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Nth-ORDER FLAT APPROXIMATION OF THE SIGNUM FUNCTION BY A POLYNOMIAL

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16. Abstract

In the interval $-\sqrt{2} \leq x \leq \sqrt{2}$, the signum function, sgn x, is demonstrated to be uniquely approximated by an odd polynomial $f_n(x)$ of order $2n-1$ whereby the approximation is nth-order flat with respect to the points (1, 1) and (-1, -1). A theorem is proved which states that, for even integers $n \geq 2$, the approximating polynomial $f_n(x)$ has a pair of nonzero real roots $\pm x_n$ such that the x_n form a monotonically decreasing sequence which converges to $\sqrt{2}$ as n approaches infinity.

For odd $n > 1$, $f_n(x)$ represents a strictly increasing monotonic function for all real x.

As n tends to infinity, $f_n(x)$ converges to sgn x uniformly in $-\sqrt{2} \leq x \leq \epsilon < 0$ and $0 < \epsilon \leq x \leq \sqrt{2}$.

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SUMMARY

In the interval $-\sqrt{2} \leq x \leq \sqrt{2}$, the signum function, sgn x, is demonstrated to be uniquely approximated by an odd polynomial $f_n(x)$ of the order $2n - 1$ whereby the approximation is nth-order flat with respect to the points (1,1) and (-1,-1).

A theorem is proved which states that, for even integers $n \geq 2$, the approximating polynomial $f_n(x)$ has a pair of nonzero real roots $\pm x_n$ such that

$$\sqrt{2 + \frac{1}{2n+1}} < x_n \leq \sqrt{3},$$

that these roots x_n form a monotonically decreasing sequence $\{x_n\}$ $(n = 2, 4 \ldots)$, and that $\lim_{n \to \infty} x_n = \sqrt{2}$.

For odd n, $f_n(x)$ represents a strictly increasing monotonic function of x in

$$-\infty < x < \infty .$$

As $n \to \infty$, $f_n(x)$ converges to sgn x uniformly in $-\sqrt{2} \leq x \leq \epsilon < 0$ and $0 < \epsilon \leq x \leq \sqrt{2}$.

DEVELOPMENT OF THE APPROXIMATION

In the process of synthesizing transfer functions which approximate constant gain and, simultaneously, constant phase response in a finite frequency band, we encounter a polynomial $f_n(x)$ which provides an nth-order flat approximation to the signum function[1]

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1. The author is indebted to Dr. Bernard A. Asner, Assistant Professor, Department of Mathematics, Robert College, Istanbul, Turkey, for valuable suggestions, encouragement, and guidance during his summer employment at MSFC.

$$\operatorname{sgn} x = \begin{cases} +1 \text{ when } x > 0 \\ 0 \text{ when } x = 0 \\ -1 \text{ when } x < 0 \end{cases}$$

in a closed interval, $-\sqrt{2} \leq x \leq \sqrt{2}$. The limit function of approximation is defined by

$$f(x) = \operatorname{sgn} x \quad \text{for } -\sqrt{2} \leq x \leq \sqrt{2} \,. \tag{1}$$

Since sgn x is an odd function, an odd polynomial in x is chosen to represent the approximating function $f_n(x)$,

$$f_n(x) = a_1x + a_3x^3 + \ldots + a_{2n-1}x^{2n-1} \quad . \tag{2}$$

Then the approximation problem can be solved by constraining $f_n(x)$, equation (2), so that $f_n(x)$ is nth-order flat at the points (1, 1) and (-1, 1); i.e.

$$f_n(\pm 1) = \pm 1$$

and

$$f_n^{(r)}(\pm 1) = 0 \quad \text{for} \quad r = 1, 2, \ldots, n-1 \,. \tag{3}$$

The second constraint in equation (3) requires that the first $n - 1$ derivatives of $f_n(x)$ vanish at $x = \pm 1$. The polynomial in equation (2) has n coefficients, and the constraint conditions in equation (3) furnish exactly n linear equations to determine uniquely the n coefficients of the polynomial. Therefore, equations (2) and (3) together yield a unique solution to the approximation problem posed.

If $f'_n(x)$ is analytic at the points $x = \pm 1$ and if and only if $f'_n(x)$ and its first $n - 2$ derivatives vanish at $x = \pm 1$, then the equation

$$f'_n(x) = 0$$

has zeros of the order $n - 1$ at $x = \pm 1$ [1]. These properties of $f'_n(x)$ are ensured if $f'_n(x)$ is chosen to be of the following form:

$$f'_n(x) = k_n(1 - x^2)^{n-1}$$

$$= k_n(1 - x)^{n-1}(1 + x)^{n-1} \tag{4}$$

where $f'_n(\pm 1) = 0$ of $(n-1)$st order so that

$$f_n^{(r)}(\pm 1) = \frac{d^{r-1}}{dx^{r-1}} f'_n(x) \Big|_{x = \pm 1} = 0 \text{ for } r = 1, 2, \ldots, n - 1.$$

We obtain $f_n(x)$ by integrating equation (4) from 0 to x,

$$f_n(x) = k_n \int_0^x (1 - t^2)^{n-1} dt \tag{5}$$

where the constant k_n is determined from the first condition in equation (3),

$$f_n(1) = 1$$

or

$$\frac{1}{k_n} = \int_0^1 (1 - t^2)^{n-1} dt \ .$$

$dt = \frac{1}{2} u^{-1/2} du$

$t = u^{\frac{1}{2}}$

To evaluate this integral we make the change of variable, $t^2 = u$, and obtain [2]

$$\frac{1}{k_n} = \int_0^1 (1 - t^2)^{n-1} dt = \frac{1}{2} \int_0^1 u^{-\frac{1}{2}} (1 - u)^{n-1} \cancel{dt}\ du$$

$$= \frac{1}{2} \frac{\Gamma(\frac{1}{2}) \ \Gamma(n)}{\Gamma(n + \frac{1}{2})}$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma(n)}{\Gamma(n + \frac{1}{2})}$$

$$= \frac{(n - 1)! \ 2^{n-1}}{1 \cdot 3 \cdot 5 \cdots (2n - 1)} \quad (n > 0, \text{ integer}) \quad (6)$$

so that

$$k_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^{n-1} (n-1)!} \ (n > 0) \ . \quad (7)$$

Considering the inequality [3],

$$n^{\frac{1}{2}} \leq \frac{\Gamma\,(n + 1)}{\Gamma\,(n + \frac{1}{2})} \leq (n + 1)^{\frac{1}{2}} \quad ,$$

(where n is a natural number) together with equation (6), the following useful inequality for k_n is obtained:

$$\frac{2}{\sqrt{\pi}} \quad \frac{n^{\frac{1}{2}}}{(1 + \frac{1}{n})^{\frac{1}{2}}} \leq k_n \leq \frac{2}{\sqrt{\pi}} n^{\frac{1}{2}} \quad . \tag{8}$$

Using the binomial expansion of $(1 - t^2)^{n-1}$ on the right of equation (5) and carrying out the integration term-by-term results in the following polynomial representation of $f_n(x)$:

$$f_n(x) = k_n \sum_{r=o}^{n-1} \binom{n-1}{r} (-1)^r \frac{x^{2r+1}}{2r + 1} \quad (n > 0) \tag{9}$$

where k_n is given by equation (7).

The behavior of $f_n(x)$ is illustrated in Figures 1 and 2 for even and odd integers n, respectively. We note that $f_n(x)$ behaves quite differently, depending on whether n is even or odd. When n is even, $f_n(x)$ has a maximum (minimum) at $x = 1$ $(x = -1)$; when n is odd, $f_n(x)$ has inflection points with zero slope at $x = \pm 1$.

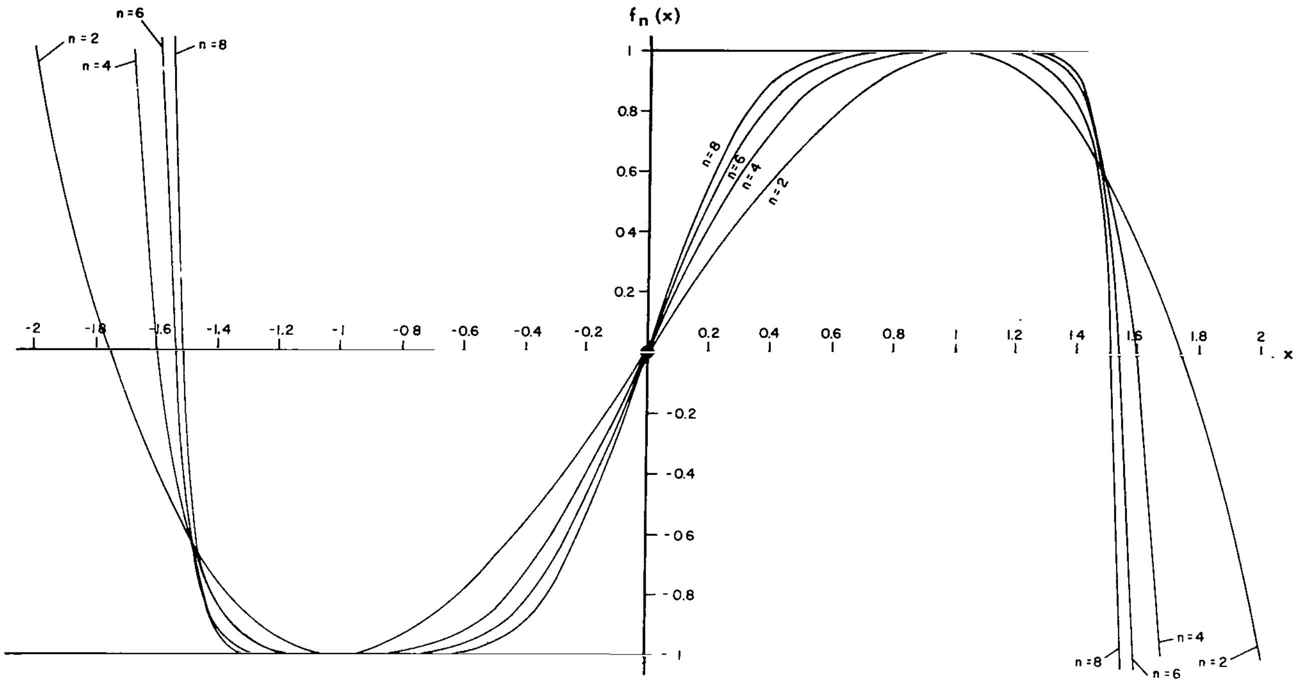

Figure 1. $f_n(x)$, for even n.

To demonstrate this behavior of $f_n(x)$ analytically, we note that $f_n(x)$ is an odd function as seen from equation (9). Therefore,

$$f_n(x) = -f_n(-x)$$

and (10)

$$f_n(0) = 0,$$

and it suffices to conduct the demonstration for $x > 0$ since $f_n(x)$, as an odd function, has rotational symmetry about the origin.

The slope of $f_n(x)$, $f'_n(x)$ as given by equation (4) is positive for $0 \leq x < 1$. Therefore, considering equations (3) and (10), $f_n(x)$ increases monotonically from 0 to 1 as x increases from 0 to 1. It follows from the second constraint in equation (3) that the slope of $f_n(x)$ is equal to zero at $x = 1$, $f'_n(1) = 0$.

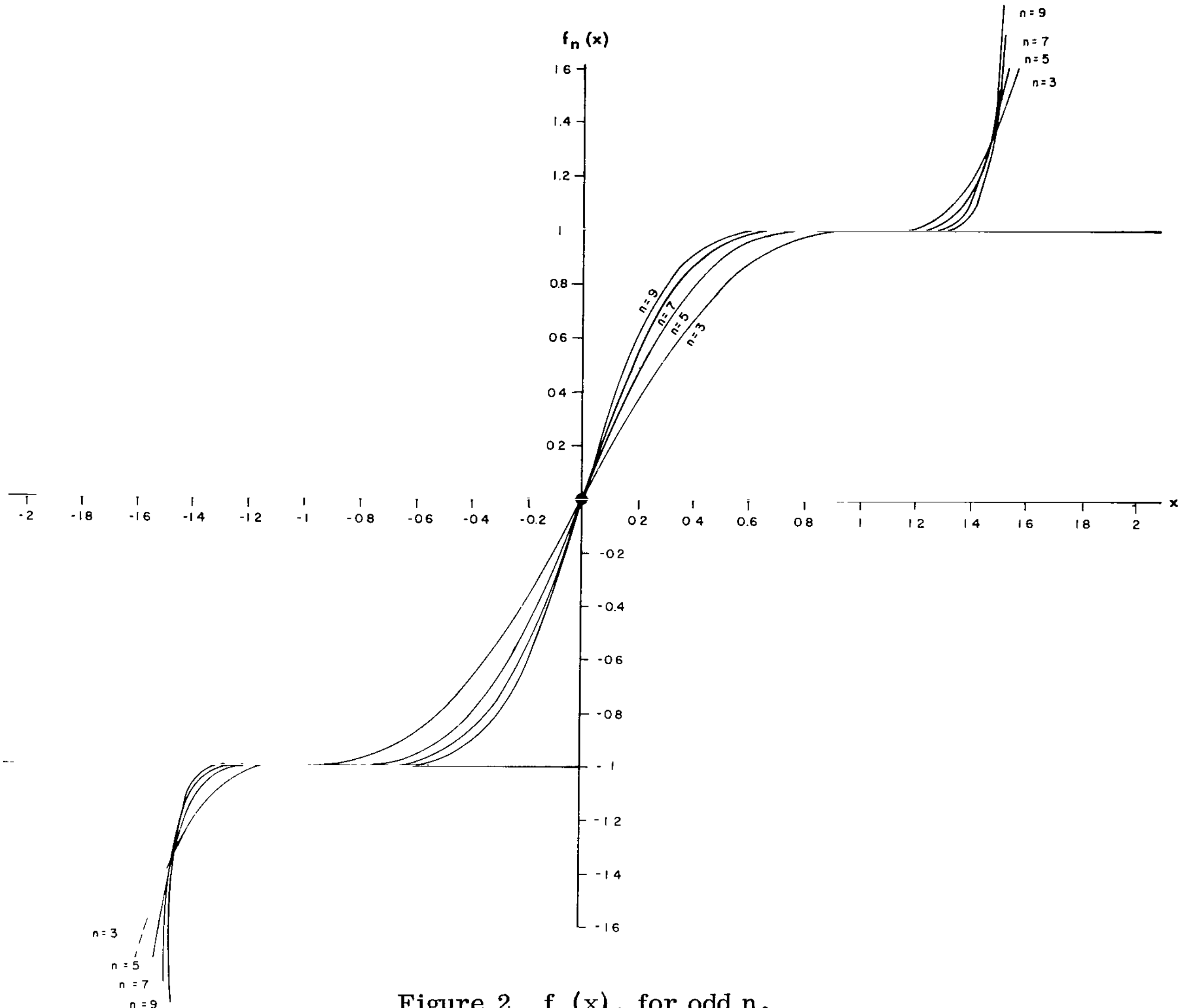

Figure 2. $f_n(x)$, for odd n.

To decide whether $f_n(1) = 1$ represents a maximum, we form higher derivatives of $f_n(x)$. The rth derivative of $f_n(x)$ is

$$f_n^{(r)}(x) = \frac{d^{r-1}}{dx^{r-1}} f_n'(x) \quad .$$

Using equation (4) and Leibnitz's product rule gives

$$f_n^{(r)}(x)$$

$$= k_n \sum_{m=o}^{r-2} \binom{r-1}{m} \binom{n-1}{m} \binom{n-1}{r-m-1} m! \ (r-m-1)! \ (-1)^{n-1} (x-1)^{n-m-1} (x+1)^{n+m-r}$$

$$+ k_n \binom{n-1}{r-1} (r-1)! \ (-1)^{n-1} (x-1)^{n-r} (x+1)^{n-1} \tag{11}$$

where $n > 1$.

For $x = 1$ we have

$$(x - 1)^{\alpha} = \begin{cases} 0 \text{ if } \alpha > 0 \\ 1 \text{ if } \alpha = 0 \end{cases}$$

and note that the last term on the right of equation (11) contains the lowest power of $(x - 1)$. Therefore the right side of equation (11) is equal to zero for $x = 1$ as long as $r \leq n - 1$ so that $f_n^{(r)}(1) = 0$ for $r = 1, 2, \ldots, n - 1$. However, for $r = n$, all terms on the right of equation (11) vanish for $x = 1$ except the last term and we obtain

$$f_n^{(n)}(1) = k_n (n-1)! \ (-1)^{n-1} 2^{n-1} . \tag{12}$$

Then a Taylor-series expansion of $f_n(x)$ about $x = 1$ yields

$$f_n(1 + h) - f_n(1) = \frac{h^n}{n!} f_n^{(n)}(1 + \vartheta h), \quad 0 < \vartheta < 1 , \tag{13}$$

because, by the second constraint in equation (3), the first $n - 1$ derivatives of $f_n(x)$ vanish at $x = 1$.

For extremely small values of h the sign of $f_n^{(n)}(1+\vartheta k)$, $0<\vartheta<1$, is the same as that of $f_n^{(n)}(1)$,

$$\operatorname{sgn}\{f_n^{(n)}(1+\vartheta k)\} = \operatorname{sgn}\{f_n^{(n)}(1)\} = \operatorname{sgn}\{(-1)^{n-1}\} \quad .$$

Then equation (13) yields

$$\operatorname{sgn}\{f_n(1+h) - f_n(1)\} = \operatorname{sgn}\{(-1)^{n-1}h^n\} \quad ,$$

and we conclude that, for even $n \geq 2$, $f_n(1) = 1$ represents a maximum because

$$f_n(1+h) - f_n(1) < 0 \text{ for } h \gtrless 0 \quad (\text{even } n \geq 2) \tag{14}$$

while, for odd $n > 1$, $f_n(1) = 1$ represents a point of inflection because

$$f_n(1+h) - f_n(1) \begin{cases} < 0 \text{ for } h < 0 \\ > 0 \text{ for } h > 0 \end{cases} \quad (\text{odd } n > 1) \; . \tag{15}$$

For $x > 0$, $f_n(x)$ has only one maximum or only one point of inflection depending on whether n is even or odd because, for $x > 0$, $f_n'(x)$ has only one zero (of multiplicity $n-1$) which occurs at $x = 1$.

This establishes the behavior of $f_n(x)$ for $x > 0$. Its behavior for $x < 0$ can be readily inferred by the odd symmetry of $f_n(x)$, first equation (10).

We further note by inspecting Figures 1 and 2 that, for even n, $f_n(x) = 0$ has a pair of nonzero real roots $\pm x_n$ which tend to $\pm\sqrt{2}$ as n increases and that, for odd n, $f_n(x)$ is an increasing function for all x. Furthermore, we note that, as n increases, $f_n(x)$ approaches the shape of the signum function, sgn (x), in the interval, $-\sqrt{2} \leq x \leq +\sqrt{2}$. These additional properties are stated in the following theorem.

THEOREM AND PROOF

Theorem

Let

$$f_n(x) = k_n \int_0^x (1 - t^2)^{n-1} dt \quad ,$$

where the integer $n > 1$ and where

$$k_n = \left(\int_0^1 (1 - t^2)^{n-1} dt \right)^{-1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n-1}\,(n-1)\,!} \quad .$$

Then:

1. If n is even, then $f_n(x) = 0$ has only one pair of nonzero real roots $\pm x_n$, such that

$$\sqrt{2 + \frac{1}{2n + 1}} < x_n < \sqrt{3}\,, \quad x_{n+2} < x_n \quad (n = 2, 4, 6, \ldots) \quad ,$$

$$\lim_{n \to \infty} x_n = \sqrt{2} \quad ,$$

and

$$f_n(x) > 0 \text{ for } x < -x_n \text{ and } 0 < x < x_n$$

$$f_n(x) < 0 \text{ for } x > x_n \text{ and } -x_n < x < 0 \ ;$$

2. If n is odd, then $f_n(x)$ is a strictly increasing monotonic function of x and

$$f_n(x) \begin{cases} > 0 \text{ for } x > 0 \\ < 0 \text{ for } x < 0 \end{cases} ;$$

3. If $n \to \infty$, then $f_n(x) \to f(x) = \operatorname{sgn} x$, uniformly in

$$-\sqrt{2} \le x \le \epsilon < 0 \quad \text{and} \quad 0 < \epsilon \le x \le \sqrt{2} \ .$$

Proof

We restrict the proof to values of $x > 0$ since the statements for $x < 0$ can be inferred from the odd symmetry of $f_n(x)$, equation (10). We shall prove part 3 of the theorem first.

In the interval J: $0 < x \le \sqrt{2}$, the limiting function $f(x)$ is equal to one,

$$f(x) = \operatorname{sgn}(x) = +1 \text{ for } 0 < x \le \sqrt{2} \ .$$

We shall demonstrate that, as $n \to \infty$,

$$f_n(x) \to f(x) = +1 \ ,$$

uniformly in the closed subinterval

$$J': \epsilon \leq x \leq \sqrt{2}\ , \ \epsilon > 0.$$

Using equation (5) we write

$$f_n(x) = k_n \int_0^x (1 - t^2)^{n-1} dt$$

$$= k_n \int_0^1 (1 - t^2)^{n-1} dt + k_n \int_1^x (1 - t^2)^{n-1} dt$$

where, according to the first constraint in equation (3),

$$f_n(1) = k_n \int_0^1 (1 - t^2)^{n-1} dt = 1$$

so that

$$f_n(x) = 1 + k_n \int_1^x (1 - t^2)^{n-1} dt \tag{16}$$

or, with the abbreviation

$$I_n(x) = k_n \int_1^x (1 - t^2)^{n-1} dt$$

$$f_n(x) = 1 + I_n(x) \quad .$$

To demonstrate that, as $n \to \infty$,

$$f_n(x) \to 1, \text{ uniformly in } J'$$

we must show that, as $n \to \infty$,

$$I_n(x) \to 0, \text{ uniformly in } J' .$$

This will be shown separately in the two closed subintervals

$$J'': 0 < \epsilon \leq x \leq 1 \quad \text{and} \quad J''': 1 \leq x \leq \sqrt{2} .$$

For $0 < \epsilon < 1$, we select the following inequality for the integrand of $I_n(x)$:

$$(1 - t^2)^{n-1} \leq (1 - \epsilon^2)^{n-1} \qquad (0 < \epsilon \leq t \leq 1; \; n = 1, 2, \ldots) ,$$

which establishes an upper bound for $\left| I_n(x) \right|$,

$$\left| I_n(x) \right| = k_n \left| \int_1^x (1-t^2)^{n-1} dt \right| \leq k_n \, (1-\epsilon) \, (1-\epsilon^2)^{n-1} \qquad (0 < \epsilon < x \leq 1;$$
$$n = 1, 2, \ldots).$$

By inequality (8)

$$k_n \leq \frac{2}{\sqrt{\pi}} \, n^{\frac{1}{2}} \; (n = 1, 2, \ldots) .$$

so that $k_n\ (1 - \epsilon)\ (1 - \epsilon^2)^{n-1} < \dfrac{2}{\sqrt{\pi}} n^{\frac{1}{2}}\ (1 - \epsilon^2)^{n-1} \quad (n = 1, 2, \dots)$

and we can write

$$\left|I_n(x)\right| < \frac{2}{\sqrt{\pi}} n^{\frac{1}{2}} (1 - \epsilon^2)^{n-1} \qquad (0 < \epsilon \leq x \leq 1;\ n = 1, 2, \dots)$$

where the upper bound of $\left|I_n(x)\right|$ is independent of x in J''.

Now $n^{\frac{1}{2}} (1 - \epsilon^2)^{n-1} = e^{\frac{1}{2} \ln n + (n-1) \ln (1-\epsilon^2)}$

and $\ln\ (1 - \epsilon^2) < 0$ for $0 < \epsilon < 1$.

Then we can choose a positive integer N sufficiently large and a positive number $M(N)$ so that

$$\frac{1}{2} \ln N + (N - 1) \ln (1 - \epsilon^2) < - M(N) < 0$$

and we obtain

$$e^{\frac{1}{2} \ln n + (n-1) \ln(1-\epsilon^2)} < e^{-M(N)} \text{ for } n > N \quad .$$

Therefore, as $n \rightarrow \infty$,

$$\left|I_n(x)\right| \rightarrow 0, \text{ uniformly in } J'':\ 0 < \epsilon \leq x \leq 1 .$$

To show that, as $n \to \infty$,

$$I_n(x) \to 0, \quad \text{uniformly in } J''': \ 1 \leq x \leq \sqrt{2} ,$$

we establish the following inequality:

$$e^{t-\sqrt{2}} \geq t^2 - 1 \quad \text{for} \quad 1 \leq t \leq \sqrt{2} . \tag{17}$$

It is a direct consequence of the mean-value theorem that

$$e^y = 1 + y + \frac{y^2}{2} e^{\theta y} \qquad (0 < \theta < 1)$$

and therefore the following inequality holds for all y:

$$e^y \geq 1 + y .$$

Then, upon letting

$$y = t - \sqrt{2}$$

we proceed to show that

$$e^{t - \sqrt{2}} \geq 1 - \sqrt{2} + t \geq t^2 - 1$$

by showing that

$$1 - \sqrt{2} + t \geq t^2 - 1$$

or

$$t^2 - t + \sqrt{2} - 2 \le 0 \quad \text{for} \quad 1 \le t \le \sqrt{2} \ .$$

Writing the left side of this inequality in factored form,

$$(t - 1 + \sqrt{2})\ (t - \sqrt{2}) \le 0 \ ,$$

we note that it is satisfied for

$$1 - \sqrt{2} < t \le \sqrt{2}$$

and, since $1 - \sqrt{2} < 1$, also for $1 \le t \le \sqrt{2}$. This establishes inequality (17). Then, using inequalities (8) and (17), we can majorize the integrand of

$$\left| I_n(x) \right| = \int_1^x k_n\ (t^2 - 1)^{n-1} \quad \text{in } 1 \le x \le \sqrt{2}$$

as follows:

$$k_n\ (t^2 - 1)^{n-1} \le \frac{2}{\sqrt{\pi}}\, n^{\frac{1}{2}}\, e^{(t - \sqrt{2})\ (n-1)} \quad \text{for} \quad 1 \le t \le \sqrt{2} \ .$$

Since the integrands are positive and increasing with t in $1 \le t \le \sqrt{2}$, we obtain

$$\left| I_n(x) \right| = \int_1^x k_n (t^2-1)^{n-1} dt \le \frac{2}{\sqrt{\pi}} \int_1^x n^{\frac{1}{2}} e^{(t-\sqrt{2})\ (n-1)} dt \le \frac{2}{\sqrt{\pi}} \int_1^{\sqrt{2}} n^{\frac{1}{2}} e^{(t-\sqrt{2})\ (n-1)} dt$$

in the subinterval J''': $1 \leq x \leq \sqrt{2}$. But,

$$\frac{2}{\sqrt{\pi}} \int_1^{\sqrt{2}} n^{\frac{1}{2}} e^{(t-\sqrt{2})(n-1)} dt < \frac{4}{\sqrt{\pi}} n^{-\frac{1}{2}}, \quad n > 1 .$$

Therefore,

$$\left| I_n(x) \right| < \frac{4}{\sqrt{\pi}} n^{-\frac{1}{2}} \quad \text{for } n > 1 \text{ in J'''} .$$

Choosing

$$\epsilon = \frac{4}{\sqrt{\pi}} N^{-\frac{1}{2}},$$

we obtain

$$\left| I_n(x) \right| < \frac{4}{\sqrt{\pi}} n^{-\frac{1}{2}} < \epsilon \text{ for } n > N, \text{ independently of } x \text{ in } 1 \leq x \leq \sqrt{2} .$$

Thus, as $n \rightarrow \infty$,

$$I_n(x) \rightarrow 0, \text{ uniformly in J''': } 1 \leq x \leq \sqrt{2} .$$

This completes the proof of part 3 of the theorem.

To prove part 1 of the theorem, we begin by showing that the equation

$$f_n(x) = 0$$

has only one positive real root x_n.

Equation (16) yields

$$f_n(x) = 1 + k_n \int_1^x (1 - t^2)^{n-1} dt$$

which, for $x > 1$ and even $n \geq 2$, can be rewritten as

$$f_n(x) = 1 - k_n \int_1^x (t^2 - 1)^{n-1} dt \qquad (x \geq 1;\ n \geq 2,\ \text{even})$$

where the integrand is positive,

$$(t^2 - 1)^{n-1} > 0 \text{ for } 1 < t < x$$

and monotonically increasing with t, so that the integral,

$$\int_1^x (t^2 - 1)^{n-1} dt > 0 \quad ,$$

is also positive and monotonically increasing with $x > 1$. Therefore, for even $n \geq 2$ and $x > 1$, $f_n(x)$ is monotonically decreasing and the equation

$$f_n(x) = 1 - k_n \int_1^x (t^2 - 1)^{n-1} dt = 0 \quad (x > 1, \ n = 2, 4, 6, \ldots) \tag{18}$$

has only one real root, $x_n > 1$.

In the special case, $n = 2$, equation (7) yields $k_2 = \frac{3}{2}$, and equation (18) can be solved in closed form to yield the positive root

$$x_2 = \sqrt{3} \ . \tag{19}$$

From equation (18) we conclude that

$$x_n > 1 \quad (n = 2, 4, 6, \ldots) \ .$$

However, we shall demonstrate that

$$x_n > \sqrt{2 + \frac{1}{2n + 1}} \ , \quad (n = 2, 4, 6, \ldots) \ . \tag{20}$$

For even $n \geq 2$, we know that $f_n(x)$ has a maximum at $x = 1$, $f_n(1) = 1$ and that, by equation (18), $f_n(x)$ is monotonically decreasing for $x > 1$ so that we have

$$f_n(x) \begin{cases} > 0 \text{ for } 0 < x < x_n \\ = 0 \text{ for } x = x_n \\ < 0 \text{ for } x > x_n \end{cases} \ . \tag{21}$$

Therefore, to demonstrate inequality (20), we must show that

$$f_n(x) > 0 \quad \text{for} \quad x = \sqrt{2 + \frac{1}{2n+1}} \quad . \tag{22}$$

Since $f_n(x) > 0$ for $0 < x < x_n$, inequality (21), we conclude that inequality (20) holds,

$$\sqrt{2 + \frac{1}{2n+1}} < x_n \quad ,$$

if inequality (22) is satisfied.

It follows from equation (18) that the following inequality is equivalent to inequality (22):

$$k_n \int_1^x (t^2 - 1)^{n-1} dt < 1 \quad \text{for} \quad x = \sqrt{2 + \frac{1}{2n+1}} \quad .$$

By defining the equation for k_n,

$$\frac{1}{k_n} = \int_0^1 (1 - t^2)^{n-1} dt \quad ,$$

this can be rewritten as

$$\int_1^\alpha (t^2 - 1)^{n-1} dt < \int_0^1 (1 - t^2)^{n-1} dt \tag{23}$$

with the abbreviation

$$\alpha = \sqrt{2 + \frac{1}{2n + 1}} \quad .$$

The change of variable

$$t = \frac{\alpha - u}{\alpha - 1}$$

enables us to make the limits of integration of the integral on the right of inequality (23) the same as those on the left, and we obtain the inequality,

$$\int_1^\alpha \frac{1}{\alpha - 1} \left[1 - \left(\frac{\alpha - t}{\alpha - 1}\right)^2\right]^{n-1} dt > \int_1^\alpha (t^2 - 1)^{n-1} dt, \quad (n = 2, 4, \ldots). \quad (24)$$

We proceed by noting that this inequality is satisfied if the following inequality between the corresponding integrands is satisfied for

$$\alpha = \sqrt{2 + \frac{1}{2n + 1}} \quad \text{and} \quad 1 \leq t \leq \alpha \ :$$

$$\frac{1}{\alpha - 1} \left[1 - \left(\frac{\alpha - t}{\alpha - 1}\right)^2\right]^{n-1} \geq (t^2 - 1)^{n-1} \quad (n = 2, 4, \ldots). \quad (25)$$

Both sides of inequality (25) contain the factor $(t - 1)^{n-1}$ because

$$(t^2 - 1) = (t + 1)\ (t - 1)$$

and

$$\left[1-\left(\frac{\alpha - t}{\alpha - 1}\right)^2\right] = \frac{1}{(\alpha - 1)^2} (2\alpha - 1 - t)\ (t - 1)$$

so that inequality (25) becomes

$$\frac{1}{(\alpha - 1)^{2n-1}} (2\alpha - 1 - t)^{n-1} (t - 1)^{n-1} \geq (t + 1)^{n-1} (t - 1)^{n-1} \ .$$

However, since $(t - 1)^{n-1} \geq 0$ for $1 \leq t \leq \alpha$, the problem is reduced to showing that

$$\frac{1}{(\alpha - 1)^{2n-1}} (2\alpha - 1 - t)^{n-1} > (t + 1)^{n-1}$$

or

$$A^{n-1}(t) > B^{n-1}(t) \quad \text{for} \quad 1 \leq t \leq \alpha = \sqrt{2 + \frac{1}{2n + 1}} \quad (n = 2, 4, \ldots)$$

where

$$A(t) = \frac{1}{(\alpha - 1)^{\frac{2n - 1}{n - 1}}} (2\alpha - 1 - t)$$

$$B(t) = t + 1 \ .$$

We note that

$$\left.\begin{array}{l} A(t) > 0 \\ B(t) > 0 \end{array}\right\} \text{ for } 1 \le t \le \alpha \quad .$$

Therefore we can continue by showing that

$$A(t) > B(t)$$

or

$$A(t) - B(t) > 0 \text{ for } 1 \le t \le \alpha = \sqrt{2 + \frac{1}{2n + 1}} \quad .$$

The left side of this inequality is linear in t. Then it suffices to show that the inequality is satisfied for the endpoints, $t = 1$ and $t = \alpha = \sqrt{2 + \frac{1}{2n + 1}}$, of the interval. This reduces the problem to showing that the following two inequalities are satisfied:

$$A(1) - B(1) > 0$$

or

$$\frac{2\,(\alpha - 1)^{\frac{2n - 1}{n - 1}}}{(\alpha - 1)} - 2 > 0 \quad \left(\alpha = \sqrt{2 + \frac{1}{2n + 1}}\,,\ n = 2, 4, \ldots\right) \tag{26}$$

and

$$A(\alpha) - B(\alpha) > 0$$

or

$$\frac{(\alpha - 1)}{(\alpha - 1)^{\frac{2n - 1}{n - 1}}} - (\alpha + 1) > 0 \; (\alpha = \sqrt{2 + \frac{1}{2n + 1}}, \quad n = 2, 4, \ldots) \; . \quad (27)$$

We reduce inequality (26) to

$$(\alpha - 1)^{\frac{n}{n - 1}} < 1$$

$$(\alpha - 1) < 1^{\frac{n - 1}{n}} = 1 \quad (n = 2, 4, \ldots) \; .$$

Then substitution of

$$\alpha = \sqrt{2 + \frac{1}{2n + 1}}$$

yields

$$\frac{1}{2n + 1} < 2 \quad (n = 2, 4, \ldots) \; .$$

Thus inequality (26) is satisfied for all even $n \geq 2$.

Inequality (27) can be reduced to $(\alpha^2 - 1)^{n-1} (\alpha - 1) < 1$.

Since $\alpha = \sqrt{2 + \frac{1}{2n + 1}}$, this becomes $(1 + \frac{1}{2n + 1})^{n-1} (\sqrt{2 + \frac{1}{2n + 1}} - 1) < 1$

or

$$(1 + \frac{1}{2n+1})^{n-1} (\sqrt{2 + \frac{1}{2n+1}} - 1) < (1 + \frac{1}{2n+1})^{n-1} (\sqrt{2.2} - 1) < 1$$

$$(n = 2, 4, \ldots) .$$

Then, since

$$\frac{1}{\sqrt{2.2} - 1} > 2 ,$$

it suffices to show that

$$(1 + \frac{1}{2n+1})^{n-1} < 2 < \frac{1}{\sqrt{2.2} - 1} . \tag{28}$$

To verify inequality (28) we start with the inequality [3]

$$(1 + \frac{1}{2n+1})^{2n+1} \leq e .$$

Dividing both sides by $(1 + \frac{1}{2n+1})^{n+2}$, we obtain

$$(1 + \frac{1}{2n+1})^{n-1} \leq \frac{e}{(1 + \frac{1}{2n+1})^{n+2}} = \frac{e}{(1 + \frac{1}{2n+1})^{n-1} (1 + \frac{1}{2n+1})^{3}} .$$

But

$$\frac{e}{(1+\frac{1}{2n+1})^{n-1}(1+\frac{1}{2n+1})^{3}} < \frac{e}{(1+\frac{1}{2n+1})^{n-1}} \qquad (n = 2,\ 4,\ \ldots)$$

so that we obtain

$$(1+\frac{1}{2n+1})^{n-1} < \frac{e}{(1+\frac{1}{2n+1})^{n-1}}$$

or

$$(1+\frac{1}{2n+1})^{n-1} < e^{\frac{1}{2}} < 2 \quad .$$

This verifies inequality (28) and, in turn, inequality (27) and completes the proof of inequality (20).

Next we shall demonstrate that the roots x_n form a bounded decreasing sequence $\{x_n\}$, $n = 2, 4, \ldots$. We start by showing that the following monotonicity condition is satisfied:

$$x_{n+2} < x_n \qquad (n = 2,\ 4,\ \ldots) \quad . \tag{29}$$

We have shown that the equation

$$f_n(x) = 0$$

for even $n \geq 2$ has only one real root $x_n > 1$ in $0 < x < \infty$.

Then, since $n + 2$ is also an even integer, the equation

$$f_{n+2}(x) = 0 \quad , \quad \text{for even } n \geq 2 \ , \tag{30}$$

must also have only one real root,

$$x_{n+2} > 1 \quad \text{in} \quad 0 < x < \infty \ .$$

Upon replacing n by $n + 2$ in equation (5) and letting $x = x_{n+2}$, equation (30) yields

$$\int_0^{x_{n+2}} (1 - t^2)^{n+1} dt = 0 \quad (n = 2, 4, \dots) \ .$$

If we integrate first from 0 to x_n and then from x_n to x_{n+2}, this equation can be rewritten as follows:

$$\int_0^{x_n} (1 - t^2)^{n+1} dt + \int_{x_n}^{x_{n+2}} (1 - t^2)^{n+1} dt = 0 \ .$$

Now we apply the mean-value theorem for integrals to the second integral because $(1 - t^2)^{n+1}$ is continuous on $[x_n, x_{n+2}]$ and obtain

$$x_{n+2} - x_n = - \frac{\int_0^{x_n} (t^2 - 1)^{n+1} dt}{(\bar{x}^2 - 1)^{n+1}} \quad (n = 2, 4, \dots) \tag{31}$$

where $\bar{x} = x_n + \vartheta (x_{n+2} - x_n)$; $0 < \vartheta < 1$, and $\bar{x} > 1$ since $x_n > 1$ and $x_{n+2} > 1$.

To prove inequality (29), we must demonstrate that the right side of equation (31) is negative. We note that

$$\bar{x} > 1$$

and therefore the denominator

$$(\bar{x}^2 - 1)^{n+1} > 0 \quad .$$

Then it remains to be shown that

$$\int_0^{x_n} (t^2 - 1)^{n+1} dt > 0 \qquad (n = 2, 4, \ldots) \; .$$

After performing integration by parts twice, we obtain

$$\int_0^{x_n} (t^2 - 1)^{n+1} dt = \left\{ \frac{t(t^2 - 1)^{n+1}}{2n + 3} - \frac{(2n + 2) t (t^2 - 1)^n}{(2n + 3)(2n + 1)} \right\} \Bigg|_0^{x_n}$$

$$+ \frac{4(n + 1) n}{(2n + 3)(2n + 1)} \int_0^{x_n} (t^2 - 1)^{n-1} dt \quad .$$

But, by definition,

$$\int_0^{x_n} (t^2 - 1)^{n-1} dt = - f_n(x_n) = 0 \qquad (n = 2, 4, \ldots) \quad .$$

Then we have

$$\int_0^{x_n} (t^2 - 1)^{n+1} dt = \frac{(2n + 1)\, x_n\, (x_n^2 - 1)^n}{(2n + 3)\,(2n + 1)} \left\{ x_n^2 - \left(2 + \frac{1}{2n + 1}\right) \right\}$$

where the right side is positive because

$$x_n > \sqrt{2 + \frac{1}{2n + 1}}\,, \quad \text{inequality (20)} \quad .$$

Therefore

$$\int_0^{x_n} (t^2 - 1)^{n+1} > 0$$

and, by equation (31),

$$x_{n+2} - x_n < 0$$

which proves inequality (29). Then, by inequality (29), $\{x_n\}$ $(n = 2, 4, \ldots)$ is a monotonically decreasing sequence with a lower bound and, by equation (19) and inequality (20), x_n is bounded by the inequality,

$$\sqrt{2 + \frac{1}{2n + 1}} < x_n \le \sqrt{3}\ .$$

We shall demonstrate that the sequence $\{x_n\}$ converges to $\sqrt{2}$,

$$\lim_{n \to \infty} x_n = \sqrt{2} \quad . \tag{32}$$

Since x_n is a root of the equation,

$$f_n(x) = 0 \qquad (n = 2, 4, \ldots) \quad ,$$

equation (5) yields

$$f_n(x_n) = k_n \int_0^{x_n} (1 - t^2)^{n-1} dt = 0$$

or, upon splitting the interval of integration,

$$k_n \int_0^{\sqrt{2}} (1 - t^2)^{n-1} dt + k_n \int_{\sqrt{2}}^{x_n} (1 - t^2)^{n-1} dt = 0 \quad . \tag{33}$$

Now, applying the mean-value theorem to the second integral results in

$$k_n \int_{\sqrt{2}}^{x_n} (1 - t^2)^{n-1} dt = - k_n (x_n - \sqrt{2}) \; (1 - \bar{x}^2)^{n-1} \qquad (n = 2, 4, \ldots) \tag{34}$$

with $\bar{x} = \sqrt{2} + \vartheta \, (x_n - \sqrt{2})$, $0 < \vartheta < 1$.

Furthermore, for $x = \sqrt{2}$, equation (5) yields

$$f_n(\sqrt{2}) = k_n \int_0^{\sqrt{2}} (1 - t^2)^{n-1} dt \quad . \tag{35}$$

Then, substituting equations (34) and (35) into equation (33) gives

$$x_n - \sqrt{2} = \frac{f_n(\sqrt{2})}{k_n (\bar{x}^2 - 1)^{n-1}} \qquad (n = 2, 4, \ldots) \tag{36}$$

with $\sqrt{2} < \bar{x} < x_n \quad (x_n \leq \sqrt{3})$.

We obtain an upper and lower bound for $(x_n - \sqrt{2})$ by substituting $\bar{x} = \sqrt{2}$ and $\bar{x} = \sqrt{3}$ into equation (36), respectively:

$$\frac{f_n(\sqrt{2})}{k_n 2^{n-1}} < (x_n - \sqrt{2}) < \frac{f_n(\sqrt{2})}{k_n} \qquad (n = 2, 4, \ldots) \tag{37}$$

since, by inequality (21), $f_n(\sqrt{2}) > 0 \quad (0 < \sqrt{2} < x_n)$.

From inequality (8), we obtain

$$\frac{\sqrt{\pi}}{2n^{\frac{1}{2}}} \leq \frac{1}{k_n} \leq \frac{\sqrt{\pi}}{2n^{\frac{1}{2}}} \left(1 + \frac{1}{n}\right)^{\frac{1}{2}} \qquad (n = 1, 2, \ldots) \quad .$$

Then inequality (37) can be replaced by

$$\frac{\sqrt{\pi}\, f_n(\sqrt{2})}{n^{\frac{1}{2}}\, 2^n} < (x_n - \sqrt{2}) < \frac{\sqrt{\pi}\,(1+\frac{1}{n})^{\frac{1}{2}} f_n(\sqrt{2})}{2n^{\frac{1}{2}}} \quad .$$

We have shown in the proof of part 3 that, as $n \to \infty$,

$$f_n(x) \to 1, \text{ uniformly in } 0 < \epsilon \leq x \leq \sqrt{2} \ .$$

Therefore,

$$\lim_{n \to \infty} f_n(\sqrt{2}) = 1$$

and, as $n \to \infty$,

$$(x_n - \sqrt{2}) \to 0$$

or $\lim_{n \to \infty} x_n = \sqrt{2}$.

This completes the proof of part 1 of the theorem. To prove part 2 of the theorem, we note that, by equation (10), $f_n(x)$ is an odd function and that, by equation (4),

$$f'_n(x) \geq 0 \text{ for } -\infty < x < +\infty \text{ and odd } n > 1 \ .$$

Therefore, for odd $n > 1$, $f_n(x)$ is a strictly increasing monotonic function of x in $-\infty < x < \infty$. This completes the proof of the theorem.

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