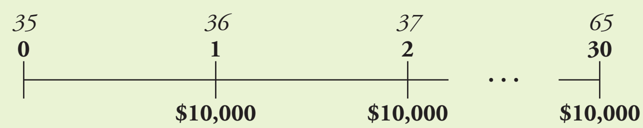
Problem

Ellen is 35 years old and she has decided it is time to plan seriously for her retirement. At the end of each year until she is 65, she will save $10,000 in a registered retirement savings account (RRSP). If the account earns 10% per year, how much will Ellen have saved at age 65?

Solution

Plan

As always, we begin with a timeline. In this case, it is helpful to keep track of both the dates and Ellen's age:



Ellen's savings plan looks like an annuity of \$10,000 per year for 30 years. (*Hint:* It is easy to become confused when you just look at age, rather than at both dates and age. A common error is to think there are only 65 − 36 = 29 payments. Writing down both dates and age avoids this problem.)

To determine the amount Ellen will have in the bank at age 65, we'll need to compute the future value of this annuity.

Execute

$$FV = \$10{,}000 \times \frac{1}{0.10}(1.10^{30} - 1)$$

$$= \$1{,}644,940.23 \text{ million at age 65}$$

Using a financial calculator or Excel:

	N	I/Y	PV	PMT	FV
Given:	30	10	0	−10,000	
Solve for:					**1,644,940**

Excel Formula: =*FV*(*RATE*,*NPER*,*PMT*,*PV*)=*FV*(0.10,30,−10000,0)

Evaluate

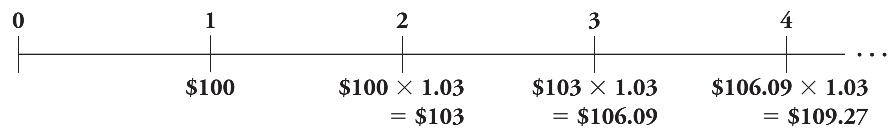
By investing \$10,000 per year for 30 years (a total of \$300,000) and earning interest on those investments, the compounding will allow her to retire with about \$1.645 million in her RRSP.

Growing Cash Flows

So far, we have considered only cash flow streams that have the same cash flow every period. If, instead, the cash flows are expected to grow at a constant rate in each period, we can also derive a simple formula for the present value of the future stream.

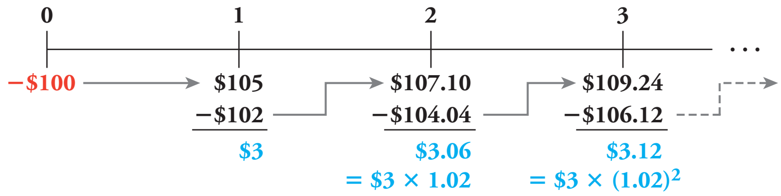
growing perpetuity A stream of cash flows that occurs at regular intervals and grows at a constant rate forever.

Growing Perpetuity. A **growing perpetuity** is a stream of cash flows that occurs at regular intervals and grows at a constant rate forever. For example, a growing perpetuity with a first payment of \$100 that grows at a rate of 3% has the following timeline:



To derive the formula for the present value of a growing perpetuity, we follow the same logic used for a regular perpetuity: Compute the amount you would need to deposit today to create the perpetuity yourself. In the case of a regular perpetuity, we created a constant payment forever by withdrawing the interest earned each year and reinvesting the principal. To increase the amount we can withdraw each year, the principal that we reinvest each year must grow. We therefore withdraw less than the full amount of interest earned each period, using the remaining interest to increase our principal.

Let's consider a specific case. Suppose you want to create a perpetuity growing at 2%, so you invest \$100 in a bank account that pays 5% interest. At the end of one year, you will have $105 in the bank—your original $100 plus $5 in interest. If you withdraw only $3, you will have $102 to reinvest—2% more than the amount you had initially. This amount will then grow to $\$102 \times 1.05 = \$107.10$ in the following year, and you can withdraw $\$3 \times 1.02 = \$3.06$, which will leave you with principal of $\$107.10 - \$3.06 = \$104.04$. Note that $\$102 \times 1.02 = \$104.04$. That is, both the amount you withdraw and the principal you reinvest grow by 2% each year. On a timeline, these cash flows look like this:



By following this strategy, you have created a growing perpetuity that starts at $3 and grows 2% per year. This growing perpetuity must have a present value equal to the cost of $100.

We can generalize this argument. If we want to increase the amount we withdraw from the bank each year by g, then the principal in the bank will have to grow by the same factor g. That is, instead of reinvesting P in the second year, we should reinvest $P(1 + g) = P + gP$. In order to increase our principal by gP, we need to leave gP of the interest in the account, so of the total interest of rP, we can only withdraw $rP - gP = P(r - g)$. We demonstrate this for the first year of our example:

Initial amount deposited	\$100	P
Interest earned	(0.05)(\$100)	rP
Amount needed to increase principal	(0.02)(\$100)	gP
Amount withdrawn	(0.05)(\$100) − (0.02)(\$100)	$rP - gP$
	= \$100(0.05 − 0.02)	$= P(r - g)$

Denoting our withdrawal as C_1, we have $C_1 = P(r - g)$. Solving this equation for P, the initial amount deposited in the bank account, gives the present value of a growing perpetuity with C_1 as the initial cash flow.[3]

Present Value of a Growing Perpetuity with first cash flow, C_1, and growth rate, g

$$PV_0 = \frac{C_1}{r - g} \tag{4.7}$$

[3]Suppose $g \geq r$. Then the cash flows grow even faster than they are discounted; each term in the sum gets larger, rather than smaller. In this case, the sum is infinite! What does an infinite present value mean? Remember that the present value is the "do-it-yourself" cost of creating the cash flows. An infinite present value means that no matter how much money you start with, it is *impossible* to reproduce those cash flows on your own. Growing perpetuities of this sort cannot exist in practice because no one would be willing to offer one at any finite price. A promise to pay an amount that forever grew faster than the interest rate is also unlikely to be kept (or believed by any savvy buyer). The only viable growing perpetuities are those where the growth rate is less than the interest rate, so we assume that $g < r$ for a growing perpetuity.

EXAMPLE 4.8

Personal Finance
Endowing a Growing Perpetuity

Problem

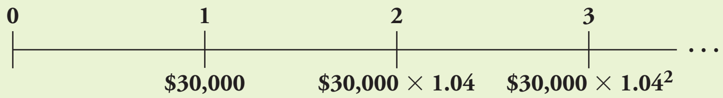
In Example 4.5, you planned to donate money to your university to fund an annual $30,000 graduation party. Given an interest rate of 8% per year, the required donation was the present value of:

$$PV_0 = \$30{,}000/0.08 = \$375{,}000 \text{ today}$$

Before accepting the money, however, the student association has asked that you increase the donation to account for the effect of inflation on the cost of the party in future years. Although $30,000 is adequate for next year's party, the students estimate that the party's cost will rise by 4% per year thereafter. To satisfy their request, how much do you need to donate now?

Solution

Plan



The cost of the party next year is $30,000, and the cost then increases 4% per year forever. From the timeline, we recognize the form of a growing perpetuity and can value it that way.

Execute

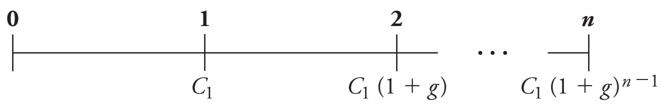
To finance the growing cost, you need to provide the present value today of:

$$PV_0 = \$30{,}000/(0.08 - 0.04) = \$750{,}000 \text{ today}$$

Evaluate

You need to double the size of your gift!

Growing Annuity. A **growing annuity** is a stream of n growing cash flows, paid at regular intervals. It is a growing perpetuity that eventually comes to an end. The following timeline shows a growing annuity with initial cash flow C_1, growing at rate g every period until period n:



As with growing perpetuities discussed earlier, we adopt the convention that the first payment occurs at date 1. Since the first payment, C_1, occurs at date 1 and the n^{th} payment, C_n, occurs at date n, there are only $n - 1$ periods of growth between these payments.

The present value of an n-period growing annuity with initial cash flow C_1, growth rate g, and interest rate r is given by

Present Value of a Growing Annuity

$$PV = C_1 \times \frac{1}{r-g}\left(1 - \left(\frac{1+g}{1+r}\right)^n\right) \tag{4.8}$$

Because the annuity has only a finite number of terms, Eq. 4.8 also works when $g > r$[7]. The process of deriving this simple expression for the present value of a growing annuity is the same as for a regular annuity. Interested readers may consult the online appendix for details.

EXAMPLE 4.9

Personal Finance
Retirement Savings with a Growing Annuity

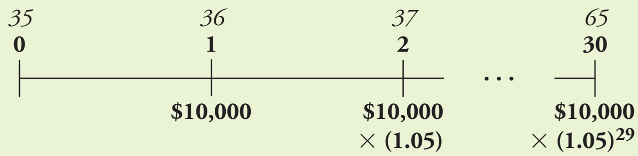
Problem

In Example 4.7, Ellen considered saving \$10,000 per year for her retirement. Although \$10,000 is the most she can save in the first year, she expects her salary to increase each year so that she will be able to increase her savings by 5% per year. With this plan, if she earns 10% per year in her RRSP, what is the present value of her planned savings and how much will Ellen have saved at age 65?

Solution

Plan

As always, we begin with a timeline. Again, it is helpful to keep track of both the dates and Ellen's age:



This example involves a 30-year growing annuity, with a growth rate of 5%, and an initial cash flow of \$10,000.

Execute

The present value of this growing annuity is given by

$$\begin{aligned} PV &= \$10{,}000 \times \frac{1}{0.10 - 0.05}\left(1 - \left(\frac{1.05}{1.10}\right)^{30}\right) \\ &= \$10{,}000 \times 15.0463 \\ &= \$150{,}463 \text{ today} \end{aligned}$$

To determine the amount she will have at age 65, we need to move this amount forward 30 years:

$$\begin{aligned} FV &= \$150{,}463 \times 1.10^{30} \\ &= \$2.625 \text{ million in 30 years} \end{aligned}$$

Unfortunately your financial calculator and Excel do not have pre-programmed functions to handle growing annuities.

Evaluate

By investing the growing amounts (starting with \$10,000) each year for 30 years, Ellen's proposed savings plan is equivalent to having \$150,463 in the bank *today*.

Ellen will have saved \$2.625 million at age 65 using the new savings plan. This sum is almost \$1 million more than she would have had in her RRSP without the additional annual increases in savings (as shown in Example 4.7).

Finding a simple formula for the future value of a growing annuity is easy, just as we saw for regular annuities. If we want to know the value n years in the future, we move the present value n periods forward on the timeline; that is, we compound the present value for n periods at interest rate r:

Future Value of a Growing Annuity

$$PV_0 = C_1 \times \frac{1}{r-g}\left(1 - \left(\frac{1+g}{1+r}\right)^n\right)$$

$$FV_n = PV_0 \times (1+r)^n = C_1 \times \frac{1}{r-g}\left(1 - \left(\frac{1+g}{1+r}\right)^n\right) \times (1+r)^n \qquad (4.9)$$

$$FV_n = C_1 \times \frac{1}{r-g}\left((1+r)^n - (1+g)^n\right)$$

Try using this formula in Example 4.9 to get the future value of Ellen's savings when she is 65.

Concept Check

9. What is the reasoning behind the fact that an infinite stream of cash flows has a finite present value?

10. How do you calculate the present value of a

a. perpetuity?

b. annuity?

c. growing perpetuity?

d. growing annuity?

4.6 Solving for Variables Other Than Present Value or Future Value

So far, we have calculated the present value or future value of a stream of cash flows. Sometimes, however, we know the present value or future value, but do not know one of the variables that so far we have been given as an input. For example, when you take out a loan, you may know the amount you would like to borrow, but may not know the loan payments that will be required to repay it. Or, if you make a deposit into a bank account, you may want to calculate how long it will take before your balance reaches a certain level. In such situations, we use the present and/or future values as inputs, and solve for the variable we are interested in. We examine several special cases in this section.

Solving for the Cash Flows

Let's consider an example where we know the present value of an investment, but do not know the cash flows. The best example is a loan—you know how much you want to borrow (the present value) and you know the interest rate, but you do not know how much you need to repay each year. Suppose you are opening a business that requires an initial investment of $100,000. Your bank manager has agreed to lend you this money. The terms of the loan state that you will make equal annual payments for the next ten years and will pay an interest rate of 8% with the first payment due one year from today. What is your annual payment?

From the bank's perspective, the timeline looks like this:

0	1	2	...	10
−$100,000	C	C		C

The bank will give you $100,000 today in exchange for ten equal payments over the next decade. You need to determine the size of the payment C that the bank will require. For the bank to be willing to lend you $100,000, the loan cash flows must have a present value of $100,000 when evaluated at the bank's interest rate of 8%. That is:

$$100{,}000 = PV(\text{10-year annuity of } C \text{ per year, evaluated at the loan rate})$$

Using the formula for the present value of an annuity,

$$100{,}000 = C \times \frac{1}{0.08}\left(1 - \frac{1}{1.08^{10}}\right) = C \times 6.71$$

solving this equation for C gives:

$$C = \frac{100{,}000}{6.71} = \$14{,}903$$

You will be required to make ten annual payments of $14,903 in exchange for $100,000 today.

We can also solve this problem with a financial calculator or Excel:

	N	I/Y	PV	PMT	FV
Given:	10	8	100,000		0
Solve for:				−14,903	
Excel Formula: $=PMT(RATE,NPER,PV,FV)=PMT(0.08,10,100000,0)$					

In general, when solving for a loan payment, think of the amount borrowed (the loan principal) as the present value of the payments. If the payments of the loan are an annuity, we can solve for the payment of the loan by inverting the annuity formula. Writing the equation for the payments formally for a loan with principal P, requiring n periodic payments of C and interest rate r, we have

Loan payment

$$C = \frac{P}{\dfrac{1}{r}\left(1 - \dfrac{1}{(1+r)^n}\right)} \tag{4.10}$$

EXAMPLE 4.10

Computing a Loan Payment

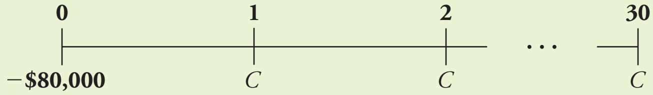
Problem

Your firm plans to buy a warehouse for $100,000. The bank offers you a 30-year loan with equal annual payments and an interest rate of 8% per year. The bank requires that your firm pay 20% of the purchase price as a down payment, so you can borrow only $80,000. What is the annual loan payment?

Solution

Plan

We start with the timeline (from the bank's perspective):



Using Eq. 4.10, we can solve for the loan payment, C, given $n = 30$, $r = 8\%$ (0.08) and $P = \$80{,}000$.

Execute

Eq. 4.10 gives the following payment (cash flow):

$$C = \frac{P}{\frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right)} = \frac{80{,}000}{\frac{1}{0.08}\left(1 - \frac{1}{(1.08)^{30}}\right)}$$

$$= \$7106.19$$

Using a financial calculator or Excel:

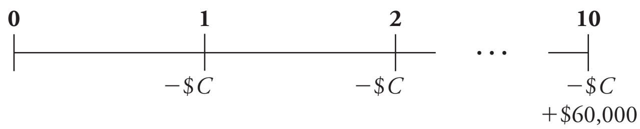
	N	I/Y	PV	PMT	FV
Given:	30	8	−80,000		0
Solve for:				**7106.19**	
Excel Formula: =*PMT*(*RATE*,*NPER*,*PV*,*FV*)=*PMT*(0.08,30,−80000,0)					

Evaluate

Your firm will need to pay $7106.19 each year to repay the loan. The bank is willing to accept these payments because the *PV* of 30 annual payments of $7106.19 at 8% interest rate per year is exactly equal to the $80,000 it is giving you today.

We can use this same idea to solve for the cash flows when we know the future value rather than the present value. As an example, suppose you have just graduated from college and you decide to be prudent and start saving for a down payment on a house. You would like to have $60,000 saved 10 years from now. If you can earn 7% per year on your savings, how much do you need to save each year to meet your goal?

The timeline for this example is



That is, you plan to save some amount C per year, and then withdraw $60,000 from the bank in ten years. Therefore, we need to find the annuity payment that has a future value of $60,000 in ten years. Use the formula for the future value of an annuity from Eq. 4.6:

$$60{,}000 = FV(\text{annuity}) = C \times \frac{1}{0.07}(1.07^{10} - 1) = C \times 13.816448$$

Therefore, $C = \frac{60{,}000}{13.816448} = \$4343.65$. Thus, you need to save $4343.65 per year. If you do, then at a 7% interest rate your savings will grow to $60,000 in 10 years when you are ready to buy a house.

Now let's solve this problem using a financial calculator or Excel:

	N	I/Y	PV	PMT	FV
Given:	10	7	0		60,000
Solve for:				−4343	
Excel Formula: =*PMT*(*RATE,NPER,PV,FV*)=*PMT*(0.07,10,0,60000)					

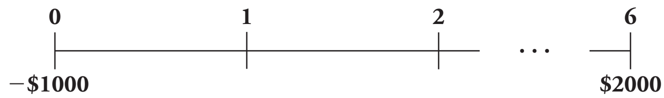
Once again, we find that you need to save \$4343 for 10 years to accumulate \$60,000.

Internal Rate of Return

internal rate of return (*IRR*) The interest rate that sets the net present value of the cash flows equal to zero.

In some situations, you know the present value and cash flows of an investment opportunity but you do not know the interest rate that equates them. This interest rate is called the **internal rate of return (*IRR*)**, defined as the interest rate that sets the net present value of the cash flows equal to zero.

For example, suppose that you have an investment opportunity that requires a \$1000 investment today and will have a \$2000 payoff in six years. This would appear on a timeline as



One way to analyze this investment is to ask the question: What interest rate, r, would you need so that the NPV of this investment is zero?

$$NPV = -1000 + \frac{2000}{(1+r)^6} = 0$$

Rearranging this calculation gives the following:

$$1000 \times (1+r)^6 = 2000$$

That is, r is the interest rate you would need to earn on your \$1000 to have a future value of \$2000 in six years. We can solve for r as follows:

$$1 + r = \left(\frac{2000}{1000}\right)^{1/6} = 1.1225$$

Or, $r = 12.25\%$. This rate is the IRR of this investment opportunity. Making this investment is like earning 12.25% per year on your money for six years.

When there are just two cash flows, as in the preceding example, it is easy to compute the IRR. Consider the general case in which you invest an amount P today, and receive FV in n years:

$$P \times (1 + IRR)^n = FV$$

$$1 + IRR = (FV/P)^{1/n}$$

Now let's consider a more sophisticated example. Suppose your firm needs to purchase a new forklift. The dealer gives you two options: (1) a price for the forklift if you pay cash and (2) the annual payments if you take out a loan from the dealer. To evaluate the loan that the dealer is offering you, you will want to compare the rate on the loan with the rate that your bank is willing to offer you. Given the loan payment that the dealer quotes, how do you compute the interest rate charged by the dealer?

In this case, we need to compute the *IRR* of the dealer's loan. Suppose the cash price of the forklift is \$40,000, and the dealer offers financing with no down payment and four annual payments of \$15,000. This loan has the following timeline:

0	1	2	3	4
\$40,000	**−\$15,000**	**−\$15,000**	**−\$15,000**	**−\$15,000**

From the timeline, it is clear that the loan is a four-year annuity with a payment of \$15,000 per year and a present value of \$40,000. Setting the *NPV* of the cash flows equal to zero requires that the present value of the payments equals the purchase price:

$$40{,}000 = 15{,}000 \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^4}\right)$$

The value of r that solves this equation, the *IRR*, is the interest rate charged on the loan. Unfortunately, in this case there is no simple way to solve for the interest rate r.[4] The only way to solve this equation is to guess at values of r until you find the right one.

Start by guessing $r = 10\%$ In this case, the value of the annuity is

$$15{,}000 \times \frac{1}{0.10}\left(1 - \frac{1}{(1.10)^4}\right) = 47{,}548$$

The present value of the payments is too large. To lower it, we need to use a higher interest rate. We guess 20% this time:

$$15{,}000 \times \frac{1}{0.20}\left(1 - \frac{1}{(1.20)^4}\right) = 38{,}831$$

Now the present value of the payments is too low, so we must pick a rate between 10% and 20%. We continue to guess until we find the right rate. Let's try 18.45%:

$$15{,}000 \times \frac{1}{0.1845}\left(1 - \frac{1}{(1.1845)^4}\right) = 40{,}000$$

The interest rate charged by the dealer is 18.45%.

An easier solution than guessing the *IRR* and manually calculating values is to use a spreadsheet or calculator to automate the guessing process. When the cash flows are an annuity, as in this example, we can use a financial calculator or Excel to compute the *IRR*. Both solve (with slightly varying notation) the following equation:

$$NPV = PV + PMT \times \frac{1}{I/Y}\left(1 - \frac{1}{(1 + I/Y)^n}\right) + \frac{FV}{(1 + I/Y)^n} = 0$$

The equation ensures that the *NPV* of investing in the annuity is zero. When the unknown variable is the interest rate, it will solve for the interest rate that sets the *NPV*

[4] With five or more periods and general cash flows, there is *no* general formula to solve for r; trial and error and interpolation (by hand or computer) is the *only* way to compute the *IRR*.

USING EXCEL

Computing *NPV* and *IRR*

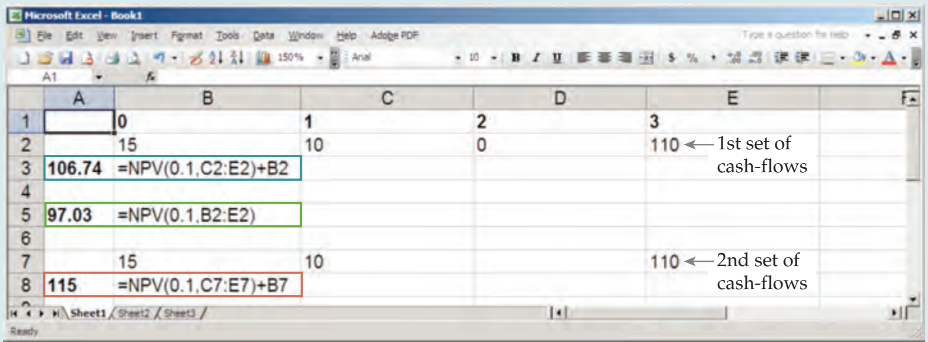
Here we discuss how to use Microsoft® Excel to solve for *NPV* and *IRR*. We also identify some pitfalls to avoid when using Excel.

NPV Function: Leaving Out Date 0

Excel's *NPV* function has the format, *NPV* (rate, value1, value2, . . .) where "rate" is the interest rate per period used to discount the cash flows, and "value1", "value2", etc., are the cash flows (or ranges of cash flows). The *NPV* function computes the present value of the cash flows *assuming the first cash flow occurs at date 1.* Therefore, if a project's first cash flow occurs at date 0, we cannot use the *NPV* function by itself to compute the *NPV*. We can use the *NPV* function to compute the present value of the cash flows from date 1 onwards, and then we must add the date 0 cash flow to that result to calculate the *NPV*. The screenshot below shows the difference. The first *NPV* calculation (outlined in blue) is correct: we used the *NPV* function for all of the cash flows occurring at time 1 and later and then added on the first cash flow occurring at time 0 since it is already in present value. The second (outlined in green) is incorrect: we used the *NPV* function for all of the cash flows, but the function assumed that the first cash flow occurs in period 1 instead of immediately.

NPV Function: Ignoring Blank Cells

Another pitfall with the *NPV* function is that cash flows that are left blank are treated differently from cash flows that are equal to zero. If the cash flow is left blank, *both the cash flow and the period are ignored.* For example, the second set of cash flows below is equivalent to the first—we have simply left the cash flow for date 2 blank instead of entering a "0." However, the *NPV* function ignores the blank cell at date 2 and assumes the cash flow is 10 at date 1 and 110 at date 2, which is clearly not what is intended and produces an incorrect answer (outlined in red).



	A	B	C	D	E
1		0	1	2	3
2		15	10	0	110 ← 1st set of cash-flows
3	106.74	=NPV(0.1,C2:E2)+B2			
4					
5	97.03	=NPV(0.1,B2:E2)			
6					
7		15	10		110 ← 2nd set of cash-flows
8	115	=NPV(0.1,C7:E7)+B7			

Because of these idiosyncrasies, we avoid using Excel's *NPV* function. It is more reliable to compute the present value of each cash flow separately in Excel, and then sum them to determine the *NPV*.

IRR Function

Excel's *IRR* function has the format *IRR* (values, guess), where "values" is the range containing the cash flows, and "guess" is an optional starting guess where Excel begins its search for an *IRR*. Two things to note about the *IRR* function:

1. The values given to the *IRR* function should include all of the cash flows of the project, including the one at date 0. In this sense, the *IRR* and *NPV* functions in Excel are inconsistent.
2. Like the *NPV* function, the *IRR* ignores the period associated with any blank cells.

equal to zero—that is, the *IRR*. For this case, you could use a financial calculator or Excel, as follows:

	N	I/Y	PV	PMT	FV
Given:	4		40,000	−15,000	0
Solve for:		**18.45**			

Excel Formula: =RATE(*NPER,PMT,PV,FV*)=RATE(4,−15000,40000,0)

Both the financial calculator and Excel correctly compute an *IRR* of 18.45%.

EXAMPLE 4.11

Personal Finance
Computing the Internal Rate of Return with a Financial Calculator

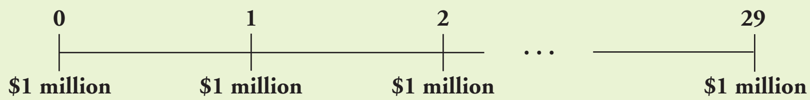
Problem

Let's return to the lottery prize in Example 4.6. How high a rate of return do you need to earn investing on your own in order to prefer the $15 million payout?

Solution

Plan

Recall that the lottery offers you the following deal: take either (a) $15 million lump sum payment immediately, or (b) 30 payments of $1 million per year starting immediately. This second option is an annuity of 29 payments of $1 million plus an initial $1 million payment.



We need to solve for the internal rate of return that makes the two offers equivalent. Anything above that rate of return would make the present value of the annuity lower than the $15 million lump sum payment, and anything below that rate of return would make it greater than the $15 million.

Execute

First, we set the present value of option (b) equal to option (a), which is already in present value since it is an immediate payment of $15 million:

$$\$15\,\text{million} = \$1\,\text{million} + \$1\,\text{million} \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^{29}}\right)$$

$$\$14\,\text{million} = \$1\,\text{million} \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^{29}}\right)$$

Using a financial calculator to solve for *r*:

	N	I/Y	PV	PMT	FV
Given:	29		−14,000,000	1,000,000	0
Solve for:		**5.72**			

Excel Formula: =RATE(*NPER,PMT,PV,FV*)=RATE(29,1000000,−14000000,0)

The *IRR* equating the two options is 5.72%.

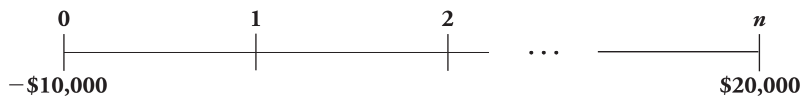
Evaluate

A rate of 5.72% makes giving up the \$15 million payment and taking the 30 installments of \$1 million exactly a zero *NPV* action. If you could earn more than 5.72% investing on your own, then you could take the \$15 million, invest it and generate 30 installments that are each more than \$1 million. If you could not earn at least 5.72% on your investments, you would be unable to replicate the \$1 million installments on your own and would be better off taking the installment plan.

Solving for *n* the Number of Periods

In addition to solving for cash flows or the interest rate, we can solve for the amount of time it will take a sum of money to grow to a known value. In this case, the interest rate, present value, and future value are all known. We need to compute how long it will take for the present value to grow to the future value.

Suppose we invest \$10,000 in an account paying 10% interest, and we want to know how long it will take for the amount to grow to \$20,000.



We want to determine n.

In terms of our formulas, we need to find n so that the future value of our investment equals \$20,000:

$$FV = \$10{,}000 \times 1.10^n = \$20{,}000 \tag{4.11}$$

One approach is to use trial and error to find n, as with the *IRR*. For example, with $n = 7$ years, $FV = \$19{,}487$, so it will take longer than 7 years. With $n = 8$ years, $FV = \$21{,}436$, so it will take between 7 and 8 years.

Alternatively, this problem can be solved on a financial calculator or Excel. In this case, we solve for n:

	N	I/Y	PV	PMT	FV
Given:		10	−10,000	0	20,000
Solve for:	7.27				

Excel Formula: =NPER(RATE,*PMT*,*PV*,*FV*)=NPER(0.10,0,−10000,20000)

It will take about 7.3 years for our savings to grow to \$20,000.

Solving for *n* Using Logarithms

The problem of solving for the number of periods can be solved mathematically as well. Dividing both sides of Eq. 4.11 by \$10,000, we have:

$$1.10^n = 20{,}000 \,/\, 10{,}000 = 2$$

To solve for an exponent, we take the logarithm of both sides, and use the fact that $\ln(X^y) = y\ln(X)$:

$$n\ln(1.10) = \ln(2)$$
$$n = \ln(2)/\ln(1.10) = 0.6931/0.0953 \approx 7.3 \text{ years}$$

EXAMPLE 4.12

Personal Finance
Solving for the Number of Periods in a Savings Plan

Problem

Let's return to your savings for a down payment on a house. Imagine that some time has passed and you have \$10,050 saved already, and you can now afford to save \$5000 per year at the end of each year. Also, interest rates have increased so that you now earn 7.25% per year on your savings. How long will it take you to get to your goal of \$60,000?

Solution

Plan

The timeline for this problem is:

We need to find *n* so that the future value of your current savings plus the future value of your planned additional savings (which is an annuity consisting of *n* payments) equals your desired amount. There are two contributors to the future value: the initial lump sum of \$10,050 that will continue to earn interest, and the annuity contributions of \$5000 per year that will earn interest as they are contributed. Thus, we need to find the future value of the lump sum plus the future value of the annuity.

Execute

We can solve this problem using a financial calculator or Excel:

	N	I/Y	PV	PMT	FV
Given:		7.25	−10,050	−5000	60,000
Solve for:	7.00				

Excel Formula: =NPER(RATE, *PMT*, *PV*, *FV*) = NPER(0.0725, −5000, −10050, 60000)

There is also a mathematical solution. We can calculate the future value of the initial cash flow by using Eq. 4.1 and the future value of the annuity using Eq. 4.6:

$$10{,}050 \times 1.0725^n + 5000 \times \frac{1}{0.0725}(1.0725^n - 1) = 60{,}000$$

Rearranging the equation to solve for *n*,

$$1.0725^n = \frac{60{,}000 \times 0.0725 + 5000}{10{,}050 \times 0.0725 + 5000} = 1.632$$

we can then solve for *n*:

$$n = \frac{\ln(1.632)}{\ln(1.0725)} = 7 \text{ years}$$

Evaluate

It will take seven years to save the down payment.

We began this chapter with the goal of developing the tools a financial manager needs to be able to apply the Valuation Principle by computing the net present value of a decision. Starting from the fundamental concept of the time value of money—a dollar today is worth more than a dollar tomorrow—we learned how to calculate the equivalent value of future cash flows today and today's cash flows in the future. We then learned some shortcuts for handling common sets of regular cash flows such as those found in perpetuities and loans. As we have seen, the discount rate is a critical input to any of our present value or future value calculations. Throughout this chapter, we have taken the discount rate as given.

What determines these discount rates? The Valuation Principle shows us that we must rely on market information to assess the value of cash flows across time. In the next chapter, we will learn the drivers of market interest rates as well as how they are quoted. Understanding interest rate quoting conventions will also allow us to extend the tools we developed in this chapter to situations where the interest rate is compounded more frequently than once per year.

Concept Check

11. How do you calculate the cash flow of an annuity?

12. What is the internal rate of return, and how do you calculate it?

13. How do you solve for the number of periods to pay off an annuity?

MyFinanceLab

Here is what you should know after reading this chapter. MyFinanceLab will help you identify what you know, and where to go when you need to practice.

Key Points and Equations	Terms	Online Practice Opportunities
4.1 The Timeline ◗ Timelines are a critical first step in organizing the cash flows in a financial problem.	stream of cash flows, p. 88 timeline, p. 88	MyFinanceLab Study Plan 4.1
4.2 Valuing Cash Flows at Different Points in Time ◗ There are three rules of valuing cash flows: a. Only cash flows that occur at the same point in time can be compared or combined. b. To calculate a cash flow's future value, you must compound it. c. To calculate a cash flow's present value, you must discount it. ◗ The future value in n years of a cash flow C today is: $C = (1 + r)^n$ (4.1) ◗ The present value today of a cash flow C received in n years is: $C \div (1 + r)^n$ (4.2)	compounding, p. 92 compound interest, p. 92 discounting, p. 94 future value, p. 92	MyFinanceLab Study Plan 4.2

4.3 Valuing a Stream of Cash Flows

- The present value of a cash flow stream is

$$PV = C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_n}{(1+r)^n} \quad (4.3)$$

MyFinanceLab Study Plan 4.3

4.4 The Net Present Value of a Stream of Cash Flows

- The *NPV* of an investment opportunity is *PV*(benefits − costs).

MyFinanceLab Study Plan 4.4

4.5 Perpetuities, Annuities, and Other Special Cases

- A perpetuity is a constant cash flow *C* paid every period, forever. The present value of a perpetuity is

$$PV_0 = \frac{C}{r} \quad (4.4)$$

- An annuity is a constant cash flow *C* paid every period for *n* periods. The present value of an annuity is

$$PV_0 = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right) \quad (4.5)$$

- The future value of an annuity at the end of the annuity is

$$FV_n = C \times \frac{1}{r}\left((1+r)^n - 1\right) \quad (4.6)$$

- In a growing perpetuity, the cash flows grow at a constant rate *g* each period. The present value of a growing perpetuity is

$$PV_0 = \frac{C_1}{r-g} \quad (4.7)$$

- A growing annuity is a growing series of cash flows, starting with C_1 paid every year for *n* years. The present value of a growing annuity is

$$PV_0 = C_1 \times \frac{1}{r-g}\left(1 - \left(\frac{1+g}{1+r}\right)^n\right) \quad (4.8)$$

- The future value of a growing annuity is

Cash flow in an Annuity (Loan Payment)

$$C_1 \times \frac{1}{r-g}\left((1+r)^n - (1+g)^n\right) \quad (4.9)$$

annuity, p. 105
consol, p. 102
growing perpetuity, p. 110
perpetuity, p. 102

MyFinanceLab Study Plan 4.5

Interactive Annuity Calculator

4.6 Solving for Variables Other Than Present Value or Future Value ◗ The annuity and perpetuity formulas can be used to solve for the annuity payments when either the present value or the future value is known. ◗ The periodic payment on an n-period loan with principal P and interest rate r is: $$\frac{P}{\frac{1}{r}\left(1-\frac{1}{(1+r)^n}\right)} \qquad (4.10)$$ ◗ The internal rate of return (IRR) of an investment opportunity is the interest rate that sets the NPV of the investment opportunity equal to zero. ◗ The annuity formulas can be used to solve for the number of periods it takes to save a fixed amount of money.	internal rate of return (IRR), p. 117	MyFinanceLab Study Plan 4.6 Using Excel: Computing *NPV*s and *IRR*

Review Questions

1. Why is a cash flow in the future worth less than the same amount today?
2. What is compound interest?
3. What is the reasoning behind the geometric growth in interest?
4. What is a discount rate?
5. What is the reasoning behind the fact that the present value of a stream of cash flows is just the sum of the present values of each individual cash flow?
6. What must be true about the cash flow stream in order for us to be able to use the shortcut formulas?
7. What is the difference between an annuity and a perpetuity?
8. What is an internal rate of return?

Problems

All problems in this chapter are available in MyFinanceLab. An asterisk (*) *indicates problems with a higher level of difficulty.*

The Timeline

1. You have just taken out a five-year loan from a bank to buy an engagement ring. The ring costs $5000. You plan to put down $1000 and borrow $4000. You will need to make annual payments of $1000 at the end of each year. Show the timeline of the loan from your perspective. How would the timeline differ if you created it from the bank's perspective?

2. You currently have a one-year-old loan outstanding on your car. You make monthly payments of $300. You have just made a payment. The loan has four years to go (i.e., it had an original term of five years). Show the timeline from your perspective. How would the timeline differ if you created it from the bank's perspective?

Valuing Cash Flows at Different Points in time

3. Calculate the future value of $2000 in
a. 5 years at an interest rate of 5% per year.
b. 10 years at an interest rate of 5% per year.
c. 5 years at an interest rate of 10% per year.
d. Why is the amount of interest earned in part (a) less than half the amount of interest earned in part (b)?

4. What is the present value of $10,000 received
a. 12 years from today when the interest rate is 4% per year?
b. 20 years from today when the interest rate is 8% per year?
c. 6 years from today when the interest rate is 2% per year?

5. Your brother has offered to give you either $5000 today or $10,000 in 10 years. If the interest rate is 7% per year, which option is preferable?

6. Your cousin is currently 12 years old. She will be going to university in six years. Your aunt and uncle would like to have $100,000 in a savings account to fund her education at that time. If the account promises to pay a fixed interest rate of 4% per year, how much money do they need to put into the account today to ensure that they will have $100,000 in six years?

7. Your mom is thinking of retiring. Her retirement plan will pay her either $250,000 immediately on retirement or $350,000 five years after the date of her retirement. Which alternative should she choose if the interest rate is
a. 0% per year?
b. 8% per year?
c. 20% per year?

8. Your grandfather put some money in an account for you on the day you were born. You are now 18 years old and are allowed to withdraw the money for the first time. The account currently has $3996 in it and pays an 8% interest rate.
a. How much money would be in the account if you left the money there until your twenty-fifth birthday?
b. What if you left the money until your sixty-fifth birthday?
c. How much money did your grandfather originally put in the account?

Valuing a Stream of Cash Flows

9. You have just received a windfall from an investment you made in a friend's business. She will be paying you $10,000 at the end of this year, $20,000 at the end of the following year, and $30,000 at the end of the year after that (three years from today). The interest rate is 3.5% per year.
a. What is the present value of your windfall?
b. What is the future value of your windfall in three years (on the date of the last payment)?

10. You have a loan outstanding. It requires making three annual payments of $1000 each at the end of the next three years. Your bank has offered to allow you to skip making the next two payments in lieu of making one large payment at the end of the loan's term in three years. If the interest rate on the loan is 5%, what final payment will the bank require you to make so that it is indifferent to the two forms of payment?

11. You are wondering whether going to university would be worthwhile. You figure that the total cost of going to university for four years, including lost wages, is $40,000 per year. However, you feel that if you get a university degree, the present value of your lifetime wages from graduation onward will be $300,000 greater than if you did not go to university. If your discount rate is 9%, what is the *NPV* of going to university?

The Net Present Value of a Stream of Cash Flows

12. You have been offered a unique investment opportunity. If you invest $10,000 today, you will receive $500 one year from now, $1500 two years from now, and $10,000 ten years from now.
    a. What is the *NPV* of the opportunity if the interest rate is 6% per year? Should you take the opportunity?
    b. What is the *NPV* of the opportunity if the interest rate is 2% per year? Should you take it now?

13. Renuka Gupta owns her own business and is considering an investment. If she undertakes the investment, it will pay $4000 at the end of each of the next three years. The opportunity requires an initial investment of $1000 plus an additional investment at the end of the second year of $5000. What is the *NPV* of this opportunity if the interest rate is 2% per year? Should Renuka take it?

Perpetuities, Annuities, and Other Special Cases

14. Your friend majoring in mechanical engineering has invented a money machine. The main drawback of the machine is that it is slow. It takes one year to manufacture $100. However, once built, the machine will last forever and will require no maintenance. The machine can be built immediately, but it will cost $1000 to build. Your friend wants to know if she should invest the money to construct it. If the interest rate is 9.5% per year, what should your friend do?

15. How would your answer to Problem 14 change if the machine takes one year to build (so the $1000 outflow is today but the first $100 inflow occurs two years from today)?

16. The British government has a consol bond outstanding paying £100 per year forever. Assume the current interest rate is 4% per year.
    a. What is the value of the bond immediately after a payment is made?
    b. What is the value of the bond immediately before a payment is made?

17. What is the present value of $1000 paid at the end of each of the next 100 years if the interest rate is 7% per year?

*18. When you purchased your car, you took out a five-year annual-payment loan with an interest rate of 6% per year. The annual payment on the car is $5000. You have just made a payment and have now decided to pay the loan off by repaying the outstanding balance. What is the payoff amount if
    a. you have owned the car for one year (so there are four years left on the loan)?
    b. you have owned the car for four years (so there is one year left on the loan)?

19. Your grandmother has been putting $1000 into a savings account on every birthday since your first (that is, when you turned one). The account pays an interest rate of 3%. How much money will be in the account on your eighteenth birthday immediately after your grandmother makes the deposit on that birthday?

20. Assume that your parents wanted to have $160,000 saved for university by your eighteenth birthday and they started saving on your first birthday. If they saved the same amount each year on your birthday and earned 8% per year on their investments,

a. how much would they have to save each year to reach their goal?

b. if they think you will take five years instead of four to graduate and decide to have $200,000 saved just in case, how much more would they have to save each year to reach their new goal?

21. A rich relative has bequeathed you a growing perpetuity. The first payment will occur in a year and will be $1000. Each year after that, you will receive a payment on the anniversary of the last payment that is 8% larger than the last payment. This pattern of payments will go on forever. If the interest rate is 12% per year,

a. what is today's value of the bequest?

b. what is the value of the bequest immediately after the first payment is made?

***22.** You are thinking of building a new machine that will save you $1000 in the first year. The machine will then begin to wear out so that the savings decline at a rate of 2% per year forever. What is the present value of the savings if the interest rate is 5% per year?

23. You work for a pharmaceutical company that has developed a new drug. The patent on the drug will last 17 years. You expect that the drug's profits will be $2 million in its first year and that this amount will grow at a rate of 5% per year for the next 17 years. Once the patent expires, other pharmaceutical companies will be able to produce the same drug and competition will likely drive profits to zero. What is the present value of the new drug if the interest rate is 10% per year?

24. A rich aunt has promised you $5000 one year from today. In addition, each year after that, she has promised you a payment (on the anniversary of the last payment) that is 5% larger than the last payment. She will continue to show this generosity for 20 years, giving a total of 20 payments. If the interest rate is 5%, what is her promise worth today?

***25.** You are running a hot internet company. Analysts predict that its earnings will grow at 30% per year for the next five years. After that, as competition increases, earnings growth is expected to slow to 2% per year and continue at that level forever. Your company has just announced earnings of $1 million. What is the present value of all future earnings if the interest rate is 8%? (Assume all cash flows occur at the end of the year.)

***26.** In 2000, when Alex Rodriguez signed his contract to join the Texas Rangers baseball team, he received a lot of attention for his "$252 million" contract (the total of the payments promised was $252 million). Assume the following:

Rodriguez was set to earn $16 million in the first year, $17 million per year in years 2 through 4, $19 million in each of years 5 and 6, $23 million in year 7, and $27 million per year in years 8 through 10. He would also receive his $10 million signing bonus spread equally over the first 5 years ($2 million per year). His deferred payments were to begin in 2011. The deferred payment amounts total $33 million and are $5 million, then $4 million, then 8 amounts of $3 million (ending in 2020). However, the actual payouts will be different. All of the deferred payments will earn 3% per year until they are paid. For example, the $5 million is deferred from 2001 to 2011, or 10 years, meaning that it will actually be $6.7196 million when paid. Assume that the $4 million payment deferred to 2012 is deferred from 2002 (each payment is deferred 10 years).

The contract is a 10-year contract, but each year has a deferred component so that cash flows are paid out over a total of 20 years. The contractual payments, signing bonus, and deferred components are given below. Note that, by contract, the deferred components are not paid in the year they are earned, but instead are paid (plus interest) 10 years later.

2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
$16M	$17M	$17M	$17M	$19M	$19M	$23M	$27M	$27M	$27M
$2M	$2M	$2M	$2M	$2M					
Deferred									
$5M	$4M	$3M	$3M	$3M	$3M	$3M	$3M	$3M	$3M

Assume that an appropriate discount rate for A-Rod to apply to the contract payments is 7% per year.

a. Calculate the true promised payments under this contract, including the deferred payments with interest.
b. Draw a timeline of all of the payments.
c. Calculate the present value of the contract.
d. Compare the present value of the contract to the quoted value of $252 million. What explains the difference?

***27.** You are trying to decide how much to save for retirement. Assume you plan to save $5000 per year with the first investment made one year from now. You think you can earn 10% per year on your investments and you plan to retire in 43 years, immediately after making your last $5000 investment.

a. How much will you have in your retirement account on the day you retire?
b. If, instead of investing $5000 per year, you wanted to make one lump-sum investment today for your retirement, how much would that lump sum need to be?
c. If you hope to live for 20 years in retirement, how much can you withdraw every year in retirement (starting one year after retirement) so that you will just exhaust your savings with the twentieth withdrawal (assume your savings will continue to earn 10% in retirement)?
d. If, instead, you decide to withdraw $300,000 per year in retirement (again with the first withdrawal one year after retiring), how many years will it take until you exhaust your savings?
e. Assuming the most you can afford to save is $5000 per year, but you want to retire with $1 million in your investment account, how high a return do you need to earn on your investments?

28. Your brother has offered to give you $1000 in one year, and after that, yearly payments growing at 3% each year and ending in 20 years. If you deposit each of these payments into your TD Bank account, how much will you accumulate 20 years from today if the interest rate is 7% per year?

Solving for Variables Other Than Present Value or Future Value

29. You have decided to buy a perpetuity. The bond makes one payment at the end of every year forever and has an interest rate of 5%. If you initially put $1000 into the bond, what is the payment every year?

30. You are thinking of purchasing a house. The house costs $350,000. You have $50,000 in cash that you can use as a down payment on the house, but you need to borrow the rest of the purchase price. The bank is offering a 30-year mortgage that requires annual payments and has an interest rate of 7% per year. What will your annual payment be if you sign up for this mortgage?

***31.** You are thinking about buying a piece of art that costs \$50,000. The art dealer is proposing the following deal: He will lend you the money, and you will repay the loan by making the same payment every two years for the next 20 years (i.e., a total of 10 payments). If the interest rate is 4% per year, how much will you have to pay every two years?

***32.** You would like to buy the house and take the mortgage described in Problem 30. You can afford to pay only \$23,500 per year. The bank agrees to allow you to pay this amount each year, yet still borrow \$300,000. At the end of the mortgage (in 30 years), you must make a balloon payment; that is, you must repay the remaining balance on the mortgage. How much will this balloon payment be?

***33.** You are saving for retirement. To live comfortably, you decide you will need to save \$2 million by the time you are 65. Today is your twenty-second birthday, and you decide, starting today and continuing on every birthday up to and including your sixty-fifth birthday, that you will put the same amount into a savings account. If the interest rate is 5%, how much must you set aside each year to make sure that you will have \$2 million in the account on your sixty-fifth birthday?

***34.** You realize that the plan in Problem 33 has a flaw. Because your income will increase over your lifetime, it would be more realistic to save less now and more later. Instead of putting the same amount aside each year, you decide to let the amount that you set aside grow by 7% per year. Under this plan, how much will you put into the account today? (Recall that you are planning to make the first contribution to the account today.)

35. You have an investment opportunity that requires an initial investment of \$5000 today and will pay \$6000 in one year. What is the *IRR* of this opportunity?

36. You are shopping for a car and read the following advertisement in the newspaper: "Own a new Spitfire! No money down. Four annual payments of just \$10,000." You have shopped around and know that you can buy a Spitfire for cash for \$32,500. What is the interest rate the dealer is advertising (what is the *IRR* of the loan in the advertisement)? Assume that you must make the annual payments at the end of each year.

37. A local bank is running the following advertisement in the newspaper: "For just \$1000 we will pay you \$100 forever!" The fine print in the ad says that for a \$1000 deposit, the bank will pay \$100 every year in perpetuity, starting one year after the deposit is made. What interest rate is the bank advertising (what is the *IRR* of this investment)?

***38.** The Laiterie de Coaticook in the Eastern Townships of Quebec produces several types of cheddar cheese. It markets this cheese in 4 varieties: aged 2 months, 9 months, 15 months, and 2 years. At the producer's store, 2 kg of each variety sells for the following prices: \$7.95, \$9.49, \$10.95, and \$11.95, respectively. Consider the cheese maker's decision whether to continue to age a particular 2-kg block of cheese. At 2 months, he can either sell the cheese immediately or let it age further. If he sells it now, he will receive \$7.95 immediately. If he ages the cheese, he must give up the \$7.95 today to receive a higher amount in the future. What is the *IRR* (expressed in percent per month) of the investment of giving up \$79.50 today by choosing to store 20 kg of cheese that is currently 2 months old and instead selling 10 kg of this cheese when it has aged 9 months, 6 kg when it has aged 15 months, and the remaining 4 kg when it has aged 2 years?

*39. Your grandmother bought an annuity from Great-West Life Insurance Company for $200,000 when she retired. In exchange for the $200,000, Great-West Life will pay her $25,000 per year until she dies. The interest rate is 5%. How long must she live after the day she retired to come out ahead (that is, to get more in value than what she paid in)?

*40. You are thinking of making an investment in a new plant. The plant will generate revenues of $1 million per year for as long as you maintain it. You expect that the maintenance costs will start at $50,000 per year and will increase 5% per year thereafter. Assume that all revenue and maintenance costs occur at the end of the year. You intend to run the plant as long as it continues to make a positive cash flow (as long as the cash generated by the plant exceeds the maintenance costs). The plant can be built and become operational immediately. If the plant costs $10 million to build, and the interest rate is 6% per year, should you invest in the plant?

*41. You have just turned 22 years old, have just received your bachelor's degree, and have accepted your first job. Now you must decide how much money to put into your registered retirement savings plan (RRSP). The RRSP works as follows: Every dollar in the RRSP earns 7% per year. You will not make withdrawals until you retire on your sixty-fifth birthday. After that point, you can make withdrawals as you see fit. You decide that you will plan to live to 100 and work until you turn 65. You estimate that to live comfortably in retirement, you will need $100,000 per year, starting at the end of the first year of retirement (i.e., when you turn 66) and ending on your one-hundredth birthday. You will contribute the same amount to the RRSP at the end of every year that you work. How much do you need to contribute each year to fund your retirement?

*42. Problem 41 is not very realistic because most people do not contribute a fixed amount to their RRSP each year. Instead, they are more likely to contribute a fixed percentage of their salary. Assume that your starting salary is $45,000 per year and it will grow 3% per year until you retire. Assuming everything else stays the same as in Problem 41, what percentage of your income do you need to contribute to the RRSP every year to fund the same retirement income?

Data Case

Assume today is July 30, 2010. Natasha Kingery is 30 years old and has a bachelor of science degree in computer science. She is currently employed at Open Text Corporation in Ottawa, and earns $38,000 a year that she anticipates will grow at 3% per year. Natasha hopes to retire at age 65 and has just begun to think about the future.

Natasha has $75,000 that she recently inherited from her aunt. She invested this money in 10-year Government of Canada bonds. She is considering whether she should further her education and would use her inheritance to pay for it.

She has investigated a couple of options and is asking for your help as a financial planning intern to determine the financial consequences associated with each option. Natasha has already been accepted to both of these programs, and could start either one soon.

One alternative that Natasha is considering is attaining a certification in network design. This certification would automatically promote her to a new position and give her a raise of $10,000. This salary differential will grow at a rate of 3% per year as long as she keeps working. The certification program requires the completion of 20 web-based courses and a score of 80% or better on an exam at the end of the course work. She has learned that the average amount of time necessary to finish the program is one year. The total cost of the program is $5000, due when she enrolls in the program. Because she will do all the work for the certification on her own time, Natasha does not expect to lose any income during the certification.

Another option is going back to school for an MBA degree. With an MBA degree, Natasha expects to be promoted to a managerial position at Open Text. The managerial position pays $20,000 a year more than her current position. She expects that this salary differential will also grow at a rate of 3% per year for as long as she keeps working. The evening program, which will take three years to complete, costs $25,000 per year, due at the beginning of each of her three years in school. Because she will attend classes in the evening, Natasha doesn't expect to lose any income while she is earning her MBA if she chooses to undertake the MBA.

1. Determine the interest rate she is currently earning on her inheritance by going to the Bank of Canada website (http://www.bankofcanada.ca/en/) and clicking on Interest Rates under the Rates and Statistics tab. Then click on "Canadian Bonds." Once there, click on "10-year lookup." Enter the appropriate date you want, July 30, 2010, and check off the desired bond, "10 year" (daily prices) to obtain the average yield or interest rate that she is earning. Use this interest rate as the discount rate for the remainder of this problem.
2. Create a timeline in Excel for her current situation, as well as the certification program and MBA degree options, using the following assumptions:
   a. Salaries for the year are paid only once, at the end of each year of your timeline.
   b. The salary increase becomes effective immediately upon graduating from the MBA program or being certified. That is, because the increases become effective immediately but salaries are paid at the end of the year, the first salary increase will be paid exactly one year after graduation or certification.
3. Calculate the present value of the salary differential for completing the certification program. Subtract the cost of the program to get the *NPV* of undertaking the certification program.
4. Calculate the present value of the salary differential for completing the MBA degree. Calculate the present value of the cost of the MBA program. Based on your calculations, determine the *NPV* of undertaking the MBA.
5. Based on your answers to Questions 3 and 4, what advice would you give to Natasha? What if the two programs are mutually exclusive? If Natasha undertakes one of the programs, there is no further benefit to undertaking the other program. Would your advice change?

Chapter 4 APPENDIX Using a Financial Calculator

Specifying Decimal Places

Make sure you always have plenty of decimal places displayed!

HP-10BII

DISP 8

TI BAII Plus Professional

2ND • 8 ENTER

Toggling Between the Beginning and End of a Period

You should always make sure that your calculator is in *end-of-period* mode.

HP-10BII

MAR

TI BAII Plus Professional

2ND PMT

Set the Number of Periods per Year

You will avoid a lot of confusion later if you always set your periods per year "P/Y" to 1:

HP-10BII

PMT 1

TI BAII Plus Professional

2ND I/Y 1 ENTER

General TVM Buttons

HP-10BII

N I/YR PV PMT FV

TI BAII Plus Professional

N I/Y PV PMT FV

Solving for the Present Value of a Single Future Cash Flow (Example 4.1)

You are considering investing in a Government of Canada bond that will make one payment of $15,000 in 10 years. If the competitive market interest rate is fixed at 6% per year, what is the bond worth today? [Answer: $8375.92]

HP-10BII

	C					Press [Orange Shift] and then the [C] button to clear all previous entries.
1	0	N				Enter the Number of periods.
6	I/YR					Enter the market annual interest rate.
1	5	0	0	0	FV	Enter the Value you will receive in 10 periods.
0	PMT					Indicate that there are no payments.
PV						Solve for the Present Value.

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2ND	FV					Press [2ND] and then the [FV] button to clear all previous entries.
3	0	N				Enter the Number of periods.
1	0	I/Y				Enter the market annual interest rate.
1	0	0	0	0	PMT	Enter the payment amount per period.
0	PV					Indicate that there is no initial amount in the retirement account.
CPT	FV					Solve for the Future Value.

Solving for the Future Value of an Annuity (Example 4.7)

Ellen is 35 years old, and she has decided it is time to plan seriously for her retirement. At the end of each year until she is 65, she will save $10,000 in a retirement account. If the account earns 10% per year, how much will Ellen have saved at age 65? [Answer: $1,644,940]

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	C					Press [Orange Shift] and then the [C] button to clear all previous entries.
3	0	N				Enter the Number of periods.
1	0	I/YR				Enter the market annual interest rate.
1	0	0	0	0	PMT	Enter the Payment amount per period.
0	PV					Indicate that there is no initial amount in the retirement account.
FV						Solve for the Future Value.

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[2ND] [FV]	Press [2ND] and then the [FV] button to clear all previous entries.
[1] [0] [N]	Enter the Number of periods.
[6] [I/Y]	Enter the market annual interest rate.
[1] [5] [0] [0] [0] [FV]	Enter the Value you will receive in 10 periods.
[0] [PMT]	Indicate that there are no payments.
[CPT] [PV]	Solve for the Present Value.

Solving for the Internal Rate of Return

If you have an initial cash outflow of $2000 and one cash inflow per year for the following four years of $1000, $400, $400, and $800, what is the internal rate of return on the project per year? [Answer: 12.12%]

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[ ] [C]	Press [Orange Shift] and then the [C] button to clear all previous entries.
[2] [0] [0] [0] [+/–] [CF_j]	Enter the initial cash outflow.
[1] [0] [0] [0] [CF_j]	Enter the first cash inflow.
[4] [0] [0] [CF_j]	Enter the second cash inflow.
[2] [ ] [CF_j]	Enter the number of consecutive periods the second cash inflow occurs.
[8] [0] [0] [CF_j]	Enter the fourth cash inflow.
[ ] [CST]	Press [Orange Shift] and then the [CST] button to calculate the IRR/year.

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[CF]	Access Cash Flow Worksheet.
[2ND] [CE|C]	Press [2ND] and then the [CE/C] button to clear all previous entries.
[2] [0] [0] [0] [+/–] [ENTER]	Enter the initial cash outflow.
[↓] [1] [0] [0] [0] [ENTER]	Enter the first cash inflow.
[↓]	Leave the frequency of the initial cash inflow at 1 (Default Setting).
[↓] [4] [0] [0] [ENTER]	Enter the second cash inflow.
[↓] [2] [ENTER]	Enter the frequency of the second cash inflow as 2.
[↓] [8] [0] [0] [ENTER]	Enter the fourth cash inflow.
[↓]	Leave the frequency of the fourth cash inflow at 1 (Default Setting).
[IRR] [CPT]	Solve for the IRR.