Number Sense Tricks

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Contents

Introduction

As most who are reading this book already know, the UIL Number Sense exam is an intense 10 minute test composed of 80 mental math problems which assesses a student's knowledge of topics ranging from simple multiplication, geometry, algebraic manipulation, to calculus. Although the exam is grueling (with 7.5 seconds per problem, it is hard to imagine it being easy!), there are various tricks to alleviate some of the heavy computations associated with the test. The purpose of writing this book is to explore a variety of these "shortcuts" as well as their applications in order to better prepare students taking the Number Sense test. In addition, this book is a source of practice material for many different types of problems so that better proficiency of the more straight-forward questions can be reached, leaving more time for harder and unique test questions.

This book is divided into three sections: Numerical Tricks, Necessary Memorizations (ranging from conversions to formulas), and Miscellaneous Topics. The difficulty of tricks discussed range from some of the most basic (11's trick, Subtracting Reverses, etc...) to the more advanced that are on the last column of the most recent exams. Most of the material is geared towards High School participants, however, after looking through some recent Middle School exams, a lot of the tricks outlined in this manual are appropriate for that contest as well (albeit, more simplified computations are used). Although this book will provide, hopefully, adequate understanding of a wide variety of commonly used shortcuts, it is **not** a replacement for practicing and discovering methods that you feel most comfortable with. In order to solidify everything exhibited in this book, regular group and individual practice sessions are recommended as well as participation in multiple competitions. For further material, you can find free practice tests for both Middle and High School levels on my website at the following URLS:
Middle School Exams: `http://bryantheath.com/middle-school-number-sense-practice-tests/`
High School Exams: `http://bryantheath.com/number-sense-practice-tests/`

The best way to approach this book is to read through all the instructional material first (and, if you are a Middle School student, skip certain sections – such as the Calculus stuff – that are not applicable to your exam) then go back and do the practice problems in each section. The reason why this is needed is because many sections deal with combinations of problems which are discussed later in the book. Also, all problems in **bold** reflect questions taken from the state competition exams. Similarly, to maintain consistent nomenclature, all (*) problems are approximation problems where $\pm 5\%$ accuracy is needed.

It should be noted that the tricks exhibited here could very easily not be the fastest method for doing the problems. I wrote down tricks and procedures that I follow, and because I am only human, there could very easily be faster, more to-the-point tricks that I haven't noticed. In fact, as I've been gleaning past tests to find sample problems, I've noticed faster methods on how to do problems and I've updated the book accordingly. One of the reasons why Number Sense is so great is that there is usually a variety of methods which can be used to get to the solution! This is apparent mostly in the practice problems. I tried to choose problems which reflects the procedures outlined in each section but sometimes you can employ different methods and come up with an equally fast (or possibly faster!) way of solving the problems.

Finally, I just want to say that although Number Sense might seem like a niche competition with limited value, there are a variety of real-world applications where being able to calculate quickly or estimate accurately can benefit you immensely both now and in your future career. One of the most immediate benefit you'll see is that your standardized math test scores will probably improve (if you can do the rote calculations quickly, it leaves more time to *really think* about the more difficult problems). Even fifteen years past my last competition, the ability to make good back-of-the-envelope calculations in my head quickly has given me an edge when it comes to on-the-fly interpretations of data I see regularly in my career. Although you'll be competing in Number Sense for just a few years in Middle and High School, the skills you acquire will last a lifetime.

-Bryant Heath

1 Numerical Tricks

1.1 Introduction: FOILing/LIOFing When Multiplying

Multiplication is at the heart of every Number Sense test. Slow multiplication hampers how far you are able to go on the test as well as making you prone to making more errors. To help beginners learn how to speed up multiplying, the concept of FOILing, learned in beginning algebra classes, is introduced as well as some exercises to help in speeding up multiplication. What is nice about the basic multiplication exercises is that *anyone* can make up problems, so practice is unbounded.

When multiplying two two-digit numbers ab and cd swiftly, a method of FOILing – or more accurately named LIOFing (Last-Inner+Outer-First) – is used. To understand this concept better, lets take a look at what we do when we multiply $ab \times cd$:

$$ab = 10a + b \text{ and } cd = 10c + d$$

$$(10a + b) \times (10c + d) = 100(ac) + 10(ad + bc) + bd$$

A couple of things can be seen by this:

1. The one's digit of the answer is simply bd or the *Last* digits (by *Last* I mean the least significant digit) of the two numbers multiplied.
2. The ten's digit of the answer is $(ad + bc)$ which is the sum of the *Inner* digits multiplied together plus the *Outer* digits multiplied.
3. The hundred's digit is ac which are the *First* digits (again, by *First* I mean the most significant digit) multiplied with each other.
4. If in each step you get more than a single digit, you carry the extra (most significant digit) to the next calculation. For example:

$$74 \times 23 = \begin{array}{ll} \text{Units:} & 3 \times 4 = \mathit{1}\mathbf{2} \\ \text{Tens:} & 3 \times 7 + 2 \times 4 + \mathit{1} = \mathit{3}\mathbf{0} \\ \text{Hundreds:} & 2 \times 7 + \mathit{3} = \mathbf{17} \\ \text{Answer:} & \mathbf{1702} \end{array}$$

Where the bold represents the answer and the italicized represents the carry.

Similarly, you can extend this concept of LIOFing to multiply any n-digit number by m-digit number in a procedure I call "moving down the line." Let's look at an example of a 3-digit multiplied by a 2-digit:

$$493 \times 23 = \begin{array}{ll} \text{Ones:} & 3 \times 3 = \mathbf{9} \\ \text{Tens:} & 3 \times 9 + 2 \times 3 = \mathit{3}\mathbf{3} \\ \text{Hundreds:} & 3 \times 4 + 2 \times 9 + \mathit{3} = \mathit{3}\mathbf{3} \\ \text{Thousands:} & 2 \times 4 + \mathit{3} = \mathbf{11} \\ \text{Answers:} & \mathbf{11339} \end{array}$$

As one can see, you just continue multiplying the two-digit number "down the line" of the three-digit number, writing down what you get for each digit then moving on (always remembering to carry when necessary). The following are exercises to familiarize you with this process of multiplication:

Problem Set 1.1:

$95 \times 30 =$	$90 \times 78 =$	$51 \times 11 =$	$83 \times 51 =$
$64 \times 53 =$	$65 \times 81 =$	$92 \times 76 =$	$25 \times 46 =$
$94 \times 92 =$	$27 \times 64 =$	$34 \times 27 =$	$11 \times 77 =$
$44 \times 87 =$	$86 \times 63 =$	$54 \times 92 =$	$83 \times 68 =$
$72 \times 65 =$	$81 \times 96 =$	$57 \times 89 =$	$25 \times 98 =$
$34 \times 32 =$	$88 \times 76 =$	$22 \times 11 =$	$36 \times 69 =$
$35 \times 52 =$	$15 \times 88 =$	$62 \times 48 =$	$56 \times 40 =$
$62 \times 78 =$	$57 \times 67 =$	$28 \times 44 =$	$80 \times 71 =$
$51 \times 61 =$	$81 \times 15 =$	$64 \times 14 =$	$47 \times 37 =$
$79 \times 97 =$	$99 \times 87 =$	$49 \times 54 =$	$29 \times 67 =$
$38 \times 98 =$	$75 \times 47 =$	$77 \times 34 =$	$49 \times 94 =$
$71 \times 29 =$	$85 \times 66 =$	$13 \times 65 =$	$64 \times 11 =$
$62 \times 15 =$	$43 \times 65 =$	$74 \times 72 =$	$49 \times 41 =$
$23 \times 70 =$	$72 \times 75 =$	$53 \times 59 =$	$82 \times 91 =$
$14 \times 17 =$	$67 \times 27 =$	$85 \times 25 =$	$25 \times 99 =$
$137 \times 32 =$	$428 \times 74 =$	$996 \times 47 =$	$654 \times 45 =$
$443 \times 39 =$	$739 \times 50 =$	$247 \times 87 =$	$732 \times 66 =$
$554 \times 77 =$	$324 \times 11 =$	$111 \times 54 =$	$885 \times 78 =$
$34 \times 655 =$	$52 \times 532 =$	$33 \times 334 =$	$45 \times 301 =$
$543 \times 543 =$	$606 \times 212 =$	$657 \times 322 =$	$543 \times 230 =$
$111 \times 121 =$	$422 \times 943 =$	$342 \times 542 =$	$789 \times 359 =$
$239 \times 795 =$	$123 \times 543 =$	$683 \times 429 =$	$222 \times 796 =$

1.2 Multiplying: The Basics

1.2.1 Multiplying by 11 Trick

The simplest multiplication trick is the 11's trick. It is a mundane version of "moving down the line," where you add consecutive digits and record the answer. Here is an example:

$$523 \times 11 = \begin{array}{ll} \text{Ones:} & 1 \times 3 = \mathbf{3} \\ \text{Tens:} & 1 \times 2 + 1 \times 3 = \mathbf{5} \\ \text{Hundreds:} & 1 \times 5 + 1 \times 2 = \mathbf{7} \\ \text{Thousands:} & 1 \times 5 = \mathbf{5} \\ \text{Answer:} & \mathbf{5753} \end{array}$$

As one can see, the result can be obtained by subsequently adding the digits along the number you're multiplying. Be sure to keep track of the carries as well:

$$6798 \times 11 = \begin{array}{ll} \text{Ones:} & \mathbf{8} \\ \text{Tens:} & 9 + 8 = \mathit{1}\mathbf{7} \\ \text{Hundreds:} & 7 + 9 + \mathit{1} = \mathit{1}\mathbf{7} \\ \text{Thousands:} & 6 + 7 + \mathit{1} = \mathit{1}\mathbf{4} \\ \text{Ten Thousands:} & 6 + \mathit{1} = \mathbf{7} \\ \text{Answer:} & \mathbf{74778} \end{array}$$

The trick can also be extended to 111 or 1111 (and so on). Where as in the 11's trick you are adding pairs of digits "down the line," for 111 you will be adding triples:

$$6543 \times 111 = \begin{array}{ll} \text{Ones:} & \mathbf{3} \\ \text{Tens:} & 4 + 3 = \mathbf{7} \\ \text{Hundreds:} & 5 + 4 + 3 = \mathit{1}\mathbf{2} \\ \text{Thousands:} & 6 + 5 + 4 + \mathit{1} = \mathit{1}\mathbf{6} \\ \text{Ten Thousands:} & 6 + 5 + \mathit{1} = \mathit{1}\mathbf{2} \\ \text{Hun. Thousands:} & 6 + \mathit{1} = \mathbf{7} \\ \text{Answer:} & \mathbf{726273} \end{array}$$

Another common form of the 11's trick is used in reverse. For example:

$$1353 \div 11 =$$
$$\text{or}$$
$$11 \times x = 1353$$

Ones Digit of x is equal to the Ones Digit of 1353:		**3**
Tens Digit of x is equal to:	$5 = 3 + x_{tens}$	**2**
Hundreds Digit of x is equal to:	$3 = 2 + x_{hund}$	**1**
Answer:		**123**

Similarly you can perform the same procedure with 111, and so on. Let's look at an example:

$$46731 \div 111 =$$
$$\text{or}$$
$$111 \times x = 46731$$

Ones Digit of x is equal to the Ones Digit of 46731:		**1**
Tens Digit of x is equal to:	$3 = 1 + x_{tens}$	**2**
Hundreds Digit of x is equal to:	$7 = 2 + 1 + x_{hund}$	**4**
Answer:		**421**

The hardest part of the procedure is knowing when to stop. The easiest way I've found is to think about how many digits the answer *should* have. For example, with the above expression, we are dividing a 5-digit number by a roughly 100, leaving an answer which should be 3-digits, so after the third-digit you know you are done.

The following are some more practice problems to familiarize you with the process:

Problem Set 1.2.1.:

1. $11 \times 54 =$ __________
2. $11 \times 72 =$ __________
3. $11 \times 38 =$ __________
4. $462 \times 11 =$ __________
5. $11 \times 74 =$ __________
6. $66 \times 11 =$ __________
7. $1.1 \times 2.3 =$ __________
8. $52 \times 11 =$ __________
9. $246 \times 11 =$ __________
10. $111 \times 456 =$ __________
11. $198 \div 11 =$ __________
12. $357 \times 11 =$ __________
13. $275 \div 11 =$ __________
14. $321 \times 111 =$ __________
15. $1.1 \times .25 =$ __________
16. $111 \times 44 =$ __________
17. $374 \div 11 =$ __________
18. $87 \times 111 =$ __________
19. $286 \div 11 =$ __________
20. $111 \times 53 =$ __________
21. $297 \div 11 =$ __________
22. $\mathbf{2233 \div 11} =$ __________
23. $198 \times 11 =$ __________
24. $297 \div 11 =$ __________
25. $111 \times 41 =$ __________
26. $111 \times 35 =$ __________
27. $111 \times 345 =$ __________
28. $2003 \times 111 =$ __________
29. $3 \times 5 \times 7 \times 11 =$ __________
30. $121 \times 121 =$ __________
31. $\mathbf{33} \times \mathbf{1111} =$ __________
32. $22 \times 32 =$ __________
33. $36963 \div 111 =$ __________
34. $20.07 \times 1.1 =$ __________

35. 11% of 22 is: ____________________% (dec.)

36. $13 \times 121 =$ ____________________

37. $27972 \div 111 =$ ____________________

38. $2006 \times 11 =$ ____________________

39. $11^4 =$ ____________________

40. $33 \times 44 =$ ____________________

41. $2 \times 3 \times 11 \times 13 =$ ____________________

42. $121 \times 22 =$ ____________________

43. $44 \times 55 =$ ____________________

44. $2 \times 3 \times 5 \times 7 \times 11 =$ ____________________

45. $2553 \div 111 =$ ____________________

46. $\mathbf{114} \times \mathbf{121} =$ ____________________

47. $44 \times 25 \times 11 =$ ____________________

48. $55 \times 33 =$ ____________________

49. (*) $32 \times 64 \times 16 \div 48 =$ ____________________

50. $2002 \div 11 =$ ____________________

51. $77 \times 88 =$ ____________________

52. (*) $44.4 \times 33.3 \times 22.2 =$ ____________________

53. $11 \times 11 \times 11 \times 11 =$ ____________________

54. $\mathbf{25553} \div \mathbf{1111} =$ ____________________

55. $11 \times 13 \times 42 =$ ____________________

56. $1111 \times 123 =$ ____________________

57. $11 \times 7 \times 5 \times 3 \times 2 =$ ____________________

58. $121 \times 124 =$ ____________________

59. (*) $33 \times 44 \times 55 =$ ____________________

1.2.2 Multiplying by 101 Trick

In the same spirit as the multiplying by 11's trick, multiplying by 101 involves adding gap connected digits. Let's look at an example:

	Ones:	1×8	**8**
	Tens:	1×3	**3**
$438 \times 101 =$	Hundreds:	$1 \times 4 + 1 \times 8$	*1***2**
	Thousands:	$1 \times 3 + \mathit{1}$	**4**
	Tens Thousands:	1×4	**4**
	Answer:	**44238**	

So you see, immediately you can write down the ones/tens digits (they are the same as what you are multiplying 101 with). Then you sum gap digits and move down the line. Let's look at another example:

	Ones/Tens:	**34**	**34**
	Hundreds:	$2 + 4$	**6**
$8234 \times 101 =$	Thousands:	$8 + 3$	*1***1**
	Tens Thousands:	$2 + \mathit{1}$	**3**
	Hundred Thousands:	**8**	**8**
	Answer:	**831634**	

Problem Set 1.2.2

1. $1234 \times 101 =$ ______
2. $10.1 \times 234 =$ ______
3. $369 \times 101 =$ ______
4. $34845 \div 101 =$ ______
5. $22422 \div 101 =$ ______
6. $202 \times 123 =$ ______
7. If 6 balls cost \$6.06, then 15 balls cost: \$ ______
8. $\mathbf{404} \times \mathbf{1111} =$ ______
9. (*) $(48 + 53) \times 151 =$ ______
10. (*) $8888 \times 62.5\% \times \frac{5}{11} =$ ______

1.2.3 Multiplying by 25 Trick

The trick to multiplying by 25 is to think of it as $\frac{100}{4}$. So the strategy is to take what ever you are multiplying with, divide it by 4 then move the decimal over to the right two places. Here are a couple of examples:

$$84 \times 25 = \frac{84}{4} \times 100 = 21 \times 100 = \mathbf{2100}$$
$$166 \times 25 = \frac{166}{4} \times 100 = 41.5 \times 100 = \mathbf{4150}$$

In a similar manner, you can use the same principle to divide numbers by 25 easily. The difference is you multiply by 4 and then move the decimal over to the left two places

$$\frac{415}{25} = \frac{415}{\frac{100}{4}} = \frac{415 \times 4}{100} = \frac{1660}{100} = 16.6$$

Problem Set 1.2.3

1. $240 \times 25 =$ ______
2. $25 \times 432 =$ ______
3. $\mathbf{2.6} \times \mathbf{2.5} =$ ______
4. $148 \times 25 =$ ______
5. $25 \times 33 =$ ______
6. $64 \div 25 =$ ______
7. $25 \times 147 =$ ______
8. $418 \times 25 =$ ______
9. $616 \div 25 =$ ______
10. $2.5 \times 40.4 =$ ______
11. $1.1 \div 2.5 =$ ______
12. $3232 \times 25 =$ ______

13. (*) $97531 \div 246 =$ ____

14. Which is smaller: $\frac{6}{25}$ or .25? ____

15. $209 \times 25 =$ ____

16. $(18 + 16)(9 + 16) =$ ____

17. (*) $334455 \div 251 =$ ____

18. 21.4 is ____ % of 25.

19. $404 \div 25 =$ ____

20. $303 \times 25 =$ ____

21. (*) $97531 \div 246 =$ ____

22. Which is larger: $\frac{7}{25}$ or .25? ____

23. $2006 \div 25 =$ ____

24. $25 \times 307 =$ ____

25. 32 is $2\frac{1}{2}\%$ of: ____

26. (*) $47985 \div 246 =$ ____

27. $\mathbf{25} \times \mathbf{2003} =$ ____

28. $15 \times 25 \times 11 =$ ____

29. $11 \times 24 \times 25 =$ ____

30. $11 \times 18 \times 25 =$ ____

31. (*) $248 \times 250 \times 252 =$ ____

1.2.4 Multiplying by 75 trick

In a similar fashion, you can multiply by 75 by treating it as $\frac{3}{4} \cdot 100$. So when you multiply by 75, first divide the number you're multiplying by 4 then multiply by 3 then move the decimal over two places to the right.

$$76 \times 75 = \frac{76 \cdot 3}{4} \cdot 100 = 19 \times 3 \times 100 = 5700$$

$$42 \times 75 = \frac{42 \cdot 3}{4} \cdot 100 = 10.5 \times 3 \times 100 = 3150$$

Again, you can use the same principle to divide by 75 as well, only you multiply by $\frac{4}{3}$ then divide by 100 (or move the decimal place over two digits to the left).

$$\frac{81}{75} = \frac{81}{\frac{3 \cdot 100}{4}} = \frac{81 \cdot 4}{3 \cdot 100} = \frac{27 \cdot 4}{100} = 1.08$$

Problem Set 1.2.4

1. $48 \times 75 =$ ____

2. $64 \times 75 =$ ____

3. $66 \div 75 =$ ____

4. $84 \times 75 =$ ____

5. (*) $443322 \div 751 =$ ____

6. $28 \times 75 =$ ______

7. $75 \times 24 =$ ______

8. (*) 7532×1468 ______

9. $48 \div 75 =$ ______

10. (*) $566472 \div 748 =$ ______

11. $\mathbf{96 \div 75} =$ ______

12. $75 \times 11 \times 24 =$ ______

13. $4800 \div 75 =$ ______

14. $75 \times 48 \times 15 =$ ______

15. $8.8 \times 7.5 \times 1.1 =$ ______

1.2.5 Multiplying by Any Fraction of 100, 1000, etc...

You can take what we learned from the 25's and 75's trick (converting them to fractions of 100) with a variety of potential fractions. $\frac{1}{8}$'s are chosen often because:

$$125 = \frac{1}{8} \cdot 1000 \qquad 37.5 = \frac{3}{8} \cdot 100 \qquad 6.25 = \frac{5}{8} \cdot 10$$

In addition, you see $\frac{1}{6}$'s, $\frac{1}{3}$'s, $\frac{1}{9}$'s, and sometimes even $\frac{1}{12}$'s for approximation problems (because they do not go evenly into $100, 1000$, etc..., they have to be approximated usually).

$$223 \approx \frac{2}{9} \cdot 1000 \qquad 8333.3 \approx \frac{5}{6} \cdot 10000 \approx \frac{1}{12} \cdot 100000 \qquad 327 \approx \frac{1}{3} \cdot 1000$$

For approximations you will rarely ever see them equate to almost exactly to the correct fraction. For example you might use $\frac{2}{3} \cdot 1000$ for any value from $654 - 678$. Usually you can tell for the approximation problems what fraction the test writer is really going for.

Problem Set 1.2.5

1. $125 \times 320 =$ ______

2. (*) $8333 \times 24 =$ ______

3. $138 \div 125 =$ ______

4. (*) $57381 \div 128 =$ ______

5. (*) $245632 \div 111 =$ ______

6. (*) $\mathbf{16667 \div 8333 \times 555} =$ ______

7. $625 \times 320 =$ ______

8. (*) $774447 \div 111 =$ ______

9. (*) $62.5 \times 3248 =$ ______

10. $12.5 \times 480 =$ ______

11. (*) $17304 \div 118 =$ ______

12. (*) 87% of $5590 =$ ______

13. (*) $457689 \div 111 =$ ______

14. (*) $625 \times 648 =$ ______

15. $375 \times 408 =$ ______

16. (*) $359954 \div 1111 =$ ______

17. $88 \times 12.5 \times .11 =$ ______

18. (*) $719 \times 875 =$ ______

19. (*) $428571 \times 22 =$ ______

20. (*) $\mathbf{85714.2 \div 714.285} =$ ______

21. $488 \times 375 =$ ______

22. (*) $6311 \times 1241 =$ ______

23. (*) $884422 \div 666 =$ ______

24. (*) 106.25% of 640 = ______

25. (*) $\mathbf{6388 \times 3.75} =$ ______

26. $240 \times 875 =$ ______

27. (*) $12.75 \times 28300 \div 102 =$ ______

28. $375 \times 24.8 =$ ______

29. (*) $857142 \times 427 =$ ______

30. $.0625 \times .32 =$ ______

31. (*) $16667 \times 369 =$ ______

32. (*) $918576 \div 432 =$ ______

33. (*) $456789 \div 123 =$ ______

34. (*) 106% of 319 = ______

35. (*) $571428 \times .875 =$ ______

36. (*) **123% of 882** = ______

37. (*) $95634 \div 278 =$ ______

38. (*) $273849 \div 165 =$ ______

39. (*) $\mathbf{5714.28 \times 85} =$ ______

40. (*) 9.08% of 443322 = ______

41. (*) $8333 \times 23 =$ ______

42. $.125 \times 482 =$ ______

43. (*) $714285 \times .875 =$ ______

44. (*) 87% of 789 = ______

45. (*) $16667 \times 49 =$ ______

46. (*) $123456 \div 111 =$ ______

47. (*) $\mathbf{875421 \div 369} =$ ______

48. (*) $71984 \times 1.371 =$ ______

49. (*) 63% of 7191 = ______

50. (*) $5714.28 \times 83 =$ ______

51. (*) $1428.57 \times 62 =$ ______

52. (*) $80520 \div 131 =$ ______

53. (*) $142.857 \times 428.571 =$ ______

54. (*) $12509 \times 635 =$ ______

55. (*) $1234 \times 567 =$ ______

56. (*) $789123 \div 456 =$ ______

57. $\mathbf{625} \times \mathbf{65} =$ __________

58. (*) $1428.57 \times 73 =$ __________

59. (*) $7142.85 \times 34.2 =$ __________

60. (*) $333 \times 808 \times 444 =$ __________

61. (*) $571428 \times 34 =$ __________

62. (*) $833 \times 612 =$ __________

63. (*) $8333 \times (481 + 358)$ __________

64. (*) $234678 \div 9111 =$ __________

65. (*) $428.571 \times 87.5 =$ __________

66. (*) $375.1 \times 83.33 \times 1.595 =$ __________

67. (*) $\mathbf{8333} \div \mathbf{6666} \times \mathbf{4444} =$ __________

68. (*) $\mathbf{8333} \times \mathbf{12\frac{1}{2}}\% \times \mathbf{.12} =$ __________

69. (*) $639 \times 375 \div 28 =$ __________

70. (*) $6250 \div 8333 \times 8888 =$ __________

71. (*) $416666 \div 555 \times 76 =$ __________

72. (*) $375 \div 833 \times 555 =$ __________

73. (*) $438 \div 9\frac{1}{11}\% \times 11.1 =$ __________

74. (*) $857142 \div 428571 \times 7777 =$ __________

75. (*) $546 \div 45\frac{5}{11}\% \times 10.8 =$ __________

76. (*) $54.5454 \times 66.6 \times 58 =$ __________

77. (*) $456 \div 18.75\% \times \frac{1}{4}$ __________

78. (*) $818 \div 44\frac{4}{9}\% \times 12.5 =$ __________

79. (*) $62.5 \div 83.3 \times 888 =$ __________

80. (*) $797 \div 87.5\% \times \frac{7}{10}$ __________

81. (*) $888 \times 87.5\% \div \frac{7}{11}$ __________

82. (*) $1250 \div 1666 \times 4444 =$ __________

83. (*) $85858 \div 585 =$ __________

84. (*) $(51597 \div 147)^2 =$ __________

1.2.6 Double and Half Trick

This trick involves multiplying by a clever version of 1. Let's look at an example:

$$\begin{aligned} 15 \times 78 &= \frac{2}{2} \times 15 \times 78 \\ &= (15 \times 2) \times \frac{78}{2} \\ &= 30 \times 39 = \mathbf{1170} \end{aligned}$$

So the procedure is you double one of the numbers and half the other one, then multiply. This trick is exceptionally helpful when multiplying by 15 or any two-digit number ending in 5. Another example is:

$$35 \times 42 = 70 \times 21 = \mathbf{1470}$$

It is also good whenever you are multiplying an even number in the teens by another number:

$$18 \times 52 = 9 \times 104 = \mathbf{936}$$

or

$$14 \times 37 = 7 \times 74 = \mathbf{518}$$

The purpose of this trick is to save time on calculations. It is a lot easier to multiply a single-digit number than a two-digit number.

Problem Set 1.2.6

1. $1.5 \times 5.2 =$ __________
2. $4.8 \times 15 =$ __________
3. $64 \times 1.5 =$ __________
4. $15 \times 48 =$ __________
5. $14 \times 203 =$ __________
6. $14 \times 312 =$ __________
7. $\mathbf{24} \times \mathbf{35} =$ __________
8. $312 \times 14 =$ __________
9. A rectangle has a length of 2.4 and a width of 1.5. Its area is __________
10. $18 \times 112 =$ __________
11. $27 \times 14 =$ __________
12. $\mathbf{21} \times \mathbf{15} \times \mathbf{14} =$ __________
13. $33.75 = 1.5\times$ __________
14. $345 \times 12 =$ __________
15. $\mathbf{1.2} \times \mathbf{1.25} =$ __________
16. 24% of 44 = __________
17. $14 \times 25 + 12.5 \times 28 =$ __________

1.2.7 Multiplying Two Numbers Near 100

Let's look at two numbers over 100 first.
Express $n_1 = (100 + a)$ and $n_2 = (100 + b)$ then:

$$\begin{aligned} n_1 \cdot n_2 &= (100 + a) \cdot (100 + b) \\ &= 10000 + 100(a + b) + ab \\ &= 100(100 + a + b) + ab \\ &= 100(n_1 + b) + ab = 100(n_2 + a) + ab \end{aligned}$$

1. The Tens/Ones digits are just the difference the two numbers are above 100 multiplied together (ab)
2. The remainder of the answer is just n_1 plus the amount n_2 is above 100, or n_2 plus the amount n_1 is above 100.

	Tens/Units:	8×3	**24**
$103 \times 108 =$	Rest of Answer:	$103 + 8$ or $108 + 3$	**111**
	Answer:		**11124**

Now let's look at two numbers below 100.

$n_1 = (100 - a)$ and $n_2 = (100 - b)$ so:

$$\begin{aligned} n_1 \cdot n_2 &= (100 - a) \cdot (100 - b) \\ &= 10000 - 100(a + b) + ab \\ &= 100(100 - a - b) + ab \\ &= 100(n_1 - b) + ab = 100(n_2 - a) + ab \end{aligned}$$

1. Again, Tens/Ones digits are just the difference the two numbers are above 100 multiplied together (ab)
2. The remainder of the answer is just n_1 minus the difference n_2 is from 100, or n_2 minus the difference n_1 is from 100.

	Tens/Ones:	$(100 - 97) \times (100 - 94) = 3 \times 6$	**18**
$97 \times 94 =$	Rest of Answer:	$97 - 6$ or $94 - 3$	**91**
	Answer:		**9118**

Now to multiply two numbers, one above and one below is a little bit more tricky.
Let $n_1 = (100 + a)$ which is the number above 100 and $n_2 = (100 - b)$ which is the number below 100, then:

$$\begin{aligned} n_1 \cdot n_2 &= (100 + a) \cdot (100 - b) \\ &= 10000 + 100(a - b) + ab \\ &= 100(100 + a - b) - ab \\ &= 100(100 + a - b - 1) + (100 - ab) \\ &= 100(n_1 - b - 1) + (100 - ab) \end{aligned}$$

To see what this means, it is best to use an example:

	Tens/Ones:	$100 - 3 \times 6$	**82**
$103 \times 94 =$	Rest of Answer:	$103 - 6 - 1$	**96**
	Answer:		**9682**

So the trick is:

1. The Tens/Ones is just the difference the two numbers are from 100 multiplied together then subtracted from 100.
2. The rest of the answer is just the number that is larger than 100 minus the difference the smaller number is from 100 minus an additional 1

Let's look at another example to solidify this:

	Tens/Ones:	$100 - 8 \times 7$	**44**
$108 \times 93 =$	Rest of Answer:	$108 - 7 - 1$	**100**
	Answer:		**10044**

It should be noted that you can extend this trick to not just integers around 100 but 1000, 10000, and so forth. For the extension, you just need to keep track how many digits each part is. For example, when we are multiplying two numbers over 100 (say 104×103) the first two digits would be $4 \times 3 = 12$, however if we were doing two numbers over 1000 (like 1002×1007) the first *three* digits would be $2 \times 7 =$ **014** not 14 like what you would be used to putting. Let's look at the example presented above and the procedure:

	Hundreds/Tens/Ones:	2×7	**014**
$1002 \times 1007 =$	Rest of Answer:	$1002 + 7 = 1007 + 2$	**1009**
	Answer:		**1009014**

The best way to remember to include the "extra" digit is to think that when you multiply 1002×1007 you are going to *expect* a seven digit number. Now adding $1002 + 7 = 1009$ gives you four of the digits, so you need the first part to produce three digits for you.

Let's look at an example of two numbers below 1000:

	Hundreds/Tens/Ones:	7×6	**042**
$993 \times 994 =$	Rest of Answer:	$993 - 6 = 994 - 7$	**987**
	Answer:		**987042**

The following are some practice problems so that you can fully understand this trick:

Problem Set 1.2.7

1. $89 \times 97 =$ __________
2. $96 \times 97 =$ __________
3. $103 \times 109 =$ __________
4. $93 \times 97 =$ __________
5. $103 \times 107 =$ __________
6. $93 \times 89 =$ __________
7. $\mathbf{102} \times \mathbf{108} =$ __________
8. $109 \times 107 =$ __________
9. $96 \times 89 =$ __________
10. $92 \times 97 =$ __________
11. $103 \times 104 =$ __________
12. $102 \times 103 =$ __________
13. $92 \times 93 =$ __________
14. $106 \times 107 =$ __________
15. $97 \times 89 =$ __________
16. $\mathbf{94} \times \mathbf{98} =$ __________
17. $94 \times 91 =$ __________
18. $91 \times 98 =$ __________
19. $993 \times 994 =$ __________
20. $103 \times 96 =$ __________
21. $93 \times 103 =$ __________
22. $\mathbf{991} \times \mathbf{989} =$ __________
23. $1009 \times 1004 =$ __________
24. $97 \times 107 =$ __________
25. $\mathbf{93} \times \mathbf{104} =$ __________
26. $96 \times 103 =$ __________
27. $991 \times 991 =$ __________
28. $\mathbf{104} \times \mathbf{97} =$ __________
29. $1003 \times 1008 =$ __________
30. (*) $98^2 + 97^2 =$ __________
31. $19^2 \times 3^2 \times 2^2 =$ __________

1.2.8 Squares Ending in 5 Trick

Here is the derivation for this trick. Let $a5$ represent any number ending in 5 (a could be any integer, not just restricted to a one-digit number).

$$\begin{aligned}(a5)^2 &= (10a+5)^2 \\ &= 100a^2 + 100a + 25 \\ &= 100a(a+1) + 25\end{aligned}$$

So you can tell from this that and number ending in 5 squared will have its last two digits equal to 25 and the remainder of the digits can be found from taking the leading digit(s) and multiplying it by one greater than itself. Here are a couple of examples:

	Tens/Ones:		**25**
$85^2 =$	Thousand/Hundreds:	$8 \times (8+1)$	**72**
	Answer:		**7225**

The next example shows how to compute 15^4 by applying the square ending in 5 trick twice, one time to get what 15^2 is then the other to get that result squared.

	Tens/Ones:	**25**		Tens/Ones:	**25**
$15^2 =$	Thousands/Hundreds:	$1 \times (1+1) = \mathbf{2}$	$225^2 =$	Rest of Answer:	$22 \times (23) = 11 \times 46 = \mathbf{506}$
	Answer:	**225**		Answer:	**50625**

In the above trick you *also* use the double/half trick *and* the 11's trick. This just shows that for some problems using multiple tricks might be necessary.

Problem Set 1.2.8

1. 25% of $25 =$ ______
2. $.35 \times 3.5 =$ ______
3. $12^2 + 2 \times 12 \times 13 + 13^2 =$ ______
4. $(115)^2 =$ ______
5. $\mathbf{f(x) = 9x^2 - 12x + 4,\ f(19) =}$ ______
6. $\mathbf{45\%}$ **of** $\mathbf{45 - 45 =}$ ______
7. (*) $12^4 =$ ______
8. $505 \times 505 =$ ______
9. A square has an area of 12.25 sq. cm. It's perimeter is: ______

1.2.9 Squares from 41-59

There is a quick trick for easy computation for squares from $41 - 59$. Let k be a 1-digit integer, then any of those squares can be expressed as $(50 \pm k)$:

$$\begin{aligned}(50 \pm k)^2 &= 2500 \pm 100 \cdot k + k^2 \\ &= 100(25 \pm k) + k^2\end{aligned}$$

What this means is that:

1. The tens/ones digits is just the difference the number is from 50, squared (k^2).
2. The remainder of the answer is taken by *adding* (if the number is greater than 50) or *subtracting* (if the number is less than 50) that difference from 25.

3. Note: You could extend this concept to squares outside the range of $41 - 59$ as long as you keep up with the carry appropriately.

Let's illustrate with a couple of examples:

	Tens/Ones:	$(50-46)^2 = 4^2$	**16**
$46^2 =$	Rest of Answer:	$25 - 4$	**21**
	Answer:		**2116**

	Tens/Ones:	$(57-50)^2 = 7^2$	**49**
$57^2 =$	Rest of Answer:	$25 + 7$	**32**
	Answer:		**3249**

	Tens/Ones:	$(61-50)^2 = 11^2$	*1***21**
$61^2 =$	Rest of Answer:	$25 + 11 + \mathit{1}$	**37**
	Answer:		**3721**

Problem Set 1.2.9

1. $58^2 =$ ____________________
2. $(510)^2 =$ ____________________
3. $47 \times 47 =$ ____________________
4. $53^2 =$ ____________________
5. (*) $48 \times 49 \times 50 =$ ____________________
6. $56^2 =$ ____________________
7. $59 \times 59 =$ ____________________
8. $41^2 =$ ____________________

1.2.10 Multiplying Two Numbers Equidistant from a Third Number

To illustrate this concept, let's look at an example of this type of problem: 83×87.
Notice that both 83 and 87 are 2 away from 85. So:

$$83 \times 87 = (85 - 2) \times (85 + 2)$$

Which notice this is just the difference of two squares:

$$(85 - 2) \times (85 + 2) = 85^2 - 2^2 = 7225 - 4 = \mathbf{7221}$$

So the procedure is:

1. Find the middle number between the two numbers being multiplied and square it.
2. Subtract from that the difference between the middle number and one of two numbers squared.

For most of these types of problems, the center number will be a multiple of 5, making the computation of its square relatively simple (See Section 1.2.7, Square's Ending in 5 Trick). The following illustrates another example:

$$61 \times 69 = 65^2 - 4^2 = 4225 - 16 = \mathbf{4209}$$

Problem Set 1.2.10:

1. $84 \times 86 =$ ______

2. $53 \times 57 =$ ______

3. $48 \times 52 =$ ______

4. $62 \times 58 =$ ______

5. $6.8 \times 7.2 =$ ______

6. $88 \times 82 =$ ______

7. $36 \times 24 =$ ______

8. $\mathbf{7.6} \times \mathbf{8.4} =$ ______

9. $5.3 \times 4.7 =$ ______

10. $51 \times 59 + 16 =$ ______

11. $96 \times 104 =$ ______

12. $81 \times 89 + 16 =$ ______

13. $34 \times 36 + 1 =$ ______

14. $73 \times 77 + 4 =$ ______

15. $62 \times 68 + 9 =$ ______

16. $32 \times 38 + 9 =$ ______

17. $18 \times 24 + 9 =$ ______

18. $61 \times 69 + 16 =$ ______

19. $43 \times 47 + 4 =$ ______

20. $88 \times 82 + 9 =$ ______

21. $57 \times 53 + 4 =$ ______

22. $38 \times 28 =$ ______

23. $41 \times 49 - 9 =$ ______

24. $\mathbf{77} \times \mathbf{73} + \mathbf{4} =$ ______

25. $65 \times 75 - 33 =$ ______

26. $33 \times 27 + 9 =$ ______

27. $71 \times 79 + 16 =$ ______

28. $72 \times 78 + 9 =$ ______

29. $53 \times 57 + 4 =$ ______

30. $105 \times 95 =$ ______

31. $62 \times 68 - 16 =$ ______

32. $36 \times 26 =$ ______

33. $83 \times 87 - 21 =$ ______

34. $23 \times 27 + 4 =$ ______

35. $29 \times 37 =$ ______

36. $\mathbf{21} - \mathbf{83} \times \mathbf{87} =$ ______

37. $112 \times 88 =$ ______

38. (*) $52 \times 48 + 49 \times 51 =$ ______

39. (*) $4.9^3 \times 3.3^3 =$ ______

40. (*) $72 \times 68 + 71 \times 69 =$ ______

41. (*) $42 \times 38 + 41 \times 39 =$ ______

42. (*) $4.8^3 \times 6.3^3 =$ ______

43. (*) $4000 + 322 \times 318 =$ ____________

44. $\mathbf{118 \times 122 + 4 =}$ ____________

45. (*) $5.1^3 \times 7.9^3 =$ ____________

46. (*) $\mathbf{34 \times 36 \times 34 \times 36} =$ ____________

1.2.11 Multiplying Reverses

The following trick involves multiplying two, two-digit numbers whose digits are reverse of each other.

$$\begin{aligned} ab \times ba &= (10a + b) \cdot (10b + a) \\ &= 100(a \cdot b) + 10(a^2 + b^2) + a \cdot b \end{aligned}$$

Here is what we know from the above result:

1. The Ones digit of the answer is just the two digits multiplied together.
2. The Tens digit of the answer is the sum of the squares of the digits.
3. The Hundreds digit of the answer is the two digits multiplied together.

Let's look at an example:

$53 \times 35 =$	Ones:	3×5	*1***5**
	Tens:	$3^2 + 5^2 + \mathit{1}$	*3***5**
	Hundreds:	$3 \times 5 + \mathit{3}$	**18**
	Answer:		**1855**

Here are some more problems to practice this trick:

Problem Set 1.2.11

1. $43 \times 34 =$ ____________
2. $\mathbf{23 \times 32} =$ ____________
3. $31 \times 13 =$ ____________
4. $21 \times 12 =$ ____________
5. $27 \times 72 =$ ____________
6. $61 \times 16 =$ ____________
7. $15 \times 51 =$ ____________
8. $14 \times 41 =$ ____________
9. $\mathbf{18 \times 81} =$ ____________
10. $36 \times 63 =$ ____________
11. $42 \times 24 =$ ____________
12. $26 \times 62 =$ ____________

1.3 Standard Multiplication Tricks

1.3.1 Extending Foiling

You can extend the method of FOILing to quickly multiply two three-digit numbers in the form $cba \times dba$. The general objective is you treat the digits of ba as one number, so after foiling you would get:

	Ones/Tens:	$(ba)^2$
$cba \times dba =$	Hundreds/Thousands:	$(c+d) \times (ba)$
	Rest of Answer:	$c \times d$

Let's look at a problem to practice this extension:

	Ones/Tens:	$(12)^2$	$\mathit{1}\mathbf{44}$
$412 \times 612 =$	Hundreds/Thousands:	$(4+6) \times (12) + \mathit{1}$	$\mathit{1}\mathbf{21}$
	Rest of Answer:	$4 \times 6 + \mathit{1}$	**25**
	Answer:		**252144**

By treating the last two digits as a single entity, you reduce the three-digit multiplication to a two-digit problem. The last two digits need not be the same in the two numbers (usually I do see this as the case though) in order to apply this method, let's look at an example of this:

	Ones/Tens:	08×11	**88**
$211 \times 808 =$	Hundreds/Thousands:	$08 \times 2 + 11 \times 8$	$\mathit{1}\mathbf{04}$
	Rest of Answer:	$2 \times 8 + \mathit{1}$	**17**
	Answer:		**170488**

The method works the best when the last two digits don't exceed 20 (after that the multiplication become cumbersome). Another good area where this approach is great for is squaring three-digit numbers:

	Ones/Tens:	06×06	**36**
$606^2 = 606 \times 606$	Hundreds/Thousands:	$06 \times 6 + 6 \times 06 = 2 \times 6 \times 6$	**72**
	Rest of Answer:	6×6	**36**
	Answer:		**367236**

In order to use this procedure for squaring, it would be beneficial to have squares of two-digit numbers memorized. Take for example this problem:

	Ones/Tens:	31×31	$\mathit{9}\mathbf{61}$
$431^2 = 431 \times 431$	Hundreds/Thousands:	$31 \times 4 + 4 \times 31 + \mathit{9} = 2 \times 4 \times 31 + \mathit{9}$	$\mathit{2}\mathbf{57}$
	Rest of Answer:	$4 \times 4 + \mathit{2}$	**18**
	Answer:		**185761**

If you didn't have 31^2 memorized, you would have to calculate it in order to do the first step in the process (very time consuming). However, if you have it memorized you would not have to do the extra steps, thus saving time.

Here are some practice problems to help with understanding FOILing three-digit numbers.

Problem Set 1.3.1

1. $202^2 =$ ____________________
2. $406 \times 406 =$ ____________________
3. $503 \times 503 =$ ____________________
4. $607^2 =$ ____________________
5. $208^2 =$ ____________________
6. $306^2 =$ ____________________
7. $\mathbf{509 \times 509} =$ ____________________
8. $804^2 =$ ____________________
9. $704 \times 704 =$ ____________________

10. $408^2 =$ ________________

11. $\mathbf{602 \times 602} =$ ________________

12. $303^2 =$ ________________

13. $\mathbf{909^2} =$ ________________

14. $402^2 =$ ________________

15. $707^2 =$ ________________

16. $301 \times 113 =$ ________________

17. $803 \times 803 =$ ________________

18. $404^2 =$ ________________

19. $512^2 =$ ________________

20. $122 \times 311 =$ ________________

21. $\mathbf{612^2} =$ ________________

22. $321 \times 302 =$ ________________

23. $714^2 =$ ________________

24. $234 \times 211 =$ ________________

25. $112 \times 211 =$ ________________

26. $\mathbf{214 \times 314} =$ ________________

27. $203 \times 123 =$ ________________

28. $121 \times 411 =$ ________________

29. $412 \times 112 =$ ________________

30. $\mathbf{505 \times 404} =$ ________________

31. $311 \times 113 =$ ________________

32. $124 \times 121 =$ ________________

33. $\mathbf{918^2} =$ ________________

34. $124 \times 312 =$ ________________

35. $311 \times 122 =$ ________________

36. $524^2 =$ ________________

37. $\mathbf{133 \times 311} =$ ________________

38. $141 \times 141 =$ ________________

39. $511 \times 212 =$ ________________

40. $122 \times 212 =$ ________________

41. $\mathbf{(12012)(12012)} =$ ________________

42. $667^2 =$ ________________

1.3.2 Factoring of Numerical Problems

In many of the intermediate problems, there are several examples where factoring can make the problem a lot easier. Outlined in the next couple of tricks are times when factoring would be beneficial towards calculation. We'll start off with some standard problems:

$$\begin{aligned} 21^2 + 63^2 &= 21^2 + (3 \cdot 21)^2 \\ &= 21^2 \cdot (1 + 9) \\ &= \mathbf{4410} \end{aligned}$$

This is a standard trick of factoring that is common in the middle section of the test. Another factoring procedure is as followed:

$$\begin{aligned} 48 \times 11 + 44 \times 12 &= 11 \cdot (48 + 4 \times 12) \\ &= 11 \cdot (96) \\ &= \mathbf{1056} \end{aligned}$$

Factoring problems can be easily identified because, at first glance, they look like they require dense computation. For example, the above problem would require two, two-digit multiplication *and then* their addition. Whereas when you factor out the 11 you are left with a simple addition and a multiplication using the 11's trick.

Another thing is that factoring usually requires the knowledge of another trick. For instance, the first problem required the knowledge of a square (21^2) while the second example involved applying the 11's trick.

The following are examples when factoring would lessen the amount of computations:

Problem Set 1.3.2

1. $\mathbf{8^2 + 24^2} =$ ____________
2. $27^2 + 9^2 =$ ____________
3. $15 \times 12 + 9 \times 30 =$ ____________
4. $28 \times 6 - 12 \times 14 =$ ____________
5. $33^2 + 11^2 =$ ____________
6. $48 \times 22 - 22 \times 78 =$ ____________
7. $3.9^2 + 1.3^2 =$ ____________
8. $2004 + 2004 \times 4 =$ ____________
9. $32 \times 16 + 16 \times 48 =$ ____________
10. $19^2 + 19 =$ ____________
11. $2005 \times 5 + 2005 =$ ____________
12. $27 \times 33 - 11 \times 81 =$ ____________
13. $21 \times 38 - 17 \times 21 =$ ____________
14. $40 \times 12 + 20 \times 24 =$ ____________
15. $51^2 + 51 \times 49 =$ ____________
16. $30 \times 11 + 22 \times 15 =$ ____________
17. $21^2 + 7^2 =$ ____________
18. $2006 - 2006 \times 6 =$ ____________
19. $12 \times 16 + 8 \times 24 =$ ____________
20. $\mathbf{1.2^2 + 3.6^2} =$ ____________
21. $14 \times 44 - 14 \times 30 =$ ____________
22. $60 \times 32 - 32 \times 28 =$ ____________
23. $\mathbf{45 \times 22 - 44 \times 15} =$ ____________
24. $(20 \times 44) - (18 \times 22) =$ ____________
25. $49^2 + 49 =$ ____________
26. $29^2 + 29 =$ ____________

27. $16 \times 66 - 16 \times 50 =$ ________________

28. $59^2 + 59 =$ ________________

29. $14 \times 38 - 14 \times 52 =$ ________________

30. $\mathbf{41 \times 17 - 17 \times 24} =$ ________________

31. $17 \times 34 - 51 \times 17 =$ ________________

32. $15 \times 36 + 12 \times 45 =$ ________________

33. $69^2 + 69 =$ ________________

34. $13 \times 77 + 91 \times 11 =$ ________________

35. $11^3 - 11^2 =$ ________________

36. $\mathbf{12 \times 90 + 72 \times 15} =$ ________________

37. $79^2 + 79 =$ ________________

38. $54 \times 11 + 99 \times 6 =$ ________________

39. $10 \cdot 11 + 11 \cdot 11 + 12 \cdot 11 =$ ________________

40. $\mathbf{119^2 + 119} =$ ________________

41. $39^2 + 39 =$ ________________

42. $18 \times 36 - 18 \times 54 =$ ________________

43. $\mathbf{22 \times 75 + 110 \times 15} =$ ________________

44. $99 \times 99 + 99 =$ ________________

45. $45 \times 16 - 24 \times 30 =$ ________________

46. $\mathbf{11^2 - 11^3} =$ ________________

47. $\mathbf{25 \times 77 + 25 \times 34} =$ ________________

48. $15 \times 18 + 9 \times 30 =$ ________________

49. $24 \times 13 + 24 \times 11 =$ ________________

50. $\mathbf{129 \times 129 + 129} =$ ________________

51. $13 \times 15 + 11 \times 65 =$ ________________

52. (*) $33 \times 31 + 31 \times 29 =$ ________________

53. $31 \times 44 + 44 \times 44 =$ ________________

54. $12^2 + 24^2 =$ ________________

55. (*) $\mathbf{73 \times 86 + 77 \times 84} =$ ________________

56. (*) $63 \times 119 + 121 \times 17 =$ ________________

57. $48 \times 11 + 44 \times 12 =$ ________________

58. $109^2 + 109 =$ ________________

59. (*) $38 \times 107 + 47 \times 93 =$ ________________

60. $\mathbf{64 \times 21 - 42 \times 16} =$ ________________

61. (*) $23 \times 34 + 43 \times 32 =$ ________________

62. $72 \times 11 + 99 \times 8 =$ ________________

63. (*) $43 \times 56 + 47 \times 54 =$ ________________

64. $15 \times 75 + 45 \times 25 =$ ________________

65. $42 \times 48 + 63 \times 42 =$ ________________

66. $14^2 - 28^2 =$ ________________

67. (*) $31 \times 117 + 30 \times 213 =$ ________________

68. $48 \times 28 + 27 \times 28 =$ ________________

69. $34 \times 56 + 55 \times 34 =$ ________________

70. (*) $34 \times 45 + 54 \times 43 =$ ________________

1.3.3 Sum of Consecutive Squares

Usually when approached with this problem, one of the squares ends in 5 making the squaring of the number relatively trivial. You want to use the approach of factoring to help aid in these problems. For example:

$$35^2 + 36^2 = 35^2 + (35+1)^2 = 2 \cdot 35^2 + 2 \cdot 35 + 1^2 = 2 \cdot 1225 + 70 + 1 = \mathbf{2521}$$

This is a brute force technique, however, it is a lot better than squaring both of the numbers and then adding them together (which you can get lost very easily doing that).

Here are some more practice problems to familiarize yourself with this procedure.

Problem Set 1.3.3

1. $35^2 + 36^2 =$ ________________
2. $12^2 + 13^2 =$ ________________
3. $15^2 + 16^2 =$ ________________
4. $25^2 + 26^2 =$ ________________
5. $40^2 + 41^2 =$ ________________
6. $80^2 + 81^2 =$ ________________

1.3.4 Sum of Squares: Factoring Method

Usually on the 3^{rd} of 4^{th} column of the test you will have to compute something like: $(30^2 - 2^2) + (30+2)^2$ (with the subtracting and additions might be reversed). Instead of memorizing a whole bunch of formulas for each individual case, it is probably just best to view these as factoring problems and using the techniques of FOILing to aid you. So for our example:

$$(30^2 - 2^2) + (30+2)^2 = 2 \cdot 30^2 + 2 \cdot 30 \cdot 2 + 2^2 - 2^2 = 1800 + 120 = \mathbf{1920}$$

Usually the number needing to be squared is relatively simple (either ending in 0 or ending in 5), making the computations a lot easier. Other times, another required step of converting a number to something more manageable will be necessary. For example:

$$19^2 + (10^2 - 9^2) = (10+9)^2 + (10^2 - 9^2) = 2 \cdot 10^2 + 2 \cdot 10 \cdot 9 + 9^2 - 9^2 = 200 + 180 = \mathbf{380}$$

or, if you have your squares memorized and noticed you also have a difference of squares (Section 1.3.6):

$$19^2 + (10^2 - 9^2) = 361 + (10-9) \cdot (10+9) = 361 + 19 = 380$$

The following are some more problems to give you practice with this technique:

Problem Set 1.3.4

1. $(11+10)^2 + (11^2 - 10^2) =$ ________________
2. $(30+2)^2 + (30^2 - 2^2) =$ ________________
3. $(10+9)^2 + (10^2 - 9^2) =$ ________________
4. $(30+2)^2 - (30^2 - 2^2) =$ ________________
5. $24^2 - (20^2 + 4^2) =$ ________________
6. $31^2 - (29^2 - 2^2) =$ ________________

7. $(30^2 - 2^2) + (30+2)^2 =$ ____________

8. $81^2 + (80+1)(80-1) =$ ____________

9. $55^2 - (50^2 - 5^2) =$ ____________

10. $47^2 + 40^2 - 7^2 =$ ____________

11. $(55+3)^2 + 55^2 - 3^2 =$ ____________

12. $30^2 - (28^2 - 2^2) =$ ____________

13. $38^2 + (30+8)(30-8) =$ ____________

14. $42^2 + (40^2 - 2^2) =$ ____________

15. $32^2 - (30^2 - 2^2) =$ ____________

16. $(28+2)^2 + (28^2 - 2^2) =$ ____________

17. $22^2 + 20^2 - 2^2 =$ ____________

18. $45^2 - (40^2 - 5^2) =$ ____________

19. $\mathbf{55^2 - 50^2 + 5^2} =$ ____________

20. $(30+2)^2 - (30^2 - 2^2) =$ ____________

21. $53 \times 53 + 50 \times 50 - 3 \times 3 =$ ____________

22. $\mathbf{46^2 - (21^2 - 25^2)} =$ ____________

1.3.5 Sum of Squares: Special Case

There is a special case of the sum of squares that have repeatedly been tested. In order to apply the trick, these conditions must be met:

1. Arrange the two numbers so that the unit's digit of the first number is one greater than the ten's digit of the second number.
2. Makes sure the sum of the ten's digit of the first number and the one's digit of the second number add up to ten.
3. If the above conditions are met, the answer is the sum of the squares of the digits of the first number times 101.

Let's look at an example: $72^2 + 13^2$.

1. The unit's digit of the first number (2) is one greater than the ten's digit of the second number (1).
2. The sum of the ten's digit of the first number (7) and the unit's digit of the second number (3) is 10.
3. The answer will be $(7^2 + 2^2) \times 101 = \mathbf{5353}$.

It is important to arrange the numbers accordingly for this particular trick to work. For example, if you see a problem like: $34^2 + 64^2$, it looks like a difficult problem where this particular trick won't apply. However, if you switch the order of the two numbers you get $34^2 + 64^2 = 64^2 + 34^2 = (6^2 + 4^2) \times 101 = \mathbf{5252}$.

Generally this trick is on the third column, and it is relatively simple to notice when to apply it because if you were having to square the two numbers *and* add them together it would take a long time. That should tip you off immediately that there is trick that you should apply!

The following are some practice problems:

Problem Set 1.3.5

1. $93^2 + 21^2 =$ ____________

2. $12^2 + 19^2 =$ ____________

3. $72^2 + 13^2 =$ ____________

4. $82^2 + 12^2 =$ ____________

5. $45^2 + 46^2 =$ ____________

6. $36^2 + 57^2 =$ ____________

7. $55^2 + 56^2 =$ ____________

8. $37^2 + 67^2 =$ ____________

1.3.6 Difference of Squares

Everybody should know that $x^2 - y^2 = (x-y)(x+y)$. You can easily apply this trick when asked to find the difference between squares of numbers. For example:

$$54^2 - 55^2 = (54-55)(54+55) = -\mathbf{109}$$

This is a pretty basic trick and is easily recognizable on the test.

The following are some more practice to give you a better feel of the problems:

Problems Set 1.3.6

1. $73^2 - 72^2 =$ ____________

2. $36^2 - 34^2 =$ ____________

3. $57^2 - 58^2 =$ ____________

4. $67^2 - 66^2 =$ ____________

5. $\mathbf{69^2} - \mathbf{67^2} =$ ____________

6. $54^2 - 55^2 =$ ____________

7. $67^2 - 65^2 =$ ____________

8. $88^2 - 87^2 =$ ____________

9. $48^2 - 49^2 =$ ____________

10. $97^2 - 96^2 =$ ____________

11. $77^2 - 76^2 =$ ____________

12. $54^2 - 53^2 =$ ____________

13. $42^2 - 44^2 =$ ____________

14. $4.7^2 - 3.3^2 =$ ____________

15. $1.3^2 - 2.6^2 =$ ____________

16. $65^2 - 64^2 + 63^2 - 62^2 =$ ____________

17. $24^2 - 6^2 =$ ____________

18. $56^2 - 55^2 + 54^2 - 53^2 =$ ____________

19. $\mathbf{76^2} - \mathbf{74^2} =$ ____________

20. $3.5^2 - 6.5^2 =$ ____________

21. $22^2 - 23^2 + 24^2 - 25^2 =$ ____________

22. $55^2 - 50^2 =$ ____________

23. $83^2 - 82^2 + 81^2 - 80^2 =$ ____________

24. $55^2 - 52^2 =$ __________

25. $44^2 - 43^2 + 42^2 - 41^2 =$ __________

26. $\mathbf{111^2 - 110^2 + 109^2 - 108^2} =$ __________

27. $11^2 - 22^2 =$ __________

28. $77^2 - 76^2 + 75^2 - 74^2 =$ __________

29. $63^2 - 57^2 =$ __________

30. $56^2 - 55^2 + 54^2 - 53^2 =$ __________

31. $59^2 - 71^2 =$ __________

32. $16^2 - 17^2 + 18^2 - 19^2 =$ __________

33. $41^2 - 42^2 + 43^2 - 44^2 =$ __________

34. $18^2 - 6^2 =$ __________

35. If $x^2 + 16^2 = 19^2$, then $x^2 =$ __________

36. $4.5^2 - 1.5^2 =$ __________

37. $21^2 - 20^2 + 19^2 - 18^2 =$ __________

38. $58^2 - 59^2 + 60^2 - 61^2 =$ __________

39. $72^2 - 78^2 =$ __________

40. $24^2 - 22^2 + 20^2 - 18^2 =$ __________

41. $89^2 - 86^2 + 83^2 - 80^2 =$ __________

42. $48^2 - 62^2 =$ __________

43. $74^2 - 76^2 + 78^2 - 80^2 =$ __________

44. $\mathbf{38^2 - 27^2} =$ __________

45. $31^2 - 33^2 + 35^2 - 37^2 =$ __________

46. $\mathbf{48^2 - 44^2 + 40^2 - 36^2} =$ __________

47. $79^2 - 76^2 + 73^2 - 70^2 =$ __________

1.3.7 Multiplying Two Numbers Ending in 5

This is helpful trick for multiplying two numbers ending in 5. Let's look at its derivation, let $n_1 = a5 = 10a+5$ and $n_2 = b5 = 10b + 5$ then:

$$\begin{aligned} n_1 \times n_2 &= (10a + 5) \cdot (10b + 5) \\ &= 100(ab) + 50(a + b) + 25 \\ &= 100(ab + \frac{a + b}{2}) + 25 \end{aligned}$$

So what does this mean:

1. If $a + b$ is even then the last two digits are 25.
2. If $a + b$ is odd then the last two digits are 75.
3. The remainder of the answer is just $a \cdot b + \lfloor \frac{a+b}{2} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Let's look at an example in each case:

	Ones/Tens:	Since $4 + 8$ is even	**25**
$45 \times 85 =$	Rest of Answer:	$4 \times 8 + \frac{4+8}{2} = 32 + 6$	**38**
	Answer:		**3825**

	Ones/Tens:	Since $3+8$ is odd	**75**
$35 \times 85 =$	Rest of Answer:	$3 \times 8 + \lfloor \frac{3+8}{2} \rfloor = 24 + 5$	**29**
	Answer:		**2975**

Problem Set 1.3.7

1. $35 \times 45 =$ ______________________
2. $95 \times 45 =$ ______________________
3. $35 \times 65 =$ ______________________
4. $85 \times 55 =$ ______________________
5. $\mathbf{65 \times 45} =$ ______________________
6. $35 \times 85 =$ ______________________
7. $65 \times 95 =$ ______________________
8. $\mathbf{55 \times 95} =$ ______________________

1.3.8 Multiplying Mixed Numbers

There are two major tricks involving the multiplication of mixed numbers. The first isn't really a trick at all as it is only using technique of FOILing. Let's illustrate with an example:

$$\begin{aligned} 8\frac{1}{8} \times 24\frac{1}{8} &= (8 + \frac{1}{8}) \times (24 + \frac{1}{8}) \\ &= 8 \cdot 24 + (8 + 24) \cdot \frac{1}{8} + \frac{1}{8} \cdot \frac{1}{8} \\ &= \mathbf{196\frac{1}{64}} \end{aligned}$$

For the most part, both of the whole numbers in the mixed numbers are *usually* divisible by the fraction you are multiplying by (in our example both 8 and 24 are divisible by 8), which means you can just write down the fractional part of the answer immediately and then continue with the problem.

The other trick for mixed numbers occur when the sum of the fractional part is 1 and the two whole numbers are the same. For example:

$$\begin{aligned} 9\frac{1}{3} \times 9\frac{2}{3} &= (9 + \frac{1}{3}) \times (9 + \frac{1}{3}) \\ &= 9^2 + (9 \cdot 2 + 9) \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} \\ &= 9^2 + 9 + \frac{2}{9} \\ &= 9(9+1) + \frac{2}{9} \\ &= \mathbf{90\frac{2}{9}} \end{aligned}$$

So the trick is:

1. The fractional part of the answer is just the two fractions multiplied together.
2. If the whole part in the problem is n then the whole part of the answer is just $n \cdot (n+1)$

Here is another example problem to show the procedure:

$$7\frac{2}{5} \times 7\frac{3}{5} = \quad \begin{array}{lll} \text{Fractional Part:} & \frac{2}{5} \cdot \frac{3}{5} & \mathbf{\frac{6}{25}} \\ \text{Whole Part:} & 7 \cdot (7+1) & \mathbf{56} \\ \text{Answer:} & & \mathbf{56\frac{6}{25}} \end{array}$$

Although these tricks are great (especially FOILing the mixed numbers) sometimes FOILing is very complicated, so the best method is to convert the mixed numbers to improper fractions and see what cancels. For example, you *don't* want to FOIL these mixed numbers:

$$4\frac{7}{12} \times 2\frac{2}{5} = \frac{7}{12} \cdot \frac{2}{5} + 4 \cdot \frac{2}{5} + 2 \cdot \frac{7}{12} + 4 \cdot 2$$

The above is *really* difficult to compute. Instead convert the numbers to improper fractions:

$$4\frac{7}{12} \times 2\frac{2}{5} = \frac{55}{12} \times \frac{12}{5} = \mathbf{11}$$

Usually the best method is to see if you can FOIL the numbers relatively quickly, and if you notice a stumbling block try to convert to improper fractions, then multiply.

Here are more practice problems to help you with this trick:

Problem Set 1.3.8

1. $4\frac{1}{4} \times 8\frac{1}{4} =$ ____________
2. $8\frac{2}{3} \times 8\frac{1}{3} =$ ____________
3. $3\frac{4}{5} \times 3\frac{1}{5} =$ ____________
4. $4\frac{2}{3} \times 6\frac{1}{4} =$ ____________
5. $\mathbf{12\frac{1}{4} \times 8\frac{1}{4}} =$ ____________
6. $15\frac{1}{6} \times 9\frac{1}{6} =$ ____________
7. $6\frac{1}{6} \times 12\frac{1}{6} =$ ____________
8. $11\frac{1}{11} \times 22\frac{1}{11} =$ ____________
9. $25\frac{2}{5} \times 5\frac{2}{5} =$ ____________
10. $5.2 \times 10.2 =$ ____________
11. $8\frac{2}{3} \times 4\frac{2}{3} =$ ____________
12. $7\frac{1}{7} \times 14\frac{1}{7} =$ ____________
13. $\mathbf{5\frac{1}{5} \times 10\frac{1}{5}} =$ ____________
14. $5\frac{1}{5} \times 25\frac{1}{5} =$ ____________
15. $\left(5\frac{2}{5}\right)^2 =$ ____________
16. $8\frac{1}{8} \times 16\frac{1}{8} =$ ____________
17. $\mathbf{10\frac{5}{6} \times 12\frac{4}{5}} =$ ____________
18. $11 \times 11\frac{10}{11} =$ ____________
19. $6\frac{2}{3} \times 9\frac{2}{3} =$ ____________
20. $\left(12\frac{2}{3}\right)^2 =$ ____________

21. $7\frac{1}{7} \times 49\frac{1}{7} =$ ______

22. $3\frac{3}{4} \times 2\frac{2}{5} =$ ______

23. $4.3 \times 2.1 =$ ______

24. $6 \times 6\frac{5}{6} =$ ______

25. $\left(6\frac{2}{3}\right)^2 =$ ______

26. $15.2 \times 5.2 =$ ______

27. $4\frac{3}{5} \times 4\frac{2}{3} =$ ______

28. $3.125 \times 1.6 =$ ______

29. $2.375 \times 2.4 =$ ______

30. $2\frac{2}{5} \times 5\frac{2}{5} =$ ______

1.3.9 $a \times \frac{a}{b}$ Trick

The following is when you are multiplying an integer times a fraction in the form $a \times \frac{a}{b}$. The derivation of the trick is not of importance, only the result is:

$$a \times \frac{a}{b} = [a + (a - b)] + \frac{(a-b)^2}{b}$$

Let's look at a couple of examples:

$$\begin{aligned} 11 \times \frac{11}{13} &= 11 + (11 - 13) + \frac{(11-13)^2}{13} \\ &= 11 - 2 + \frac{4}{13} \\ &= \mathbf{9\frac{4}{13}} \end{aligned}$$

It also works for multiplying by fractions larger than 1:

$$\begin{aligned} 13 \times \frac{13}{12} &= 13 + (13 - 12) + \frac{(13-12)^2}{12} \\ &= 13 + 1 + \frac{1}{12} \\ &= \mathbf{14\frac{1}{12}} \end{aligned}$$

As you can see, when you are multiplying by a fraction less than 1 you will be *subtracting* the difference between the numerator and denominator while when you are multiplying by a fraction greater than 1 you will be *adding* the difference.

It should be noted that there are exceptions (usually on the fourth column) where applying this trick is relatively difficult and it is much easier to just convert to improper fractions then subtract. An example of this is:

$$7 \times \frac{7}{15} - 7 = (7-8) + \frac{8^2}{15} - 7 = -8 + \frac{64}{15} = -8 + 4 + \frac{4}{15} = \mathbf{-3\frac{11}{15}}$$

The above expression was relatively difficult to compute, however if we convert to improper fractions:

$$7 \times \frac{7}{15} - 7 = \frac{7 \cdot 7}{15} - \frac{7 \cdot 15}{15} = \frac{7 \cdot (7 - 15)}{15} = \frac{-56}{15} = \mathbf{-3\frac{11}{15}}$$

This method is a lot less cumbersome and gets the answer relatively swiftly. However, it should be noted that the majority of times the trick is applicable and should *definitely* be used.

The following are more examples to illustrate this trick:

Problem Set 1.3.9

1. $11 \times \frac{11}{14} =$ ____________
2. $22 \times \frac{22}{25} =$ ____________
3. $19 \times \frac{19}{23} =$ ____________
4. $27 \times \frac{27}{32} =$ ____________
5. $16 \times \frac{16}{19} =$ ____________
6. $29 \times \frac{29}{34} =$ ____________
7. $31 \times \frac{31}{34} =$ ____________
8. $14 \times \frac{14}{17} - 3 =$ ____________
9. $11 \times \frac{11}{14} + 3 =$ ____________
10. $\mathbf{13 \times \frac{13}{16} + 13 =}$ ____________
11. $13 \times \frac{13}{17} + 4 =$ ____________
12. $13 \times \frac{13}{14} - 13 =$ ____________
13. $17 \times \frac{17}{18} - 17 =$ ____________
14. $\mathbf{22 \times \frac{22}{25} - 22 =}$ ____________
15. $14 \times \frac{14}{17} - 14 =$ ____________
16. $17 \times 1\frac{17}{21} =$ ____________
17. $13 \times \frac{13}{16} - 13 =$ ____________
18. $11 \times \frac{11}{12} - 11 =$ ____________
19. $\mathbf{7 \times \frac{7}{15} - 7 =}$ ____________
20. $14 \times \frac{14}{17} - 14 =$ ____________
21. $15 \times \frac{15}{17} - 15 =$ ____________
22. $\mathbf{35 \times 1\frac{35}{38} =}$ ____________
23. $13 \times \frac{13}{15} - 13 =$ ____________

1.3.10 Combination of Tricks

The following are a practice set of combination of some of the multiplication tricks already mentioned in the book. Most are approximations which occur on the third or fourth columns of the test.

Problem Set 1.3.10

1. (*) $12 \times 14 \times 16 =$ ____________
2. (*) $21 \times 31 \times 41 =$ ____________
3. (*) $13 \times 15 \times 17 =$ ____________
4. (*) $14 \times 16 \times 28 =$ ____________

5. (*) $146 \times 5 \times 154 =$ ______

6. (*) $24 \times 34 \times 44 =$ ______

7. (*) $24 \times 36 \times 48 =$ ______

8. (*) $44 \times 25 \times 11^2 =$ ______

9. (*) $22 \times 25 \times 28 =$ ______

10. (*) $83 \times 87 \times 91 =$ ______

11. (*) $43 \times 47 \times 51 =$ ______

12. (*) $\mathbf{27} \times \mathbf{29} \times \mathbf{31} \times \mathbf{33} =$ ______

13. (*) $23 \times 33 \times 43 =$ ______

14. (*) $29 \times 127 + 31 \times 213 =$ ______

15. (*) $41 \times 44 \times 47 =$ ______

16. (*) $31 \times 42 \times 53 =$ ______

17. (*) $22 \times 44 \times 66 =$ ______

18. (*) $39 \times 40 \times 41 =$ ______

19. (*) $\sqrt[3]{1329} \times \sqrt{171} \times 15 =$ ______

20. (*) $42 \times 48 \times 45 =$ ______

21. (*) $\mathbf{52} \times \mathbf{55} \times \mathbf{58} =$ ______

22. (*) $18 \times 20 \times 22 =$ ______

23. (*) $24 \times 34 \times 44 =$ ______

24. (*) $80 \times 82 \times 84 =$ ______

25. (*) $28 \times 30 \times 32 =$ ______

26. (*) $66 \times 68 \times 70 =$ ______

27. (*) $\mathbf{63} \times \mathbf{65} \times \mathbf{67} =$ ______

28. (*) $41 \times 43 \div 51 \times 53 =$ ______

29. (*) $67 \times 56 + 65 \times 76 =$ ______

30. (*) $56 \times 45 + 54 \times 65 =$ ______

31. (*) $\mathbf{112} \times \mathbf{123} + \mathbf{132} \times \mathbf{121} =$ ______

32. (*) $29 \times 11 + 31 \times 109 =$ ______

33. (*) $75^2 \div 25^2 \times 50^4 =$ ______

34. (*) $18^3 \times 15^3 \div 9^3 =$ ______

35. (*) $50^5 \div 25^5 \times 5^5 =$ ______

36. (*) $24^3 \times 21^3 \div 4^4 =$ ______

37. (*) $21^3 \times 18^2 \div 9^3 =$ ______

38. (*) $75^4 \div 50^3 \times 25^2 =$ ______

39. $\mathbf{24^2} \times \mathbf{18^3} \div \mathbf{6^4} =$ ______

40. (*) $\sqrt[3]{3380} \times \sqrt{\mathbf{223}} \times \mathbf{16} =$ ______

1.4 Dividing Tricks

Most of these tricks concern themselves with finding the remainders when dividing by certain numbers.

1.4.1 Finding a Remainder when Dividing by 4, 8, etc...

Everybody knows that to see if a number is divisible by 2 you just have to look at the last digit, and if that is divisible by 2 (i.e. any even number) then the entire number is divisible by 2. Similarly, you can extend this principle to see if any integer is divisible by $4, 8, 16$, etc... For divisibility by 4 you look at the last two digits in the number, and if that is divisible by 4, then the entire number is divisible by 4. With 8 it is the last three digits, and so on. Let's look at some examples:

$123456 \div 4$ has what remainder? Look at last two digits: $56 \div 4 = r\mathbf{0}$

$987654 \div 8$ has what remainder? Look at last three digits: $654 \div 8 = r\mathbf{6}$

Here are some practice problems to get you familiar with this procedure:

Problem Set 1.4.1

1. $364 \div 4$ has what remainder? __________

2. **$\mathbf{1324354 \div 4}$ has what remainder?** ______

3. $246531 \div 8$ has what remainder? __________

4. $81736259 \div 4$ has what remainder? ________

5. $124680 \div 8$ has what remainder? __________

6. $214365 \div 8$ has what remainder? __________

7. Find k so that the five digit number $5318k$ is divisible by 8: ______________

1.4.2 Finding a Remainder when Dividing by 3, 9, etc...

In order to find divisibility with 3, you can sum up all the digits and see if that result is divisible by 3. Similarly, you can do the same thing with 9. Let's look at two examples:

$34952 \div 3$ has what remainder? Sum of the Digits: $(3 + 4 + 9 + 5 + 2) = 23$ $23 \div 3 = r\mathbf{2}$

$112321 \div 9$ has what remainder? Sum of the Digits: $(1 + 1 + 2 + 3 + 2 + 1) = 10$ $10 \div 9 = r\mathbf{1}$

For some examples, you can employ faster methods by using modular techniques in order to get the results quicker (see Section 3.4 Modular Arithmetic). For example, if we were trying to see the remainder of 366699995 when dividing by 3, rather than summing up all the digits (which would be a hassle) and then seeing the remainder when that is divided by 3, you can look at each digit and figure out what it's remainder is when dividing by 3 then summing *those*. So for our example:

$366699995 \cong (0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 2) \cong 2 \pmod 3$ therefore it leaves a remainder of **2**.

Here is a set of practice problems:

Problem Set 1.4.2

1. $24680 \div 9$ has a remainder of: __________

2. $6253178 \div 9$ has a remainder of: __________

3. $2007 \div 9$ has a remainder of: __________

4. $13579 \div 9$ has a remainder of: __________

5. $2468 \div 9$ has a remainder of: __________

6. Find the largest integer k such that $3k7$ is divisible by 3: __________

1.4.3 Finding a Remainder when Dividing by 11

Finding the remainder when dividing by 11 is very similar to finding the remainder when dividing by 9 with one catch: you add up alternating digits (beginning with the ones digits) then subtract the sum of the remaining digits. Let's look at an example to illustrate the trick:

$13542 \div 11$ has what remainder?	
Sum of the Alternating Digits:	$(2 + 5 + 1) = 8$
Sum of the Remaining Digits:	$(4 + 3) = 7$
Remainder:	$8 - 7 = \mathbf{1}$

Sometimes adding then subtracting "down the digits" will be easier than finding two explicit sums then subtracting. For example, if we were finding the remainder of $3456789 \div 11$, instead of doing $(9 + 7 + 5 + 3) - (8 + 6 + 4) = 24 - 18 = \mathbf{6}$ it might be easier to do $9 - 8 + 7 - 6 + 5 - 4 + 3 = 1 + 1 + 1 + 3 = \mathbf{6}$. That is what is so great about number sense tricks, is there are always methods and approaches to making them faster!

Problem Set 1.4.3

1. $7653 \div 11$ has a remainder of: __________

2. $745321 \div 11$ has a remainder of: __________

3. **$\mathbf{142536 \div 11}$ has a remainder of:** __________

4. $6253718 \div 11$ has a remainder of: __________

5. $87125643 \div 11$ has a remainder of: __________

6. $325476 \div 11$ has a remainder of: __________

7. Find k so that $23578k$ is divisible by 11: __________

8. Find k so that $1482065k5$ is divisible by 11: __________

9. Find k so that $456k89$ is divisible by 11: __________

10. Find k so that $377337k$ is divisible by 11: __________

1.4.4 Finding Remainders of Other Integers

Another popular question on number sense tests include finding the remainder when dividing by 6 or 12 or some combination of the tricks mentioned above. When dividing seems trivial, sometimes it is best to just long divide to get the remainder (for example $1225 \div 6 = r\mathbf{1}$ from obvious division), however, when this seems tedious, you can use a combination of the two of the tricks mentioned above (depending on the factors of the number you are dividing). Let's look at an example:

$556677 \div 6$ has what remainder?

Dividing by 2: $r\mathbf{1}$

Dividing by 3: $(5+5+6+6+7+7) = 36 \div 3$ $r\mathbf{0}$

So now the task is to find an appropriate remainder (less than 6) such that it is odd (has a remainder of 1 when dividing by 2) and is divisible by 3 (has a remainder of 0 when dividing by 3). From this information, you get $r = \mathbf{3}$. Let's look at another example to solidify this procedure:

$54259 \div 12$ has what remainder?

Dividing by 4: $59 \div 4$ $r\mathbf{3}$

Dividing by 3: $(5+4+2+5+9) = 25 \div 3$ $r\mathbf{1}$

So for this instance, we want an appropriate remainder (less than 12) that has a remainder of 3 when dividing by 4, and a remainder of 1 when dividing by 3. Running through the integers of interest $(0 - 11)$, you get the answer $r = \mathbf{7}$.

The best way of getting faster with this trick is through practice and familiarization of the basic principles. The following are some more practice questions:

Problem Set 1.4.4

1. $2002 \div 6$ has a remainder of: ______
2. $2006 \div 6$ has a remainder of: ______
3. $112358 \div 6$ has a remainder of: ______
4. If $852k$ is divisible by 6 then the largest value for k is: ______
5. $13579248 \div 6$ has a remainder of: ______
6. **$322766211 \div 6$ has a remainder of:** ______
7. $563412 \div 6$ has a remainder of: ______
8. Find k, $k > 0$ so that the 4-digit number $567k$ is divisible by 6: ______
9. **If 86k6 is divisible by 6 then the largest value for k is:** ______
10. $423156 \div 12$ has a remainder of: ______
11. If $555k$ is divisible by 6 then the largest value for k is: ______
12. **Find k$>$ 4 so that the 6-digit number 3576k2 is divisible by 12:** ______
13. $735246 \div 18$ has a remainder of: ______
14. $6253718 \div 12$ has a remainder of: ______
15. Find k, $k > 0$ so that the 5-digit number $8475k$ is divisible by 6: ______

1.4.5 Remainders of Expressions

Questions like $(4^3 - 15 \times 43) \div 6$ has what remainder, are very popular and appear anywhere from the 2^{nd} to the 4^{th} column. This problem has its root in modular arithmetic (See Section 3.4: Modular Arithmetic), and the procedure for solving it is simply knowing that "the remainders after algebra is equal to the algebra of the remainders." So instead of actually finding what $4^3 - 15 \times 43$ is and then dividing by 6, we can figure out what the remainder of each term is when dividing by 6, then do the algebra. So:

$$(4^3 - 15 \times 43) \div 6 \cong (4 - 3 \times 1) \div 6 = r\mathbf{1}$$

It should be noted that if a negative value is computed as the remainder, addition of multiples of the number which you are dividing by are required. Let's look at an example:

$$(15 \times 43 - 34 \times 12) \div 7 \cong (1 \times 1 - 6 \times 5) \div 7 = -29 \Rightarrow -29 + 5 \cdot (7) = r\mathbf{6}$$

So in the above question, after computing the algebra of remainders, we get an unreasonable remainder of -29. So to make this a reasonable remainder (a positive integer such that $0 \leq r < 7$), we added a multiple of 7 (in this case 35) to get the correct answer.

You can use this concept of "negative remainders" to your benefit as well. For example, if we were trying to see the remainder of $13^8 \div 14$, the long way of doing it would be noticing that $13^2 = 169 \div 14 = r1 \Rightarrow 1^4 \div 14 = r\mathbf{1}$ or you could use this concept of negative remainders (or congruencies if you are familiar with that term) to say that $13^8 \div 14 \Rightarrow (-1)^8 \div 14 = r\mathbf{1}$.

The following are some practice problems to solidify using the "algebra of remainders" method:

Problem Set 1.4.5

1. $(31 \times 6 - 17) \div 8$ has a remainder of: ______
2. $(34 \times 27 + 13) \div 4$ has a remainder of: ______
3. $(44 \times 34 - 24) \div 4$ has a remainder of: ______
4. $(33 + 23 \times 13) \div 3$ has a remainder of: ______
5. $(23 + 33 \times 43) \div 4$ has a remainder of: ______
6. $(\mathbf{24} \times \mathbf{34} - \mathbf{44}) \div \mathbf{7}$ **has a remainder of:** ______
7. $(11^2 + 9 \times 7) \div 5$ has a remainder of: ______
8. $(15 \times 3 - 6^2) \div 9$ has a remainder of: ______
9. $(12 \times 9 - 2^3) \div 8$ has a remainder of: ______
10. $(65 \times 4 - 3^2) \div 10$ has a remainder of: ______
11. $(34 \times 56 - 12) \div 9$ has a remainder of: ______
12. $(65 - 4 \times 3) \div 6$ has a remainder of: ______
13. $(34 \times 56 - 12) \div 9$ has a remainder of: ______
14. $(2 \times 3^4 + 5^6) \div 7$ has a remainder of: ______
15. $(23 - 4 \times 5 + 6) \div 7$ has a remainder of: ______
16. $(34 \times 5 - 6) \div 7$ has a remainder of: ______
17. $(1 + 2 - 3 \times 4^5) \div 6$ has a remainder of: ______
18. $(\mathbf{8}^{\mathbf{2}} + \mathbf{4} \times \mathbf{6} - \mathbf{10}) \div \mathbf{3}$ **has a remainder of:** ______
19. $(12 \times 5 + 18 + 15) \div 8$ has a remainder of: ______
20. $(7^3 + 8^2 - 9^1) \div 6$ has a remainder of: ______
21. $(20 + 4 \times 6^2) \div 8$ has a remainder of: ______
22. $(72 \times 64 - 83) \div 7$ has a remainder of: ______
23. $(15 \times 30 - 45) \div 7$ has a remainder of: ______

24. $(6^4 \times 5^3 - 4^2) \div 3$ has a remainder of: ______

25. **$(\mathbf{2^4 \times 3^6 - 5^{10}}) \div \mathbf{4}$ has a remainder of:** ___

26. $(9^2 - 7 \times 5) \div 4$ has a remainder of: ______

27. $(8^2 \times 6 - 4) \div 3$ has a remainder of: ______

28. $(12 \times 34 - 56) \div 7$ has a remainder of: ______

1.4.6 Dividing by 9 Trick

From Section 1.4.2 it is explained how a remainder can be found when dividing by 9. However, you can continue this process of adding *select* digits to get the complete answer when dividing by 9. The following is the result when you divide a four digit number *abcd* by 9 without carries. The details of the proof is omitted, only the result is shown:

	Fractional Part:	$\frac{a+b+c+d}{9}$
	Ones:	$a+b+c$
$abcd \div 9 =$	Tens:	$a+b$
	Hundreds:	a

I think the gist of the trick is self explanatory, let's look at a simple example:

	Fractional Part:	$\frac{1+1+2+3}{9}$	$\mathbf{\frac{7}{9}}$
	Ones:	$1+2+3$	**6**
$3211 \div 9 =$	Tens:	$2+3$	**5**
	Hundreds:	3	**3**
	Answer:		$\mathbf{356\frac{7}{9}}$

Here is a little bit more complicated of a problem involving a larger number being divided as well as incorporating carries:

	Fractional Part:	$\frac{7+5+2+2+3}{9}$	$\mathit{2}\mathbf{\frac{1}{9}}$
	Ones:	$5+2+2+3+\mathit{2}$	$\mathit{1}\mathbf{4}$
	Tens:	$2+2+3+\mathit{1}$	**8**
$32257 \div 9 =$	Hundreds:	$2+3$	**5**
	Thousands:	3	**3**
	Answer:		$\mathbf{3584\frac{1}{9}}$

Here are some problems to give you more practice with this trick:

Problem Set 1.4.6

1. $354 \div 9 =$ ______

2. $503 \div 9 =$ ______

3. $2003 \div 9 =$ ______

4. $321 \div 9 =$ ______

5. $456 \div 9 =$ ______

6. $1234 \div 9 =$ ______

7. $12345 \div 9 =$ ______

8. $2475 \div 45 =$ ______

1.4.7 Converting $\frac{a}{40}$ and $\frac{b}{80}$, etc... to Decimals

The following isn't necessarily a trick but more of a procedure I like to follow when I am approached with converting $\dfrac{a}{40}$ and $\dfrac{b}{80}$ into decimals (usually on the first column of problems). So for $\dfrac{a}{40}$ I treat it as:

$$\frac{a}{40} = \frac{a}{40} \times \frac{\frac{1}{4}}{\frac{1}{4}} = \frac{\frac{a}{4}}{10}$$

So the technique is to divide the numerator by 4 then shift the decimal point over. Similarly, for $\dfrac{b}{80}$ you want to divide by 8 and shift the decimal point over. Let's look at a couple of examples:

$$\frac{43}{40} = 1 + \frac{3}{40} = 1 + \frac{.75}{10} = \mathbf{1.075}$$

$$\frac{27}{80} \Rightarrow \frac{27}{8} = 3.375 \Rightarrow \frac{3.375}{10} = .\mathbf{3375}$$

Here are some practice problems of this type:

Problem Set 1.4.7

1. $\dfrac{1}{40} =$ ______ %

2. $\dfrac{3}{40} =$ ______ %

3. $\dfrac{\mathbf{7}}{\mathbf{40}} =$ ______ %

4. $\dfrac{\mathbf{21}}{\mathbf{40}} =$ ______ %

5. $\dfrac{43}{40} =$ ______ (dec.)

6. $\dfrac{3}{(2^3)(5^1)}$ ______ (dec.)

7. $.0125 =$ ______ % (frac.)

8. 48 is ______ % greater than 40

9. $\dfrac{7}{40} =$ ______ %

10. 32 is what % of 80? ______

11. $\dfrac{11}{40} =$ ______ %

12. $\dfrac{3^2}{(2^3)(5^2)} =$ ______ (dec.)

13. 72 is what % of 400? ______ %

14. $\dfrac{5}{(2^3)(5^2)} =$ ______ (dec.)

15. $4\frac{7}{20}$ = ________________ %

16. $\frac{5}{80}$ = ________________ %

17. 27.5% = ________________ (frac.)

18. $\frac{4^3}{(2^3)(5^2)}$ = ________________ (dec.)

19. 1.6 is ________________ % of 20

20. $\frac{3^4}{(2^4)(5^4)}$ = ________________ (dec.)

1.5 Adding and Subtracting Tricks

The following are tricks where adding/subtracting are required to solve the problems.

1.5.1 Subtracting Reverses

A common first column problem from the early 2000s involves subtracting two numbers whose digits are reverses of each other (like $715 - 517$ or $6002 - 2006$). Let the first number $n_1 = abc = 100a + 10b + c$ so the second number with the digits reversed would be $n_2 = cba = 100c + 10b + a$ so:

$$\begin{aligned} n_1 - n_2 &= (100a + 10b + c) - (100c + 10b + a) \\ &= 100(a - c) + (c - a) \\ &= 100(a - c) - (a - c) \end{aligned}$$

So the gist of the trick is:

1. Take the difference between the most significant and the least significant digit and multiply it by 100 if it is a three-digit number, or if it is a four digit number multiply by 1000 (however, it only works for 4-digit numbers and above if the middle digits are 0's; for example, $7002 - 2007$ the method works but $7012 - 2107$ it *doesn't* work).
2. Then subtract from that result the difference between the digits.

Let's look at an example:

	Step 1:	$(8-2) \times 100$	600
$812 - 218 =$	Step 2:	$600 - 6$	**594**
	Answer:		**594**

It also works for when the subtraction is a negative number, but you need to be careful:

	Step 1:	$(1-5) \times 100$	-400
$105 - 501 =$	Step 2:	$-400 - (1-5)$	$-\mathbf{396}$
	Answer:		$-\mathbf{396}$

Like I said, you have to be careful with negative signs, a better (and highly recommended approach outlined in the next section) is to say: $105 - 501 = -(501 - 105) = -396$. By negating and reversing the numbers, you deal with positive numbers which are naturally more manageable. After you find the solution, you negate the result because of the sign switch.

Problem Set 1.5.1

1. $654 - 456 =$ ________________
2. $256 - 652 =$ ________________
3. $4002 - 2004 =$ ________________

4. $702 - 207 =$ ______

5. $453 - 354 =$ ______

6. $5002 - 2005 =$ ______

7. $2006 - 6002 =$ ______

8. $2003 - 3002 =$ ______

9. $678 - 876 =$ ______

10. $2007 - 7002 =$ ______

1.5.2 Switching Numbers and Negating on Subtraction

Far too common, students make a mistake when subtracting two fractions whose result is a negative answer. An example of this is $4\frac{5}{6} - 10\frac{11}{12}$. Most of the time, it is incredibly easier switching the order of the subtraction then negating the answer. Taking the above problem as an example:

$$\begin{aligned} 4\frac{5}{6} - 10\frac{11}{12} &= -(10\frac{11}{12} - 4\frac{5}{6}) \\ &= -(10\frac{11}{12} - 4\frac{10}{12}) \\ &= -(\mathbf{6\frac{1}{12}}) \end{aligned}$$

Here is another example to illustrate the same point:

$$\begin{aligned} 2\frac{5}{6} - 4\frac{2}{3} &= -(4\frac{2}{3} - 2\frac{5}{6}) \\ &= -(4\frac{4}{6} - 2\frac{5}{6}) \\ &= -(\mathbf{1\frac{5}{6}}) \end{aligned}$$

Problems Set 1.5.2

1. $2\frac{2}{3} - 3\frac{5}{6} =$ ______
2. $4\frac{2}{3} - 6\frac{3}{5} =$ ______
3. $1\frac{5}{9} - 3\frac{5}{9} =$ ______
4. $2\frac{3}{4} - 4\frac{3}{5} =$ ______
5. $1\frac{3}{7} - 3 =$ ______
6. $\mathbf{2\frac{3}{8}} - \mathbf{3\frac{1}{4}} =$ ______
7. $2\frac{3}{4} - 6\frac{7}{8} =$ ______
8. $3\frac{4}{5} - 8\frac{9}{10} =$ ______
9. $3\frac{4}{9} - 5\frac{1}{3} =$ ______
10. $5\frac{6}{7} - 12\frac{13}{14} =$ ______
11. $3\frac{1}{6} - 6\frac{1}{3} =$ ______
12. $2\frac{5}{6} - 4\frac{2}{3} =$ ______
13. $4\frac{7}{8} - 12\frac{23}{24} =$ ______
14. $\mathbf{4\frac{5}{6}} - \mathbf{10\frac{11}{12}} =$ ______
15. $2\frac{3}{5} - 7\frac{1}{10} =$ ______
16. $1\frac{4}{5} - 3\frac{2}{5} =$ ______

1.5.3 $\dfrac{a}{b \cdot (b+1)} + \dfrac{a}{(b+1)\cdot(b+2)} + \cdots$

The best way to illustrate this trick is by example:

$$\begin{aligned}\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30} &= \frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6} \\ &= \frac{1+1+1+1}{2\cdot 6} \\ &= \frac{4}{12} = \mathbf{\frac{1}{3}}\end{aligned}$$

So the strategy when you see a series in the form of $\frac{a}{b\cdot(b+1)}+\frac{a}{(b+1)\cdot(b+2)}+\cdots$ is to add up all the numerators and then divide it by the smallest factor in the denominators multiplied by the largest factor in the denominators. Let's look at another series:

$$\begin{aligned}\frac{1}{42}+\frac{1}{56}+\frac{1}{72}+\frac{1}{90}+\frac{1}{110} &= \frac{1}{6\cdot 7}+\frac{1}{7\cdot 8}+\frac{1}{8\cdot 9}+\frac{1}{9\cdot 10}+\frac{1}{10\cdot 11} \\ &= \frac{1+1+1+1+1}{6\cdot 11} \\ &= \frac{5}{66}\end{aligned}$$

Problems Set 1.5.3

1. $\dfrac{1}{12}+\dfrac{1}{20}+\dfrac{1}{30}+\dfrac{1}{42} =$ ____________________

2. $\mathbf{\dfrac{1}{72}+\dfrac{1}{90}+\dfrac{1}{110}+\dfrac{1}{132}} =$ ____________________

3. $\dfrac{1}{30}+\dfrac{1}{42}+\dfrac{1}{56} =$ ____________________

4. $\mathbf{\dfrac{7}{30}+\dfrac{7}{20}+\dfrac{7}{12}} =$ ____________________

1.5.4 $\dfrac{a}{b}+\dfrac{b}{a}$ Trick

Let's look at when we add the two fractions $\dfrac{a}{b}+\dfrac{b}{a}$:

$$\begin{aligned}\frac{a}{b}+\frac{b}{a} &= \frac{a^2+b^2}{ab} \\ &= \frac{2ab}{ab}-\frac{2ab}{ab}+\frac{a^2+b^2}{ab} \\ &= 2+\frac{(a-b)^2}{ab}\end{aligned}$$

Here is an example:

$$\frac{5}{7}+\frac{7}{5} = 2+\frac{(7-5)^2}{7\cdot 5} = \mathbf{2\frac{4}{35}}$$

There are some variations to this trick. For example:

$$\frac{11}{13}+\frac{2}{11} = \left(\frac{11}{13}+\frac{13}{11}\right)-\frac{11}{11} = 2+\frac{2^2}{143}-1 = \mathbf{1\frac{4}{143}}$$

This is a popular variation that is used especially on the last column of the test because the trick is there but not as obvious.

The following are some practice problems to help you master this trick:

Problems Set 1.5.4

1. $\frac{12}{13} + \frac{13}{12} =$ ______

2. $\frac{5}{6} + \frac{6}{5} =$ ______

3. $\frac{15}{19} + \frac{19}{15} =$ ______

4. $\frac{3}{5} + \frac{5}{3} - 2 =$ ______

5. $\frac{7}{5} + \frac{5}{7} - 1 =$ ______

6. $\frac{11}{13} + \frac{2}{11} =$ ______

7. $\frac{7}{13} + \frac{6}{7} =$ ______

8. $\frac{5}{6} + 1\frac{1}{5} - 2 =$ ______

9. $\frac{13}{15} + \frac{2}{13} =$ ______

10. $\frac{5}{8} + \frac{8}{5} - \frac{9}{40} =$ ______

11. $\frac{3}{5} + \frac{5}{3} + \frac{11}{15} =$ ______

12. $\mathbf{\frac{5}{7} + \frac{7}{5} - 3} =$ ______

13. $\frac{15}{17} + \frac{2}{15} =$ ______

14. $\frac{11}{15} + \frac{4}{11} =$ ______

15. $\frac{11}{13} + \frac{2}{11} =$ ______

16. $\frac{14}{15} + \frac{1}{14} =$ ______

17. $\mathbf{1\frac{12}{13} + 1\frac{1}{12}} =$ ______

18. $\left(\frac{5}{7} + \frac{7}{5}\right) \div 2 =$ ______

19. $\frac{11}{12} + \frac{1}{11} =$ ______

20. $\mathbf{\frac{15}{22} + \frac{7}{15} - 1} =$ ______

21. $\mathbf{\frac{11}{14} + \frac{3}{11} - 2} =$ ______

1.5.5 $\frac{a}{b} - \frac{na-1}{nb+1}$

The following deals with subtracting fractions in the form $\frac{a}{b} - \frac{na-1}{nb+1}$. Most of these problems are on the 3^{rd} of 4^{th} columns, and they are relatively easy to pick out because of how absurd the problem would be if you didn't know the formula:

$$\frac{a}{b} - \frac{na-1}{nb+1} = \frac{(a+b)}{b \cdot (nb+1)}$$

So the numerator of the answer is just the sum of the numerator and denominator of the first number (e.g., the number who's numerator and denominators are small values) while the denominator of the answer is just the multiplication of the two denominators. Here is an example:

$$\frac{6}{7} - \frac{29}{36} = \frac{6+7}{7 \cdot 36} = \mathbf{\frac{13}{252}}$$

Like I said it is easy to notice when to do this problem because, if you didn't know the formula, if would be relatively difficult to solve swiftly.

There is one variation to the formula which is:

$$\frac{a}{b} - \frac{na+1}{nb-1} = \frac{-(a+b)}{b \cdot (nb-1)}$$

When approached with these problems, it is best to take time to notice which type it is. The easiest way of seeing which formula to apply is to look at the denominator of the more "complicated" number and see if it is one *greater* or one *less* than a multiple of the denominator of the "simple" number. Here's an example:

$$\frac{7}{11} - \frac{43}{65} = \frac{-(7+11)}{11 \cdot 65} = \frac{\mathbf{-18}}{\mathbf{715}}$$

So on the above question, notice that 65 is one less a multiple of 11, so you know to apply the second formula.

Here are some practice problems to help you out:

Problems Set 1.5.5

1. $\frac{4}{9} - \frac{11}{28} =$ ________________
2. $\frac{2}{7} - \frac{7}{29} =$ ________________
3. $\frac{\mathbf{4}}{\mathbf{13}} - \frac{\mathbf{11}}{\mathbf{40}} =$ ________________
4. $\frac{7}{15} - \frac{27}{61} =$ ________________
5. $\frac{8}{11} - \frac{31}{45} =$ ________________
6. $\frac{\mathbf{8}}{\mathbf{11}} - \frac{\mathbf{87}}{\mathbf{122}} =$ ________________
7. $\frac{3}{8} - \frac{26}{73} =$ ________________
8. $\frac{\mathbf{4}}{\mathbf{5}} - \frac{\mathbf{67}}{\mathbf{86}} =$ ________________
9. $\frac{8}{3} - \frac{41}{14} =$ ________________
10. $\frac{8}{9} - \frac{87}{100} =$ ________________
11. $\frac{\mathbf{67}}{\mathbf{81}} - \frac{\mathbf{17}}{\mathbf{20}} =$ ________________
12. $\frac{3}{8} - \frac{14}{41} =$ ________________
13. $\frac{7}{15} - \frac{15}{29} =$ ________________
14. $\frac{5}{8} - \frac{24}{41} =$ ________________
15. $\frac{8}{9} - \frac{31}{37} =$ ________________
16. $\frac{10}{11} - \frac{39}{45} =$ ________________
17. $\frac{11}{16} - \frac{32}{49} =$ ________________
18. $\frac{8}{11} - \frac{87}{122} =$ ________________
19. $\frac{4}{7} - \frac{35}{64} =$ ________________
20. $\frac{9}{46} - \frac{2}{9} =$ ________________
21. $\frac{3}{8} - \frac{14}{41} =$ ________________
22. $\frac{7}{11} - \frac{55}{89} =$ ________________

2 Memorizations

2.1 Important Numbers

2.1.1 Squares

In order for faster speed in taking the test, squares up to 25 should definitely be memorized with memorization of squares up to 50 being highly recommended. In the event that memorization can't be achieved, remember the tricks discussed in Section 1 of the book as well as the method of FOILing. The following table should aid in memorization:

$11^2 = 121$	$12^2 = 144$	$13^2 = 169$	$14^2 = 196$
$15^2 = 225$	$16^2 = 256$	$17^2 = 289$	$18^2 = 324$
$19^2 = 361$	$20^2 = 400$	$21^2 = 441$	$22^2 = 484$
$23^2 = 529$	$24^2 = 576$	$25^2 = 625$	$26^2 = 676$
$27^2 = 729$	$28^2 = 784$	$29^2 = 841$	$30^2 = 900$
$31^2 = 961$	$32^2 = 1024$	$33^2 = 1089$	$34^2 = 1156$
$35^2 = 1225$	$36^2 = 1296$	$37^2 = 1369$	$38^2 = 1444$
$39^2 = 1521$	$40^2 = 1600$	$41^2 = 1681$	$42^2 = 1764$
$43^2 = 1849$	$44^2 = 1936$	$45^2 = 2025$	$46^2 = 2116$
$47^2 = 2209$	$48^2 = 2304$	$49^2 = 2401$	$50^2 = 2500$

On the next page you will find practice problems concerning squares. Avoid FOILing when possible so that you can work on having automatic responses on some of the questions.

Problems Set 2.1.1

1. $28^2 =$ __________
2. $3.2^2 =$ __________
3. $29 \times 29 =$ __________
4. $16 \times 16 =$ __________
5. $\mathbf{31^2} =$ __________
6. If 2.2 cm= 1 inch, then
   2.2 in equals how many cm.? __________
7. $34 \times 34 =$ __________
8. $17 \times 17 =$ __________
9. $23 \times 23 =$ __________
10. $\mathbf{18} \times \mathbf{18} =$ __________
11. 24% of 24 is: __________
12. $23^2 =$ __________
13. $32^2 =$ __________
14. $\mathbf{14} \times \mathbf{14} =$ __________
15. $21^2 =$ __________
16. $24^2 =$ __________
17. 31% of 31 is: __________
18. What is 27% of 27: __________
19. $\mathbf{34^2} =$ __________
20. $26^2 =$ __________
21. $\mathbf{17^2} =$ __________
22. $33 \times 33 =$ __________
23. Find $x < 0$ when $x^2 = 729$: __________
24. (*) $\sqrt{1090} \times 31 =$ __________
25. (*) $\sqrt{291} \times 23 =$ __________
26. $\sqrt{-196} \times \sqrt{-256} =$ __________
27. $\frac{3}{4}$ of 24% of 1.8: __________
28. (*) $509 \times \sqrt{905} =$ __________
29. (*) $\sqrt{327} \times \sqrt{397} \times \sqrt{487} =$ __________
30. (*) $14^4 =$ __________
31. (*) $\sqrt{362} \times \sqrt{440} =$ __________
32. $959 \times \sqrt{960} =$ __________
33. (*) $13^4 =$ __________
34. (*) $\sqrt{451} \times 451 =$ __________
35. (*) $\sqrt{574} \times \sqrt{577} \times \sqrt{580} =$ __________
36. (*) $17^4 =$ __________
37. (*) $\sqrt{1025} \times \sqrt{63} =$ __________
38. (*) $28 \times 56 \times 14 \div 42 =$ __________
39. (*) $\sqrt{1030} \times 2^5 =$ __________
40. (*) $21^4 =$ __________

2.1.2 Cubes

The following cubes should also be memorized:

$5^3 = 125$	$6^3 = 216$	$7^3 = 343$	$8^3 = 512$
$9^3 = 729$	$10^3 = 1000$	$11^3 = 1331$	$12^3 = 1728$
$13^3 = 2197$	$14^3 = 2744$	$15^3 = 3375$	$16^3 = 4096$
$17^3 = 4913$	$18^3 = 5832$	$19^3 = 6859$	$20^3 = 8000$

Again, only FOIL when necessary on the practice problems on the next page.

Problem Set 2.1.2

1. $\sqrt[3]{(1728)} =$ ______

2. $11^3 =$ ______

3. $\mathbf{14} \times \mathbf{14} \times \mathbf{14} =$ ______

4. $(-343)^{\frac{1}{3}} =$ ______

5. $12^3 =$ ______

6. $16^3 =$ ______

7. $\sqrt[3]{1728} \div \sqrt{36} =$ ______

8. $11^4 \div 11 =$ ______

9. $(-12)^3 =$ ______

10. $(\mathbf{2197})^{\frac{\mathbf{1}}{\mathbf{3}}} =$ ______

11. $(-729)^{\frac{1}{3}}$ ______

12. $8^3 =$ ______

13. $\mathbf{15^3} =$ ______

14. $12 \times 12 \times 12 =$ ______

15. $(125 \div 64)^{\frac{1}{3}} =$ ______

16. $\mathbf{13^3} =$ ______

17. $7 \times 7 \times 7 =$ ______

18. $-\mathbf{1331}^{\frac{1}{3}} =$ ______

19. $6 \times 6 \times 6 =$ ______

20. $15 \times 15 \times 15 =$ ______

21. (*) $\sqrt[3]{1730} \times \mathbf{145} =$ ______

22. $(27 \div 216)^{\frac{1}{3}}$ ______

23. If $x = 7$ then $(x+3)(x^2 - 3x + 9) =$ ______

24. $\sqrt{\mathbf{676}} \div \sqrt[3]{-2197} =$ ______

25. $(1.728)^{\frac{1}{3}} =$ ______

26. $8^3 \times 5^3 =$ ______

27. $\mathbf{11^5} \div \mathbf{121} =$ ______

28. $\sqrt[3]{1.331} =$ ______

29. (*) $89 \times 90 \times 91 =$ ______

30. $\sqrt[3]{.729} =$ ______

31. (*) $(\mathbf{121})^{\mathbf{3}} =$ ______

32. $3^4 - 6^3 + 9^2 =$ ______

33. $\sqrt[3]{1728} \div \sqrt{576} =$ ______

34. $\sqrt{225} \times \sqrt[3]{3375} =$ ______

35. $8^3 - 9^3 =$ ______

36. (*) $13^3 \times 3^4 =$ ______

37. $2^3 \times 5^3 \times 7^3 =$ ______

38. (*) $119 \times 120 \times 121 =$ ______

39. (*) $14^3 \times 4^5 =$ ______

40. $8^4 =$ ______

2.1.3 Powers of $2, 3, 5$

Memorizing powers of certain integers like $2, 3, 5$, etc... can be beneficial in solving a variety of problems ranging from approximation problems to logarithm problems. In some instances, powers of integers can be calculated based on other means than memorization. For example, $7^4 = \left(7^2\right)^2 = 49^2 = 2401$ However, the following powers should be memorized for quick calculation:

$2^3 = 8$	$3^3 = 27$	$5^3 = 125$
$2^4 = 16$	$3^4 = 81$	$5^4 = 625$
$2^5 = 32$	$3^5 = 243$	$5^5 = 3125$
$2^6 = 64$	$3^6 = 729$	
$2^7 = 128$	$3^7 = 2187$	
$2^8 = 256$		
$2^9 = 512$		
$2^{10} = 1024$		

On the next page are problems concerning higher powers of certain integers.

Problem Set 2.1.3

1. $5^3 + 3^3 + 2^3 =$ ______
2. $2^3 - 3^3 - 4^3 =$ ______
3. $(\sqrt{64} - \sqrt{36})^5 =$ ______
4. $5^x = 125, x^5 =$ ______
5. $4^3 - 5^3 =$ ______
6. $2^{x+1} = 32,\ x - 1 =$ ______
7. $2^3 + 3^3 + 5^3 =$ ______
8. $5^3 - 3^3 =$ ______
9. $\sqrt[3]{125 \times 512} =$ ______
10. $2^3 + 3^3 + 4^3 - 5^3 =$ ______
11. $x^3 = 64$, so $3^x =$ ______
12. $4^5 \times 5^5 =$ ______
13. $27^2 =$ ______
14. **If $\mathbf{x^5 = -32}$, then $\mathbf{5^x =}$** ______
15. $2^5 \times 5^3 =$ ______
16. $8^4 \times 5^4 =$ ______
17. (*) $5^5 + 4^4 + 3^3 + 2^2 + 1^1 =$ ______
18. $2^6 \times 5^4 =$ ______
19. $5^{x-1} = 3125$, then $x + 1 =$ ______
20. $\mathbf{2^3} - \mathbf{3^3} - \mathbf{5^3} =$ ______
21. $\dfrac{3^4}{2^3 \cdot 5^3} =$ ______
22. $6^3 + 4^3 + 2^3 =$ ______
23. $3^4 + 4^3 = 5 \cdot x$, then $x =$ ______
24. (*) $5^1 + 4^2 + 3^3 + 2^4 + 1^5 =$ ______
25. $9^x = 243$, then $x =$ ______
26. $8^3 \times 5^3 =$ ______
27. $\mathbf{2^3} \times \mathbf{8^3} \times \mathbf{5^3} =$ ______
28. $2^5 \times 3^4 \times 5^2 =$ ______
29. $2^4 \times 7^2 \times 5^3 =$ ______
30. $\mathbf{4^2} \times \mathbf{5^2} \times \mathbf{6^2} =$ ______
31. $\mathbf{2^5} \times \mathbf{3^3} \times \mathbf{5^2} =$ ______
32. $2^3 \times 3^4 \times 5^5 =$ ______
33. $(3^3 - 2^3 + 1^3) \times 5^3 =$ ______
34. $2^5 \times 3^4 \times 5^2 =$ ______
35. $2^5 \times 3^4 \times 5^5 =$ ______
36. $2^3 \times 3^2 \times 4^2 \times 5^3 =$ ______

2.1.4 Important Fractions

The following fractions should be memorized for reasons stated in Section 1.2.5. In addition, early problems on the test typically involve converting these fractions to decimals and percentages. So if these conversions were memorized, a lot of time would be saved. Omitted are the "obvious" fractions ($\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{5}$, *etc...*).

Fraction	%	Fraction	%	Fraction	% / Decimal
$\frac{1}{6}$	$16\frac{2}{3}\%$	$\frac{1}{7}$	$14\frac{2}{7}\%$	$\frac{1}{8}$	$12\frac{1}{2}\% = .125$
$\frac{5}{6}$	$83\frac{1}{3}\%$	$\frac{2}{7}$	$28\frac{4}{7}\%$	$\frac{3}{8}$	$37\frac{1}{2}\% = .375$
		$\frac{3}{7}$	$42\frac{6}{7}\%$	$\frac{5}{8}$	$62\frac{1}{2}\% = .625$
		$\frac{4}{7}$	$57\frac{1}{7}\%$	$\frac{7}{8}$	$87\frac{1}{2}\% = .875$
		$\frac{5}{7}$	$71\frac{3}{7}\%$		
		$\frac{6}{7}$	$85\frac{5}{7}\%$		

Fraction	%	Fraction	%	Fraction	%	Fraction	%
$\frac{1}{9}$	$11\frac{1}{9}\%$	$\frac{1}{11}$	$9\frac{1}{11}\%$	$\frac{1}{12}$	$8\frac{1}{3}\%$	$\frac{1}{16}$	$6\frac{1}{4}\%$
$\frac{2}{9}$	$22\frac{2}{9}\%$	$\frac{2}{11}$	$18\frac{2}{11}\%$	$\frac{5}{12}$	$41\frac{2}{3}\%$	$\frac{3}{16}$	$18\frac{3}{4}\%$
$\frac{3}{9}$	$33\frac{3}{9}\%$	$\frac{3}{11}$	$27\frac{3}{11}\%$	$\frac{7}{12}$	$58\frac{1}{3}\%$	$\frac{5}{16}$	$31\frac{1}{4}\%$
$\frac{4}{9}$	$44\frac{4}{9}\%$	$\frac{4}{11}$	$36\frac{4}{11}\%$	$\frac{11}{12}$	$91\frac{2}{3}\%$	$\frac{7}{16}$	$43\frac{3}{4}\%$
$\frac{5}{9}$	$55\frac{5}{9}\%$	$\frac{5}{11}$	$45\frac{5}{11}\%$			$\frac{9}{16}$	$56\frac{1}{4}\%$
$\frac{6}{9}$	$66\frac{6}{9}\%$	$\frac{6}{11}$	$54\frac{6}{11}\%$			$\frac{11}{16}$	$68\frac{3}{4}\%$
$\frac{7}{9}$	$77\frac{7}{9}\%$	$\frac{7}{11}$	$63\frac{7}{11}\%$			$\frac{13}{16}$	$81\frac{1}{4}\%$
$\frac{8}{9}$	$88\frac{8}{9}\%$	$\frac{8}{11}$	$72\frac{8}{11}\%$			$\frac{15}{16}$	$93\frac{3}{4}\%$
		$\frac{9}{11}$	$81\frac{9}{11}\%$				
		$\frac{10}{11}$	$90\frac{10}{11}\%$				

Fraction	%	Fraction	%
$\frac{1}{13}$	$7\frac{9}{13}\%$	$\frac{1}{14}$	$7\frac{1}{7}\%$
$\frac{2}{13}$	$15\frac{5}{13}\%$	$\frac{3}{14}$	$21\frac{3}{7}\%$
$\frac{3}{13}$	$23\frac{1}{13}\%$	$\frac{5}{14}$	$35\frac{5}{7}\%$
$\frac{4}{13}$	$30\frac{10}{13}\%$	$\frac{9}{14}$	$64\frac{2}{7}\%$
$\frac{5}{13}$	$38\frac{6}{13}\%$	$\frac{11}{14}$	$78\frac{4}{7}\%$
$\frac{6}{13}$	$46\frac{2}{13}\%$	$\frac{13}{14}$	$92\frac{6}{7}\%$
$\frac{7}{13}$	$53\frac{11}{13}\%$		
$\frac{8}{13}$	$61\frac{7}{13}\%$		
$\frac{9}{13}$	$69\frac{3}{13}\%$		
$\frac{10}{13}$	$76\frac{12}{13}\%$		
$\frac{11}{13}$	$84\frac{8}{13}\%$		
$\frac{12}{13}$	$92\frac{4}{13}\%$		

To aid in memorization, it would first help to memorize the first fractions in each column. From, here the others can be quickly derived by multiplying the initial fraction by the required integer to get the desired results. For example, if you only had $\frac{1}{11}$ memorized as $9\frac{1}{11}\%$, but you need to know what $\frac{5}{11}$ is, then you could simply multiply by 5:

$$5 \times \frac{1}{11} = 5 \times \left(9\frac{1}{11}\%\right) = 45\frac{5}{11}\%$$

Although memorization of *all* fractions is ideal, this method will result in correctly answering the question, albeit a lot slower.
On the next page you'll find a variety of practice problems.

Problem Set 2.1.4

1. $12\frac{1}{2}\% =$ ______(frac.)
2. $\frac{11}{5} =$ ______%
3. Which is larger $\frac{5}{9}$ or .56? ______
4. **Which is larger $\mathbf{\frac{5}{8}}$ or .622** : ______
5. $\frac{17}{8} =$ ______(dec.)
6. .777 . . . − .333 . . . + .555 . . . = ______
7. $\frac{3}{5} =$ ______%
8. $\frac{1}{8} =$ ______(dec.)
9. **Which is smaller $\mathbf{\frac{9}{11}}$ or .81**? ______
10. $\frac{1}{16} =$ ______%
11. .125 − .375 − .625 = ______
12. $\frac{11}{4} =$ ______%
13. Which is larger $\frac{5}{9}$ or .555 or 55%? ______
14. .1666 . . . − .333 . . . + .8333 . . . = ______
15. The reciprocal of −1.0625 is: ______
16. Which is larger .46 or $\frac{5}{11}$? ______
17. .111 . . . − .333 . . . − .666 . . . = ______
18. 37.5% = ______(frac.)
19. Which is smaller $\frac{9}{11}$ or .8? ______
20. $\mathbf{\frac{3}{7}} =$ ______%
21. $\frac{7}{9} =$ ______%
22. .08333 . . . + .1666 . . . + .25 = ______
23. Which is smaller $\frac{7}{11}$ or .56 : ______
24. **Which is larger $\mathbf{\frac{9}{11}}$ or 81**%? ______
25. .1666 . . . + .333 . . . + .8333 . . . = ______
26. $\frac{7}{16} =$ ______% (dec.)
27. $32 \div .181818\ldots =$ ______
28. $\frac{2}{7} =$ ______%
29. Which is larger −.375 or $\frac{-5}{12}$? ______
30. **.333** . . . − **.666** . . . − **.999** . . . = ______
31. $\frac{1}{14} =$ ______%
32. .0625 + .125 + .25 = ______
33. $55\frac{5}{9}\%$ of 27 is: ______
34. 12.5% of 24 is: ______
35. Which is larger −.27 or $\frac{-2}{7}$? ______
36. $55 \div .454545\ldots =$ ______
37. .111 . . . − .1666 . . . − .333 . . . = ______

38. $\mathbf{\frac{5}{16}}$ = __________% **(dec.)** __________

39. $363 \div .272727\ldots =$ __________

40. $21\frac{3}{7}\% =$ __________(frac.)

41. $88 \times .090909\ldots =$ __________

42. $4\frac{4}{5} \div .444\ldots =$ __________

43. $\frac{3}{14} =$ __________%

44. $35\frac{5}{7}\% =$ __________(frac.)

45. $72 \times .083333\ldots =$ __________

46. $78\frac{4}{7}\% =$ __________(frac.)

47. $911 \div .090909\ldots =$ __________

48. $\frac{1}{12} =$ __________%

49. $\frac{11}{14} =$ __________%

50. 50 is 6.25% of __________

51. $242 \div .181818\ldots =$ __________

52. $16\frac{2}{3}\% \times 482 =$ __________

53. $\mathbf{75 \div .5555}\ldots =$ __________

54. $64\frac{2}{7}\% =$ __________(frac.)

55. $\mathbf{1.21 \div .09090}\ldots =$ __________

56. $1\frac{7}{8} =$ __________% (frac.)

57. $6.25\% =$ __________(frac.)

58. $\frac{17}{14} =$ __________%

59. $42\frac{6}{7}\% =$ __________(frac.)

60. $3\frac{3}{4}\% =$ __________(frac.)

61. $\mathbf{1\frac{1}{10}}\% =$ __________(frac.)

62. $\mathbf{92\frac{6}{7}}\% =$ __________(frac.)

63. $7\frac{1}{7}\% =$ __________(frac.)

64. 75 is 3.125% of __________

65. $\mathbf{6\frac{7}{8}}\% =$ __________(dec.)

66. $\frac{13}{14} =$ __________%

67. $3\frac{1}{13}\% =$ __________(frac.)

68. $\mathbf{\frac{15}{14}} =$ __________%

69. $21\frac{3}{7}\% =$ __________(frac.)

2.1.5 Special Integers

The following integers have important properties which are exploited regularly on the number sense test. They are:

$$\mathbf{999}: \quad 999 = 27 \times 37 \qquad \mathbf{77}: \quad 77 = \frac{1001}{13} \qquad \mathbf{3367}: \quad 3367 = \frac{10101}{3} \qquad \mathbf{1430}: \quad 1430 = \frac{10010}{7}$$

$$\mathbf{1073}: \quad 1073 = 29 \times 37 \qquad \mathbf{154}: \quad 154 = \frac{2002}{13} \qquad \mathbf{1443}: \quad 1443 = \frac{10101}{7}$$

$$\vdots$$

$$\mathbf{693}: \quad 693 = \frac{9009}{13}$$

The following are some examples showing how to use these special numbers:

999 Trick:

$$\begin{aligned} 333 \times \frac{1}{27} \times \frac{1}{37} &= \frac{1}{3} \times \mathbf{999} \times \frac{1}{27} \times \frac{1}{37} \\ &= \frac{1}{3} \times \frac{27 \cdot 37}{27 \cdot 37} \\ &= \frac{\mathbf{1}}{\mathbf{3}} \end{aligned}$$

1001 Trick:

$$\begin{aligned} 385 \times 13 &= 77 \times 5 \times 13 \\ &= \frac{\mathbf{1001}}{13} \times 5 \times 13 \\ &= 1001 \times 5 \\ &= \mathbf{5005} \end{aligned}$$

10101 Trick:

$$\begin{aligned} 1443 \times 56 &= \frac{\mathbf{10101}}{\mathbf{7}} \times 56 \\ &= 10101 \times \frac{56}{7} \\ &= 10101 \times 8 \\ &= \mathbf{80808} \end{aligned}$$

On the next page you'll find a wealth of problems to practice this trick.

Problem Set 2.1.5

1. $572 \times 21 =$ __________

2. $\frac{2}{37} \times 999 =$ __________

3. $33.67 \times 15 =$ __________

4. $715 \times 35 =$ __________

5. $3367 \times 21 =$ __________

6. $1073 \div 29 =$ __________

7. $715 \times 28 =$ __________

8. $429 \times 35 =$ __________

9. $63 \times 429 =$ __________

10. $1073 \div 37 =$ __________

11. $444 \times \frac{5}{37} =$ __________

12. $63 \times 572 =$ __________

13. $143 \times 49 = 1001\times$ __________

14. $29 \times 37 =$ __________

15. $42 \times 715 =$ __________

16. $715 \times 98 =$ __________

17. $27 \times 37 =$ __________

18. $715 \times 77 =$ __________

19. $\mathbf{105} \times \mathbf{715} =$ __________

20. $\mathbf{429} \times \mathbf{357} =$ __________

21. $14 \times 715 =$ __________

22. $42 \times 429 =$ __________

23. $21 \times 336.7 =$ __________

24. $36 \times 3.367 =$ __________

25. $715 \times 49 =$ __________

26. $33.67 \times 27 =$ __________

27. $707 \times 715 =$ __________

28. $429 \times 21 =$ __________

29. $\mathbf{336.7} \times \mathbf{3.3} =$ __________

30. $\mathbf{707} \times \mathbf{429} =$ __________

31. $385 \times 13 =$ __________

32. $111 \times \frac{7}{27} =$ __________

33. $539 \times 13 =$ __________

34. $666 \times \frac{2}{37} =$ __________

35. (*) $\frac{5}{37} \times 5548 =$ __________

36. $333 \times \frac{1}{27} \times \frac{1}{37} =$ __________

37. $462 \times 13 =$ __________

38. $999 \times \frac{7}{27} \times \frac{7}{37} =$ __________

39. $6006 \div 462 =$ __________

40. $444 \times \frac{4}{37} =$ ______

41. $770 \times 13 =$ ______

42. $888 \times \frac{4}{37} =$ ______

43. $666 \times \frac{16}{27} \times \frac{24}{37} =$ ______

44. $143 \times 77 =$ ______

45. $\mathbf{143} \times \mathbf{63} =$ ______

46. $84 \times 429 =$ ______

47. $143 \times 49 =$ ______

48. $444 \times \frac{5}{37} =$ ______

49. $222 \times \frac{1}{27} =$ ______

50. $63 \times 143 =$ ______

51. $555 \times \frac{6}{37} =$ ______

52. $444 \times \frac{1}{27} =$ ______

53. $143 \times 77 =$ ______

54. $888 \times \frac{4}{37} =$ ______

55. $666 \times \frac{1}{27} =$ ______

56. $777 \times \frac{7}{37} =$ ______

57. $444 \times \frac{2}{27} =$ ______

58. $999 \times \frac{3}{37} =$ ______

59. $666 \times \frac{3}{27} =$ ______

60. $888 \times \frac{24}{27} =$ ______

61. $999 \times \frac{1}{27} =$ ______

62. $\mathbf{143} \times \mathbf{13} \times \mathbf{7} =$ ______

63. $\mathbf{666} \times \mathbf{\frac{18}{37}} =$ ______

64. $\mathbf{999} \times \mathbf{\frac{5}{27}} =$ ______

65. $1001 \times 25 = 143 \times$ ______

66. $3 \times 11 \times 13 \times 21 =$ ______

67. $3 \times 5 \times 7 \times 11 \times 13 =$ ______

2.1.6 Roman Numerals

The following are the roman numerals commonly tested on the exam:

I = 1	V = 5	X = 10	L = 50
C = 100	D = 500	M = 1000	

Knowing the above table and also the fact that you arrange the numerals in order from greatest to least ($M \to I$) with the exception of one rule: you can't put four of the same numerals consecutively. For example, to express 42 in roman numerals it would *not* be 42 =XXXXII, it would be 42 =XLII. To circumvent the problem of putting four of the same numerals consecutively, you use a method of "subtraction." Anytime a numeral of lesser value is placed in front of a numeral of greater value, you subtract from the larger numeral the small numeral. So in our case 40 is represented by XL$=50 - 10 = 40$. When converting numbers, it is best to think of the number as a sum of ones, tens, hundreds, etc... units). A good example of what I mean is to express 199 in roman numerals. The way you want to look at it is $199 = 100 + 90 + 9$ then express each one as a roman numeral. So 100 = C, 90 = XC, and 9 = IX, so 199 = **CXCIX**.

Problem Set 2.1.6

1. MMXLII = ____________
2. XLIV = ____________
3. MMIII = ____________
4. CXCIX = ____________
5. MDCLXVI = ____________
6. CDXLIV = ____________
7. CCLXXVII = ____________
8. **MCDLIX** = ____________
9. CMXCIX = ____________
10. MMCCXXII = ____________
11. CXI − CC = ____________
12. MD + DC = ____________
13. CM + XC + IX = ____________
14. DC − LX − VI = ____________
15. **XIII + MMIV=** ____________
16. MIII + MIV = ____________
17. MC + DL + XIV = ____________
18. MCXI + DLV = ____________
19. MMV − DCXLI = ____________
20. MMLIX − LIII = ____________
21. MCXI − DLV = ____________
22. CMIX − CDIV = ____________
23. MDXLV − XV = ____________
24. DCII ÷ IX = ____________
25. CCCLXXIV ÷ XI = ____________
26. CDI × V = ____________
27. CCLXXX ÷ XIV = ____________
28. MMV ÷ V = ____________
29. XXVII × CXI = ____________
30. **MI** × **XI** = ____________
31. MMVII × XXV = ____________
32. MCCLX ÷ XV = ____________
33. MMVI × XI = ____________
34. **CDIV** ÷ **XL** = ____________

2.1.7 Platonic Solids and Euler's Formula

The following is a list of important characteristics of Platonic Solids which are popularly asked on the test:

Platonic Solid	Face Polygons	# of Faces	# of Vertices	# of Edges
Tetrahedron	Triangles	4	4	6
Cube	Squares	6	8	12
Octahedron	Triangles	8	6	12
Dodecahedron	Pentagons	12	20	30
Icosahedron	Triangles	20	12	30

If you ever forget one of the characteristics of the solids but remember the other two, you can always use Euler's formula of: Faces + Vertices − Edges = 2 to get the missing value.

The following is a short problem set concerning Platonic Solids. For best practice, cover up the above table!

Problem Set 2.1.7

1. A dodecahedron has ________ vertices.
2. An icosahedron has ______ congruent faces.
3. The area of the base of a tetrahedron is 4 ft^2. The total surface area is ______ ft^2.
4. A tetrahedron has ________ vertices.
5. An octahedron has __________ edges.
6. A hexahedron has __________ faces.
7. A dodecahedron is a platonic solid with 30 edges and _________ vertices.
8. An octahedron has _________ vertices.
9. An icosahedron is a platonic solid with 30 edges and ____________ vertices.
10. **A dodecahedron is a platonic solid with 30 edges and __________ vertices.**

2.1.8 π and e Approximations

Using the standard approximations of: $\pi \approx 3.1$, $e \approx 2.7$, and $e^2 \approx 7.4$ lead to the beneficial results of:

$$\pi^2 \approx 10,\ e^3 \approx 20, \text{ and } \pi \cdot e \approx 8.5$$

Knowing these values, we can approximate various powers of e and π relatively simple and within the require margin of error of $\pm 5\%$. The following is an example where these approximations are useful:

$$\begin{aligned}(e \times \pi)^4 &= e^4 \times \pi^4 \\ &= e \cdot e^3 \cdot \left(\pi^2\right)^2 \\ &\approx e \cdot 20 \cdot 100 \\ &\approx e \cdot 2000 \\ &\approx \mathbf{5400}\end{aligned}$$

The following are more practice problems concerning these approximations:

Problem Set 2.1.8

1. (*) $2\pi^4 =$ ______
2. (*) $e^2 \times \pi^4 =$ ______
3. (*) $e^4 =$ ______
4. (*) $\pi^5 =$ ______
5. (*) $(e \times \pi)^4 =$ ______
6. (*) $\pi^5 + e^4 =$ ______
7. (*) $\pi^{\mathbf{3}} \times \mathbf{e^4} =$ ______
8. (*) $(3\pi)^4 =$ ______
9. (*) $(e + 1.3)^5 =$ ______
10. (*) $[(\pi - .2)(e + .3)]^3 =$ ______
11. (*) $(\pi + \mathbf{1.9})^{\mathbf{3}}(\mathbf{e} + \mathbf{2.3})^{\mathbf{3}} =$ ______
12. (*) $(4e)^3 =$ ______
13. (*) $\mathbf{e^4}\pi^{\mathbf{4}} =$ ______
14. (*) $\pi^e e^\pi =$ ______
15. (*) $(3\pi + 2e)^4 =$ ______
16. (*) $\pi^\pi \mathbf{e^e} =$ ______

2.1.9 Distance and Velocity Conversions

The following are important conversion factors for distance and velocities:

1 mile = 5280 ft.

1 mile = 1760 yd.

1 mile/hr = $\frac{22}{15}$ ft/s

1 ft/min = $\frac{1}{5}$ in/s

1 ft/hr = $\frac{1}{5}$ in/min

1 mile/hr = $\frac{88}{5}$ in/s

1 inch = 2.54 cm.

The following is a short problem set concerning these conversions. For best practice, cover up the above table!

Problem Set 2.1.9

1. 15 miles per hour = ______ feet per second.
2. 3.5 yards = ______ inches.
3. .375 of a foot = ______ in.
4. 48 inches per second = ______ ft/min.
5. 7.5 mph = ______ inches per second.
6. $12\frac{1}{2}\%$ of a mile = ______ yards.
7. 25% of a mile = ______ yards.
8. $\frac{1}{3}$ of a mile = ______ feet.

9. $\frac{3}{4}$ of 3 yards = ____________ inches.

10. **$\frac{2}{3}$ of a mile = ____________ ft.**

11. 10 feet = ____________ yards.

12. $83\frac{1}{3}$% of a foot = ____________ inches.

13. 30 mph = ____________ ft/sec.

14. 30 feet per minute = ______ feet per second.

15. 36 in/s = ____________ inches per minute.

16. 480 inches per minute = ____________ in/s.

17. 45 mph = ____________ ft/s.

18. 33 ft/s = ____________ mph.

19. **7.5 mph = ____________ ft/s.**

2.1.10 Conversion between Distance → Area, Volume

Students find linear conversions relatively simple (for example 1ft. = 12in.), however when asked to find how many cubic inches are in cubic feet, they want to revert back to the linear conversion, which is incorrect ($1\text{ft.}^3 \neq 12\text{in.}^3$). When converting between linear distance to areas and volumes you must square or cube the conversion factor, respectively. So in our example, we know that:

$$1\text{ft.} = 12\text{in.} \Longrightarrow 1\text{ft.}^3 = (12)^3\text{in.}^3 = \mathbf{1728}\text{in.}^3$$

Another example converting ft.^2 to yd.^2 is:

$$1\text{yd.} = 3\text{ft.} \Longrightarrow 1\text{yd.}^2 = (3)^2\text{ft.}^2 = \mathbf{9}\text{ft.}$$

Problem Set 2.1.10

1. 3 cubic yards = ____________ ft.^3

2. 1 cubic foot = ____________ cubic inches.

3. 9 square yards = ____________ square feet.

4. 432 square inches = ____________ ft.^2

5. 3 square yards = ____________ square feet.

6. 243 cubic feet = ____________ cubic yards.

7. 3 cubic feet = ____________ cubic inches.

8. 4320 cubic inches = ____________ cubic feet.

9. 1 square meter = ______ square centimeters.

10. 12 square feet = ____________ square yards.

11. 216 square inches = ____________ square feet.

12. 1728 cubic inches = ____________ cubic feet.

13. **$1\frac{1}{3}$ cubic yards = ____________ cubic feet.**

14. 2 cubic feet = ____________ cubic inches.

15. 5 square decameters = ______ square meters.

2.1.11 Fluid and Weight Conversions

The following are important fluid conversions. Although some conversions can be made from others (for example, the amount of cups in a gallon doesn't need to be explicitly stated, but it would be helpful to have it memorized so you don't have to multiply how many quarts in a gallon, how many pints in a quart, and how many cups in a pint), it is recommended that everything in the table should be memorized:

1 gallon = 4 quarts

1 quart = 2 pints

1 pint = 2 cups

1 gallon = 16 cups

1 gallon = 128 oz.

1 cup = 8 oz.

1 tbsp. = .5 oz.

1 tsp. = $\frac{1}{6}$ oz.

1 gallon = 231 in^3

1 pound = 16 oz.

1 ton = 2000 lbs.

Problem Set 2.1.11

1. 1 quart = ________ cups
2. 1 quart = ________ ounces
3. 3 pints = ________ ounces
4. 3 gallons = ________ cubic inches
5. $\frac{2}{3}$ gallon = ________ cubic inches
6. $1\frac{1}{3}$ gallon = ________ cubic inches
7. 75% of 1 gallon = ________ ounces
8. 256 ounces = ________ pounds
9. **750 pounds** = ________ **% of a ton**
10. 75% of a gallon = ________ pints
11. $12\frac{1}{2}$% of a pint = ________ ounces
12. 4 pints is what % of a gallon: ________
13. 2 quarts is what % of a pint: ________
14. 6 tablespoons is ________ % of a cup
15. 9 cups is what % of a quart: ________
16. A quart is what % of a cup: ________
17. 2541 cubic inches = ________ gallons
18. 3 pints is what % of a cup: ________
19. 3 pints is what % of a gallon: ________
20. **5 gallons** = ________ **cubic inches**
21. 32 ounces = ________ pints
22. 3.5 pints = ________ quarts
23. 2.5 pints = ________ cups
24. 37.5% of a gallon is ________ pints

25. 62.5% of a gallon is ____________ quarts

26. **87.5% of a gallon is ____________ ounces**

27. 16 ounces is what part of a gallon: ________

28. 1 gallon = ________________ cubic inches

29. $\frac{3}{11}$ of a gallon = ____________ cubic inches

30. $\frac{3}{8}$ of a quart = ________________ ounces

31. $\frac{7}{11}$ of a gallon = ____________ cubic inches

32. 3 quarts and 2 pints = ____________ ounces

33. **7 quarts and 6 pints =________ gallons**

2.1.12 Celsius to Fahrenheit Conversions

These types of problems used to *always* be on the Number Sense tests in the early 1990's but have since been noticeably absent until recently. Here are the conversion factors:

Fahrenheit → Celcius: $C = \frac{5}{9}(F - 32)$

Celcius → Fahrenheit: $F = \frac{9}{5}C + 32$

There *is* a shortened trick for converting Celsius to Fahrenheit:

1. Double the given temperature in Celsius.
2. Move the decimal over to the left one and subtract that from the doubled number.
3. Add 32 to that result to get the answer.

Using this technique, lets convert 20° Celsius to Fahrenheit:

$$20°\text{C} \Rightarrow 40 - 4 = 36 \Rightarrow 36 + 32 = \mathbf{68°F}$$

A couple of important degrees which pop-up frequently that are fit for memorization are:
32°F = 0°C, 212°F = 100°C, and −40°F = −40°C.

Do the following conversions:

Problem Set 2.1.12

1. 25° C = ____________________ ° F
2. −40° C = ____________________ ° F
3. 98.6° F = ____________________ ° C

2.2 Formulas

The following are handy formulas which, when mastered, will lead to solving a large handful of problems.

2.2.1 Sum of Series

The following are special series who's sums should be memorized:

Sum of the First m Integers

$$\sum_{n=1}^{m} n = 1+2+3+\cdots+m = \frac{m\cdot(m+1)}{2}$$

Example:

$$1+2+3\cdots+11 = \frac{11\cdot 12}{2} = \mathbf{66}$$

Sum of the First m Odd Integers

$$\sum_{n=1}^{m} 2n-1 = 1+3+5+\cdots+(2m-1) = \left(\frac{(2m-1)+1}{2}\right)^2 = m^2$$

Example:

$$1+3+5+\cdots+15 = \left(\frac{15+1}{2}\right)^2 = 8^2 = \mathbf{64}$$

Sum of the First m Even Numbers

$$\sum_{n=1}^{m} 2n = 2+4+6+\cdots+2m = m\cdot(m+1)$$

Example:

$$2+4+6+\cdots+22 = \frac{22}{2}\cdot\left(\frac{22}{2}+1\right) = 11\cdot 12 = \mathbf{132}$$

Sum of First m Squares

$$\sum_{n=1}^{m} n^2 = 1^2+2^2+\cdots+m^2 = \frac{m\cdot(m+1)\cdot(2m+1)}{6}$$

Example:

$$1^2+2^2+\cdots+10^2 = \frac{10\cdot(10+1)\cdot(2\cdot 10+1)}{6} = 35\cdot 11 = \mathbf{385}$$

Sum of the First m Cubes

$$\sum_{n=1}^{m} n^3 = 1^3+2^3+\cdots+m^3 = \left(\frac{m\cdot(m+1)}{2}\right)^2$$

Example:

$$1^3+2^3+3^3+\cdots+10^3 = \left(\frac{10\cdot 11}{2}\right)^2 = 55^2 = \mathbf{3025}$$

Sum of the First m Alternating Squares

$$\sum_{n=1}^{m} (-1)^{n+1} n^2 = 1^2-2^2+3^2-\cdots\pm m^2 = \pm\frac{m\cdot(m+1)}{2}$$

Examples:

$$1^2-2^2+3^2-\cdots+9^2 = \frac{9\cdot 10}{2} = \mathbf{45}$$

$$1^2-2^2+3^2-\cdots-12^2 = -\frac{12\cdot 13}{2} = -\mathbf{78}$$

Sum of a General Arithmetic Series

$$\sum_{i=1}^{m} a_i = a_1+a_2+a_3+\cdots+a_m = \frac{(a_1+a_m)\cdot m}{2}$$

To find the number of terms: $m = \dfrac{a_m - a_1}{d} + 1$

Where d is the common difference.

Example:
$8 + 11 + 14 + \cdots + 35 =$

$m = \dfrac{35 - 8}{3} + 1 = 10$

So $\sum = \dfrac{(8+35)\cdot 10}{2} = 43 \cdot 5 = \mathbf{215}$

<u>Sum of an Infinite Geometric Series</u>

$$\sum_{n=0}^{\infty} a_1 \cdot (d)^n = a_1(1 + d + d^2 + \cdots) = \frac{a_1}{1-d}$$

Where d is the common ratio with $|d| < 1$ and a_1 is the first term in the series.

Examples:
$3 + 1 + \dfrac{1}{3} + \cdots = \dfrac{3}{1 - \frac{1}{3}} = \dfrac{3}{\frac{2}{3}} = \mathbf{\dfrac{9}{2}}$

$4 - 2 + 1 - \dfrac{1}{2} + \cdots = \dfrac{4}{1 - \left(\frac{-1}{2}\right)} = \dfrac{4}{\frac{3}{2}} = \mathbf{\dfrac{8}{3}}$

<u>Special Cases: Factoring</u>

Sometimes simple factoring can lead to an easier calculation. The following are some examples:

$$\begin{aligned} 3 + 6 + 9 + \cdots + 33 &= 3 \cdot (1 + 2 + \cdots + 11) \\ &= 3\left(\frac{11 \cdot 12}{2}\right) \\ &= 18 \cdot 11 = \mathbf{198} \end{aligned}$$

$$\begin{aligned} 11 + 33 + 55 + \cdots + 99 &= 11 \cdot (1 + 3 + 5 + \cdots + 9) \\ &= 11 \cdot \left(\frac{1+9}{2}\right)^2 \\ &= 11 \cdot 25 = \mathbf{275} \end{aligned}$$

Another important question involving sum of integers are word problems which state something similar to: The sum of three consecutive odd numbers is 129, what is the largest of the numbers?

In order to solve these problems it is best to know what you are adding. You can represent the sum of the three odd numbers by: $(n-2) + n + (n+2) = 129$. From this you can see that if you divide the number by 3, you will get that the *middle* integer is 43, thus making the largest integer $43 + 2 = \mathbf{45}$.

Here is another example problem: The sum of four consecutive even numbers is 140, what is the smallest?

For this one you can represent the sum by $(n-2) + (n) + (n+2) + (n+4) = 140$, so dividing the number by 4 will get you the integer *between* the second and third even number. So $140 \div 4 = 35$, so the two middle integers are 34 and 36, making the smallest integer **32**.

So from this we learned that you can divide the sum by the number of consecutive integers you are adding, and if the number of terms are odd, you get the middle integer, and if the number of terms are even, you get the number between the two middle integers.

The following are some more practice problems concerning the sum of series:

Problem Set 2.2.1

1. $2+4+6+8+\cdots+22=$ __________

2. $1+2+3+4+\cdots+21=$ __________

3. $1+3+5+7+\cdots+25=$ __________

4. The 25^{th} term of $3, 8, 13, 18, \cdots$: __________

5. $6+4+\frac{8}{3}+\frac{16}{9}+\cdots=$ __________

6. $2+4+6+8+\cdots+30=$ __________

7. $1+3+5+7+\cdots+19=$ __________

8. $\frac{3}{5}-\frac{3}{10}+\frac{3}{20}-\cdots=$ __________

9. The 20^{th} term of $1, 6, 11, 16, \cdots$: __________

10. $22+20+18+16+\cdots+2=$ __________

11. $1+3+5+\cdots+17=$ __________

12. $2+4+6+\cdots+44=$ __________

13. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=$ __________

14. $1^3+2^3+3^3+\cdots+6^3=$ __________

15. $6+12+18+\cdots+66=$ __________

16. $3+5+7+9+\cdots+31=$ __________

17. $2+1+\frac{1}{2}+\frac{1}{4}+\cdots=$ __________

18. $-\frac{3}{2}+\frac{1}{2}-\frac{1}{6}+\frac{1}{18}-\cdots=$ __________

19. $3+5+7+9+\cdots+23=$ __________

20. $\frac{4}{7}+\frac{8}{49}+\frac{16}{343}+\cdots=$ __________

21. $1+4+7+\cdots+25=$ __________

22. $4+1+\frac{1}{4}+\frac{1}{16}+\cdots=$ __________

23. $2+\frac{2}{5}+\frac{2}{25}+\cdots=$ __________

24. $3+9+15+21+\cdots+33=$ __________

25. $7+14+21+28+\cdots+77=$ __________

26. The 11^{th} term in the arithmetic sequence $12, 9.5, 7, 4.5\cdots$ is: __________

27. $4+8+12+\cdots+44=$ __________

28. $8+16+24+32+\cdots+88=$ __________

29. $5^1-5^0+5^{-1}-5^{-2}+\cdots=$ __________

30. $(x)+(x+2)+(x+4)=147$, then $(x)+(x+4)=$

31. $6+12+18+24+\cdots+36=$ __________

32. $3+8+13+18+\cdots+43=$ __________

33. $1^2+2^2+3^2+4^2+5^2+6^2=$ __________

34. $5+1+\frac{1}{5}+\frac{1}{25}+\cdots=$ __________

35. $\frac{2}{3} + \frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \cdots =$ ______

36. $3 + 5 + 7 + 9 + \cdots + 31 =$ ______

37. $7 + 14 + 21 + 28 + 35 + 42 =$ ______

38. $8 + 10 + 12 + \cdots + 20 =$ ______

39. $10 + 15 + 20 + 25 + \cdots 105 =$ ______

40. $8 + 4 + 2 + 1 + \cdots =$ ______

41. $4 + 8 + 12 + 16 + \cdots + 44 =$ ______

42. (*) $1^3 + 2^3 + 3^3 + \cdots + 6^3 =$ ______

43. $\mathbf{6 + 12 + 18 + 24 + \cdots + 66} =$ ______

44. $2 + 6 + 10 + \cdots + 42 =$ ______

45. $1^3 - 2^3 + 3^3 - 4^3 + 5^3 =$ ______

46. $3 + 1\frac{1}{2} + \frac{3}{4} + \cdots =$ ______

47. $14 + 28 + 42 + 56 + 70 + 84 =$ ______

48. $\mathbf{121 + 110 + 99 + \cdots + 11} =$ ______

49. $2 + 9 + 16 + 23 + \cdots + 44 =$ ______

50. $13 + 26 + 39 + 52 + 65 + 78 =$ ______

51. $\mathbf{36 + 32 + 28 + \cdots + 12} =$ ______

52. $88 + 80 + 72 + \cdots + 8 =$ ______

53. The sum of 3 consecutive odd integers is 105. The largest integer: ______

54. $4^1 - 4^0 + 4^{-1} - 4^{-2} + \cdots =$ ______

55. (*) $(1 + 2 + 3 + \cdots + 29)^2 =$ ______

56. (*) $1^3 + 2^3 + 3^3 + \cdots + 11^3 =$ ______

57. $\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \cdots + 1\frac{4}{5} + 2 =$ ______

58. $(6^3 + 4^3 + 2^3) - (5^3 + 3^3 + 1^3) =$ ______

59. $3 - 1 - \frac{1}{3} - \frac{1}{9} - \frac{1}{27} - \cdots =$ ______

60. $\mathbf{\frac{1}{3} + \frac{2}{3} + 1 + 1\frac{1}{3} + \cdots + 2\frac{1}{3}} =$ ______

61. $3^3 - 4^3 - 2^3 + 5^3 =$ ______

62. $\mathbf{6 - 1 - \frac{1}{6} - \frac{1}{36}} - \cdots =$ ______

63. $2 + 5 + 8 + \cdots + 20 =$ ______

64. (*) $1^3 + 2^3 + 3^3 + \cdots + 13^3 =$ ______

65. $\frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \cdots =$ ______

66. $\frac{1}{4} + \frac{3}{4} + \frac{5}{4} + \cdots + \frac{15}{4} =$ ______

67. (*) $(3 + 6 + 9 + \cdots + 30)^2 =$ ______

68. (*) $1^3 + 2^3 + 3^3 + \cdots + 8^3 =$ ______

2.2.2 Fibonacci Numbers

It would be best to have the Fibonacci numbers memorized up to F_{15} because they crop up every now and then on the number sense test. In case you are unaware, the fibonacci sequence follows the recursive relationship of $F_n = F_{n-1} + F_{n-2}$. The following is a helpful table:

$F_1 = 1$	$F_2 = 1$	$F_3 = 2$	$F_4 = 3$
$F_5 = 5$	$F_6 = 8$	$F_7 = 13$	$F_8 = 21$
$F_9 = 34$	$F_{10} = 55$	$F_{11} = 89$	$F_{12} = 144$
$F_{13} = 233$	$F_{14} = 377$	$F_{15} = 610$	

The most helpful formula to memorize concerning Fibonacci Numbers is the the sum of the first n Fibonacci Numbers is equal to $F_{n+2} - 1$.

A common problem asked on the latter parts of the number sense test is:

Find the sum of the first eight terms of the Fibonacci sequence $2, 5, 7, 12, 19, \ldots$.

Now there are two methods of approach for doing this. The first requires knowledge of large Fibonacci numbers:

Method 1:

The sum of a the first n-terms of a general Fibonacci sequence $a, b, a+b, a+2b, 2a+3b, \ldots$ is

$$\sum = a \cdot (F_{n+2} - 1) + d \cdot (F_{n+1} - 1)\,.\ \text{Where } d = (b - a)$$

So for our example:

$$\sum = 2 \cdot (F_{10} - 1) + (5 - 2) \cdot (F_9 - 1) = 2 \cdot 54 + 3 \cdot 33 = 108 + 99 = \mathbf{207}$$

Method 2:

The other method of doing this sum requires memorization of knowing a formula for each particular sum. The following is a list of the sums of a general Fibonacci sequence $a, b, a+b, a+2b, 2a+3b, \ldots$ for 1-12 terms (the number of terms which have been on the exam):

n	Fibonacci Number	Sum of First F_n Numbers	Formula
1	a	a	$a = F_1$
2	b	$a+b$	$a+b = F_3$
3	$a+b$	$2a+2b$	$2(a+b) = 2 \cdot F_3$
4	$a+2b$	$3a+4b$	$4(a+b) - a = 4 \cdot F_3 - a$
5	$2a+3b$	$5a+7b$	$7(a+b) - 2a = 7 \cdot F_3 - 2a$
6	$3a+5b$	$8a+12b$	$4(2a+3b) = 4 \cdot F_5$
7	$5a+8b$	$13a+20b$	$4(3a+5b) + a = 4 \cdot F_6 + a$
8	$8a+13b$	$21a+33b$	$7(3a+5b) - 2b = 7 \cdot F_6 - 2b$
9	$13a+21b$	$34a+54b$	$7(5a+8b) - (a+2b) = 7 \cdot F_7 - F_4$
10	$21a+34b$	$55a+88b$	$11(5a+8b) = 11 \cdot F_7$
11	$34a+55b$	$89a+143b$	$11(8a+13b) + a = 11 \cdot F_8 + a$
12	$55a+89b$	$144a+232b$	$18(8a+13b) - b = 18 \cdot F_8 - b$

So in our case, we are summing the first 8 terms, which is just $7 \cdot F_6 - 2b$, where F_6 represents the sixth term in the sequence of $2, 5, 7, 12, 19, \ldots$ (which is 31), so $7 \cdot 31 - 2 \cdot 5 = 217 - 10 = \mathbf{207}$.

So in solving it this way you have to calculate what the 6^{th} term in the sequence as well as knowing the formula. Usually it will be required to calculate a middle term in the sequence, and then apply the formula.

These type of questions are usually computationally intense, so it is recommended to skip them and come back to work on them after the completion of all other problems. The following are some more practice problems:

Problem Set 2.2.2

1. The sum of the first 11 terms of the Fibonacci Sequence $2, 4, 6, 10, 16, 26, \ldots$: ______
2. The sum of the first 9 terms of the Fibonacci Sequence $3, 5, 8, 13, 21, \ldots$: ________
3. The sum of the first 9 terms of the Fibonacci Sequence $4, 7, 11, 18, 29, \ldots$: _______
4. The sum of the first 10 terms of the Fibonacci Sequence $4, 5, 9, 14, 23, \ldots$: ________
5. The sum of the first 11 terms of the Fibonacci Sequence $1, 5, 6, 11, 17, 28, \ldots$: ______
6. The sum of the first 12 terms of the Fibonacci Sequence $1, 2, 3, 5, 8, 13, 21, \ldots$: _____
7. The sum of the first 11 terms of the Fibonacci Sequence $2, 5, 7, 12, 19, 31, \ldots$: ______
8. **The sum of the first 9 terms of the Fibonacci Sequence $\mathbf{3, 8, 11, 19, \ldots}$:** ______

9. The sum of the first 9 terms of the Fibonacci Sequence $2, 4, 6, 10, 16, \ldots$: ________

10. The sum of the first 9 terms of the Fibonacci Sequence $1, 5, 6, 11, 17, \ldots$: ________

11. The sum of the first 9 terms of the Fibonacci Sequence $3, 5, 8, 13, 21, \ldots$: ________

12. The sum of the first 9 terms of the Fibonacci Sequence $-3, 4, 1, 5, 6, \ldots$: ________

13. The sum of the first 9 terms of the Fibonacci Sequence $1, 1, 2, 3, 5, \ldots$: ________

14. The sum of the first 9 terms of the Fibonacci Sequence $-3, 2, -1, 1, 0, \ldots$: ________

15. **The sum of the first 9 terms of the Fibonacci Sequence** $\mathbf{1, 3, 4, 7, 11, \ldots}$: ________

16. $1 + 1 + 2 + 3 + 5 + 8 + \cdots + 55 =$ ________

17. $1 + 3 + 4 + 7 + 11 + 18 + \cdots + 123 =$ ________

18. $3 + 6 + 9 + 15 + 24 + \cdots + 267 =$ ________

19. $\mathbf{4 + 6 + 10 + 16 + 26 + \cdots + 288 =}$ ________

2.2.3 Integral Divisors

The following are formulas dealing with integral divisors. On all the formulas, it is necessary to prime factorize the number of interest such that: $n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots p_n^{e_n}$.

Number of Prime Integral Divisors
Number of prime integral divisors can be found by simply prime factorizing the number, and count how many distinct prime numbers $(p_1, p_2, \ldots)$ you have in it's representation.

Example:
Find the number of prime integral divisors of 120.
$120 = 2^3 \cdot 3 \cdot 5 \Rightarrow \#$ of prime divisors $= (1 + 1 + 1) = \mathbf{3}$

Number of Integral Divisors
Number of Integral Divisors $= (e_1 + 1) \cdot (e_2 + 1) \cdot (e_3 + 1) \cdots (e_n + 1)$

Example:
Find the number of integral divisors of 48.
$48 = 2^4 \cdot 3^1 \Rightarrow (4 + 1) \cdot (1 + 1) = \mathbf{10}$

Sum of the Integral Divisors

$$\sum = \frac{p_1^{e_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{e_2+1} - 1}{p_2 - 1} \cdots \frac{p_n^{e_n+1} - 1}{p_n - 1}$$

Example:
Find the sum of the integral divisors of 36.
$36 = 2^2 \cdot 3^2$

$$\sum = \frac{2^3 - 1}{2 - 1} \cdot \frac{3^3 - 1}{3 - 1} = \frac{7}{1} \cdot \frac{26}{2} = 7 \cdot 13 = \mathbf{91}$$

Number of Relatively Prime Integers less than n

Number of Relatively Prime $= (p_1 - 1) \cdot (p_2 - 1) \cdots (p_n - 1) \cdot (p_1^{e_1 - 1}) \cdot (p_2^{e_2 - 1}) \cdots (p_n^{e_n - 1})$

or

Number of Relatively Prime $= \dfrac{p_1 - 1}{p_1} \cdot \dfrac{p_2 - 1}{p_2} \cdots \dfrac{p_n - 1}{p_n} \times n$

Both techniques are relatively (no pun intended) quick and you should do whichever you feel comfortable with. Here is an example to display both method:

Example:
Find the number of relatively prime integers less than 20.
$20 = 2^2 \cdot 5$

$\#$ of Relatively Prime Integers $= (2-1) \cdot (5-1) \cdot (2^{2-1}) \cdot (5^{1-1}) = 4 \cdot 2 = \mathbf{8}$

or

$\#$ of Relatively Prime Integers $= \dfrac{1}{2} \cdot \dfrac{4}{5} \times 20 = \mathbf{8}$

Sum of Relatively Prime Integers less than n

$\sum = (\#$ of Relatively Prime Integers$) \times \dfrac{n}{2}$

Example:
Find the sum of the relatively prime integers less than 24.
$24 = 2^3 \cdot 3$

$\#$ of Relatively Prime Integers $= \dfrac{1}{2} \cdot \dfrac{2}{3} \times 24 = 8$

$\sum = 8 \times \dfrac{24}{2} = 8 \cdot 12 = \mathbf{96}$

We should introduce a distinction between proper and improper integral divisors here. A proper integral divisor is any positive integral divisor of the number excluding the number itself. So for example, the number 14 has 4 total integral divisors $(1, 2, 7, 14)$, but only 3 proper integral divisors $(1, 2, 7)$. Some number sense questions will ask for the sum of proper integral divisors or the number of proper integral divisors of number. When those are asked, you need to be aware to *exclude* the number itself from those calculations. For example, the sum of the proper integral divisors of $22 = 3 \times 12 - 22 = 36 - 22 = \mathbf{14}$.

In addition, on the questions asking for the number of co-prime (or relatively prime) within a range of values, it is best to calculate the total number of relatively prime integers and then start excluding ones that are out of range. For example, to calculate the number of integers greater than 3 which are co-prime to 20 you would find the number of co-prime integers less than 20 which is $(2-1)(5-1)(2^{(2-1)})(5^{1-1}) = 8$ then you can exclude the numbers 1 and 3. So the number of integers greater than three which are co-prime to 20 would be $8 - 2 = 6$. The quickest way of finding whether or not an integer is co-prime to another integer, is to put it in fraction form and see if the fraction is reducible. For example, 3 is co-prime to 20 because $\dfrac{3}{20}$ is irreducible.

With integral divisor problems it is best to get a lot of practice so that better efficiency can be reached. The following are some sample practice problems:

Problem Set 2.2.3

1. 30 has how many positive prime integral divisors: ____________

2. 36 has how many positive integral divisors: ____________

3. The sum of the positive integral divisors of 42 is: ________

4. The number of prime factors of 210 is: ____

5. The number of positive integral divisors of 80 is: ________

6. The number of positive integral divisors of $2^4 \times 5$ is: ________

7. The sum of the distinct prime factors of 75 total: ________

8. The number of positive integral divisors of 96 is: ________

9. **The number of positive integral divisors of 100 is:** ________

10. The sum of the positive integral divisors 48 is: ________

11. The sum of the proper positive integral divisors of 24 is: ________

12. The sum of the positive integral divisors of 28 is: ________

13. The number of positive integral divisors of $6^1 \times 3^2 \times 2^3$: ________

14. The sum of the proper positive integral divisors of 30 is: ________

15. How many positive integral divisors does 81 have: ________

16. How many positive integral divisors does 144 have: ________

17. The sum of the positive integral divisors $3 \times 5 \times 7$ is: ________

18. The number of positive integral divisors of $6^5 \times 4^3 \times 2^1$: ________

19. The sum of the positive integral divisors of 20 is: ________

20. The number of positive integral divisors of 24 is: ________

21. The sum of the positive integral divisors of 28 is: ________

22. The number of positive integral divisors of $2^3 \times 3^4 \times 4^5$: ________

23. The number of positive integral divisors of 64 is: ________

24. **The sum of the proper positive integral divisors of 36 is:** ________

25. The number of positive integral divisors of $2^4 \times 3^6 \times 5^{10}$ is: ________

26. The number of positive integral divisors of $5^3 \times 3^2 \times 2^1$: ________

27. How many positive integers less than 90 are relatively prime to 90: ________

28. Sum of the proper positive integral divisors of 18 is: ________

29. The sum of the positive integers less than 18 that are relatively prime to 18: ________

30. The number of positive integral divisors of $12 \times 3^3 \times 2^4$: ________

31. **How many positive integers less than $\mathbf{16 \times 25}$ are relatively prime to $\mathbf{16 \times 25}$:** ________

32. How many integers between 30 and 3 are relatively prime to 30: ________

33. How many positive integer less than 9×8 are relatively prime to 9×8: ________

34. How many integers between 1 and 20 are relatively prime to 20: ________

35. **The number of positive integral divisors of $\mathbf{50 \times 5^4 \times 2^3}$:** ________

36. **The sum of the positive integral divisors of 48:** ________

2.2.4 Number of Diagonals of a Polygon

The formula for the number of diagonals in a polygon is derived by noticing that from each of the n vertices in an n-gon, you can draw $(n-3)$ diagonals creating $n \cdot (n-3)$ diagonals, however each diagonal would be drawn twice, so the total number of diagonals is:

$$\#\text{ of Diagonals} = \frac{n \cdot (n-3)}{2}$$

As an example lets look at the number of diagonals in a hexagon:

$$\#\text{ of Diagonals in a Hexagon} = \frac{6 \cdot 3}{2} = 9$$

Here are some problems for you to practice this formula:

Problem Set 2.2.4

1. The number of diagonals a 5-sided regular polygon has: ________

2. If a regular polygon has 27 distinct diagonals, then it has how many sides: ________

3. A pentagon has how many diagonals: ________

4. A nonagon has how many diagonals: ________

5. An octagon has how many diagonals: ________

6. **A decagon has how many diagonals:** ________

7. A rectangle has how many diagonals: ________

8. A septagon has how many diagonals: ________

2.2.5 Exterior/Interior Angles

When finding the exterior, interior, or the sum of exterior or interior angles of a regular n-gon, you can use the following formulas:

Sum of Exterior Angles:	$360°$
Exterior Angle:	$\dfrac{360°}{n}$
Interior Angle:	$180° - \dfrac{360°}{n} = \dfrac{180°(n-2)}{n}$
Sum of Interior Angles:	$n \cdot \dfrac{180°(n-2)}{n} = 180(n-2)$

If you were to only remember one of the above formulas, let it be that the sum of the exterior angles of every regular polygon be equal to 360. From there you can derive the rest relatively swiftly (although it is *highly* recommended that you have all formulas memorized).

Example: Find the sum of the interior angles of an octagon.
Solution: $\sum = 180(8-2) = \mathbf{1080}$.

In order to find the interior angle from the exterior angle, you used the fact that they are supplements. Both supplements and complements of angles appear on the number sense test every now and then, so here are their definitions:

Complement of $\theta = 90^\circ - \theta$
Supplement of $\theta = 180^\circ - \theta$

Here are some practice problems on both exterior/interior angles as well as supplement/complement:

Problem Set 2.2.5

1. A regular nonagon has an interior angle of: __________
2. An interior angle of a regular pentagon has a measure of: __________
3. The supplement of an interior angle of a regular octagon measures: __________
4. The angles in a regular octagon total: __________
5. The measure of an interior angle of a regular hexagon measures: __________
6. The sum of the angles in a regular decagon is: __________
7. The supplement of a 47° angle is: __________
8. The sum of the interior angles of a regular pentagon is: __________

2.2.6 Triangular, Pentagonal, etc... Numbers

We are all familiar with the concept of square numbers $1, 4, 9, 16, \ldots, n^2$ and have a vague idea of how they can be viewed geometrically (n^2 can be represented by n rows of dots by n columns of dots). This same concept of translating "dots to numbers" can extend to any regular polygon. For example, the idea of a triangular number is the amount of dots which can be arranged into an equilateral triangle $(1, 3, 6, \ldots)$. The following are formulas for these "geometric" numbers:

$$\text{Triangular: } T_n = \frac{n(n+1)}{2}$$

$$\text{Square: } S_n = \frac{n(2n-0)}{2} = n^2$$

$$\text{Pentagonal: } P_n = \frac{n(3n-1)}{2}$$

$$\text{Hexagonal: } H_n = \frac{n(4n-2)}{2}$$

$$\text{Heptagonal: } E_n = \frac{n(5n-3)}{2}$$

$$\text{Octagonal: } O_n = \frac{n(6n-4)}{2}$$

$$\text{M-Gonal: } M_n = \frac{n[(M-2)n-(M-4)]}{2}$$

As one can see, only the last formula is necessary for memorization (all the others can be derived from that one).

Some other useful formulas:

Sum of Consecutive Triangular Numbers: $T_{n-1} + T_n = n^2$

Sum of First m Triangular Numbers: $\sum_{n=1}^{m} T_n = T_1 + T_2 + \cdots + T_m = \frac{m(m+1)(m+2)}{6}$

Sum of the Same Triangular and Pentagonal Numbers: $T_n + P_n = 2n^2$

Examples:

1. The 6th Triangular Number? $\frac{6(6+1)}{2} = \mathbf{21}$
2. The 4th Octagonal Number? $\frac{4(6 \cdot 4 - 4)}{2} = \frac{4 \cdot 20}{2} = \mathbf{40}$
3. The 5th Pentagonal Number? $\frac{5(3 \cdot 5 - 1)}{2} = \frac{5 \cdot 14}{2} = \mathbf{35}$
4. The Sum of the 6th and 7th Triangular Numbers? $7^2 = \mathbf{49}$

Problem Set 2.2.6

1. The 7^{th} pentagonal number: ____________
2. The 4^{th} octagonal number: ____________
3. The 5^{th} pentagonal number: ____________
4. The 8^{th} octagonal number: ____________
5. **The $\mathbf{12^{th}}$ hexagonal number:** ________
6. The 7^{th} septagonal number is: ____________
7. **The $\mathbf{5^{th}}$ pentagonal number is:** ________
8. The 6^{th} pentagonal number is: ____________
9. The 5^{th} hexagonal number is: ____________
10. The 11^{th} triangular number is: ____________
11. The 12^{th} triangular number is: ____________
12. **The $\mathbf{6^{th}}$ hexagonal number is:** ________
13. The sum of the 5^{th} triangular and the 6^{th} triangular numbers: ____________
14. The sum of the 3^{rd} triangular and the 3^{rd} pentagonal numbers: ____________

2.2.7 Finding Sides of a Triangle

A popular triangle question gives two sides of a triangle and asks for the minimum/maximum value for the other side conforming to the restriction that the triangle is right, acute or obtuse. The two sets of formulas which will aid in solving these questions are:

Triangle Inequality: $a + b > c$

	Right Triangle:	$a^2 + b^2 = c^2$
Variations on the Pythagorean Theorem:	Acute Triangle:	$a^2 + b^2 > c^2$
	Obtuse Triangle:	$a^2 + b^2 < c^2$

If you don't have the Pythagorean relationships for acute/obtuse triangle memorized, the easiest way to think about the relationship on the fly is remembering that an equilateral triangle is acute so $a^2 + a^2 > a^2$.

Let's look at some examples:

An acute triangle has integer sides of $4, x,$and 9. What is the largest value of x?

Solution: Using the Pythagorean relationship we know: $4^2 + 9^2 > x^2$ or $97 > x^2$. Knowing this and the fact that x is an integer, we know that the largest value of x is **9**.

An acute triangle has integer sides of $4, x,$and 9. What is the smallest value of x?

Solution: For this we use the triangle inequality. We want 9 to be the largest side (so x would have to be less than 9), so apply the inequality knowing this: $4 + x > 9$ which leads to the smallest integer value of x is **6**

An obtuse triangle has integer sides of $7, x,$ and 8. What is the smallest value of x?

Solution: For this, we want the largest value in the obtuse triangle to be 8 then apply the Triangle Inequality: $7 + x > 8$ with x being an integer. This makes the smallest value of x to be **2**.

An obtuse triangle has integer sides of $7, x,$ and 8. What is the largest value of x?

Solution: Here, x is restricted by the Triangle Inequality (if we used the Pythagorean Theorem for obtuse, we would get an unbounded result for x: $7^2 + 8^2 < x^2$ makes x unbounded). So we know from that equation: $7 + 8 > x$ so the largest integer value for x is **14**.

Another important type of triangle problem involves being given one side of a right triangle and having to compute the other sides. For example, the sides of a right triangle are integers, one of its sides is 9, what is the hypotenuse?

Where this gets it's foundation is from the Pythagorean Theorem which states that $a^2 + b^2 = c^2$. If the smallest side is given (call it a), then we can express $a^2 = c^2 - b^2 = (c - b)(c + b)$. Now is where the trick comes into play. The goal becomes to find two numbers that when subtracted together from each other multiplied with them added to each other is the smallest side squared. When the smallest side squared gives an odd number (in our case 81 is odd), the goal is reduced considerably by thinking of taking consecutive integers (so $c - b = 1$) and $c + b = a^2$. The easiest way to find two consecutive integers whose sum is a third number is to divide, the third number by 2, and the integers straddle that mixed number. So in our case $9^2 = 81 \div 2 = 40.5$ so $b = 40$ and $c = \mathbf{41}$, and we're done. Let's look at another example:

The sides of a right triangle are integers, one of its sides is 11, what is the other side?

Solution: $11^2 = 121$ which is odd, so $121 \div 2 = 60.5$ so the other side is **60**.

Very seldom do they give you a side who's square is even. In that case let's look at the result:

The sides of a right triangle are integers, one of its sides is 10, what is the hypotenuse?

Solution: The easiest way of solving these problems is divide the number they give you by a certain amount to get an odd number, then perform the usual procedure on that odd number (outlined above), then when you get the results multiply each side by the number you originally divided by. Let's look at what happens in our example. So to get an odd number we must divide 10 by 2 to get 5. Now to find the other side/hypotenuse with smallest side given is 5 you do: $5^2 \div 2 = 12.5 \Rightarrow b = 12$ and $c = 13$. Now to get the correct side/hypotenuse lengths, we must multiply by what we originally divided by (2) so $b = 12 \cdot 2 = 24$ and $= 13 \cdot 2 = \mathbf{26}$. As you can see there are a couple of mores steps to this procedure, and you have to remember what you divided by at the beginning so you can multiply the side/hypotenuse by that same amount at the end.

There are some variations to this, say they tell you that the hypotenuse is 61 and ask for the smallest side. Since half of the smallest side squares is roughly the hypotenuse, you will be looking for squares who are near $61 \cdot 2 = 122$, so you know that $s = \mathbf{11}$.

In addition, there are some algebraic applications that frequently ask the same thing. For example, if it is given that $x^2 - y^2 = 53$ and asks you to solve for y. You do the same procedure: $(x + y)(x - y) = 53$, since 53 is odd, you are concerned with consecutive numbers adding up to 53, so $53 \div 2 = 26.5 \Rightarrow x = 27$ and $y = \mathbf{26}$.

Getting practice with these problems are critical so that you can immediately know which formula to apply and which procedure to follow. Complete the following:

Problem Set 2.2.7

1. An obtuse triangle has integral sides of $3, x$, and 7. The largest value for x is: ____
2. The sides of a right triangle are integers. If one leg is 9 then the other leg is: ________
3. x, y are positive integers with $x^2 - y^2 = 53$. Then y= ________________
4. A right triangle with integer sides has a hypotenuse of 113. The smallest leg is: ____
5. An acute triangle has integer side lengths of $4, 7$, and x. The smallest value for x is: ___
6. An acute triangle has integer side lengths of $4, 7$, and x. The largest value for x is: ____
7. **x,y are integers with $\mathbf{x^2 - y^2 = -67}$ then x is:** ____________
8. **An obtuse triangle has integer side lengths of $x, 7$, and 11. The smallest value of x is:** __________
9. $a^2 + b^2 = 113^2$ where $0 < a < b$ and a, b are integers. Then $a =$ __________
10. The sides of a right triangle are $x, 7$, and 11. If $x < 7$ and $x = a\sqrt{2}$ then $a =$ _
11. An acute triangle has integer sides of $2, 7$, and x. The largest value of x is: ________
12. An obtuse triangle has integer sides of 6,x, and 11. The smallest value of x is: ______
13. An acute triangle has integer sides of 7, 11, and x. The smallest value of x is: ________
14. **An obtuse triangle has integer sides of $8, 15$, and x. The smallest value of x is:** _
15. The sides of a right triangle are integral. If one leg is 13, find the length of the other leg: _
16. A right triangle has integer side lengths of $7, x$, and 25. Its area is: __________

2.2.8 Equilateral Triangle Formulas

Area of an Equilateral Triangle when knowing the side-length s:

$$\text{Area} = \frac{s^2 \cdot \sqrt{3}}{4}$$

Area of an Equilateral Triangle when knowing the height h:

$$\text{Area} = \frac{h^2 \cdot \sqrt{3}}{3}$$

Finding the height when given the side length s:

$$\text{Height} = \frac{s \cdot \sqrt{3}}{2}$$

Example:

An equilateral triangle's perimeter is 12. Its area is $4k \cdot \sqrt{3}$. What is k?

$$s = \frac{12}{3} = 4 \Rightarrow A = \frac{4^2 \cdot \sqrt{3}}{4} = 4\sqrt{3} \Rightarrow k = \mathbf{1}$$

Example:

An equilateral triangle has a height of 4, what is its side length?

$$h = 4 = \frac{\sqrt{3} \cdot s}{2} \Rightarrow s = \frac{4 \cdot 2}{\sqrt{3}} = \frac{\mathbf{8 \cdot \sqrt{3}}}{\mathbf{3}}$$

Here are some practice problems for this formula:

Problem Set 2.2.8

1. The sides of an equilateral triangle are $2\sqrt{3}$ cm, then its height is: ____________
2. The area of an equilateral triangle is $9\sqrt{3}\text{cm}^2$, then its side length is: ____________
3. If the area of an equilateral triangle is $3\sqrt{3}\text{ft}^2$ then its side length is: ____________
4. The height of an equilateral triangle is 12 in. Its area is $4k\sqrt{3}$, $k =$ ____________
5. The perimeter of an equilateral triangle is 12 cm. Its area is $k\sqrt{3}$ cm^2. $k =$ ____________
6. Find the perimeter of an equilateral triangle whose area is $9\sqrt{3}\text{cm}^2$: ____________
7. The area of an equilateral triangle is $3\sqrt{3}\text{in}^2$. Its height is: ____________
8. An equilateral triangle has an area of $27\sqrt{3}\text{cm}^2$. Its height is: ____________

2.2.9 Formulas of Solids

Usually basic formulas for spheres, cubes, cones, and cylinders are fair game for the Number Sense test. In order to solve these problems, memorize the following table:

Type of Solid	Volume	Surface Area
Cube	s^3	$6s^2$
Sphere	$\frac{4}{3}\pi r^3$	$4\pi r^2$
Cone	$\frac{1}{3}\pi r^2 h$	$\pi rl + \pi r^2$
Cylinder	$\pi r^2 h$	$2\pi rh$

(In the above formulas, s is the side-length, r is the radius, h is the height, and l is the slant height.)

In addition to knowing the above formulas, a couple of other ones are:

Face Diagonal of a Cube $= s\sqrt{2}$

Body Diagonal of a Cube $= s\sqrt{3}$

Problem Set 2.2.9

1. Find the surface area of a cube who's side length is 11 in.: __________
2. Find the surface area of a sphere who's radius is 6 in.: __________
3. If the radius of a sphere is tripled, then the volume is multiplied by: __________
4. The total surface area of a cube with an edge of 4 inches is: __________
5. A cube has a volume of 512 cm^2. The area of the base is: __________
6. A cube has a surface area of 216 cm^2. The volume of the cube is: __________
7. If the total surface area of a cube is 384 cm^2, then the volume of the cube is: ___
8. Find the volume of a cube with an edge of 12 cm.: __________
9. A tin can has a diameter of 8 and a height of 14. The volume is $k\pi, k =$ __________

2.2.10 Combinations and Permutations

For most, this is just a refresher on the definitions of Combinations (${}_n\mathrm{C}_k$) and Permutations (${}_n\mathrm{P}_k$):

$$ {}_n\mathrm{C}_k = \frac{n!}{k! \cdot (n-k)!} $$

$$ {}_n\mathrm{P}_k = \frac{n!}{(n-k)!} $$

Here is an example:

$$ {}_7\mathrm{C}_4 = \frac{7!}{4!(7-4)!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = \mathbf{35} $$

With combinations and permutations (and factorials in general) you want to look at ways of canceling factors from the factorial to ease in calculation. In addition, the following is a list of the factorials which should be memorized for quick access:

$3! = 6$	$4! = 24$	$5! = 120$	$6! = 720$
$7! = 5040$	$8! = 40320$	$9! = 362880$	$10! = 3628800$

Another often tested principle on Combinations is that:

$$_n\mathrm{C}_k \ = \ _n\mathrm{C}_{n-k}$$

The above will show up in the form of questions like this:

$$_5\mathrm{C}_2 \ = \ _5\mathrm{C}_k \rightarrow k = ?$$

Solution: Using the above formula, you know that $k = 5 - 2 = \mathbf{3}$.

Another often tested question on Combinations and Permutations is when you divide one by another:

$$\frac{_n\mathrm{C}_k}{_n\mathrm{P}_k} = \frac{1}{k!} \ or \ \frac{_n\mathrm{P}_k}{_n\mathrm{C}_k} = k!$$

Problem Set 2.2.10

1. $_5\mathrm{P}_3 =$ ______
2. $_5\mathrm{C}_3 =$ ______
3. $_6\mathrm{C}_3 =$ ______
4. $_7\mathrm{C}_4 =$ ______
5. $_7\mathrm{P}_4 =$ ______
6. $_6\mathrm{P}_2 =$ ______
7. $_8\mathrm{C}_6 =$ ______
8. $_5\mathrm{C}_2 =$ ______
9. $_8\mathrm{P}_3 =$ ______
10. $_8\mathrm{C}_3 =$ ______
11. $\mathbf{_9C_2} =$ ______
12. $_4\mathrm{P}_2 \div {_4\mathrm{C}_2} =$ ______
13. $_6\mathrm{P}_3 \div {_6\mathrm{C}_3} =$ ______
14. $_7\mathrm{P}_4 \div {_7\mathrm{C}_3} =$ ______
15. $_8\mathrm{C}_5 \div {_8\mathrm{P}_5} =$ ______
16. $\mathbf{_9P_3 \div {_9C_3}} =$ ______
17. $_4\mathrm{P}_3 \div {_3\mathrm{P}_2} =$ ______
18. $_4\mathrm{C}_3 \times {_3\mathrm{C}_2} =$ ______
19. $_5\mathrm{P}_3 \times {_4\mathrm{P}_2} =$ ______
20. $\mathbf{_6C_3 \div {_6P_3}} =$ ______
21. $_6\mathrm{C}_1 + {_4\mathrm{P}_1} =$ ______
22. $({_5\mathrm{C}_2})({_5\mathrm{P}_2}) =$ ______

2.2.11 Trigonometric Values

Trigonometry problems have been increasingly popular for writers of the number sense test. Not only are they testing the basics of sines, cosines, and tangents of special angles ($30°, 45°, 60°, 90°$ – and variations in each quadrant) but also the trigonometric reciprocals (cosecant, secant, and cotangent).

First, let's look at the special angles in the first quadrant where all values of the trigonometric functions are positive. In the table, each trigonometric function is paired below with it's reciprocal:

Trig Function	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
csc	Undefined	2	$\sqrt{2}$	$\frac{\sqrt{3}\cdot 2}{3}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
sec	1	$\frac{\sqrt{3}\cdot 2}{3}$	$\sqrt{2}$	2	Undefined
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
cot	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

All of those can be derived using the memorable "SOHCAHTOA" technique to special right triangles (it is assume one can do this, so it is omitted in this text. If help is needed, see any elementary geometry book.). In addition, it is clear that the values at the reciprocal trigonometric function is just the multiplicative inverse (that's why they are called *reciprocal* trigonometric functions!).

Now to find the values of trigonometric functions in any quadrant it is essential to remember two things. The first is you need to get the sign straight of the values depending on what quadrant you are in. The following plot and mnemonic device will help with getting the sign correct:

Sin	**A**ll
II	I
- - -	- - -
Tan	**C**os
III	IV

The above corresponds to which trigonometric functions (and their reciprocals) are positive in which quadrants. Now if you forget this, you can take the first letter of each function in their respected quadrants and remember the mnemonic device of "**A**ll **S**tudents **T**ake **C**alculus" to remember where each function is positive.

The second challenge to overcome in computing each Trigonometric Function at any angle is to learn how to reference each angle to its first quadrant angle, so that the chart above could be used. The following chart will help you find the appropriate reference angle depending on what quadrant you are in. Assume that you are given an angle θ which resides in each of the quadrants mentioned. The following would be it's reference angle:

	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
Reference Angle:	θ	$180^\circ - \theta$	$\theta - 180^\circ$	$360^\circ - \theta$

So now we have enough information to compute any trigonometric function at any angle. Let's look at a couple of problems:

Problem: $\sin(210^\circ)$
Solution: Now you know the angle is in Quadrant-III, so the result will be negative (only cosine is positive in Q-III). Now to find the reference angle is is just $\theta - 180^\circ = 210^\circ - 180^\circ = 30^\circ$. So the $\sin(30^\circ)$ from the table is $\frac{1}{2}$ so the answer is: $\sin(210^\circ) = -\mathbf{\frac{1}{2}}$.

Problem: $\cot(135^\circ)$
Solution: So the cot/tan function is negative in Q-II. To find the reference angle, it is simply $180^\circ - \theta = 180^\circ - 135^\circ = 45^\circ$. Now the $\cot(45^\circ) = 1$ (from the table) so the answer is: $\cot(135^\circ) = \mathbf{-1}$.

Problem: $\cos(-30^\circ)$
Solution: So an angle of $-30^\circ = 330^\circ$ which is in Q-IV where cosine is positive. Now to find the reference angle you just do $360^\circ - \theta = 360^\circ - 330^\circ = 30^\circ$, and $\cos(30^\circ) = \frac{\sqrt{3}}{2}$. So the answer is just $\cos(-30^\circ) = \mathbf{\frac{\sqrt{3}}{2}}$.

It should be noted that all of these problems have been working with degrees. Students should familiarize themselves with using radians as well using the conversion rate of: $\pi = 180^\circ$. So an angle (given in radians) of $\frac{\pi}{6} = \frac{180^\circ}{6} = 30^\circ$.

It is great for all students to practice solving these types of problems. The following are some practice problems. If more are needed, just consult any elementary geometric textbook or pre-calculus textbook.

Problem Set 2.2.11

1. $\sin(-30^\circ) =$ ______________
2. $\cos\theta = .375$ then $\sec\theta =$ ______________
3. $\sin(3\pi) =$ ______________
4. $\tan(225^\circ) =$ ______________
5. $\sin\left(\sin^{-1}\frac{1}{2}\right) =$ ______________
6. $\sin\theta = -.1$ then $\csc\theta =$ ______________
7. $\sin\frac{11\pi}{6} =$ ______________
8. $\cos(-5\pi) =$ ______________
9. $\frac{\pi}{18} =$ ______________ $^\circ$
10. $\cos(\sec^{-1} 3) =$ ______________
11. $\frac{5\pi}{8} =$ ______________ $^\circ$
12. $\frac{\pi}{5} =$ ______________ $^\circ$
13. $\cos(\sin^{-1} 1) =$ ______________
14. $\tan(-45^\circ) =$ ______________
15. $\sin(-\pi) =$ ______________
16. $\cos(-300^\circ) =$ ______________

17. $\sin^{-1}(\sin 1) =$ ______

18. $\csc(-150°) =$ ______

19. $\sec(\mathbf{120°}) =$ ______

20. $\tan(-225°) =$ ______

21. $\dfrac{3\pi}{5} =$ ______°

22. $\tan(-45°) =$ ______

23. $\tan(315°) =$ ______

24. If $0° < x < 90°$ and $\tan x = \cot x$, $x =$ ______

25. $280° = k\pi$ then $k =$ ______

26. $\tan \dfrac{\mathbf{5}\pi}{\mathbf{4}} =$ ______

27. $\cos\theta = .08333\ldots$ then $\sec\theta =$ ______

28. $\sin(5\pi) + \cos(5\pi) =$ ______

29. $\sec(60°) =$ ______

30. $12° = \dfrac{\pi}{k}, k =$ ______

31. $\cos\theta = -.25$ then $\sec\theta =$ ______

32. $\tan^2 60° =$ ______

33. $\mathbf{1.25}\pi =$ ______°

34. $\cot^2 60° =$ ______

35. $\sin\left[\cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right)\right] =$ ______

36. $\cos(-3\pi) - \sin(-3\pi) =$ ______

37. $\cos\left(\dfrac{-4\pi}{3}\right) + \sin\left(\dfrac{-5\pi}{6}\right) =$ ______

38. $2\sin 120° \cos 30° =$ ______

39. $\cos(240°) - \sin(150°) =$ ______

40. $\sin\left(\cos^{-1}\dfrac{\sqrt{3}}{2}\right) =$ ______

41. $\sin(\cos^{-1} 1) =$ ______

42. If $\csc\theta = -3$, where $270° < \theta < 300°$, then $\sin\theta =$ ______

43. $\sin\left(\dfrac{-\mathbf{7}\pi}{\mathbf{6}}\right) - \cos\left(\dfrac{-\mathbf{2}\pi}{\mathbf{3}}\right) =$ ______

44. $\sec\theta = \mathbf{-3}$, θ **is in QIII, then** $\cos\theta =$ ______

45. $\cos\dfrac{5\pi}{6} \times \sin\dfrac{2\pi}{3} =$ ______

46. $\sin\dfrac{3\pi}{4} \times \cos\dfrac{5\pi}{4} =$ ______

47. $\sin 30° + \cos 60° = \tan x$
$0° \le x \le 90°, x =$ ______

48. $\cos\left(\sin^{-1}\dfrac{\sqrt{3}}{2}\right) =$ ______

49. $\sin\left(\dfrac{-\pi}{3}\right) \times \sin\left(\dfrac{\pi}{3}\right) =$ ______

50. $\cos(120°) \times \cos(120°) =$ ______

51. $216° = k\pi, k =$ ______

52. $\cos\left(\dfrac{-2\pi}{3}\right) \times \cos\left(\dfrac{4\pi}{3}\right) =$ ______

53. $\tan(30°) \times \cot(60°) =$ ______

54. $\cos\left(\dfrac{-\pi}{3}\right) \times \cos\left(\dfrac{\pi}{3}\right) =$ ______

55. $\sin\dfrac{\pi}{\mathbf{6}} + \cos\dfrac{\pi}{\mathbf{3}} = \tan\dfrac{\pi}{\mathbf{k}}$
then k $=$ ______

56. $\cos^{-1} .8 + \cos^{-1} .6 = k\pi$ then $k =$ ____________

57. $\sin(300°) \times \cos(330°) =$ ____________

58. $\sin\left(\frac{-\pi}{6}\right) \times \cos\left(\frac{\pi}{3}\right) =$ ____________

59. $630° = k\pi$, $k =$ ____________

2.2.12 Trigonometric Formulas

Recently, questions involving trigonometric functions have encompassed some basic trigonometric identities. The most popular ones tested are included here:

The Fundamental Identities

$\sin^2 + \cos^2 = 1$
$1 + \cot^2 = \csc^2$
$\tan^2 + 1 = \sec^2$

Sum to Difference Formulas

$\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$
$\cos(a \pm b) = \cos(a) cos(b) \mp \sin(a)\sin(b)$

Double Angle Formulas

$\sin(2a) = 2\sin(a)\cos(a)$
$\cos(2a) = \cos^2(a) - \sin^2(a)$ with variants:
$\cos(2a) = 1 - 2\sin^2(a)$
$\cos(2a) = 2\cos^2(a) - 1$

Sine $\rightarrow$ Cosine

$\sin(90° - \theta) = \cos(\theta)$

Most of the time, using trigonometric identities will not only aid in speed but will also be necessary. Take for example:

$\sin(10°)\cos(20°) + \sin(20°)\cos(10°)$

Without using the sum to difference formula, this would be impossible to calculate, however after using the formula you get:

$\sin(10°)\cos(20°) + \sin(20°)\cos(10°) = \sin(10° + 20°) = \sin(30°) = \mathbf{\frac{1}{2}}$

The following are some practice problems using these identities:

Problem Set 2.2.12

1. $\cos^2 30° + \sin^2 30° =$ ____________

2. $\cos^2 30° - \sin^2 30° =$ ____________

3. $2\sin 15° \cos 15° =$ ____________

4. $2\sin 30^\circ \sin 30^\circ - 1 =$ ______

5. $1 - \sin^2 30^\circ =$ ______

6. $\cos 22^\circ = \sin\theta, 0^\circ < \theta < 90^\circ, \theta =$ ______

7. $[2\sin\frac{\pi}{3}\cos\frac{\pi}{3}]^2 =$ ______

8. $2\sin 15^\circ \cos 15^\circ - 1 =$ ______

9. $3\csc^2 45^\circ - 3\cot^2 45^\circ =$ ______

10. $\cos^2 30^\circ - \sin^2 30^\circ =$ ______

11. $\sin \mathbf{105}^\circ \cos \mathbf{105}^\circ =$ ______

12. $\sin 38^\circ = \cos\theta, 270^\circ < \theta < 360^\circ, \theta =$ ______

13. $\sin 30^\circ \cos 60^\circ - \sin 60^\circ \cos 30^\circ =$ ______

14. $2\cos^2\frac{\pi}{6} - 1 =$ ______

15. $(1 - \sin 60^\circ)(1 + \sin 60^\circ) =$ ______

16. $2\sin 15^\circ \sin 75^\circ =$ ______

17. $(\sin\frac{\pi}{3} - \cos\frac{\pi}{3})(\sin\frac{\pi}{3} + \cos\frac{\pi}{3}) =$ ______

18. If $\sin(A) = \frac{3}{5}$, then $\cos(2A) =$ ______

19. $\mathbf{1 - 2\sin^2\frac{\pi}{6}} =$ ______

20. $\cos 75^\circ \sin 75^\circ =$ ______

21. $\sin 15^\circ \cos 45^\circ - \sin 45^\circ \cos 15^\circ =$ ______

22. $2 - 4\sin^2 30^\circ =$ ______

23. $\cos \mathbf{95}^\circ \cos \mathbf{25}^\circ - \sin \mathbf{95}^\circ \sin \mathbf{25}^\circ =$ ______

24. $\sin\frac{\pi}{6} + \cos\frac{\pi}{3} =$ ______

25. $\cos 15^\circ \sin 45^\circ - \cos 45^\circ \sin 15^\circ =$ ______

26. $(\sin\frac{\pi}{6} - \cos\frac{\pi}{6})(\sin\frac{\pi}{6} + \cos\frac{\pi}{6}) =$ ______

27. $2\tan^2\theta - 2\sec^2\theta =$ ______

2.2.13 Graphs of Sines/Cosines

Popular questions for the last column involve determining amplitudes, periods, phase shifts, and vertical shifts for plots of sines/cosines. If you haven't been introduced this in a pre-calculus class, use the following as a rough primer:

The general equation for any sine/cosine plot is:

$$y = A\sin[B(x - C)] + D$$

Amplitude:	$\lvert A\rvert$
Period:	$\frac{2\pi}{B}$
Phase Shift:	C
Vertical Shift:	D (Up if > 0, Down if < 0)

Example: Find the period of $y = 3\sin(\pi x - 2) + 8$.

Solution: We need the coefficient in front of x to be 1, so we need to factor out π, making the graph:

$y = 3\sin[\pi(x - \frac{2}{\pi})] + 8$. Now we can apply the above table to see that the period= $\frac{2\pi}{\pi} = \mathbf{2}$. The other characteristics of the graph is that the amplitude= 3, the phase shift= $\frac{2}{\pi}$, and it is vertically shifted by 8 units.

Here are some more practice problems:

Problem Set 2.2.13

1. What is the amplitude of $y = 4\cos(2x) + 1$: __
2. The graph of $y = 2 - 3\cos[2(x - 5)]$ has a horizontal displacement of: ________
3. The graph of $y = 2 - 2\cos[3(x - 5)]$ has a vertical shift of: ________________
4. What is the amplitude of $y = 2 - 3\cos[4(x+5)]$:
5. The period of $y = 5\cos\left[\frac{1}{4}(x + 3\pi)\right] + 2$ is $k\pi, k =$ ________________________
6. The phase shift of $y = 5\cos[4(x + 3)] - 2$ is: __
7. The amplitude of $y = 2 - 5\cos[4(x - 3)]$ is: __
8. The vertical displacement of $y = 5\cos[4(x + 3)] - 2$ is: ________________
9. The phase shift of $f(x) = 2\sin(3x - \frac{\pi}{2})$ is $k\pi, k =$ ____________________________
10. **The period of $\mathbf{y = 2 - 3}\cos(\mathbf{4\pi x + 2\pi})$ is:**
11. The period of $y = 2 + 3\sin(\frac{x}{5})$ is: ________
12. The graph of $y = 1 - 2\cos(3x + 4)$ has an amplitude of: __________________

2.2.14 Vertex of a Parabola

This question was much more popular on tests from the 90's, but it is being resurrected on some of the more recent tests. When approached with a parabola in the form of $f(x) = Ax^2 + Bx + x$, the coordinate of the vertex is:

$(h, k) = \left(\frac{-B}{2A}, f\left(\frac{-B}{2A}\right)\right)$.

Example: Find the y-coordinate of the vertex of the parabola who's equation is $y = 3x^2 - 12x + 16$.

Solution: $x = \dfrac{-(-12)}{2 \cdot 3} = 2 \Rightarrow y = 3 \cdot 2^2 - 12 \cdot 2 + 16 = \mathbf{4}$.

It should be noted that if the parabola is in the form $x = ay^2 + by + c$, then the vertex is:

$(h, k) = \left(f\left(\frac{-b}{2a}\right), \frac{-b}{2a}\right)$. (Due to a rotation of axis).

The following are some practice problems:

Problem Set 2.2.14

1. The vertex of the parabola
   $y = 2x^2 + 8x - 1$ is $(h, k), k =$ ____________

2. The vertex of $y = x^2 - 2x - 4$ is $(h, k), k =$ ___

3. If $g(x) = 2 - x - x^2$, then
   the axis of symmetry is $x =$ ____________

2.2.15 Discriminant and Roots

A very popular question is, when given a quadratic equation, determining the value of an undefine coefficient so that the roots are distinct/equal/complex. Take the following question:

Find the value for k such that the quadratic $3x^2 - x - 2k = 0$ has equal roots.

Well we know from the quadratic equation that the roots of a general polynomial $ax^2 + bx + c = 0$ can be determined from:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So we know from this that:

Distinct Roots:	$b^2 - 4ac > 0$
Equal Roots:	$b^2 - 4ac = 0$
Complex Conjugate Roots:	$b^2 - 4ac < 0$

So in our case we need to find the value of k such that the discriminant ($b^2 - 4ac$) is equal to zero.

$$b^2 - 4ac = 1^2 - 4 \cdot 3 \cdot (-2k) = 0 \Rightarrow k = \frac{-1}{4 \cdot 3 \cdot 2} = \mathbf{\frac{-1}{24}}$$

The following are some more practice problems:

Problem Set 2.2.15

1. For $2x^2 - 4x - k = 0$ to have 2 equal roots, the smallest value of k is: ____________

2. For $3x^2 - x - 2k = 0$ to have equal roots k has to be: ____________

3. **For $\mathbf{3x^2 - 2x + 1 - k = 0}$ to have equal roots, k has to be:** ________

4. The discriminant of $2x^2 - 3x = 1$ is: ________

5. For what value of k does $3x^2 + 4x + k = 0$ have equal roots: ________

6. For $x^2 - 2x - 3k = 0$ to have one real solution k has to be: ____________

3 Miscellaneous Topics

3.1 Random Assortment of Problems

3.1.1 GCD and LCM

How finding the Greatest Common Divisor (or GCD) is taught in classes usually involves prime factorizing the two numbers and then comparing powers of exponents. However, this is not the most efficient way of doing it during a number sense competition. One of the quickest way of doing it is by employing Euclid's Algorithm who's method won't be proven here (if explanation is necessary, just Google to find the proof). The following outlines the procedure:

1. Arrange the numbers so that $n_1 < n_2$ then find the remainder when n_2 is divided by n_1 and call it r_1.
2. Now divide n_1 by r_1 and get a remainder of r_2.
3. Continue the procedure until any of the remainders are 0 and the number you are dividing by is the GCD or when you notice what the GCD of any pair of numbers is.

Let's illustrate with some examples:

Problem: GCD$(36, 60)$

Solution: Well, when 60 is divided by 36 it leaves a remainder of 24. So, GCD$(36, 60)$ =GCD$(24, 36)$. Continuing the procedure, when 36 is divided by 24 it leaves a remainder of 12. So, GCD$(36, 60)$ =GCD$(24, 36)$ =GCD$(12, 24)$, which from here you can tell the GCD is **12**. You could also have stopped after the first step when you notice that the GCD$(24, 36)$ is 12, and you wouldn't have to continue the procedure.

Problem: GCD$(108, 140)$

Solution: $\text{GCD}(108, 140) \to \text{GCD}(32, 108) \to \text{GCD}(12, 32) \to \text{GCD}(8, 12) \to \text{GCD}(4, 8) = \mathbf{4}$

If at any point in that process you notice what the GCD of the two numbers is by observation, you can cut down on the amount of steps in computation.

For computing the LCM between two numbers a and b, I use the formula:

$$\text{LCM}(a, b) = \frac{a \times b}{\text{GCD}(a, b)}$$

So to find what the LCM is, we must first compute the GCD. Using a prior example, let's calculate the LCM$(36, 60)$:

$$\text{LCM}(36, 60) = \frac{36 \times 60}{12} = 3 \times 60 = \mathbf{180}$$

The procedure is simple enough, let's do one more example.

Problem: Find the LCM of 44 and 84.

Solution: $\text{GCD}(44, 84) = \text{GCD}(40, 44) = \text{GCD}(4, 40) = 4 \Rightarrow \text{LCM}(44, 84) = \frac{44 \times 84}{4} = 11 \times 84 = \mathbf{924}$

It should be noted that there are some questions concerning the GCD of more than two numbers (usually not ever more than three). The following outlines the procedure which should be followed:

1. Find the GCD of two of the numbers.
2. Find the LCM of those two numbers by using the GCD and the above formula.

3. Calculate the GCD of the LCM of those two numbers and the third number.

It should be noted that usually one of the numbers is a multiple of another, thus leaving less required calculations (because the LCM between two numbers which are multiples of each other is just the larger of the two numbers).

The following are some more practice problems for finding GCDs and LCMs using this method:

Problem Set 3.1.1

1. The GCF of 35 and 63 is: ____________
2. The LCM of 64 and 20 is: ____________
3. The LCM of 27 and 36 is: ____________
4. The GCF of 48 and 72 is: ____________
5. The GCD of 27 and 36 is: ____________
6. The LCM of 63 and 45 is: ____________
7. The GCD of 132 and 156 is: ____________
8. **The LCM of 57 and 95 is:** ____________
9. The GCD of 52 and 91 is: ____________
10. The LCM of 52 and 28 is: ____________
11. The GCD of 48 and 54 is: ____________
12. The GCD of 54 and 36 is: ____________
13. The LCM of 27 and 36 is: ____________
14. The LCM of 108 and 81 is: ____________
15. The GCD of 28 and 52 is: ____________
16. The LCM of 51 and 34 is: ____________
17. **The LCM of $\mathbf{2^3 \times 3^2}$ and $\mathbf{2^2 \times 3^3}$ is:** ____
18. The LCM of 28 and 42 is: ____________
19. The LCM of 54 and 48 is: ____________
20. The LCM of 84 and 70 is: ____________
21. **The GCF of 132 and 187 is:** ____________
22. The LCM of 48 and 72 is: ____________
23. The GCF of $51, 68,$ and 85 is: ____________
24. The $\text{GCF}(24, 44) - \text{LCM}(24, 44) =$ ____________
25. The LCM of $16, 20,$and 32 is: ____________
26. The $\text{GCD}(15, 28)$ times $\text{LCM}(15, 28)$ is: ______
27. The LCM of $12, 18,$and 20 is: ____________
28. The LCM of $14, 21,$and 42 is: ____________
29. The LCM of $8, 18,$and 32 is: ____________
30. The $\text{GCD}(15, 21) + \text{LCM}(15, 21) =$ ____________
31. The GCF of $44, 66,$and 88 is: ____________
32. The product of the GCF and LCM of 21 and 33 is: ____________

33. The LCM of $16, 32$,and 48 is: ____________

34. The GCD$(18, 33)$ + LCM$(18, 33)$ = ________

35. The LCM of $14, 28$,and 48 is: ____________

36. **The LCM(21, 84)-GCF(21, 84)** = ________

37. The LCM of $24, 36$,and 48 is: ____________

38. The GCD$(16, 20)$ − LCM$(16, 20)$ = ________

39. The GCF of $42, 28$,and 56 is: ____________

40. **The product of the GCF and LCM of 24 and 30 is:** ________

41. The LCM of $36, 24$ and 20 is: ____________

42. **The LCM of 28, 42,and 56 is:** ________

3.1.2 Perfect, Abundant, and Deficient Numbers

For this section let's begin with the definitions of each type.

A **perfect number** has the sum of the proper divisors equal to itself. The first three perfect numbers are 6 $(1 + 2 + 3 = 6)$, 28 $(1 + 2 + 4 + 7 + 14 = 28)$, and 496 $(1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 = 496)$. Notice that there are really *only* two perfect numbers that would be reasonable to test on a number sense test (6 and 28 should be memorized as being perfect).

An **abundant number** has the sum of the proper divisors greater than itself. Examples of an abundant number is 12 $(1 + 2 + 3 + 4 + 6 = 16 > 12)$ and 18 $(1 + 2 + 3 + 6 + 9 = 21 > 18)$. An interesting property of abundant numbers is that any multiple of a perfect or abundant number is abundant. Knowing this is *very* beneficial to the number sense test.

As you can assume through the process of elimination, a **deficient number** has the sum of the proper divisors less than itself. Examples of these include any prime number (because they have only one proper divisor which is 1), 10 $(1 + 2 + 5 = 8 < 10)$, and 14 $(1 + 2 + 7 = 10 < 14)$ just to name a few. An interesting property is that any power of a prime is deficient (this is often tested on the number sense test).

3.1.3 Sum and Product of Coefficients in Binomial Expansion

From the binomial expansion we know that:

$$\begin{aligned}(ax + by)^n &= \sum_{k=0}^{n} \binom{n}{k} (ax)^{n-k} (by)^k \\ &= \binom{n}{0} a^n \cdot x^n + \binom{n}{1} a^{n-1} b^1 \cdot x^{n-1} y^1 + \cdots \binom{n}{n} b^n y^n\end{aligned}$$

From here we can see that the sum of the coefficients of the expansion is:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Where we can retrieve these sums by setting $x = 1$ and $y = 1 \Rightarrow$ Sum of the Coefficients $= (a + b)^n$!

Here is an example to clear things up:

Problem: Find the Sum of the Coefficients of $(x+y)^6$.

Solution: Let $x = 1$ and $y = 1$ which leads to the Sum of the Coefficients $= (1+1)^6 = \mathbf{64}$.

An interesting side note on this is when asked to find the Sum of the Coefficients of $(x-y)^n$ it will always be 0 because by letting $x = 1$ and $y = 1$ you get the Sum of the Coefficients $= (1-1)^n = \mathbf{0}$.

As for the product of the coefficients, there are no easy way to compute them. The best method is to memorize some of the first entries of the Pascal triangle (if you're unfamiliar with how Pascal's triangle relates to the coefficients of expansion, I suggest Googling it):

$$
\begin{array}{c}
1 \\
1\ 1 \\
1\ 2\ 1 \\
1\ 3\ 3\ 1 \\
1\ 4\ 6\ 4\ 1 \\
1\ 5\ 10\ 10\ 5\ 1 \\
1\ 6\ 15\ 20\ 15\ 6\ 1
\end{array}
$$

Here are some more practice to get acquainted with both the sum and product of coefficients:

Problem Set 3.1.3

1. The sum of the coefficients in the expansion of $(5x-9y)^3$ is: __________
2. The sum of the coefficients in the expansion of $(5x+7y)^3$ is: __________
3. The sum of the coefficients in the expansion of $(x-y)^3$ is: __________
4. The sum of the coefficients in the expansion of $(a+b)^3$ is: __________
5. The sum of the coefficients in the expansion of $(x+y)^6$ is: __________
6. The sum of the coefficients in the expansion of $(x+y)^2$ is: __________
7. **The sum of the coefficients in the expansion of $(\mathbf{a}+\mathbf{b})^5$ is:** __________
8. The sum of the coefficients in the expansion of $(a-b)^4$ is: __________
9. The sum of the coefficients in the expansion of $(3x-y)^4$ is: __________
10. The product of all the coefficients in the expansion $(x+y)^4$ is: __________
11. The product of the coefficients in the expansion of $(2a+2b)^2$ is: __________
12. The product of the coefficients in the expansion of $(a+b)^3$ is: __________
13. The product of the coefficients in the expansion of $(a-b)^4$ is: __________
14. The product of the coefficients in the expansion of $(3a+3b)^2$ is:__________

15. **The product of the coefficients in the expansion of $(\mathbf{a}+\mathbf{b})^{\mathbf{5}}$ is:** ________

16. The product of the coefficients in the expansion of $(a-b)^2$ is: ________

17. The product of the coefficients in the expansion of $(4a-3b)^2$ is: ________

18. The sum of the coefficients in the expansion of $(x^2-6x+9)^2$ is: ________

19. **The product of the coefficients in the expansion of $(\mathbf{4x}+\mathbf{5})^{\mathbf{2}}$ is:** ________

3.1.4 Sum/Product of the Roots

Define a polynomial by $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \cdots a_1 x^1 + a_0 = 0$. The three most popular questions associated with the number sense test concerning roots of polynomials are: sum of the roots, sum of the roots taken two at a time, and product of the roots. For the polynomial $p_n(x)$ these values are defined:

Sum of the roots:		$\frac{-a_{n-1}}{a_n}$
Sum of the roots taken two at a time:		$\frac{a_{n-2}}{a_n}$
Product of the roots:	If n is even	$\frac{a_0}{a_n}$
	If n is odd	$\frac{-a_0}{a_n}$

Let's see what this means for our generic quadratics/cubics:
$p_2(x) = ax^2 + bx + c = 0$ and $p_3(x) = ax^3 + bx^2 + cx = 0$

$p_2(x) = ax^2 + bx + c = 0$	Sum of the roots:	$\frac{-b}{a}$
	Product of the roots:	$\frac{c}{a}$

$p_3(x) = ax^3 + bx^2 + cx = 0$	Sum of the roots:	$\frac{-b}{a}$
	Product of the roots taken two at a time:	$\frac{c}{a}$
	Product of the roots:	$\frac{-d}{a}$

Since the quadratic only has two roots, the sum of the roots taken two at a time happens to be the product of the roots. You can extend the same procedure for polynomials of any degree, keeping in mind the alternating signs for the product of the roots. The following are practice problems:

Problem Set 3.1.4

1. The sum of the roots of $2x^2 - 3x + 1 = 0$ is: ______
2. The sum of the roots of $(x-4)(x-5) = 0$ is: ______
3. The sum of the roots of $3x^3 - 2x^2 + x - 4 = 0$ is: ______
4. The product of the roots of $x^2 + 3x = 7$ is: ______
5. The sum of the roots of $x^2 - 9 = 0$ is: ______
6. The sum of the roots of $4x^2 + 3x = 2$ is: ______
7. The sum of the roots of $(2x-3)^2 = 0$ is: ______
8. The product of the roots of $5x^3 - 8x^2 + 2x + 3 = 0$ is: ______
9. The product of the roots of $4x^3 - 3x^2 + 2x - 1 = 0$ is: ______
10. The sum of the roots of $3x^3 + 2x^2 = 9$ is: ______
11. **The sum of the roots of $\mathbf{x^3 - 13x = 12}$ is:** ______
12. Let R,S,T be the roots of $2x^3 + 4x = 5$. Then $R \times S \times T =$ ______
13. The product of the roots of $5x^3 + 4x - 3 = 0$ is: ______
14. The sum of the roots of $(3x-2)(2x+1) = 0$ is: ______
15. The sum of the product of the roots taken two at a time of $2x^3 + 4x^2 - 6x = 8$ is: ______
16. The sum of the roots of $2x^3 + 4x^2 - 3x + 5 = 0$ is: ______
17. The product of the roots of $(2x-1)(3x+2)(4x-3) = 0$ is: ______
18. Let R,S,T be the roots of $2x^3 + 4x = 5$. Then $RS + RT + ST =$ ______
19. **The equation $\mathbf{2x^3 - bx^2 + cx = d}$ has roots r,s,t and rst=3.5, then d =** ______
20. The sum of the roots of $3x^2 - bx + c = 0$ is -12 then $b =$ ______
21. If r, s, and t are the roots of the equation $2x^3 - 4x^2 + 6x = 8$ then $rs + rt + st =$ ______
22. The sum of the roots of $4x^3 + 3x^2 - 2x - 1 = 0$ is: ______
23. The product of the roots of $4x^3 - 3x^2 + 2x + 1 = 0$ is: ______
24. The sum of the roots of $5x^3 + 4x - 3 = 0$ is: ______
25. **The equation $\mathbf{2x^3 - bx^2 + cx = d}$ has roots $\mathbf{r, s, t}$. If $\mathbf{r + s + t = -2}$ then $\mathbf{b =}$** ______

3.1.5 Finding Units Digit of x^n

This is a common problem on the number sense test which seems considerably difficult, however there is a shortcut method. Without delving too much into the modular arithmetic required, you can think of this problem as exploiting patterns. For example, let's find the units digit of 3^{47}, knowing:

$3^1 =$	3	Units Digit: **3**	
$3^2 =$	9	Units Digit: **9**	
$3^3 =$	27	Units Digit: **7**	
$3^4 =$	81	Units Digit: **1**	From this you can see the units digit repeats every 4^{th} power.
$3^5 =$	243	Units Digit: **3**	
$3^6 =$	729	Units Digit: **9**	
$3^7 =$	2187	Units Digit: **7**	
$3^8 =$	6561	Units Digit: **1**	

So in order to see what is the units digit you can divide the power in question by 4 then see what the remainder r is. And in order to find the appropriate units digit, you'd then look at the units digit of 3^r. For example, the units digit for 3^5 could be found by saying $5 \div 4$ has a remainder of 1 so, the units digit of 3^5 corresponds to that of 3^1 which is **3**. So to reiterate, the procedure is:

1. For low values of n, compute what the units digit of x^n is.
2. Find out how many unique integers there are before repetition (call it m).
3. Find the remainder when dividing the large n value of interest by m (call it r)
4. Find the units digit of x^r, and that's your answer.

So for our example of 3^{47}:

$$47 \div 4 \text{ has a remainder of } 3$$

$$3^3 \text{ has the units digit of } \mathbf{7}$$

Other popular numbers of interest are:

Numbers	**Repeating Units Digits**	**Number of Unique Digits**
Anything ending in 2	$2, 4, 8, 6$	4
Anything ending in 3	$3, 9, 7, 1$	4
Anything ending in 4	$4, 6$	2
Anything ending in 5	5	1
Anything ending in 6	6	1
Anything ending in 7	$7, 9, 3, 1$	4
Anything ending in 8	$8, 4, 2, 6$	4
Anything ending in 9	$9, 1$	2

Using the above table, we can calculate the units digit of any number raised to any power relatively simple. To show this, find the units digit of 27^{63}:

$$\text{From the table, we know it repeats every } 4^{th} \text{ power, so: } 63 \div 4 \Rightarrow r = 3$$

$$r = 3 \text{ corresponds to } 7^3 \text{ which ends in a } \mathbf{3}$$

This procedure is also helpful with raising the imaginary number i to any power. Remember from Algebra:

i^1	i
i^2	-1
i^3	$-i$
i^4	1
i^5	i
i^6	-1
i^7	$-i$
i^8	1

So, after noticing that it repeats after every 4^{th} power, we can compute for example i^{114}.

$$114 \div 4 \text{ has a remainder of } 2 \Rightarrow i^2 = \mathbf{-1}$$

The following are examples of these types of problems:

Problem Set 3.1.5

1. Find the units digit of 19^7: ________
2. Find the units digit of 17^6: ________
3. Find the units digit of 8^8: ________
4. Find the units digit of 7^7: ________
5. Find the units digit of 13^{13}: ________
6. Find the units digit of 17^5: ________
7. $i^{78} =$ ________
8. $i^{66} =$ ________
9. Find the units digit of 16^5: ________

3.1.6 Exponent Rules

These problems are usually on the third column, and if you know the basics of exponential rules they are easy to figure out. The rules to remember are as followed:

$$x^a \cdot x^b = x^{a+b} \qquad \frac{x^a}{x^b} = x^{a-b} \qquad (x^a)^b = x^{ab}$$

The following are problems concerning each type:

Product Rule: Let $3^x = 70.1$, then $3^{x+2} = ?$
Solution: $3^{x+2} = 3^x \cdot 3^2 = 70.1 \cdot 9 = 630.9$

Quotient Rule: Let $5^x = 2$, represent 5^{x-2} as a decimal.
Solution: $5^{x-2} = \dfrac{5^x}{5^2} = \dfrac{2}{25} = \mathbf{.08}$

Power Rule: Let $4^x = 1.1$ then $2^{6x} = ?$
Solution: $4^x = 2^{2x} = 1.1 \Rightarrow 2^{6x} = \left(2^{2x}\right)^3 = 1.1^3 = \mathbf{1.331}$

The following are some more problems about exponent rules:

Problem Set 3.1.6

1. $6^x = 34$,then $6^{x+2} =$ __________
2. $3^x = 70.1$,then $3^{x+2} =$ __________
3. $4^{x+1} = 2$,then $4^{x-1} =$ __________
4. $6^x = 72$,then $6^{x-2} =$ __________
5. $7^x = 14$,then $7^{x-2} =$ __________
6. $4^x = .125$,then $4^{2x} =$ __________
7. $8^x = 17$,then $8^{2x} =$ __________
8. $2^x = 14.6$,then $2^{x+1} =$ __________
9. $4^x = 32$,then $x =$ __________
10. $9^x = 108$,then $3^{2x+1} =$ __________
11. $6^{2x} = 36$,then $6^{3x} =$ __________
12. $8^x = 256$,then $x =$ __________
13. $27^x = 81$, then $x =$ __________
14. $2^8 \div 4^3$ has a remainder of: __________
15. $9^x = 27^{x+2}$, then $x =$ __________
16. $n^4 = 49$,then $n^6 =$ __________
17. $16^x = 169$,then $4^x =$ __________
18. $5^{3x} = 25^{2+x}$,then $x =$ __________
19. $n^6 = 1728$,then $n^4 =$ __________
20. $4^x \div 16^x = 4^{-2}$,then $x =$ __________
21. $6^8 \div 8$ has a remainder of: __________
22. $\sqrt[3]{a^4} \times \sqrt[4]{a^3} = \sqrt[12]{a^n}$, $\mathbf{n} =$ __________

3.1.7 Log Rules

Logarithms are usually tested on the third and fourth columns of the test, however, if logarithm rules are fully understood these can be some of the simplest problems on the test. The following is a collection of log rules which are actively tested:

Definition:	$\log_a b = x$	$a^x = b$
Power Rule:	$\log_a b^n$	$n \log_a b$
Addition of Logs:	$\log_a b + \log_a c$	$\log_a(bc)$
Subtraction of Logs:	$\log_a b - \log_a c$	$\log_a(\frac{b}{c})$
Change of Bases:	$\log_a b$	$\dfrac{\log b}{\log a}$

In the above table $\log_{10} a$ is represented as $\log a$. The following are some sample problems illustrating how each one of the rules might be tested:

Example: Find $\log_4 .0625$.

Solution: Applying the definition we know that $4^x = .0625 = \dfrac{1}{16}$. Therefore, our answer is $x = -\mathbf{2}$

Example: Find $\log_8 16$.

Solution: Again, applying the definition, $8^x = 16$, which can be changed to $2^{3x} = 2^4 \Rightarrow x = \dfrac{\mathbf{4}}{\mathbf{3}}$.

Example: Find $\log_{12} 16 + \log_{12} 36 - \log_{12} 4$.

Solution: We know from the addition/subtraction of logs that the above expression can be written as $\log_{12} \dfrac{16 \cdot 36}{4} = \log_{12} 16 \cdot 9 = \log_{12} 144 \Rightarrow 12^x = 144 \Rightarrow x = \mathbf{2}$.

Example: Find $\log_5 8 \div \log_{25} 16$

Solution: These are probably the most challenging logarithm problems you will see on the exam. They involved changing bases and performing the power rule. Let's look at what happens when we change bases:

$$\log_5 8 \div \log_{25} 16 = \frac{\log 8}{\log 5} \div \frac{\log 16}{\log 25} = \frac{\log 2^3}{\log 5} \times \frac{\log 5^2}{\log 2^4} = \frac{3 \cdot \log 2}{\log 5} \times \frac{2 \cdot \log 5}{4 \cdot \log 2} = 3 \times \frac{1}{2} = \frac{\mathbf{3}}{\mathbf{2}}.$$

In addition to the above problems, there are some approximations of logarithms which pop up. For those, there are some quantities which would be nice to have memorized to compute a more accurate approximations. Those are:

$$\log_{10} 2 \approx .3 \qquad \log_{10} 5 \approx .7$$

$$\ln 2 \approx .7 \qquad \ln 10 \approx 2.3$$

Where $\ln x = \log_e x$.

The following is example of how approximations of logs can be calculated:

$$200 \log 200 = 200 \log(2 \cdot 100) = 200 \cdot (\log 2 + \log 100) \approx 200 \cdot (.3 + 2) = \mathbf{460}$$

The following are some more practice problems:

Problem Set 3.1.7

1. $-2\log_3 x = 4,\ x =$ ______

2. $\log_{12} 2 + \log_{12} 8 + \log_{12} 9 =$ ______

3. $\log_3 40 - \log_3 8 + \log_3 1.8 =$ ______

4. $\log_x 216 = 3,\ x =$ ______

5. $f(x) = \log_3 x - 4,\ f(3) =$ ______

6. $\log_8 16 =$ ______

7. $\log_3 x = 4,\ \sqrt{x} =$ ______

8. $\log_x 343 = 3,\ x =$ ______

9. If $\log .25 = 3$, then $\log 4 =$ ______

10. $(\log_5 6)(\log_6 5) =$ ______

11. $\log_3 216 \div \log_3 6 =$ ______

12. $\log_3 32 - \log_3 16 + \log_3 1.5 =$ ______

13. $\log_2 64 \div \log_2 4 =$ ______

14. $\log_4 32 + \log_4 2 - \log_4 16 =$ ______

15. $\log_5 625 \times \log_5 25 \div \log_5 125 =$ ______

16. $\log_4 8 \times \log_8 4 =$ ______

17. $\log_{\mathbf{4}} \mathbf{256} \div \log_{\mathbf{4}} \mathbf{16} \times \log_{\mathbf{4}} \mathbf{64} =$ ______

18. $\log_8 k = \frac{1}{3},\ k =$ ______

19. $\log_5 M = 2,\ \sqrt{M} =$ ______

20. $4\log_9 k = 2,\ k =$ ______

21. $\log_4 8 = N$ then $2N =$ ______

22. $\log_9 3 = W$ then $3W =$ ______

23. $\log_k 32 = 5,\ k =$ ______

24. $\log_3[\log_2(\log_2 256)] =$ ______

25. $\log_4 .5 = k,\ k =$ ______

26. $\log_5[\log_4(\log_3 81)] =$ ______

27. $\log_{16} 8 = w,\ w =$ ______

28. $\log_9 k = 2.5,\ k =$ ______

29. $\log_2[\log_3(\log_2 512)] =$ ______

30. $\log_b .5 = -.5,\ b =$ ______

31. $\log_b 8 = 3,\ b =$ ______

32. $\log_3[\log_4(\log_5 625)] =$ ______

33. $\log_4 8 = k, k =$ ______

34. $\log_4[\log_3(\log_5 125)] =$ ______

35. $\log_{\mathbf{4}} \mathbf{.125} = \mathbf{k},\ \mathbf{k} =$ ______

36. $\log_8(3x - 2) = 2,\ x =$ ______

37. $\log_4[\log_2(\log_6 36)] =$ ______

38. $\log_4 x = 3,\ \sqrt{x} =$ ______

39. $\log_5 x^2 = 4,\ \sqrt{x} =$ ______

40. (*) $300 \log 600 =$ ______

41. $\log_4 x = -.5,\ x =$ ______

42. $3\log_2 x = 6$, $\sqrt{x} =$ ____________________

43. $\log_2 x = 9$, $\sqrt[3]{x} =$ ____________________

44. $\log_x 64 = 3$, $x^{-2} =$ ____________________

45. $\log_{\mathbf{9}} \mathbf{x} = \mathbf{2}$, $\sqrt{\mathbf{x}} =$ ____________________

46. $\log_k 1728 = 3$, $k =$ ____________________

47. $\log_4 x = 3$, $\sqrt{x} =$ ____________________

48. $\log_2(\log_{10} 100) =$ ____________________

49. $\log_x 64 = 1.5$, $x =$ ____________________

50. $\log_8(\log_4 16) =$ ____________________

51. $\log_{\mathbf{9}}(\log_{\mathbf{3}} \mathbf{27}) =$ ____________________

3.1.8 Square Root Problems

A common question involves the multiplication of two square roots together to solve for (usually) an integer value. For example:

$$\begin{aligned}\sqrt{12} \times \sqrt{27} &= \sqrt{12} \times \sqrt{3} \times \sqrt{9} \\ &= \sqrt{36} \times \sqrt{9} \\ &= 6 \times 3 = \mathbf{18}\end{aligned}$$

Usually the best approach is to figure out what you can take away from one square roots and multiply the other one by it. So from the above example, notice that we can take a 3 away from 37 to multiply the 12 with, leading to just $\sqrt{36} \times \sqrt{9}$ which are easy square roots to calculate. With this method, there are really no "tricks" involved, just a method that should be practiced in order to master it. The following are some more problems:

Problem Set 3.1.8

1. $\sqrt{75} \times \sqrt{27} =$ ____________________

2. $\sqrt{75} \times \sqrt{48} =$ ____________________

3. $\sqrt{44} \times \sqrt{99} =$ ____________________

4. $\sqrt{39} \times \sqrt{156} =$ ____________________

5. $\sqrt{27} \times \sqrt{48} =$ ____________________

6. $\sqrt{98 \times 8} =$ ____________________

7. $\sqrt{44 \times 11} =$ ____________________

8. $\sqrt{96 \times 24} =$ ____________________

9. $\sqrt{72 \times 18 =} =$ ____________________

10. $\sqrt{\mathbf{45}} \div \sqrt{\mathbf{80}} =$ ____________________

11. $\sqrt{\mathbf{28}} \div \sqrt{\mathbf{63}} =$ ____________________

12. $\sqrt[3]{125 \times 512} =$ ____________________

3.1.9 Finding Approximations of Square Roots

Seeing a problem like approximating $\sqrt{1234567}$ is very common in the middle of the test. The basic trick is you want to "take out" factors of 100 under the radical. Let's look at the above example after noticing that we can roughly approximate (within the margin of error) $\sqrt{1234567} \approx \sqrt{1230000}$. Now:

$$\sqrt{1230000} = \sqrt{123 \cdot 100 \cdot 100} = 10 \cdot 10\sqrt{123}$$

Now we are left with a much simpler approximation of the $100 \cdot \sqrt{123} \approx 100 \cdot 11 = \mathbf{1100}$.

You can follow the same procedure for cubed roots as well, only you need to find factors of 1000 under the radical to take out. Let's look at the example of $\sqrt[3]{1795953}$ after making the early approximation of $\sqrt[3]{1795953} \approx \sqrt[3]{1795000}$

$$\sqrt[3]{1795000} = \sqrt[3]{1795 \cdot 1000} = 10 \cdot \sqrt[3]{1795}$$

Well we should have memorized that $12^3 = 1728$ so we can form a rough approximation:

$$10 \cdot \sqrt[3]{1795} = 10 \cdot 12.1 = \mathbf{121}$$

So the trick is if you are approximating the n^{th} root of some number, you "factor out" sets of the n-digits and then approximate a much smaller value, then move the decimal place over accordingly.

Now in some instances you are asked to find the *exact* value of the cubed root. For example: $\sqrt[3]{830584}$. Now the procedure would be as followed:

1. Figure out how many digits you are going to have by noticing how many three-digit "sets" there are. Most will only be two digit numbers, however this is not guaranteed.
2. To find out the units digit, look at the units digit of the number given and think about what number cubed would give that result.
3. After that, you want to disregard the last three digits, and look at the remaining number and find out what number cubed is the first integer *less* than that value.

So to use the procedure give above for the problem of $\sqrt[3]{830584}$:

1. Well you have two, three-digit "sets" (the sets being 584 and 830). This means that we are looking for a two-digit number in our answer.
2. The last digit is 4, so what number cubed ends in a 4? The answer is that $\mathbf{4}^3 = 64$ so the last digit of the answer is **4**.
3. Now we disregard the first set of three (584) and look at the remaining numbers (830). So what number cubed is *less* than 830. Well we know $10^3 = 1000$ and $9^3 = 729$ so **9** is the largest integer so that when cubed is less than 830. So that is the tens digit.
4. The answer is **94**.

The following are problems so that you can practice this procedure of finding approximate and exact values of square and cubed roots.

Problem Set 3.1.9

1. (*) $\sqrt{15376} =$ ________________
2. $\sqrt[3]{830584} =$ ________________
3. (*) $\sqrt{23456} =$ ________________
4. (*) $\sqrt{32905} =$ ________________
5. (*) $\sqrt{6543210} =$ ________________
6. $\sqrt[3]{658503} =$ ________________
7. (*) $\sqrt{\mathbf{6213457}} =$ ________________
8. (*) $\sqrt{173468} =$ ________________

9. (*) $\sqrt{6420135} =$ ________________

10. (*) $\sqrt{\mathbf{872143}} =$ ________________

11. (*) $\sqrt{272727} =$ ________________

12. (*) $\sqrt{38527} =$ ________________

13. (*) $\sqrt{32323} =$ ________________

14. (*) $\sqrt{18220} =$ ________________

15. (*) $\sqrt{25252} =$ ________________

16. (*) $\sqrt{265278} =$ ________________

17. (*) $\sqrt{81818} =$ ________________

18. (*) $\sqrt{262626} =$ ________________

19. (*) $\sqrt{765432} =$ ________________

20. (*) $\sqrt{80808} =$ ________________

21. (*) $\sqrt{97531} =$ ________________

22. (*) $\sqrt{\mathbf{86420}} =$ ________________

23. (*) $\sqrt{8844} \times \sqrt{6633} =$ ________________

24. (*) $\sqrt[3]{217777} \times \sqrt{3777} \times 57 =$ ________________

25. (*) $\sqrt[3]{26789} \times \sqrt{911} \times 31 =$ ________________

26. (*) $\sqrt[3]{215346} \times \sqrt{3690} \times 57 =$ ________________

27. (*) $\sqrt[3]{2006 \times 6002} =$ ________________

28. (*) $\sqrt[3]{63489} \times \sqrt{1611} \times 41 =$ ________________

29. (*) $\sqrt[4]{14643} \times \sqrt[3]{1329} \times \sqrt{\mathbf{120}} =$ ________________

3.1.10 Complex Numbers

The following is a review of Algebra-I concerning complex numbers. Recall that $i = \sqrt{-1}$. Here are important definition concerning the imaginary number $a + bi$:

Complex Conjugate:	$a - bi$
Complex Modulus:	$\sqrt{a^2 + b^2}$
Complex Argument:	$\arctan \dfrac{b}{a}$

The only questions that are usually asked on the number sense test is multiplying two complex numbers and rationalizing a complex number. Let's look at examples of both:

Multiplication: $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$

Example: $(3 - 2i) \cdot (4 + i) = a + bi$, $a + b =$
Solution: $a = 3 \cdot 4 + 2 \cdot 1 = 14$ and $b = 3 \cdot 1 + (-2) \cdot 4 = -5$. So $a + b = 14 - 5 = \mathbf{9}$.

Rationalizing: $(a + bi)^{-1} = \dfrac{a - bi}{a^2 + b^2}$

Example: $(3 - 4i)^{-1} = a + bi$, $a - b =$?
Solution: $(3 - 4i)^{-1} = \dfrac{3 + 4i}{3^2 + 4^2} \Rightarrow a = \dfrac{3}{25}$ and $b = \dfrac{4}{25}$. So $a - b = \dfrac{3}{25} - \dfrac{4}{25} = -\dfrac{\mathbf{1}}{\mathbf{25}}$.

The following are some more practice problems about Complex Numbers:

Problem Set 3.1.10

1. $(4-i)^2 = a+bi$, $a =$ __________
2. $(6-5i)(6+5i) =$ __________
3. The conjugate of $(4i-6)$ is $a+bi$, $a =$ __________
4. $(5+i)^2 = a+bi$, $a =$ __________
5. $(9-3i)(3+9i) = a+bi$, $a =$ __________
6. $(8+3i)(3-8i) = a+bi$, $a =$ __________
7. $(2+3i) \div (2i) = a+bi$, $a =$ __________
8. $(3-4i)(3+4i) =$ __________
9. $(24-32i)(24+32i) =$ __________
10. $\mathbf{(5+12i)^2 = a+bi,\ a+b =}$ __________
11. $(3-5i)(2-5i) = a+bi$, $a+b =$ __________
12. $(2-5i)(3+5i) = a+bi$, $a =$ __________
13. $(2-5i)(3-4i) = a+bi$, $a-b =$ __________
14. $(4-3i)(2-i) = a+bi$, $a-b =$ __________
15. $(2+7i)(2-7i) = a+bi$, $a-b =$ __________
16. $(2+3i)(4+5i) = a+bi$, $a =$ __________
17. $(3+4i)^2 = a+bi$, $a =$ __________
18. The modulus of $14+48i$ is: __________
19. $(2-5i)^2 = a+bi$, $a+b =$ __________
20. $(5+4i)(3+2i) = a+bi$, $a =$ __________
21. $(0+4i)^2 = a+bi$, $b =$ __________
22. $(4+5i)(4-5i) =$ __________
23. The modulus of $(11+60i)^2$ is: __________
24. $(0-3i)^5 = a+bi$, $b =$ __________
25. $(3-5i)(2+i) = a+bi$, $a+b =$ __________
26. $\mathbf{(4-2i)(3-i) = a+bi,\ a+b =}$ __________
27. $(1+i)^9 =$ __________
28. $(2+3i) \div (3-2i) = a+bi$, $b =$ __________
29. $\mathbf{(2-3i) \div (3-2i) = a+bi,\ a =}$ __________
30. $(2i)^6 =$ __________
31. $(3+4i) \div (5i) = a+bi$, $a+b =$ __________
32. The modulus of $(24+7i)^2$ is: __________
33. $(3i-2) \div (3i+2) = a+bi$, $b =$ __________
34. The modulus of $(5+12i)^2$ is: __________

3.1.11 Function Inverses

Usually on the last column you are guaranteed to have to compute the inverse of a function at a particular value. The easiest way to do this is to not *explicitly* solve for the inverse and plug in the point but rather, compute the inverse at that point as you go. For example if you are given a function $f(x) = \frac{3}{2}x - 2$ and you want to calculate $f^{-1}(x)$ at the point $x = 3$, you *don't* want to do the standard procedure for finding inverses (switch the x and y variables and solve for y) which would be:

$$x = \frac{3}{2}y - 2 \Rightarrow y = (x+2)\cdot\frac{2}{3} \text{ at x=3: } \Rightarrow y = (3+2)\cdot\frac{2}{3} = \mathbf{\frac{10}{3}}$$

Not only do you solve for the function, you have to remember the function while you're plugging in numbers. An easier way is just switch the x and y variables, then plug in the value for x, then compute y. That way you aren't solving for the inverse function for *all* points, but rather the inverse at that particular point. Let's see how doing that procedure would look like:

$$x = \frac{3}{2}y - 2 \Rightarrow 3 = \frac{3}{2}y - 2 \Rightarrow y = (3+2)\cdot\frac{2}{3} = \mathbf{\frac{10}{3}}$$

Although this might not seem like much, it does help in saving some time.

Another important thing to remember when computing inverses is a special case when the function is in the form:

$$f(x) = \frac{ax+b}{cx+d} \Rightarrow f^{-1}(x) = \frac{-dx+b}{cx-a}$$

This was a very popular trick awhile back, but slowly it's appearance has been dwindling, however that does not mean a resurgence is unlikely. The important thing to remember is to line up the x's on the numerator and denominator so it is in the require form. Here is an example problem to show you the trick:

Example: Find $f^{-1}(2)$ where $f(x) = \frac{2x+3}{4+5x}$.

Solution: $f(x) = \frac{2x+3}{4+5x} = \frac{2x+3}{5x+4} \Rightarrow f^{-1}(x) = \frac{-4x+3}{5x-2} \Rightarrow f^{-1}(2) = \frac{-4\cdot 2+3}{5\cdot 2-2} = \mathbf{\frac{-5}{8}}$

Here are some problems to give you some practice:

Problem Set 3.1.11

1. $f(x) = 3x + 2$, $f^{-1}(-2) =$ ________________
2. $f(x) = \frac{4x}{5}$, $f^{-1}(2) =$ ________________
3. $f(x) = 2 - 3x$, $f^{-1}(1) =$ ________________
4. $f(x) = x^2 - 1$ and $x > 0$, $f^{-1}(8) =$ ________
5. $f(x) = 5 + 3x$, $f^{-1}(-2) =$ ________________
6. $f(x) = 4 - 3x$, $f^{-1}(2) =$ ________________
7. $f(x) = \frac{8}{3+x}$, $f^{-1}(2) =$ ________________
8. $f(x) = \frac{3-2x}{4}$, $f^{-1}(-1) =$ ________________
9. $f(x) = \frac{x^3}{3} + 3$, $f^{-1}(-6) =$ ________________
10. $\mathbf{f(x) = 2 - \frac{3x}{4}}$, $\mathbf{f^{-1}(5) =}$ ________________
11. $f(x) = 2x + 1$, $f^{-1}(3) =$ ________________
12. $g(x) = 3x + 2$, $g^{-1}(-1) =$ ________________

13. $h(x) = 2x - 3,\ h^{-1}(-1) =$ ____________

14. $f(x) = 2(x + 3),\ f^{-1}(-4) =$ ____________

15. $f(x) = 2 - 3x,\ f^{-1}(4) =$ ____________

16. $h(x) = 5x - 3,\ h^{-1}(2) =$ ____________

17. $\mathbf{h(x) = 5 - 3x,\ h^{-1}(-2)} =$ ____________

18. $f(x) = 2x + 2,\ f^{-1}(-2) =$ ____________

19. $f(x) = 3x - 3,\ f^{-1}(-3) =$ ____________

20. $f(x) = 4 - 4x,\ f^{-1}(-4) =$ ____________

21. $f(x) = \dfrac{3x - 1}{x - 3},\ f^{-1}(1) =$ ____________

22. $\mathbf{f(x) = \dfrac{2x + 1}{x - 2}, f^{-1}(3)} =$ ____________

23. $f(x) = \dfrac{3x - 1}{x - 3},\ f^{-1}(-1) =$ ____________

24. $\mathbf{f(x) = \dfrac{1 - 3x}{x + 3},\ f^{-1}(-2)} =$ ____________

3.1.12 Patterns

There is really no good trick to give you a quick answer to most pattern problems (especially the ones on the latter stages of the test). However, it is best to try to think of common things associated between the term number and the term itself. For example, you might want to keep in mind: squares, cubes, factorials, and Fibonacci. Let's look at some example problems:

Problem: Find the next term of $1, 5, 13, 25, 41, \ldots$
Solution I: So for this, notice that you are adding to each term $4, 8, 12, 16$ respectively. So each time you are incrementing the addition by 4 so, the next term will simply be $16 + 4$ added to 41 which is **61**.
Solution II: Another way of looking at this is to notice that $1 = 1^2 + 0^2$, $5 = 2^2 + 1^2$, $13 = 3^2 + 2^2$, $25 = 4^2 + 3^2$, $41 = 5^2 + 4^2$, so the next term is equal to $6^2 + 5^2 = \mathbf{61}$

Problem: Find the next term of $0, 7, 26, 63, \ldots$
Solution: For this one, notice that each term is one less than a cube: $0 = 1^3 - 1$, $7 = 2^3 - 1$, $26 = 3^3 - 1$, $63 = 4^3 - 1$, so the next term would be equal to $5^3 - 1 = \mathbf{124}$.

Here are some more problems to give you good practice with patterns:

Problem Set 3.1.12

1. Find the next term of $48, 32, 24, 20, 18, \ldots$: ____

2. Find the next term of $1, 4, 11, 26, 57, \ldots$: ____

3. Find the next term of $1, 8, 21, 40, \ldots$: ______

4. Find the next term of $0, 1, 5, 14, 30, 55, \ldots$: ____

5. **Find the next term of:**
   $\mathbf{2, 9, 28, 65, 126}, \ldots$: ____________

6. The next term of $1, 2, 6, 24, 120, \ldots$ is: ______

7. The next term of $2, 2, 4, 6, 10, 16, \ldots$ is: ______

8. Find the 9^{th} term of $1, 2, 4, 8, \ldots$: ________

9. **Find the** 10^{th} **term of:**
   $\mathbf{2, 6, 12, 20, 30}, \ldots$: ____________

10. Find the 100^{th} term of
    $2, 6, 10, 14, 18, \ldots$: ____________

11. The 10^{th} term of $2, 5, 10, 17, 26 \ldots$ is: ________

12. The next term of $1, 4, 10, 19, 31, \ldots$ is: ________

13. The 8^{th} term of $2, 9, 28, 65, 126, \ldots$ is: ________

14. The 8^{th} term of $0, 7, 26, 63, 124, \ldots$ is: ________

15. The next term of $1, 5, 6, 11, 17, 28, \ldots$ is: ________

16. Find the next term of $.0324, .054, .09, .15, \ldots$: _

3.1.13 Probability and Odds

Usually these problems involve applying the definitions of Odds and Probability which are:

$$\text{Probability} = \frac{\text{Desired Outcomes}}{\text{Total Outcomes}}$$
$$\text{Odds} = \frac{\text{Desired Outcomes}}{\text{Undesirable Outcomes}}$$

So the *probability* of rolling snake-eyes on a dice would be $\frac{1}{36}$ while the *odds* of doing this would be $\frac{1}{35}$. Usually the problems involving odds and probability on Number Sense tests are relatively simple where desired outcomes can be computed by counting. The following are some practice problems so you can be familiar with the types of problems asked:

Problem Set 3.1.13

1. The odds of drawing a king from a 52-card deck is: ________

2. If 2 dice are tossed, what is the probability of getting a sum of 11: ________

3. A bag has a 3 red, 6 white, and 9 blue marbles. What is the probability of drawing a red one: ________

4. Three coins are tossed. Find the odds of getting 3 tails: ________

5. The odds of losing are 4-to-9. The probability of winning is: ________

6. The probability of winning is $\frac{5}{9}$. The odds of losing is: ________

7. The odds of losing is $\frac{7}{13}$. The probability of winning is: ________

8. If three dice are tossed once, what is the probability of getting three 5's: ________

9. If all of the letters in the words "NUMBER SENSE" are put in a box, what are the odds of drawing an 'E': ________

10. **The probability of success if $\mathbf{\frac{8}{17}}$. The odds of failure is:** ________

11. **If all of the letters in the words "STATE MEET" were put in a box, what is the probability of drawing an 'E':** ____

12. A pair of dice is thrown, the odds that the sum is a multiple of 5 is: ________

13. The probability of losing is $44\frac{4}{9}\%$. The odds of winning is: ________

14. **The odds of winning the game is 3 to 5. The probability of losing the game is:** _

15. A number is drawn from $\{1, 2, 3, 6, 18\}$. The probability that the number drawn is not a prime number is: ____________

16. The odds of drawing a red 7 from a standard 52-card deck is: ____________

17. A number is randomly drawn from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. What are the odds that the number drawn is odd: ______

18. A number is drawn from the set $\{1, 2, 3, 4, 5\}$. What is the probability that the number drawn is a factor of 6: ____

19. The odds of randomly drawing a prime number from the set $\{1, 2, 3, 4, 5\}$ is: ____________

20. When two dice are tossed, the probability that the sum of the faces will be 3 is: ______

21. A pair of dice is thrown. The probability that their sum is 7 is: ____________

22. A pair of dice is thrown. The odds that their sum is 7 is: ____________

23. A pair of dice is thrown. The odds that the sum is 6 or 8 is: ____________

24. Two dice are tossed. What is the probability the sum is a multiple of 4: ______

25. **Two dice are tossed. What is the probability the sum is a multiple of 5:** ____________

26. A die is rolled. What is the probability that a multiple of 2 is shown: ____________

27. A die is rolled. What is the probability that a composite number is rolled: ______

28. A die is rolled. What is the probability that a factor of 12 is shown: ____________

29. The probability of losing is 4-to-7. What are the odds of winning: ____________

30. A pair of dice are rolled. What are the odds that the same number is shown: ______

3.1.14 Sets

Questions concerning sets are by far the easiest problems on the number sense tests. The only topics that are actively questioned are the definitions of intersection, union, compliment, and subsets. Let sets $A = \{M, E, N, T, A, L\}$ and $B = \{M, A, T, H\}$ then:

Intersection: The intersection between A and B (notated as $C = A \cap B$) is defined to be elements which are in *both* sets A and B. So in our case $C = A \cap B = \{M, A, T\}$ which consists of **3** elements.

Union: The union between A and B (notated as $D = A \cup B$) is defined to be a set which contains all elements in A and all elements in B. So $D = A \cup B = \{M, E, N, T, A, L, H\}$ which consists of **7** elements.

Complement: Let's solely look at set A and define a new set $E = \{T, E, N\}$. Then the complement of E (notated a variety of ways, typically $\bar{E}$ of E') with respect to Set A consists of simply all elements in A which aren't in E. So $\bar{E} = \{M, A, L\}$, which consists of three elements.

Subsets: The number of possible subsets of a set is 2^n where n is the number of elements in the set. The number of *proper* subsets consists of all subsets which are strictly in the set. The result is that this disregards the subset of the set itself. So the number of proper subsets is $2^n - 1$. So in our example, the number of subsets of A is $2^7 = 128$ and the number of proper subsets is $2^7 - 1 = 127$. Another way to ask how many different subsets a particular set has is asking how many elements are in a set's *Power Set*. So the number of elements in the Power Set of B is simply $2^4 = \mathbf{16}$.

The following are questions concerning general set theory on the number sense test:

Problem Set 3.1.14

1. Set B has 15 proper subsets. How many elements are in B: ____________

2. The number of subsets of $\{1, 3, 5, 7, 9\}$ is: ____________

3. The number of elements in the power set of $\{M, A, T, H\}$ is: ____________

4. If the power set for A contains 32 elements, then A contains how many elements: ____________

5. The number of distinct elements of $[\{t, w, o\} \cup \{f, o, u, r\}] \cap \{e, i, g, h, t\}$ is: ____________

6. The number of distinct elements of $\{m, a, t, h\} \cap \{e, m, a, t, i, c, s\}$ is: ____________

7. The number of distinct elements of $[\{f, i, v, e\} \cap \{s, i, x\}] \cup \{t, e, n\}$ is: ____________

8. If universal set $U = \{2, 3, 5, 7, 9, 11, 13, 17, 19\}$ and $A = \{3, 7, 13, 17\}$, then A' contains how many distinct elements: ____________

9. If the universal set $U = \{n, u, m, b, e, r, s\}$ and set $A = \{s, u, m\}$ then the complement of set A contains how many distinct elements:

10. The universal set $U = \{n, u, m, b, e, r, s\}$, $A \subset U$ and $A = \{e, u\}$, then the complement of A contains how many elements: ____________

11. The number of distinct elements in $[\{z, e, r, o\} \cap \{o, n, e\}] \cup \{t, w, o\}$ is: ____________

12. The number of distinct elements in $[\{m, e, d, i, a, n\} \cap \{m, e, a, n\}] \cap \{m, o, d, e\}$ is:

13. The set $\{F, U, N\}$ has how many subsets: ____________

14. The set $\{T, W, O\}$ has how many proper subsets: ____________

15. Set A has 32 subsets. How many elements are in A: ____________

16. The set P has 63 proper subsets. How many elements are in P: ____________

17. Set A has 15 proper subsets. How many elements are in A: ____________

18. **The set A has 8 distinct elements. How many proper subsets with at least one element does A have:** ____________

19. Set $A = \{a, b, c, d\}$. How many proper subsets does set A have: ____________

20. The number of proper subsets of $\{M, A, T, H\}$ is: ____________

21. Set $A = \{o, p, q, r, s\}$ has how many improper subsets: ____________

3.2 Changing Bases

3.2.1 Converting Integers

One of the topics I've found rather difficult teaching to students is the concept of changing bases. It seems that students have the concept of a base-10 system so ingrained in their mind (almost always unbeknownst to them) that it is difficult considering other base systems. Hopefully this section will be a good introduction to the process of changing bases and doing basic operations in other number systems. First, let's observe how we look at numbers in the usual base-10 fashion.

Everyone knows that 1254 means that you have one-thousand, two-hundred, and fifty-four of something, but expressing this in an unusual manner we can say:

$$1294 = 1 \cdot 1000 + 2 \cdot 100 + 5 \cdot 10 + 4 \cdot 1 = 1 \cdot 10^3 + 2 \cdot 10^2 + 9 \cdot 10^1 + 4 \cdot 10^0$$

From this we can see where this concept of "base-10" comes from, we are adding combinations of these powers of tens (depending on what $0-9$ digit we multiply by). So, you can express any integer n in base-10 as:

$$n = a_m \cdot 10^m + a_{m-1}10^{m-1} + a_{m-2} \cdot 10^{m-2} + \cdots a_1 \cdot 10^1 + a_0 \cdot 10^0$$

Where all a_m's are integers ranging from $0-9$.

The fact that we are summing these various powers of 10 is completely an arbitrary one. We can easily change this to some other integer (like 6 for example) and develop a base-6 number system. Let's see what it would look like:

$$n = a_m \cdot 6^m + a_{m-1}6^{m-1} + a_{m-2} \cdot 6^{m-2} + \cdots a_1 \cdot 6^1 + a_0 \cdot 6^0$$

Where all a_m's are integers ranging from $0-5$.

So to use an example, let look at what the number 123_6 (where the subscript denotes we are in base-6) would look like in our usual base-10 system:

$$123_6 = 1 \cdot 6^2 + 2 \cdot 6^1 + 3 \cdot 6^0 = 1 \cdot 36 + 2 \cdot 6 + 3 \cdot 1 = 36 + 12 + 3 = \mathbf{51_{10}}$$

From this we have found the way to convert any base-n whole number to base-10!

Let's look at another example:

$$3321_4 = 3 \cdot 4^3 + 3 \cdot 4^2 + 2 \cdot 4^1 + 1 \cdot 4^0 = 3 \cdot 64 + 3 \cdot 16 + 2 \cdot 4 + 1 \cdot 1 = 192 + 48 + 8 + 1 = \mathbf{249_{10}}$$

So now that we know how to convert from base-n to base-10, let's look at the process on how to convert the opposite direction:

1. Find the highest power of n which divides the base-10 number (let's say it is the m^{th} power).
2. Figure out how many times it divides it and that will be your $(m+1)^{th}$ digit in base-n.
3. Take the remainder and figure out how many times one less than the highest power divides it (so see how many times n^{m-1} divides it). That is your $(m)^{th}$ digit.
4. Take the remainder, and continue process.

I know that this might seem complicated, but let's look at an example we have already done in the "forward" direction to illustrate how to go "backwards." Convert 51_{10} to base-6:

1. Well we know $6^2 = 36$ and $6^3 = 216$, so the highest power which divides 51 is 6^2.
2. 36 goes into 51 one time, so our 3^{rd} digit is **1**.
3. The remainder when dividing 51 by 36 is 15.
4. Now we see how many times 6^1 goes into 15 (which is 2 times, so our 2^{nd} digits is **2**).
5. The remainder when dividing 15 by 6 is 3.
6. $6^0 = 1$ divides 3 obviously 3 times, so our 1^{st} digit is **3**
7. So after conversion, $51_{10} = \mathbf{123_6}$, which corresponds to what we expected.

As a warning, some digits might be zero when you do the base conversion. Let's look at an example of this: Convert 18_{10} to base-4:

$4^2 = 16$ and $4^3 = 64$, so $4^2 = 16$ goes into 18 once with a remainder of 2:	Third Digit is **1**
Now $4^1 = 4$ doesn't go into 2:	Second Digit is **0**
4^0 goes into 2 twice:	First Digit is **2**
Answer:	$\mathbf{102_4}$

This seems like a lot of steps in making a base conversion, but after substantial practice, it will become second nature. Here are some practice problems with just converting bases from base-n to base-10 and reverse.

Problem Set 3.2.1

1. $212_5 =$ __________ $_{10}$
2. $108 =$ __________ $_4$
3. $2004_5 =$ __________ $_{10}$
4. $3^4 + 3 =$ __________ $_9$
5. $2^4 + 2 =$ __________ $_4$
6. $82 =$ __________ $_5$
7. $4^3 + 4 =$ __________ $_4$
8. $24 =$ __________ $_2$
9. $3^3 + 3 =$ __________ $_3$
10. $\mathbf{48} =$ __________ $_{\mathbf{3}}$
11. $\mathbf{4^3 + 2^3} =$ __________ $_{\mathbf{8}}$
12. $2^4 + 1 =$ __________ $_8$
13. $200_{10} =$ __________ $_7$
14. $72 + 18 + 4 =$ __________ $_6$
15. $234_{10} =$ __________ $_5$
16. $123_4 =$ __________ $_{10}$
17. $2^5 + 2 =$ __________ $_4$
18. $430_{10} =$ __________ $_5$
19. $\mathbf{540_{10}} =$ __________ $_{\mathbf{6}}$
20. $\mathbf{243 + 27 + 3} =$ __________ $_{\mathbf{9}}$
21. $200_5 =$ __________ $_{10}$
22. $200_6 =$ __________ $_{10}$
23. $4^4 + 4^2 + 4^0 =$ __________ $_4$
24. $3^3 + 3^2 + 3^0 =$ __________ $_3$
25. $216 + 108 + 30 + 5 =$ __________ $_6$
26. $44_b = 40$, then $b =$ __________
27. $123_{10} =$ __________ $_5$
28. $123_4 =$ __________ $_5$

29. $8^2 + 2^4 + 4^0 =$ ________________$_4$

30. $\mathbf{234_5} =$ ________________$_{\mathbf{4}}$

31. $\mathbf{686 + 98 + 14} =$ ________________$_{\mathbf{7}}$

32. $77_{10} =$ ________________$_7$

33. $4^3 + 4 =$ ________________$_8$

34. $234_5 =$ ________________$_{10}$

35. $3^4 + 3^2 + 3^0 =$ ________________$_3$

36. $123_{10} =$ ________________$_4$

37. $125 + 75 + 15 + 1 =$ ________________$_5$

38. $234_{10} =$ ________________$_5$

39. $1728 + 288 + 36 + 4 =$ ________________$_{12}$

40. $\mathbf{128 + 48 + 12 + 2} =$ ________________$_{\mathbf{4}}$

41. Find b when $4b_6 = 29$: ________________

42. $45_6 =$ ________________$_9$

43. $210_4 =$ ________________$_6$

44. $43_8 =$ ________________$_9$

45. $34_5 =$ ________________$_7$

3.2.2 Converting Decimals

In the similar manner of how we analyzed an integer n in base-10, we can took at decimals in base-10 as well. For example, let's look at how we see .125 in base-10

$$.125 = 1 \cdot (.1) + 2 \cdot (.01) + 5 \cdot (.001) = 1 \cdot 10^{-1} + 2 \cdot 10^{-2} + 5 \cdot 10^{-3}$$

You can display this in terms of fractions as well:

$$= \frac{1}{10} + \frac{2}{100} + \frac{5}{1000} = \frac{1}{10} + \frac{1}{50} + \frac{1}{200} = \frac{20+4+1}{200} = \frac{1}{8}$$

Similar to the previous session, we can replace the powers of ten by the power of any fraction. Let's look at converting $.321_6$ to a base-10 fraction:

$$.321_6 = \frac{3}{6} + \frac{2}{36} + \frac{1}{216} = \frac{108+12+1}{216} = \frac{121}{216}$$

Because of the complexity and calculations involved, going in the reverse direction is seldom (if ever) used on a number sense test. In addition, the test usually asks for a base-10 fraction representation (be sure to reduce!). Here are some practice problems to help you familiarize yourself with this process:

Problem Set 3.2.2

1. Change $.32_5$ to a base-10 fraction: ________________

2. Change $.34_5$ to a base-10 fraction: ________________

3. **Change $\mathbf{.111_7}$ to a base-**10 **fraction:** ________________

4. Change $.33_4$ to a base-10 fraction: ________________

5. **Change $.234_5$ to a base-10 fraction:** ______________

6. Change $.44_8$ to a base-10 fraction: ______________

7. Change $.33_6$ to a base-10 fraction: ______________

8. Change $.66_{12}$ to a base-10 fraction: ______________

9. Change $.202_5$ to a base-10 fraction: ______________

10. Change $.55_6$ to a base-10 fraction: ______________

11. **Change $.444_5$ to a base-10 fraction:** ______________

12. Change $.44_5$ to a base-10 decimal: ______________

13. Change $.14_5$ to a base-10 decimal: ______________

14. Change $\frac{9}{16}$ to a base-4 decimal: ______________

15. Change $\frac{35}{36}$ to a base-6 decimal: ______________

16. Change $\frac{15}{16}$ to a base-4 decimal: ______________

17. Change $\frac{15}{16}$ to a base-8 decimal: ______________

18. Change $\frac{11}{25}$ to a base-5 decimal: ______________

19. Change $\frac{30}{49}$ to a base-7 decimal: ______________

3.2.3 Performing Operations

For some basic operations in other bases, sometimes it is simpler to convert all numbers to base-10, perform the operations, then convert back to base-n. Let's look at an example where I would employ this technique:

$$23_4 \times 3_4 + 12_4 = 11 \times 3 + 6 = 39 = 213_4$$

However, when numbers are larger, this might not be the case, so let's look at the most popular operations on the number sense test which are addition (and subsequently subtraction) and multiplication (division is usually not tested, so I will exclude explaining this operation).

<u>Addition:</u>
For addition of two integers in base-10 we sum the digits one at a time writing down the answer digit $(0-9)$ and carrying when necessary. Other base-n work in the same manner with the only difference being the answer digits range from 0 to $(n-1)$. Let's look at an example:

	First Digit:	$4_6 + 3_6$	$\mathit{1}\mathbf{1}_6$
	Second Digit:	$5_6 + 2_6 + \mathit{1}_6$	$\mathit{1}\mathbf{2}_6$
$124_6 + 53_6 =$	Third Digit:	$1_6 + \mathit{1}_6$	$\mathbf{2}_6$
	Answer:		$\mathbf{221_6}$

<u>Subtraction:</u>
Subtraction works in a similar method, only the concept of "borrowing" when you can't subtract the digits is slightly altered. When you "borrow" in base-10 you lower the digit you are borrowing from and then

"add" 10 to the adjacent digit to aid in the subtraction. In a different base-n, you will be borrowing in the same fashion but adding n to the adjacent digit. Let's look at an example:

	First Digit:	Since you "can't" do $2-3$ you have to borrow	
		$(\mathit{4}_4 + 2_4) - 3_4$	$\mathbf{3}_4$
$122_4 - 13_4 =$	Second Digit:	$(2_4 - \mathit{1}_4) - 1_4$	$\mathbf{0}_4$
	Third Digit:	1_4	$\mathbf{1}_4$
	Answer:		$\mathbf{103}_4$

In the above expressions, everything in italics represents the borrowing process. When borrowing from the second digit, you lower it by 1 (seen by the $-\mathit{1}_4$) and then add to the adjacent digit (the first one) $\mathit{4}_4$.

Multiplication:
What I like to do for multiplication in a different base is essentially perform the FOILing procedure in base-10 then convert your appropriate result to base-n and apply appropriate carry rules. Let's look at a couple of examples (one involving carries and the other one not):

	First Digit:	$1 \times 3 = 3_{10}$	$\mathbf{3}_9$
	Second Digit:	$1 \times 1 + 2 \times 3 = 7_{10}$	$\mathbf{7}_9$
$13_9 \times 21_9 =$	Third Digit:	$2 \times 1 = 2_{10}$	$\mathbf{2}_9$
	Answer:		$\mathbf{273}_9$

The above scenario was simple because no carries were involved and converting those particular single digits from base-10 to base-9 was rather simple. Let's look at one with carries involved:

	First Digit:	$3 \times 5 = 15_{10}$	$\mathit{1}\mathbf{6}_9$
	Second Digit:	$3 \times 4 + 2 \times 5 + \mathit{1} = 23_{10}$	$\mathit{2}\mathbf{5}_9$
$45_9 \times 23_9 =$	Third Digit:	$2 \times 4 + \mathit{2} = 10_{10}$	$\mathit{1}\mathbf{1}_9$
	Fourth Digit:	$\mathit{1}$	$\mathbf{1}$
	Answer:		$\mathbf{1156}_9$

The above example shows the procedure where you do the FOILing in base-10 then convert that to base-9, write down last digit, carry any remaining digits, repeat procedure. As one can see to perform multiplication in other bases it is important to have changing bases automatic so that the procedure is relatively painless.

To practice the above three operations here are some problems:

Problem Set 3.2.3

1. $112_6 + 4_6 =$ ____________________ $_6$
2. $53_6 \times 4_6 =$ ____________________ $_6$
3. $101_2 - 11_2 =$ ____________________ $_2$
4. $44_5 \times 4_5 =$ ____________________ $_5$

5. $26_9 \div 6_9 =$ ____________________ $_9$

6. $37_8 + 56_8 =$ ____________________ $_8$

7. $88_9 + 82_9 =$ ____________________ $_9$

8. $100_6 - 44_6 =$ ____________________ $_6$

9. $104_8 - 47_8 =$ ____________________ $_8$

10. $143_5 \div 4_5 =$ ____________________ $_5$

11. $\mathbf{22_9 - 66_9} =$ ____________________ $\mathbf{_9}$

12. $\mathbf{135_7 \times 4_7} =$ ____________________ $\mathbf{_7}$

13. $132_4 - 33_4 =$ ____________________ $_4$

14. $42_5 - 34_5 + 23_5 =$ ____________________ $_5$

15. $\mathbf{123_5 \times 4_5} =$ ____________________ $\mathbf{_5}$

16. $\mathbf{33_4 \times 3_4 - 21_4} =$ ____________________ $\mathbf{_4}$

17. $22_7 \times 4_7 =$ ____________________ $_7$

18. $33_6 \times 3_6 =$ ____________________ $_6$

19. $22_6 + 33_6 + 44_6 =$ ____________________ $_6$

20. $44_8 \times 4_8 =$ ____________________ $_8$

21. $32_6 \div 5_6 \times 4_6 =$ ____________________ $_6$

22. $24_7 \div 6_7 + 24_7 =$ ____________________ $_7$

23. $23_6 + 45_6 - 50_6 =$ ____________________ $_6$

24. $\mathbf{23_5 \times 4_5 - 10_5} =$ ____________________ $\mathbf{_5}$

25. $123_4 \div 3_4 =$ ____________________ $_4$

26. $431_5 \div 4_5 =$ ____________________ $_5$

27. $222_3 \times 2_3 =$ ____________________ $_3$

28. $(21_5 - 12_5) \times 11_5 =$ ____________________ $_5$

29. $(33_4 + 22_4) \times 11_4 =$ ____________________ $_4$

30. $235_6 \div 5_6 =$ ____________________ $_6$

31. $\mathbf{543_7 \div 6_7} =$ ____________________ $\mathbf{_7}$

32. $234_5 + 432_5 =$ ____________________ $_5$

33. $33_4 \times 2_4 - 11_4 =$ ____________________ $_4$

34. $44_5 \times 2_5 + 33_5 =$ ____________________ $_5$

35. $(13_5 + 12_5) \times 11_5 =$ ____________________ $_5$

36. $11_4 \times 21_4 - 3_4 =$ ____________________ $_4$

37. $12_5 + 23_5 + 34_5 =$ ____________________ $_5$

38. $\mathbf{(22_4 + 33_4) \times 11_4} =$ ____________________ $\mathbf{_4}$

3.2.4 Changing Between Bases: Special Case

When changing between two bases m and n, the standard procedure is to convert the number from base-m to base-10 then convert that into base-n. However, there are special cases when the middle conversion into base-10 is unnecessary: when n is an integral power of m (say $n = m^a$, a an integer) or vice versa. The procedure is relatively simple, take the digits of m in groups of a and convert each group into base-n. For example, if we are converting 1001001_2 into base-4, you would take 1001001 in groups of two (since $2^{\mathbf{2}} = 4$) and converting each group into base-4. Let's see how it would look:

Convert 1001001_2 to base-4	First Digit:	01_2	$\mathbf{1}_4$
	Second Digit:	10_2	$\mathbf{2}_4$
	Third Digit:	00_2	$\mathbf{0}_4$
	Fourth Digit:	1_2	$\mathbf{1}_4$
	Answer:		$\mathbf{1021}_4$

Let's look at an example where the converting base is that of the original base cubed (so you would take it in groups of 3):

Convert 110001011_2 to base-8	First Digit:	011_2	$\mathbf{3}_8$
	Second Digit:	001_2	$\mathbf{1}_8$
	Third Digit:	110_2	$\mathbf{6}_8$
	Answer:		$\mathbf{613}_8$

Similarly, you can perform the method in reverse. So when converting from base-9 to base-3 you would take each digit in base-9 and convert it to two-digit base-3 representation. For example:

Convert 643_9 to base-3	First/Second Digits:	3_9	$\mathbf{10}_3$
	Third/Fourth Digits:	4_9	$\mathbf{11}_3$
	Fifth/Sixth Digits:	6_9	$\mathbf{20}_3$
	Answer:		$\mathbf{201110}_3$

Problem Set 3.2.4

1. $46_9 =$ ________________$_3$
2. $48_9 =$ ________________$_3$
3. $1011011_2 =$ ________________$_8$
4. $123_4 =$ ________________$_2$
5. $\mathbf{2122_3} =$ ________________$_\mathbf{9}$
6. $345_8 =$ ________________$_2$
7. $123_4 =$ ________________$_2$
8. $\mathbf{101011_2} =$ ________________$_\mathbf{4}$
9. $231_4 =$ ________________$_2$
10. $432_8 =$ ________________$_2$
11. $312_4 =$ ________________$_2$
12. $1111_2 =$ ________________$_4$
13. $1011_2 =$ ________________$_4$
14. $123_4 =$ ________________$_2$
15. $\mathbf{11011_2} =$ ________________$_\mathbf{4}$

3.2.5 Changing Bases: Sum of Powers

When asked the sum of a series of powers of two $(1+2+4+8+\cdots+2^n)$, it is best to represent the number in binary, then we can see the result. For example purposes let's look at the sum $1+2+4+8+16+32+64$. If we represented them as binary it would be:

$$1+2+4+8+16+32+64 = 1\cdot 2^0 + 1\cdot 2^1 + 1\cdot 2^2 + 1\cdot 2^3 + 1\cdot 2^4 + 1\cdot 2^5 + 1\cdot 2^6 = 1111111_2$$

$$1111111_2 = 10000000_2 - 1_2 \Rightarrow 2^7 - 1 = 128 - 1 = \mathbf{127}$$

Although this method is easiest with binary, you can apply it to other powers as well, as long as you are carefully. For example:

$$2 + 2\cdot 3 + 2\cdot 9 + 2\cdot 27 + 2\cdot 81 + 2\cdot 243 = 2\cdot 3^0 + 2\cdot 3^1 + 2\cdot 3^2 + 2\cdot 3^3 + 2\cdot 3^4 + 2\cdot 3^5 = 222222_3$$

$$222222_3 = 1000000_3 - 1 = 3^6 - 1 = \mathbf{728}$$

3.2.6 Changing Bases: Miscellaneous Topics

There are a handful of topics involving changing bases that rely on understanding other tricks previously discussed in this book. Take this problem for example:

Problem: Convert the decimal $.333\cdots_7$ into a base-10 fraction.
Solution: The above problem relies on using the formula for the sum of an infinite geometric series:

$$.333\cdots_7 = \frac{3}{7} + \frac{3}{49} + \frac{3}{343} + \cdots = \frac{\frac{3}{7}}{1-\frac{1}{7}} = \frac{3}{7}\times\frac{7}{6} = \mathbf{\frac{1}{2}}$$

Another problem which relies on understanding of how the derivation of finding the remainder of a number when dividing by 9, only in a different base is:

Problem: The number $123456_7 \div 6$ has what remainder?
Solution: The origins of this is rooted in modular arithmetic (see Section 3.4) and noticing that: $7^n \cong 1 (\text{mod } 6)$. So our integer can be represented as:

$$123456_7 = 1\cdot 7^5 + 2\cdot 7^4 + 3\cdot 7^3 + 4\cdot 7^2 + 5\cdot 7^1 + 6\cdot 7^0 \cong (1+2+3+4+5+6) = \frac{6\cdot 7}{2} = 21 \cong \mathbf{3}(\text{mod } 6)$$

So an important result is that when you have a base-n number and divide it by $n-1$, all you need to do is sum the digits and see what the remainder *that is* when dividing by $n-1$.

Problem Set 3.2.6

1. $.555\ldots_7 =$ ______________________ $_{10}$
2. The remainder when 123456_7 is divided by 6 is: ______________________
3. $.666\ldots_8 =$ ______________________ $_{10}$
4. $.777\ldots_9 =$ ______________________ $_{10}$
5. $\mathbf{.111\ldots_5} =$ ______________________ $_{\mathbf{10}}$

3.3 Repeating Decimals

The following sections are concerned with expressing repeating decimals as fractions. All of the problems of this nature have their root in sum of infinite geometric series.

3.3.1 In the form: $.aaaaa\ldots$

Any decimal in the form $.aaaaa\ldots$ can be re written as:

$$.aaaa\ldots = \frac{a}{10} + \frac{a}{100} + \frac{a}{1000} + \cdots$$

Which we can sum appropriately using the sum of an infinite geometric sequence with the common difference being $\frac{1}{10}$ (See Section 2.2.1):

$$\frac{a}{10} + \frac{a}{100} + \frac{a}{1000} + \cdots = \frac{\frac{a}{10}}{1 - \frac{1}{10}} = \frac{a}{10} \times \frac{10}{9} = \frac{\mathbf{a}}{\mathbf{9}}$$

Which is what we expected knowing what the fractions of $\frac{1}{9}$ are. For example:

$$.44444\ldots = \frac{\mathbf{4}}{\mathbf{9}}$$

3.3.2 In the form: $.ababa\ldots$

In a similar vein, fractions in the form $.ababab\ldots$ can be treated as:

$$.ababab\ldots = \frac{ab}{100} + \frac{ab}{10000} + \frac{ab}{1000000} + \cdots = \frac{\frac{ab}{100}}{1 - \frac{1}{100}} = \frac{ab}{100} \times \frac{100}{99} = \frac{\mathbf{ab}}{\mathbf{99}}$$

Where ab represents the digits (not $a \times b$). Here is an example:

$$.242424\ldots = \frac{24}{99} = \frac{\mathbf{8}}{\mathbf{33}}$$

You can extend the concept for any sort of continuously repeating fractions.
For example, $.abcabcabc\ldots = \frac{abc}{999}$, and so on.

Here are some practice problems to help you out:

Problem Set 3.3.2

1. $.272727\ldots =$ ______________________
2. $.414141\ldots =$ ______________________
3. $.212121\ldots =$ ______________________
4. $\mathbf{.818181}\ldots =$ ______________________
5. $.363636\ldots =$ ______________________
6. $.020202\ldots =$ ______________________
7. $.727272\ldots =$ ______________________
8. $.151515\ldots =$ ______________________
9. $.308308\ldots =$ ______________________
10. $.231231\ldots =$ ______________________

11. $.303303\ldots =$ ______________________ 12. $.099099099\ldots =$ ______________________

3.3.3 In the form: $.abbbb\ldots$

Fractions in the form $.abbbb\ldots$ are treated in a similar manner (sum of an infinite series) with the inclusion of one other term (the .a term). Let's see how it would look:

$$.abbb\ldots = \frac{a}{10} + \frac{b}{100} + \frac{b}{1000} + \cdots = \frac{a}{10} + \frac{\frac{b}{100}}{1 - \frac{1}{10}} = \frac{a}{10} + \frac{b}{90}$$

However we can continue and rewrite the fraction as:

$$\frac{a}{10} + \frac{b}{90} = \frac{9 \cdot a + b}{90} = \frac{(10 \cdot a + b) - a}{90}$$

Lets take a step back to see what this means. The numerator is composed of the sum $(10 \cdot a + b)$ which represents the two-digit number ab. Then you subtract from that the non-repeating digit and place that result over 90. Here is an example to show the process:

$$.27777\ldots = \frac{27 - 2}{90} = \frac{25}{90} = \mathbf{\frac{5}{18}}$$

Here are some more problems to give you more practice:

Problem Set 3.3.3

1. $.23333\ldots =$ ______________________
2. $.32222\ldots =$ ______________________
3. $.21111\ldots =$ ______________________
4. $.32222\ldots =$ ______________________
5. $\mathbf{.01222}\ldots =$ ______________________

3.3.4 In the form: $.abcbcbc\ldots$

Again, you can repeat the process above for variances. In this example we can represent $.abcbc\ldots$ can be represented in fraction form:

$$.abcbcbc\ldots = \frac{abc - a}{990}$$

Where the abc represents the three-digit number abc (not the product $a \cdot b \cdot c$). Here is an example:

$$.437373737\ldots = \frac{437 - 4}{990} = \mathbf{\frac{433}{990}}$$

It is important for the Number Sense test to reduce all fractions. This can sometimes be the tricky part. The easiest way to check for reducibility is to see if you can divide the numerator by $2, 3$,or 5. In the above example, 433 is not divisible by $2, 3, 5$ so the fraction is in its lowest form.

Here is an example where you can reduce the fraction:

$$.2474747\ldots = \frac{247 - 2}{990} = \frac{245}{990} = \mathbf{\frac{49}{198}}$$

Problem Set 3.3.4

1. $.2131313\ldots =$ ______________

2. $.1232323\ldots =$ ______________

3. $.2313131\ldots =$ ______________

4. $\mathbf{.3050505}\ldots =$ ______________

5. $.2050505\ldots =$ ______________

6. $.3141414\ldots =$ ______________

7. $.2717171\ldots =$ ______________

8. $.2353535\ldots =$ ______________

9. $.0474747\ldots =$ ______________

10. $.2141414\ldots =$ ______________

11. $.1232323\ldots =$ ______________

3.4 Modular Arithmetic

A lot has been made about the uses of modular arithmetic (for example all of the sections dealing with finding remainders when dividing by $3, 9, 11$, etc...). Here is a basic understanding of what is going on with modular arithmetic.

When dividing two numbers a and b results in a quotient q and a remainder of r we say that $a \div b = q + \frac{r}{b}$. With modular arithmetic, we are only concerned with the remainder so the expression of $a \div b = q + \frac{r}{b} \Rightarrow a \cong r(\text{mod } b)$.

So you know $37 \div 4$ has a remainder of 1, so we say $37 \cong 1(\text{mod } 4)$. As noted before, what's great about modular arithmetic is you can do the algebra of remainders (See: Section 1.4.5, Remainders of Expressions). From this phrase alone is where all of our divisibility rules come from. For example, let's see where we get our divisibility by 9 rule:

Recall we can express any base-10 number n by: $n = a_m 10^m + a_{m-1}10^{m-1} + \cdots + a_1 10^1 + a_0 10^0$

So when we are trying to see the remainder when dividing by 9, we want to find what x is in the expression:

$$n \cong x(\text{mod } 9)$$

However we do know that $10 \cong 1(\text{mod } 9)$, meaning $10^a \cong 1(\text{mod } 9)$ for all $a \geq 0$. So:

$$n = a_m 10^m + a_{m-1}10^{m-1} + \cdots + a_1 10^1 + a_0 10^0 \cong (a_m + a_{m-1} + \cdots + a_1 + a_0)(\text{mod } 9)$$

Well $a_m + a_{m-1} + \cdots + a_1 + a_0$ is just the sum of the digits, so we just proved that in order for a number n to be divisible by 9 then the sum of it's digits have to be divisible by 9.

Learning the basics in modular arithmetic is not only crucial for recognizing and forming divisibility rules but also they pop up as questions on the number sense test. For example:
Problem: Find x, $0 \leq x \leq 4$, if $x + 3 \cong 9(\text{mod } 5)$. *Solution:* Here we know that $9 \cong 4(\text{mod } 5)$, so the problem reduces to finding x restricted to $0 \leq x \leq 4$ such that $x + 3 \cong 4(\text{mod } 5)$, which simply makes $x = \mathbf{1}$.

The following are some more problems to get you some practice on modular arithmetic:

Problem Set 3.4

1. $x + 6 \cong 9 \pmod 7$,
   $0 \le x \le 6$, then $x =$ ________

2. $4^7 \div 7$ has a remainder of: ________

3. $2^5 \times 3^5 \div 5$ has a remainder of: ________

4. $2^6 \times 3^4 \div 5$ has a remainder of: ________

5. **$\mathbf{8^7 \div 6}$ has a remainder of:** ________

6. **If N is a positive integer and $\mathbf{4N \div 5}$ has a remainder of** 2 **then $\mathbf{N \div 5}$ has a remainder of:** ________

7. $x + 3 \cong 9 \pmod 5$,
   $0 \le x \le 4$, then $x =$ ________

8. $x + 4 \cong 1 \pmod 8$,
   $0 \le x \le 7$, then $x =$ ________

9. $3^8 \div 7$ has a remainder of: ________

10. $3x \cong 17 \pmod 5$,
    $0 \le x \le 5$, then $x =$ ________

11. $3x - 2 \cong 4 \pmod 7$,
    $0 \le x \le 7$, then $x =$ ________

12. $6^8 \div 7$ has a remainder of: ________

13. $3^7 \div 7$ has a remainder of: ________

14. $5^4 \div 11$ has a remainder of: ________

3.5 Fun with Factorials!

All of these problems incorporate common factorial problems.

3.5.1 $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$

The sum of $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$ is a fairly simple problem if you know the formula (its derivation is left to the reader).

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

The simplest case would be to compute sums like:

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! = (4+1)! - 1 = 120 - 1 = \mathbf{119}$$

There are slight variations which could be asked (the easiest of which would be leaving out some terms).

$$1 \cdot 1! + 3 \cdot 3! + 5 \cdot 5! = (5+1)! - 1 - 2 \cdot 2! - 4 \cdot 4! = 720 - 1 - 4 - 96 = \mathbf{619}$$

The following are some practice problems:

Problem Set

1. $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + 5 \cdot 5! =$ ________

2. $1 \cdot 1! + 2 \cdot 2! + \cdots + 6 \cdot 6! =$ ________

3. $\mathbf{1 \cdot 1! + 2 \cdot 2! + \cdots + 7 \cdot 7!} =$ ________

4. $1 \cdot 1! - 2 \cdot 2! - 3 \cdot 3! - 4 \cdot 4! =$ ________

5. $\mathbf{2 \cdot 1! + 3 \cdot 2! + 4 \cdot 3! + 5 \cdot 4!} =$ ________

3.5.2 $\frac{a! \pm b!}{c!}$

This problem has pretty much nothing to do with factorials and mostly with basic fraction simplification. Take the following example:

$$\frac{8!+6!}{7!} = \frac{8!}{7!} + \frac{6!}{7!} = \mathbf{8\frac{1}{7}}$$

Sometimes it is easier to just factor out the common factorial, for example:

$$\frac{3!+4!-5!}{3!} = \frac{3! \cdot (1+4-5\cdot 4)}{3!} = 1+4-20 = \mathbf{-15}$$

Problem Set 3.5.2

1. $\frac{8!+6!}{7!} =$ ________________
2. $\frac{10!+8!}{9!} =$ ________________
3. $\frac{7!-5!}{6!} =$ ________________
4. $\mathbf{\frac{11!-9!}{10!}} =$ ________________
5. $\mathbf{\frac{10!-11!}{9!}} =$ ________________
6. $6 \cdot 5 \cdot 4! - 5! =$ ________________
7. $(2!+3!) \div 5! =$ ________________
8. $(2! \times 3!) - 4! =$ ________________
9. $7! \div 6! - 5! =$ ________________
10. $7 \times 5! - 6! =$ ________________
11. $2! - 3! \times 5! =$ ________________
12. $8! \div 6! - 4! =$ ________________
13. $\mathbf{\frac{5! \cdot 4!}{6!}} =$ ________________
14. $\frac{4 \times 5! - 5 \times 4!}{4!} =$ ________________
15. $\frac{4 \times 5! + 5 \times 4!}{4!} =$ ________________
16. $\frac{6 \times 7! - 7 \times 6!}{6!} =$ ________________
17. $\frac{10 \times 9! - 10! \times 9}{9!} =$ ________________
18. $\frac{8! \times 7 - 8 \times 7!}{7!} =$ ________________
19. $\mathbf{\frac{11 \times 10! - 11! \times 10}{11!}} =$ ________________
20. $\mathbf{6! \div (3! \times 2!)} =$ ________________

3.5.3 Wilson's Theorem

I've seen a couple of questions in the latter stages of the number sense question which asks something along the lines of:

$$6! \cong x(\text{mod } 7),\ 0 \leq x \leq 6,\ x = ?$$

Questions like this use the result from Wilson' Theorem which states:

$$\text{For prime } p,\ (p-1)! \cong (p-1)(\text{mod } p)$$

So using the above Theorem, we know that $6! \cong x(\text{mod } 7),\ 0 \leq x \leq 6,\ x = \mathbf{6}$.

It is essentially for p to be prime Wilson's Theorem to be applicable. Usually, with factorial problems, you can lump common factors and then can check divisibility. For example:

$$4! \cong x(\text{mod } 6),\ 0 \leq x \leq 5,\ x = ?$$

Well we know that $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot 6 \cong 0(\text{mod } 6) \Rightarrow x = \mathbf{0}$.

The following are some more problems to give you some practice:

Problem Set 3.5.3

1. $(4!)(3!)(2!) \cong x(\text{mod } 8)$, $0 \leq x \leq 7$, then $x =$ ________________
2. $(4+2)! \cong x(\text{mod } 7)$, $0 \leq x \leq 6$, then $x =$ ________________
3. $(5-2)! \cong x(\text{mod } 5)$, $0 \leq x \leq 5$, then $x =$ ________________
4. $\dfrac{5! \cdot 3!}{4!} \cong k(\text{mod } 8)$, $0 \leq k \leq 7$, then $k =$ ________________
5. $\dfrac{5! \cdot 4!}{3!} \cong k(\text{mod } 9)$, $0 \leq k \leq 8$, then $k =$ ________________
6. $\mathbf{5! \cdot 3! \cong k(\text{mod } 8)}$, $\mathbf{0 \leq k \leq 7}$, **then** $\mathbf{k} =$ ________________

3.6 Basic Calculus

If you are one of the fortunate to reach the end of the fourth column, you will experience usually two or three calculus related problems which are relatively simple if you know the basics of calculus. If you haven't had basic calculus preparation, the following is a rough introduction on the computations of limits, derivatives, and integrals associated with the number sense test.

3.6.1 Limits

Usually the limits are the simplest kind where simple substitution can be used to get an appropriate answer. For example:

$$\lim_{x \to 3} 3x^2 - 4 = 3 \cdot 3^2 - 4 = \mathbf{23}$$

However certain problems, which when passing the limit, might lead to a $\frac{0}{0}$ violation. In this case, you want to see if there are any common factors that you can cancel so that passing the limit *doesn't* give you an indeterminate form. Let's look at an example:

$$\lim_{x \to 2} \frac{(x-2)(x+3)}{(x+5)(x-2)} = \lim_{x \to 2} \frac{(x+3)}{(x+5)} = \mathbf{\frac{5}{7}}$$

If we had plugged in $x = 2$ into the original problem, we would have gotten a $\frac{0}{0}$ form, however after canceling the factors, we were able to pass the limit.

The final testable question concerning limits involve l'hôpitals rule (this requires the understanding of derivatives in order to compute it, see the next section for instructions on how to compute that). L'hôpitals rule states that if you come across a limit that gives an indeterminant form (either $\frac{0}{0}$ or $\frac{\infty}{\infty}$) you can compute the limit by taking the derivative of both the numerator and the denominator then passing the limit. So:

$$\lim_{x \to n} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \Rightarrow \lim_{x \to n} \frac{f(x)}{g(x)} = \lim_{x \to n} \frac{f'(x)}{g'(x)} \Rightarrow \frac{f'(n)}{g'(n)}$$

Let's look at an example of l'hôpitals rule with computing the limit $\lim_{x\to 0}\frac{\sin x}{x}$:

$$\lim_{x\to 0}\frac{\sin x}{x} = \frac{0}{0} \Rightarrow \lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{(\sin x)'}{x'} = \lim_{x\to 0}\frac{\cos x}{1} = \mathbf{1}$$

The following are some more practice problems with limits:

Problem Set 3.6.1

1. $\lim_{x\to\infty}\frac{3x+8}{7x-4} =$ ____________________
2. $\lim_{x\to 4} 2x-6 =$ ____________________
3. $\lim_{x\to 2}\frac{x^2-4}{x-2} =$ ____________________
4. $\lim_{x\to 2}\frac{x^3-8}{x^2-4} =$ ____________________
5. $\lim_{x\to\infty}\frac{3x-1}{x} =$ ____________________
6. $\lim_{x\to 3}\frac{x^3-27}{x-3} =$ ____________________
7. $\lim_{x\to 0}\frac{x^2-3x}{x} =$ ____________________
8. $\lim_{x\to 3}\frac{x^3-27}{x^2-9} =$ ____________________

3.6.2 Derivatives

Usually on the number sense test, there is guaranteed to be a derivative (or double derivative) of a polynomial. Almost every single time, the use of the power rule is all that is required, so let's see how we can take the derivative of a polynomial:

$$\text{Define: } f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 x^0$$
$$\textit{then}$$
$$f'(x) = a_n(n)x^{n-1} + a_{n-1}(n-1)x^{n-2} + \cdots + a_1(1)x^0$$

So the procedure is you multiply the coefficient by the power and then lower the power (notice that a constant after differentiating disappears). Let's look at an example:

Problem: Let $f(x) = x^3 - 3x^2 + x - 3$, solve for $f'(2)$.
Solution: $f'(x) = 1\cdot 3x^2 - 3\cdot 2x + 1 \Rightarrow f'(2) = 1\cdot 3\cdot 2^2 - 3\cdot 2\cdot 2 + 1 = \mathbf{1}$

When approached with taking double derivatives ($f''(x)$), then just follow the procedure twice:

Problem: Let $f(x) = 5x^3 + 3x^2 - 7$, solve for $f''(1)$.
Solution: $f'(x) = 5\cdot 3x^2 + 3\cdot 2x = 15x^2 + 6x \Rightarrow f''(x) = 15\cdot 2x + 6 \Rightarrow f''(1) = 30\cdot 1 + 6 = \mathbf{36}$

In the off chance that the derivative of sine/cosine or the $e^x/\ln x$ is needed (like for using l'hôpitals rule), here is a chart showing these functions and their derivatives:

Function	**Derivative**
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\frac{1}{x}$

For more derivative rules, consult a calculus textbook (it would be good to be familiar with more derivative rules for the math test, but unlikely those rules will be applied to the number sense test).

Here are some problems to practice taking derivatives:

Problem Set 3.6.2

1. $f(x) = 3x^2 + x - 5,\ f'(-2) =$ ____________
2. $f(x) = x^2 - 2x + 22,\ f'(2) =$ ____________
3. $g(x) = 2x^2 - 3x + 1,\ g'(2) =$ ____________
4. $f(x) = 3x^3 - 3x + 3,\ f'(-3) =$ ____________
5. $f(x) = 4x^3 + 2x^2,\ f''(-.5) =$ ____________
6. $f(x) = x^3 - 3x + 3,\ f'(3) =$ ____________
7. $f(x) = x^4 - 4x + 4,\ f'(4) =$ ____________
8. $f(x) = 3x^2 + 4x - 5,\ f'(-6) =$ ____________
9. $f(x) = 2x^3 - 3x^4,\ f''(-1) =$ ____________
10. $f(x) = 4x^3 - 3x^2 + 1,\ f'(-1) =$ ____________
11. $f(x) = x^2 - 3x + 4,\ f''(-1) =$ ____________
12. $f(x) = 3x + 5x^2 - 7x^4,\ f'(1) =$ ____________
13. $f(x) = 3x^3 - 2x^2 + x,\ f''(1) =$ ____________
14. $f(x) = 2x^3 - 4x^2 + 6x,\ f'(1) =$ ____________
15. $f(x) = x^5 + x^3 - x,\ f''(2) =$ ____________
16. $f(x) = 4x^3 - 3x^2 + x,\ f'(-1) =$ ____________
17. $f(x) = x^3 - 3x^2 + 5x,\ f''(2) =$ ____________
18. $\mathbf{f(x) = 4x^3 - 3x^2 + 2x,\ f''(1) =}$ ____________
19. $f(x) = 2x^2 - 3x + 4,\ f'(-1) =$ ____________
20. $f(x) = 4 - 3x - 2x^2,\ f'(-1) =$ ____________
21. $g(x) = x^3 - 3x - 3,\ g'(-3) =$ ____________
22. $g(x) = 2x^3 + 3x^2 + 5,\ g''(4) =$ ____________
23. $h(x) = 1 + 2x^2 - 3x^3,\ h''(4) =$ ____________
24. $f(x) = 4 - 3x^2 + 2x^3,\ f''(5) =$ ____________
25. $f(x) = x^3 - 3x + 3,\ f'(-3) =$ ____________
26. $f(x) = x^4 - 4x^2 + 4,\ f'(-4) =$ ____________
27. $f(x) = 3x^3 + 3x - 3,\ f'(-3) =$ ____________
28. $f(x) = 3x^2 - 4x + 2,\ f'(\frac{1}{3}) =$ ____________

3.6.3 Integration

3.6.4 Integration

Again, only basic integration is required for the number sense test. The technique for integrating is essentially taking the derivative backwards (or anti-derivative) and then plugging in the limits of integration. The following shows a generic polynomial being integrated:

$$\int_a^b a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x^1+a_0x^0dx = F(x) = \left(\frac{a_n}{n+1}x^{n+1}+\frac{a_{n-1}}{n}x^n+\cdots+\frac{a_1}{2}x^2+\frac{a_0}{1}x^1\right)_a^b = F(b)-F(a)$$

Let's look at an example:

Problem: Evaluate $\displaystyle\int_0^2 3x^2 - x\,dx$.

Solution: $\displaystyle\int_0^2 3x^2 - x\,dx = \left(x^3-\frac{1}{2}x^2\right)_0^2 = (2^3-\frac{1}{2}2^2)-(0^3-\frac{1}{2}\cdot 0) = \mathbf{6}$

Again, you can apply the table in the previous section for computing integrals of functions (just go in reverse).

To end this section on Integration, there is one special case when integrating, such that the integral is trivial, and that is:

$$\int_{-a}^{a} \text{Odd Function}\ \ dx = 0$$

So when you are integrating an odd function who's limits are negatives of each other, the result is zero. Let's look at an example of where to apply this:

$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin(x)\,dx = 0$$

Since sine is an odd function, the integral (with the appropriate negative limits) is simply zero!

The following are some more practice problems concerning integration:

Problem Set 3.6.3

1. $\displaystyle\int_0^2 x^2+3\,dx =$ ________________
2. $\displaystyle\int_2^4 2x-3\,dx =$ ________________
3. $\displaystyle\int_1^4 2x\,dx =$ ________________
4. $\displaystyle\int_{-3}^3 x^2\,dx =$ ________________
5. $\displaystyle\int_0^4 \frac{x}{2}\,dx =$ ________________
6. $\displaystyle\int_0^1 x^{\frac{3}{4}}\,dx =$ ________________
7. $\displaystyle\int_1^3 (x^2-2)\,dx =$ ________________
8. $\displaystyle\int_{-2}^4 x+1\,dx =$ ________________
9. $\displaystyle\int_0^\pi \sin x\,dx =$ ________________
10. $\displaystyle\int_{\mathbf{0}}^\pi \cos\mathbf{x}\,\mathbf{dx} =$ ________________
11. $\displaystyle\int_0^3 \frac{x}{3}\,dx =$ ________________
12. $\displaystyle\int_1^3 x^2\,dx =$ ________________
13. $\displaystyle\int_1^3 \frac{3x}{2}\,dx =$ ________________
14. $\displaystyle\int_1^3 x^{-2}\,dx =$ ________________
15. $\displaystyle\int_{\mathbf{1}}^{\mathbf{\frac{3}{2}}} \mathbf{x^{-2}\,dx} =$ ________________
16. $\displaystyle\int_0^1 1-x^2\,dx =$ ________________

17. $\int_0^4 \sqrt{x}\,dx =$ ______

18. $\int_{-1}^2 4x\,dx =$ ______

19. $\int_0^3 x^2\,dx =$ ______

20. $\int_1^e \frac{2}{x}\,dx =$ ______

21. $\int_0^4 x-1\,dx =$ ______

22. $\int_0^2 x^3\,dx =$ ______

23. $\int_1^e \frac{-3}{x}\,dx =$ ______

24. $\int_{\mathbf{0}}^{\mathbf{3}} \mathbf{2x+1\,dx} =$ ______

25. $\int_0^1 x^{\frac{2}{3}}\,dx =$ ______

26. $\int_{\mathbf{0}}^{\mathbf{14}} \mathbf{13-x\,dx} =$ ______

27. $\int_{-1}^1 x+1\,dx =$ ______

28. $\int_0^1 \sqrt{x}\,dx =$ ______

29. $\int_0^1 \sqrt[3]{x}\,dx =$ ______

30. $\int_{-1}^2 3x^2\,dx =$ ______

31. $\int_2^4 \frac{3}{5}x\,dx =$ ______

32. $\int_1^2 x^3\,dx =$ ______

33. $\int_0^2 x^3\,dx =$ ______

34. $\int_{\mathbf{0}}^{\mathbf{2}} \mathbf{x^3+1\,dx} =$ ______

35. $\int_0^2 x\,dx =$ ______

36. $\int_{-1}^2 2x\,dx =$ ______

37. $\int_0^4 3-x\,dx =$ ______

38. $\int_0^2 \frac{3x}{4}\,dx =$ ______

39. $\int_0^3 \frac{4x}{3}\,dx =$ ______

4 Tricks Added with 2018 Revision

The following is an assortment of tricks that can be used to solve problems from more recent Number Sense Exams. Some tricks are variations or extensions of already mentioned shortcuts (which I'll reference) while others are entirely new ones. They are broken out into rough categories in order to better organize them.

4.1 Multiplication

4.1.1 Multiplying Three-Digit Number by Two-Digit Number

We briefly touched on how to apply FOILing/LIOFing principles in Section 1.1 – chiefly concerning ourselves with two-digit number multiplication – but it seems that more recent exams have really emphasized the multiplication of three-digit numbers, starting around the third column. We'll start by illustrating how to perform a multiplication of a three-digit number, $n_1 = abc$, by a two-digit number $n_2 = ef$, where a, b, c, e, f are digits.

When doing a three-digit by two-digit multiplication, it's best to break it down into a two-digit multiplication (while keeping track of carries) followed by a two-digit and one-digit multiplication. The reason this is possible is that you can treat n_2 as being a three-digit number, with it's leading digit being a 0. After that, you then "group" the digits bc and ef together (and treat each collection as an individual unit) and perform a normal FOIL/LIOF twice. To understand this concept better, lets take a look at what we do when we multiply $abc \times 0ef$:

$$abc = 100a + (bc) \text{ and } 0ef = 100 \cdot 0 + (ef)$$

$$[100a + (bc)] \times [100 \cdot 0 + (ef)] = 100a \cdot 0 + 100a(ef) + 0 \cdot (bc) + (bc)(ef)$$

Which simplifies to:

$$100a(ef) + (bc)(ef)$$

Now what does this tell us:

1. The one's and ten's digit of the answer is simply the last two digits when performing the multiplication of the groups of bc and ef.
2. Almost always there will be a carry – possibly a two-digit carry – when performing this multiplication.
3. The remainder of the answer is just the leading digit of the three-digit number, a, multiplied by the two-digit number, ef, plus the carry.

Here is a simple example:

	Units and Tens:	$17 \times 15 = \mathit{2}\mathbf{55}$
$117 \times 15 =$	Remaining	$1 \times 15 + \mathit{2} = \mathbf{17}$
	Answer:	**1755**

In an nutshell: you perform the 17 by 15 multiplication first to get the last two digits and keep track of the carry, then you perform the 1 and 15 multiplication to get the remaining digits (including the previously calculated carry). Here is a slightly more difficult problem.

	Units and Tens:	$33 \times 37 = \mathit{12}\mathbf{21}$
$233 \times 37 =$	Remaining	$2 \times 37 + \mathit{12} = \mathbf{86}$
	Answer:	**8621**

In this example, the carry is actually a two-digit carry because the first multiplication produces a four-digit number. Also, don't be surprised if these straightforward multiplications require the use of other tricks. For the first step, you can use the Multiplying Two Numbers Equidistant from a Third Number (Section 1.2.10) and the Squares Ending in 5 (Section 1.2.8) tricks to do $33 \times 37 = 35^2 - 2^2 = 1221$ in order to quickly perform that step. After that is done, the rest is pretty straightforward.

The following are exercises to familiarize you with performing these more involved multiplications. Note: sometimes other shortcuts can be used, so be on the lookout!

Problem Set 4.1.1

1. $314 \times 17 =$ ________________
2. $143 \times 91 =$ ________________
3. $202 \times 34 =$ ________________
4. $13 \times 332 =$ ________________
5. $202 \times 53 =$ ________________
6. $112 \times 13 =$ ________________
7. $221 \times 23 =$ ________________
8. $123 \times 45 =$ ________________
9. $231 \times 31 =$ ________________
10. $202 \times 76 =$ ________________
11. $321 \times 19 =$ ________________
12. $35 \times 122 =$ ________________
13. $123 \times 98 =$ ________________
14. $135 \times 79 =$ ________________
15. $17 \times 289 =$ ________________
16. $121 \times 81 =$ ________________
17. $48 \times 152 =$ ________________
18. $751 \times 18 =$ ________________
19. $16 \times 216 =$ ________________
20. $345 \times 67 =$ ________________
21. $765 \times 43 =$ ________________

4.1.2 Multiplying Three-Digit Number by Three-Digit Number

This is an extension of the previous section and the procedure is the same but requires a little more multiplication and bookkeeping. Let $n_1 = abc$ and $n_2 = def$, where a, b, c, d, e, f are digits. You'll want to do the groups of n_1 as a and bc and n_2 as d and ef and perform the FOIL/LOIFing.

$$abc = 100a + (bc) \text{ and } def = 100d + (ef)$$
$$[100a + (bc)] \times [100d + (ef)] = 10000ad + 100[a(ef) + d(bc)] + (bc)(ef)$$

This shows us that:

1. Again, the ones and tens digit of the answer is simply the last two digits when performing the multiplication of the groups of bc and ef.
2. Again, carries are common, so keep track!

3. The next *two* digits (e.g., the thousands and hundreds) is the addition of the *Inner* and *Outer* multiplications between the two-digit groups with their one-digit counterpart on the opposing number, plus the carry.

4. The remainder of the answer is just the two leading digits multiplied together, plus the carry.

Here is a simple example:

$211 \times 416 =$	Units and Tens:	$16 \times 11 = \mathit{1}\mathbf{72}$
	Hundreds and Thousands:	$16 \times 2 + 11 \times 4 + \mathit{1} = \mathbf{77}$
	Remaining:	$4 \times 2 = \mathbf{8}$
	Answer:	$\mathbf{87772}$

Now most of the time, the digits of the three-digit by three-digit multiplication are pretty low which makes the *actual* multiplication part pretty easy – so the challenge is just keeping track of everything in your head appropriately. Here is a more difficult problem that requires more concentration concerning the actual multiplication:

$217 \times 245 =$	Units and Tens:	$17 \times 45 = \mathit{7}\mathbf{65}$
	Hundreds and Thousands:	$45 \times 2 + 17 \times 2 + \mathit{7} = \mathit{1}\mathbf{31}$
	Remaining:	$2 \times 2 + \mathit{1} = \mathbf{5}$
	Answer:	$\mathbf{53165}$

Here, you had to basically do a FOIL/LIOF on two, two-digit numbers before proceeding to the simple multiplication with bookkeeping. Additionally, if you would rather just treat each digit as a separate entity and just move down the line, as explained in Section 1.1, by all means! This is just an alternative way of producing the same result in, possibly, a quicker amount of time.

The following are exercises to familiarize you with performing these more involved multiplications:

Problem Set 4.1.2

1. $212 \times 311 =$ ______________________
2. $208^2 =$ ______________________
3. $404^2 =$ ______________________
4. $331 \times 122 =$ ______________________
5. $707^2 =$ ______________________
6. $131 \times 223 =$ ______________________
7. $402^2 =$ ______________________
8. $804^2 =$ ______________________
9. $234 \times 211 =$ ______________________
10. $909^2 =$ ______________________
11. $123 \times 321 =$ ______________________
12. $306^2 =$ ______________________
13. $222 \times 203 =$ ______________________
14. $317 \times 245 =$ ______________________
15. $204^2 =$ ______________________
16. $408^2 =$ ______________________
17. $344 \times 522 =$ ______________________

18. $121 \times 411 =$ ______________________

19. $221 \times 141 =$ ______________________

20. $131 \times 212 =$ ______________________

21. $124 \times 312 =$ ______________________

22. $311 \times 122 =$ ______________________

23. $412 \times 112 =$ ______________________

24. $123 \times 301 =$ ______________________

25. $511 \times 212 =$ ______________________

26. $151 \times 115 =$ ______________________

27. $213 \times 331 =$ ______________________

28. $141 \times 114 =$ ______________________

29. $243 \times 151 =$ ______________________

30. $215 \times 152 =$ ______________________

31. $132 \times 214 =$ ______________________

32. $135 \times 152 =$ ______________________

33. $344 \times 522 =$ ______________________

34. $126 \times 214 =$ ______________________

35. $415 \times 312 =$ ______________________

36. $215 \times 321 =$ ______________________

37. $113 \times 314 =$ ______________________

38. $414 \times 325 =$ ______________________

4.1.3 Multiplying Two Numbers Whose Units Add to 10 and the Rest is the Same

This is a more generalized version of the Squares Ending in 5 Trick (Section 1.2.8). Take $n_1 = ab$ and $n_2 = ac$, with $b + c = 10$. Then:

$$ab \times ac = (10a + b)(10a + c) = 10a(10a + b + c) + bc$$

Since $b + c = 10$, then:

$$10a(10a + b + c) + bc = 10a(10a + 10) + bc = 100a(a + 1) + bc$$

So from this, you can see that the last two digits are just the units digits multiplied together, and the remainder of the digits can be found from taking the leading digit(s) and multiplying it by one greater than itself. (Note: the Squares Ending in 5 trick uses this fact, knowing always that $bc = 5 \times 5 = 25$, so you can automatically just write down 25 as the last two digits). Here are some examples:

	Tens/Ones:	8×2	**16**
$68 \times 62 =$	Remaining:	$6 \times (6 + 1)$	**42**
	Answer:		**4216**

	Tens/Ones:	3×7	**21**
$173 \times 177 =$	Remaining:	$17 \times (17 + 1)$	**306**
	Answer:		**30621**

Now you can just as easily combine the Multiplying Two Numbers Equidistant from a Third Number Trick (Section 1.2.10) with the Squares Ending in 5 Trick – making the first problem be $68 \times 62 = 65^2 - 3^2 = 4225 - 9 = 4216$ – but this "new" trick cuts down on doing the subtraction. The following are a few practice problems to help you with this alternative method:

Problem Set 4.1.3

1. $71 \times 79 =$ ________________
2. $112 \times 118 =$ ________________
3. $44 \times 46 =$ ________________
4. What's the area of a rectangle with sides 64 and 66 ________________
5. $192 \times 198 =$ ________________
6. $111 \times 119 =$ ________________
7. $333 \times 337 =$ ________________
8. $221 \times 229 =$ ________________

4.1.4 Binomial Approximation

I have seen a few questions that uses the well known first-order binomial approximation of:

$$(1+x)^n \approx 1 + xn, \text{ if } |xn| \ll 1$$

These will typically be approximation questions (as the identity itself is an approximation) and, because $xn \ll 1$, the questions usually has this value multiplied by a large integer in order to give a sufficient range of answers. An example question would be:

$$1000(1.0002)^{50} \approx 1000[1 + (.0002 \times 50)] = 1000(1.01) = 1010$$

This answer is incredibly close to the exact answer of $1010.049\ldots$. A natural question that arises is how much does $|xn|$ need to be less than 1 in order to use it? There is no easy answer to this, but I'd figure that if the test writers have a problem that *looks* like you'd be able to use the approximation, then you are probably OK to use it!

4.1.5 Multiplying by Fraction Close to 1

This trick – which is more like clever factoring – is used whenever you see a whole number being multiplied by a fraction close to 1. Here is an example:

$$14 \times \frac{15}{16} = 14 \times (1 - \frac{1}{16}) = 14 - \frac{14}{16} = \mathbf{14\frac{7}{8}}$$

Compared with trying to reduce a large improper fraction by doing the straightforward multiplication, treating the fraction as (1 – a small number) is easier to compute the whole and fractional components of the answer. You can also use this procedure if the fraction is slightly above 1:

$$17 \times \frac{13}{11} = 17 \times (1 + \frac{2}{11}) = 17 + \frac{34}{11} = \mathbf{20\frac{1}{11}}$$

Here you can tell that it's important to perform the fractional multiplication first as it might affect adding/subtracting values from the whole number portion. In the above problem, the fractional part is improper, so you must reduce it to a mixed number in order to get a correct answer.

Don't forget that if the whole number and the numerator of the fraction are of the same value, you can apply the $a \times \frac{a}{b}$ trick (Section 1.3.9) for possibly a quicker solution, but you can always apply this method if you forget that trick as well. Here are some more practice problems so you can get better with this technique:

Problem Set 4.1.5

1. $6 \times \frac{7}{8} + 5 =$ ____________ (mixed number)
2. $13 \times \frac{14}{15} =$ ____________ (mixed number)
3. $18 \times \frac{19}{20} =$ ____________ (mixed number)
4. $12 \times \frac{13}{14} =$ ____________ (mixed number)
5. $13 \times \frac{13}{14} + 13 =$ ____________ (mixed number)
6. $11 \times \frac{12}{13} =$ ____________ (mixed number)
7. $11 \times \frac{14}{17} =$ ____________ (mixed number)
8. $13 \times \frac{13}{14} - 13 =$ ____________
9. $17 \times \frac{17}{18} - 17 =$ ____________
10. $14 \times \frac{14}{17} - 14 =$ ____________
11. $13 \times \frac{13}{16} - 13 =$ ____________
12. $15 \times \frac{15}{17} - 15 =$ ____________
13. $14 \times \frac{14}{17} - 14 =$ ____________

4.1.6 $n^2 + n = (n+1)^2 - (n+1)$

This section is in response to a type of problem I've seen crop up on some of the most recent tests. Basically, by restating the problem in a slightly different way leads to the same factoring of the expression but an easier time calculating. Notice that:

$$n^2 + n = n^2 + 2n + 1 - n - 1 = (n+1)^2 - (n+1)$$

You can use this identity to solve the problem of things like $89^2 + 89 =$

$$89^2 + 89 = 90^2 - 90 = 8100 - 90 = \mathbf{8010}$$

Typically the problem will involve an integer one away from a multiple of 10 or 5, making the squaring and the subtraction relatively easy. You can also apply the trick in reverse:

$$66^2 - 66 = 65^2 + 65 = 4225 + 65 = \mathbf{4290}$$

Here are a few more practice problems for you so that you can start to notice when to use this trick:

Problem Set 4.1.6

1. $49^2 + 49 =$ ____________
2. $79^2 + 79 =$ ____________
3. $59^2 + 59 =$ ____________
4. $69^2 + 69 =$ ____________
5. $24^2 + 24 =$ ____________
6. $99 \times 99 + 99 =$ ____________

4.2 Memorizations

The following sections detail some additional memorizations that will need to be practiced in order to better prepare for questions from more recent Number Sense Exams. These memorizations are in addition to the highly detailed Section 2.0 and are a supplement *not* a replacement.

4.2.1 Conversions, Part 2

The following are some additional conversions that have been seen on recent exams:

1 sq. mile = 640 acres

1 mile = 320 rods

1 rod = 16.5 feet

1 bushel = 4 pecks

1 day = 1440 minutes

The rod conversions are especially help as this can easily be used to solve for unusual fractions of miles. Knowing that a 1 mile = 320 rods $\times$ 16.5 feet/rod, through the Double and Half Trick (Section 1.2.6), you can see that you can reduce 1 mile to 160×33 feet. This is extremely helpful if you are asked something like how many feet $\frac{2}{11}$'s of a mile is. Additionally, knowing that a day is $12^2 \times 10$ minutes can lead to quick reductions as well.

The following are some problems detailing these relatively obscure conversions:

Problem Set 4.2.1

1. How many bushels are 15 pecks? __________
2. $\dfrac{1}{11}$ of a mile is how many feet? __________
3. 44 bushels are how many pecks? __________
4. 160 acres is ____________________sq. miles
5. $\dfrac{1}{8}$ miles is ________________________rods
6. $\dfrac{1}{44}$ of a mile is how many feet? __________
7. (*) 2 days 7 hours 12 minutes = ____ minutes

4.2.2 Exotic Definitions of Numbers

Here are some additional classifications of numbers, similar to what is found in Section 3.1.2:

1. A **happy number** is a number whose sum of the squares of the individual digits eventually leads to a chain that terminates to 1. For example 19 is a happy number because $19 \Rightarrow 1^2 + 9^2 = 82 \Rightarrow 8^2 + 2^2 = 68 \Rightarrow 6^2 + 8^2 = 100 \Rightarrow 1^2 + 0^2 + 0^2 = \mathbf{1}$. The first handful of happy numbers are 1, 7, 10, 13, 19, 23, 28, 31, 32, 44, 49, 68, 70, 79, 82, 86, 91, 94, 97, and 100.
2. An **extravagant/wasteful number** is a number whose prime factorization has more digits than the number itself (treating both the base and exponents as individual digits). For example 18 is an extravagant number because $18 = 2 \times 3^2 \Rightarrow 18$ contains 2 digits and its prime factorization contains 3 digits. The first handful of extravagant numbers are 4, 6, 8, 9, 12, 18, 20, 22, 24, 26, 28, 30, 33, 34, 36, 38, 39, 40, 42, 44, 45, 46, 48, 50.

3. An **economical/frugal number** is the opposite of an extravagant number: its prime factorization contains less digits than the number itself. For example 128 is an economical number because $128 = 2^7 \Rightarrow 128$ contains 3 digits and its prime factorization contains 2 digits. Unsurprisingly, cubes and higher powers of 2 and 3 are economical.

4. An **odious number** is a non-negative number whose binary representation has an odd number of 1s. An example is $7 = 111_2$ which has 3 ones in its binary representation.

5. An **evil number** is the opposite of an odious number: it has an even number of 1s. An example is $9 = 1001_2$ which has 2 ones.

You can find all these crazy definitions of numbers (and more!) from the On-Line Encyclopedia of Integer Sequences (OEIS). It's a pretty cool resource that you can spend hours just browsing cool sequences of numbers.

4.2.3 Square Root of Small Integers

The following are square roots of small integers that are good to have memorized at least to the truncated thousandths place:

$$\sqrt{2} = 1.414\ldots \qquad \sqrt{3} = 1.732\ldots \qquad \sqrt{5} = 2.236\ldots$$

$$\sqrt{6} = 2.449\ldots \qquad \sqrt{7} = 2.645\ldots \qquad \sqrt{8} = 2.828\ldots$$

There have been straightforward approximation questions asking things like what is the hundredths digit of $\sqrt{5}$ as well as approximation questions which you can use the above truncated decimals to help with the calculations. Here are a few examples:

Problem Set 4.2.3

1. Round $2\sqrt{3}$ to the nearest tenth. __________
2. The greatest integer less than $12\sqrt{2}$ is ______
3. Truncate $\sqrt{2} + \sqrt{3} + \sqrt{5}$ to one decimal place. __________ (decimal)
4. The greatest integer function is written as $f(x) = [x]$. Find $[\sqrt{7} - \sqrt{3}]$. __________
5. The greatest integer function is written as $f(x) = [x]$. Find $[\sqrt{6} + \sqrt{7}]$. __________
6. The greatest integer less than $12\sqrt{2}$ is ______
7. The greatest integer function is written as $f(x) = [x]$. Find $[2\sqrt{3} - \pi]$. __________
8. Round $(5\sqrt{2} + 4\sqrt{3})$ to the nearest whole number. __________
9. Round $(\sqrt{5} + 6\sqrt{7}$ to the nearest whole number. __________

4.2.4 Approximations Using Phi

In addition to the π and e Approximations found in Section 2.1.8, ϕ is a constant that has occassionally been asked in recent exams. Here are a few convenient properties:

$\phi = 1.618\ldots$	$\phi^2 \approx 2.6$	$\phi^3 \approx 4.2$
$\phi^5 \approx 11$	$\phi^\phi \approx 2.2$	$\pi \times \phi \approx 5$
$e \times \phi \approx 4.4$	$\pi \times e \times \phi \approx 13.8$	

Here are few problems I could find on recent tests that use some of the above approximations:

Problem Set 4.2.4

1. (*) $3.1\pi \times 2.7e \times 1.6\phi =$ ________________

2. The greatest integer function is written as $f(x) = [x]$. Find $[\pi + e + \phi]$. ____________

4.2.5 Standard Fibonacci Numbers

It's becoming *crucial* to have at least the first fourteen Fibonacci numbers memorized as the test writers tend to ask at least two questions on each test that stem from directly knowing them (it will also help immensely with reducing the work load concerning the calculations required in Section 2.2.2 and the upcoming Section 4.3). Below is a table of the first fourteen Fibonacci numbers:

$F_1 = 1$	$F_2 = 1$	$F_3 = 2$	$F_4 = 3$
$F_5 = 5$	$F_6 = 8$	$F_7 = 13$	$F_8 = 21$
$F_9 = 34$	$F_{10} = 55$	$F_{11} = 89$	$F_{12} = 144$
$F_{13} = 233$	$F_{14} = 377$		

4.3 Properties of Fibonacci Numbers

Because the number of questions concerning Fibonacci Numbers has exploded in recent years, I decided to put together an entire section detailing how to solve current as well as possible future questions involving these numbers. Oftentimes, having the first fourteen Fibonacci Numbers (as detailed in Section 4.2.5) is essential in order to tackle solving these problems. So be sure you have a firm memorization of those numbers!

One thing to note: I will refer to the well known sequence of 1, 1, 2, 3, 5, 8, ... as the **Standard Fibonacci Sequence**, denoted by F_n and *any* sequence defined by the recursive relation $S_n = S_{n-1} + S_{n-2}$ as **Arbitrary Fibonacci Sequences**, denoted as A_n. It is important to differentiate between the two as several tricks involve using both types.

4.3.1 Adding Consecutive Terms of Arbitrary Fibonacci Sequence, Method 1

This is a common question where the test writer will give the beginning and end terms of an Arbitrary Fibonacci Sequence and ask for the sum of a subset of the terms. An example is finding the sum of 4, 7, 11, ..., 47, and 76.

The trick uses the telescoping properties of the recursion relation of the Fibonacci sequence. Knowing that

$A_n = A_{n-1} + A_{n-2}$, rearranging yields $A_{n-2} = A_n - A_{n-1}$. From here you can see the following:

$$\begin{aligned} A_1 &= A_3 - A_2 \\ A_2 &= A_4 - A_3 \\ A_3 &= A_5 - A_4 \\ A_4 &= A_6 - A_5 \\ A_5 &= A_7 - A_6 \\ A_6 &= A_8 - A_7 \\ A_7 &= A_9 - A_8 \end{aligned}$$

You can sum both the left and the right-hand side of all these equations to produce your telescoping series:

$$A_1 + A_2 + A_3 + \ldots + A_7 = (A_3 - A_2) + (A_4 - A_3) + (A_5 - A_4) + \ldots + (A_9 - A_8) = A_9 - A_2$$

So in general when you are summing up an Arbitrary Fibonacci Sequence (starting from the first term), then sum of the first n terms is simply $A_{n+2} - A_2$. Using that fact and applying it to our example question, we just need to find $A_9 - A_2$. We are given up to A_5, so all you have to keep straight is appropriately summing up to the ninth term:

$$A_6 = 47 \text{ and } A_7 = 76 \Rightarrow A_8 = 76 + 47 = 123 \Rightarrow A_9 = 123 + 76 = 199$$

Therefore the sum is $= 199 - 7 = \mathbf{192}$. You can see, the real difficultly with these problems is keeping your previous two Fibonacci numbers in your head in order to find the next term. There is unquestionably a lot of bookkeeping involved, so this method is best if the test writter explicitly writes *most* of the sequence in the problem statement. That way, you only have to compute two or three additional terms before applying the formula to find the sum. You can find a few practice problems below.

Problem Set 4.3.1

1. $2 + 1 + 3 + 4 + 7 + 11 + \ldots + 29 + 47 =$ _____
2. $1 + 1 + 2 + 3 + 5 + 8 + \ldots + 24 + 55 =$ _____
3. $5 + 7 + 12 + 19 + 31 + \ldots + 131 + 212 =$ _____
4. $3 + 4 + 7 + 11 + 18 + 29 + \ldots + 123 =$ _____
5. $3 + 7 + 10 + 17 + 27 + \ldots + 115 + 186 =$ _____
6. $15 + 18 + 33 + 51 + 84 + 135 + 219 + 354 =$ ___

4.3.2 Adding Consecutive Terms of Arbitrary Fibonacci Sequence, Method 2

The other way of doing the sum of the terms of an Arbitrary Fibonacci Sequence – especially if you are given few terms in the problem statement – involves using your knowledge of the Standard Fibonacci Sequence, F_n. I'll leave out the derivation (because it is lengthy), but you can calculate the sum of the first n terms of an Arbitrary Fibonacci Sequence (A_n) using the following formula:

$$\sum = A_1 \times F_n + A_2 \times (F_{n+1} - 1)$$

So taking our example from the previous section, you can find the sum of the first 7 terms of 4, 7, 11, ..., 47, 76 by:

$$\sum = 4 \times F_7 + 7 \times (F_8 - 1) = 4 \times 13 + 7 \times (21 - 1) = 52 + 140 = \mathbf{192}$$

As you can see, this method is calculation-intensive (you have to have your Standard Fibonacci Numbers memorized, perform two multiplications, and then sum everything up), but you don't have to worry about actually *finding* any terms in the sequence. So yeah, either way is difficult, so it's best if you find the one that works for you and really practice it well! Here are some more problems for you:

Problem Set 4.3.2

1. The sum of the first eight terms of the Fibonacci sequence 2, 5, 7, 12, 19, ... is ______
2. The sum of the first nine terms of the Fibonacci sequence 1, 5, 6, 11, 17, 28, ... is ______
3. The sum of the first eight terms of the Fibonacci sequence 3, 4, 7, 11, 18, ... is ______
4. The sum of the first nine terms of the Fibonacci sequence 2, 4, 6, 10, 16, ... is ______
5. The sum of the first nine terms of the Fibonacci sequence 1, 5, 6, 11, 17, ... is ______
6. The sum of the first nine terms of the Fibonacci sequence 4, 7, 11, 18, 29, ... is ______
7. The sum of the first ten terms of the Fibonacci sequence 2, 5, 7, 12, 19, ... is ______
8. The sum of the first nine terms of the Fibonacci sequence 3, 5, 8, 13, 21, ... is ______
9. The sum of the first nine terms of the Fibonacci sequence -3, 4, 1, 5, 6, ... is ______
10. The sum of the first nine terms of the sequence 4, 6, 10, 16, 26, ... is ______
11. The sum of the first nine terms of the Fibonacci sequence 1, 1, 2, 3, 5, ... is ______
12. The sum of the first ten terms of the sequence 4, 6, 10, 16, 26, ... is ______
13. The sum of the first ten terms of the Fibonacci sequence 3, 6, 9, 15, 24, ... is ______
14. The sum of the first ten terms of the Fibonacci sequence 0, 3, 3, 6, 9, 15, ... is ______
15. The sum of the first ten terms of the Fibonacci sequence 4, 5, 9, 14, 23, ... is ______
16. The sum of the first eleven terms of the Fibonacci sequence 2, 4, 6, 10, 16, ... is ______
17. The sum of the first ten terms of the Lucas sequence 3, 4, 7, 11, 18, ... is ______
18. The sum of the first ten terms of the sequence 1, 4, 5, 9, 14, ... is ______
19. The sum of the first twelve terms of the Fibonacci sequence 1, 2, 3, 5, 8, ... is ______

4.3.3 Adding Odd of Even Terms of Arbitrary Fibonacci Sequence

Again, the derivations are pretty lengthy, so lets just look at the results.

For the sum of the odd terms (e.g., A_1, A_3, etc...) of an Arbitrary Fibonacci Sequence:

$$\begin{aligned}\sum_{i=1}^{n} A_{2i-1} &= A_1 + A_3 + A_5 + \ldots + A_{2n-1} \\ &= A_{2n} - (A_2 - A_1)\end{aligned}$$

What this means is that the sum is equal to the next term in the complete sequence (which will be an *even* term) with the difference between the first and second terms subtracted from it. Using our previous example sequence of 4, 7, 11, 18, 29, 47, ..., then the sum of the first 3 odd terms (4, 11, 29) is:

$$\sum = A_6 - (A_2 - A_1) = 47 - (7 - 4) = 44$$

For the sum of the even terms (e.g., A_2, A_4, etc...) of an Arbitrary Fibonacci Sequence:

$$\begin{aligned}\sum_{i=1}^{n} A_{2i} &= A_2 + A_4 + A_6 + \ldots + A_{2n} \\ &= A_{2n+1} - A_1\end{aligned}$$

What this means is that the sum is equal to the next term in the complete sequence (which will be an *odd* term) with the first term subtracted from it. Using our previous example sequence of 4, 7, 11, 18, 29, 47, ..., then the sum of the first 3 even terms (7, 18, 47) is:

$$\sum = A_7 - A_1 = 76 - 4 = 72$$

Now these are trivial examples where the sum is simple to compute. In order to use the formulas, you'll need to either have a long list of terms given in the problem statement *or* they'll ask about the Standard Fibonacci Sequence which you'd then have the next term in the sequence memorized to help with the calculations. For example:

The sum of the first 7 odd terms of the Standard Fibonacci Sequence:

$$\sum = F_1 + F_3 + \ldots + F_{13} = F_{14} - (F_2 - F_1) = 377 - (1 - 1) = \mathbf{377}$$

The sum of the first 5 even terms of the Standard Fibonacci Sequence:

$$\sum = F_2 + F_4 + \ldots + F_{10} = F_{11} - F_1 = 89 - 1 = \mathbf{88}$$

Now I haven't explicitly seen any problems that use these sequences, but it wouldn't hurt to be familiar with these procedures if you suddenly see these types of problems make an appearance on either the Number Sense or Mathematics exams.

4.3.4 Sum of the Squares of Arbitrary Fibonacci Sequence

For the sum of the squares of the first n terms of an Arbitrary Fibonacci Sequence ($A_1^2 + A_2^2 + A_3^2 + \ldots A_n^2$) the formula is:

$$A_1^2 + A_2^2 + \ldots + A_n^2 = A_n \times A_{n+1} - A_1(A_2 - A_1)$$

Notice that if a Standard Fibonacci Sequence is used, the last term cancels because $F_2 - F_1 = 0$, so the sum is simply $F_n \times F_{n+1}$. A quick example is:

$$1^2 + 1^2 + 2^2 + 3^2 + \ldots + 21^2 + 34^2 = 34 \times 55 = \mathbf{1870}$$

Here are a handful of questions involving this formula:

Problem Set 4.3.4

1. $2^2 + 1^2 + 3^2 + 4^2 + 7^2 =$ ________________

2. $1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 =$ ________________

3. $2^2 + 3^2 + 5^2 + 8^2 + 13^2 =$ ________________

4. $1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 =$ ________________

5. $2^2 + 1^2 + 3^2 + 4^2 + 7^2 + 11^2 =$ ________________

4.4 Additional Formulas

4.4.1 $\frac{a}{b} - \frac{na-1}{nb-1}$

The following is a supplement to the formulas given in Section 1.5.5 and deals with subtracting expressions in the form $\frac{a}{b} - \frac{na-1}{nb-1}$. Here is the formula:

$$\frac{a}{b} - \frac{na-1}{nb-1} = \frac{(b-a)}{b \cdot (nb-1)}$$

So the numerator of the answer is just the difference between the denominator and the numerator of the first number (e.g., the number whose numerator and denominator are small values) while the denominator of the answer is just the multiplication of the denominators of the two numbers. Here is an example:

$$\frac{3}{5} - \frac{17}{29} = \frac{5-3}{5 \cdot 29} = \frac{\mathbf{2}}{\mathbf{145}}$$

There is another variation to the above formula which is:

$$\frac{a}{b} - \frac{na+1}{nb+1} = \frac{-(b-a)}{b \cdot (nb+1)}$$

So basically, it's the same procedure as above only you'll negate the answer. Because there are now *four* total types of these problems, when you come across a similar question on the exam it is best to take some time to notice which type it is and then apply the correct formula. The easiest way of seeing which formula to apply is to look at both the numerator and the denominator of the more "complicated" number and determine if they are one greater or one fewer than a multiple of their respective numerator and denominator of the "simpler" number. The formulas are very similar, so it's best to have a lot practice!

Problem Set 4.4.1

1. $\frac{5}{7} - \frac{9}{13} =$ ________________

2. $\frac{5}{7} - \frac{11}{15} =$ ________________

3. $\frac{7}{11} - \frac{55}{87} =$ ________________

4. $\frac{3}{5} - \frac{59}{99} =$ ________________

5. $\frac{2}{7} - \frac{21}{71} =$ ________________

6. $\frac{29}{35} - \frac{5}{6} =$ ________________

7. $\frac{43}{49} - \frac{7}{8} =$ ________________

8. $\frac{23}{25} - \frac{229}{249} =$ ________________

4.4.2 Factorizations

We'll start by showing two of the most common "obscure" factorizations that are asked about on the Number Sense exam:

$$x^3 + y^3 = [(x+y)^2 - 3xy](x+y)$$

$$x^3 - y^3 = [(x-y)^2 + 3xy](x-y)$$

Usually, these questions will give you the values of $(x \pm y)$ and xy and will ask what $x^3 \pm y^3$ is equal to. Here is an example:

Problem: $x + y = 5$ and $xy = 3$, then $x^3 + y^3 =$

Solution: Applying the first formula: $x^3 + y^3 = (5^2 - 3 \cdot 3)(5) = \mathbf{80}$

Oftentimes, the problem will make it difficult to mentally calculate the exact values of x and y, so knowing the formula is required in order to come up with a correct answer.

There are a host of other really interesting factorizations that aren't commonly taught in schools that might, eventually, wind up on the Number Sense or (more likely) the Mathematics exam so I thought I'd share them. Here are a few of my favorites:

1. Sophie Germain Identity: $x^4 + 4y^4 = [(x+y)^2 + y^2][(x-y)^2 + y^2] = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$
2. Vieta/Newton Factorization, squares: $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
3. $(xy + x + y + 1) = (x+1)(y+1)$
4. $(xy - x - y + 1) = (x-1)(y-1)$

I can see the Sophie Germain Identity being used if the test writer gives $x^2 + 2y^2$ and xy in the problem statement. The Vieta/Newton Factorization is useful if you are discussing properties of roots a, b, c of a cubic polynomial (you can tell that the sum of the squares or the roots is related to the sum of the roots and the sum of the roots taken two at a time: $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab + bc + ca)$). The final two factorizations are helpful if you ever come across the expression $xy \pm x \pm y$ and are trying to factor. Similar to "completing the square", you can "complete the rectangle" just adding 1 on both sides of the equation and then factor. I can definitely see this being used in future exams.

As for practice problems, I'll just stick to the first two identities which have *actually* been seen on the exam so far.

Problem Set 4.4.2

1. If $xy = 1$ and $x + y = -2$, then $x^3 + y^3 =$ ____
2. If $xy = 3$ and $x - y = -1$, then $x^3 - y^3 =$ ____
3. If $x + y = 1$ and $xy = 3$, then $x^3 + y^3 =$ ____
4. If $x - y = 2$ and $xy = 5$, then $x^3 - y^3 =$ ____
5. If $xy = \dfrac{5}{3}$ and $x + y = 4$, then $x^3 + y^3 =$ ____
6. If $xy = -3$ and $x - y = -2$, then $x^3 - y^3 =$ ____

4.4.3 Sum of the Reciprocals of Triangular Numbers

This is a very interesting problem whose solution is really independent of knowing the triangular numbers, T_n, themselves, but rather just knowning what term n they are in the sequence. Here is the formula:

$$\frac{1}{T_n} + \frac{1}{T_{n+1}} + \frac{1}{T_{n+2}} + \ldots + \frac{1}{T_m} = 2\left(\frac{1}{n} - \frac{1}{m+1}\right)$$

Here's an example, with the formula applied:

$$1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} = 2\left(\frac{1}{1} - \frac{1}{4+1}\right) = 2\left(1 - \frac{1}{5}\right) = \mathbf{\frac{8}{5}}$$

All you had to know what that the sequence started with the reciprocal of the first ($n = 1$) Triangular number and ended with the fourth ($m = 4$) Triangular number. Also, the sequence doesn't have to start from $n = 1$, you can have it start from an arbitrary term:

$$\frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} = 2\left(\frac{1}{3} - \frac{1}{6+1}\right) = \mathbf{\frac{8}{21}}$$

All that matters is that you need to know what term the first and last Triangular numbers in the sequence are (which you can back-track using the formulas supplied in Section 2.2.6). Here are a few more practice problems:

Problem Set 4.4.3

1. $\frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} =$ ________________

2. $\frac{1}{3} + \frac{1}{6} + \frac{1}{10} =$ ________________

3. $\frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} =$ ________________

4. $\frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} =$ ________________

5. $\frac{1}{6} + \frac{1}{10} + \frac{1}{15} =$ ________________

4.4.4 Geometric and Harmonic Means

Geometric, G_n, and harmonic, H_n, means are starting to be asked on Number Sense exams in a variety of ways. It's best to begin with the formulas:

$$G_n = \sqrt[n]{x_1 x_2 \cdots x_n} \text{ where, } x_1, x_2, \ldots, x_n \text{ are values}$$

$$H_n = \frac{n}{\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)} \text{ where, } x_1, x_2, \ldots, x_n \text{ are values}$$

Questions asking about geometric means are pretty straightforward: What is the geometric mean of 6, 4, and 9? $\Rightarrow \sqrt[3]{6 \times 4 \times 9} = \sqrt[3]{216} = \mathbf{6}$. One interesting thing to note (for the UIL Mathematics Exam) is that there are several applications of the geometric mean with right triangles. For instance, the altitude of a right triangle to its hypotenuse is the geometric mean of the two segments the hypotenuse is split into. Anyways, I highly suggest looking up various physical interpretations of the geometric mean online.

As for the harmonic mean I have seen two different types of questions. The first involves asking what the harmonic mean of the roots of a cubic polynomial are. Assuming the roots are r, s, and t, applying the formula yields:

$$H_3 = \frac{3}{\left(\frac{1}{r} + \frac{1}{s} + \frac{1}{t}\right)} = \frac{3pqr}{pq + pr + qr}$$

So you can relate the harmonic mean of the roots to the product of the roots and the sum of roots taken two at time (similar to what we found with the Vieta/Newton Factorization in Section 4.4.2). No doubt, you'll need to familiarize yourself with Section 3.1.4 in order to determine what these sums are. Here is an example:

Problem: What is the harmonic mean of the roots of $x^3 + 2x^2 - 3x + 7 = 0$

Solution: Applying the formula: $H_3 = \dfrac{3 \cdot (-7)}{-3} = \mathbf{7}$

The second interpretation of harmonic mean is the classic dual-labor problem. No doubt you've come across a problem like this before: Joe can paint a house in 5 hours and Jane can paint a house in 3 hours; how many hours does it take for both of them to paint a house? The answer is simply one-half of the harmonic mean!

$$\text{Together: } \frac{1}{2} \times \frac{2}{\left(\frac{1}{5} + \frac{1}{3}\right)} = \mathbf{1\frac{7}{8}} \text{ hours}$$

This interpretation is also known as the "Crossed Ladder Problem" (e.g., two ladders are crossed, what is the height at the crossing point). Anyways, feel free to look up other applications of the harmonic mean as well. Below are a few more practice problems:

Problem Set 4.4.4

1. The harmonic mean of 5 and 7 is ________
2. If $x^3 - 11x^2 + 38x = 40$, then the harmonic mean of the roots is ________
3. If $x^3 + 3x^2 + 2x + 1 = 0$, then the harmonic mean of the roots is ________
4. If $x^3 + 4x^2 + 13x + 7 = 0$, then the harmonic mean of the roots is ________
5. If $x^3 - 9x^2 + 26x - 24 = 0$, then the harmonic mean of the roots is ________
6. If $x^3 - \frac{13}{12}x^2 - \frac{5}{12}x + \frac{1}{2} = 0$, then then the harmonic mean of the roots is ________
7. The harmonic mean of the roots of $x^3 + Bx^2 + 3x + D = 0$ is 4. Find D. ________
8. If $2x^2 + 7x - 4 = 0$, then the harmonic mean of the roots is ________
9. The positive geometric mean of 8 and 18 is ____

4.4.5 Distance Between a Point and a Line

Finding the distance between a point and a line is a very computationally intense problem, so you'll typically see it on the last column. Assuming the equation of the line is $ax + by + c = 0$ and the point is (x_0, y_0), the formula is:

$$\text{Distance} = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

So the numerator is just inserting the (x, y) coordinates of the point into the equation of the line while the denominator is a little more involved requiring the square root of the sum of squares of coefficients of the line. Typically, the equation for the line will be an easy Pythagorean triple ($a = 3, b = 4$) making the problem a little bit less intense. Here is an sample problem:

Problem: What is the distance between the point $(3, 1)$ and the line $3x + 4y = -2$

Solution: Applying the formula: $d = \dfrac{|3(3) + 4(1) + 2|}{\sqrt{3^2 + 4^2}} = \dfrac{15}{5} = 3$

In this case you had to shift the constant to the left of the equals sign in order to have the line in the correct form before applying the formula. Here are a couple of practice problems:

Problem Set 4.4.5

1. The distance between the line $3x - 4y = 6$ and the point $(5, 1)$ is ______
2. The distance between the line $3x - 4y = 3$ and the point $(4, 1)$ is ______
3. The distance between the line $3x - 4y = -3$ and the point $(-2, -3)$ is ______
4. The distance between the point $(3, 1)$ and the line $5x - 12y = 1$ is ______
5. The distance between the line $3x + 4y = 5$ and the point $(1, 1)$ is ______
6. The distance between the point $(2, 1)$ and the line $3x + 4y = 5$ is ______
7. The distance between the line $3x + 4y = 1$ and the point $(-2, 2)$ is ______
8. The distance between the line $4x + 3y = 11$ and the point $(-2, 3)$ is ______

4.4.6 Distance Between Two Parallel Lines

The distance between two parallel lines whose equations are $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is simply:

$$\text{Distance} = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

Again, the equation for the line is usually an easy Pythagorean triple ($a = 3, b = 4$) making the problem a lot more simple to compute. Also, because you are taking the absolute values of the difference, if the constants are both on the right-side of the equals sign you don't have to move them to exactly match the equation of the lines given (oftentimes, this leads to dealing with negative numbers that can lead to more mistakes). Here are a couple of problems to give you practice:

Problem Set 4.4.5

1. The distance between the lines $3x - 4y = 8$ and $3x - 4y = 3$ is ______
2. The distance between the lines $3x + 4y = -1$ and $3x + 4y = 6$ is ______
3. The distance between the lines $5x + 12y = 2$ and $5x + 12y = 9$ is ______
4. The distance between the lines $3x + 4y = 9$ and $3x + 4y = -1$ is ______
5. The distance between the lines $3x - 4y = 7$ and $3x - 4y = 10$ is ______
6. The distance between the lines $\sqrt{2}x + \sqrt{7}y = 2$ and $\sqrt{2}x + \sqrt{7}y = 5$ is ______
7. The distance between the lines $\sqrt{11}x + \sqrt{5}y = 5$ and $\sqrt{11}x + \sqrt{5}y = -2$ is ______
8. The distance between the lines $\sqrt{2}x + \sqrt{7}y = 23$ and $\sqrt{2}x + \sqrt{7}y = 2$ is ______

4.5 Miscellaneous Topics

4.5.1 More on Sets

I've seen a lot of questions asking for the number of subsets containing a set amount of elements (e.g., in a set with 5 elements, how many subsets contain exactly 3 elements). Although this looks like a set problem, it is more of a combinatorics problem involving the formulas described in Section 2.2.10. Because we don't care about the specific order of the elements in the subset, you apply the ${}_n\mathrm{C}_k$ formula with n being the number of total elements in the set and k being the elements in the requested sub-set. Here's an example:

Problem: How many subsets containing only 2 elements does the set $\{N, U, M, B, E, R\}$ have?

Solution: Applying the formula: ${}_6\mathrm{C}_2 = \dfrac{6!}{2! \cdot (6-2)!} = \dfrac{6 \times 5}{2} = \mathbf{15}$

The only way to really complicate this is if they ask for the number of subsets containing either 2 or 3 elements (or whatever those arbitrary values are). In this case, you just apply the combination formula twice and then add (${}_6\mathrm{C}_2$ + ${}_6\mathrm{C}_3$). Here are some more practice problems:

Problem Set 4.5.1

1. The set $\{a, b, c\}$ has ______ 2-element subsets
2. The set $\{s, l, o, p, e\}$ has ___ 3-element subsets
3. The set $\{a, b, c, d\}$ has _____ 3-element subsets
4. The set $\{l, i, n, e, a, r\}$ has _ 4-element subsets
5. The set $\{a, b, c, d\}$ has _____ 2-element subsets
6. The set $\{t, e, x, a, s\}$ has ___ 3-element subsets
7. How many three element subsets does a five element set contain? ____________________
8. How many two element subsets does a six element set contain? ____________________
9. How many four element subsets does $\{m, o, n, d, a, y\}$ have? ____________________
10. How many subsets containing only 4 elements does the set $\{d, e, c, i, m, a, l, s\}$ have? ______
11. How many subsets containing only 2 or 3 elements does the set $\{s, q, u, a, r, e\}$ have? ____

4.5.2 Repeating Decimals in Reverse

Another problem that has been in vogue in recent years is applying all the procedures used with repeating decimals (Section 3.3), but in reverse. Instead of giving you a decimal and asking for a fraction, they give you a fraction and ask for the first few digits of the decimal. Here is an example:

$$\frac{23}{90} = 0.\_\_\_\_\_\_\_\_ \text{ (first four digits)}$$

There is really nothing new with these types of problems, you just have to work in reverse. Knowing the denominator is 90, the repeating fraction is in the form $.abbb\ldots$. From here, you are trying to find an a and b such that $ab - a = 23$, where ab is a two-digit number (not $a \times b$). Here, it's quick to see that $ab = 25$ and you got your answer of **2555**.

The only tricky thing a test writer can do is give you a fraction where some reduction has taken place. Take for example:

$$\frac{17}{45} = 0.\_\_\_\_\_\_\_\_ \text{ (first four digits)}$$

Here, the denominator has been reduced by a factor of 2, so you need to multiply by $\frac{2}{2}$ to get the fraction in a form you can work with:

$$\frac{17}{45} = \frac{34}{90} \Rightarrow ab - a = 34 \Rightarrow ab = 37 \Rightarrow \text{Answer: } \mathbf{3777}$$

Here are some more practice problems that go through all of the different types of repeating decimals outlined in Section 3.3:

Problem Set 4.5.2

1. The first four digits of the decimal for $\frac{16}{90}$ is 0. ________________
2. The first 3 digits of $\frac{13}{33}$ is 0. ________________
3. The first four digits of the decimal for $\frac{17}{45}$ is 0. ________________
4. The first four digits of the decimal for $\frac{17}{90}$ is 0. ________________
5. The first 4 digits of the decimal of $\frac{43}{90}$ is 0. ________________
6. The first four digits of the decimal for $\frac{11}{45}$ is 0. ________________
7. The first 4 digits of $\frac{31}{90}$ is 0. ________________
8. The first four digits of the decimal for $\frac{71}{330}$ is 0. ________________
9. The first 3 digits of the decimal of $\frac{42}{99}$ is 0. ________________
10. The first 4 digits of $\frac{13}{45}$ is 0. ________________

4.5.3 Repeating Decimals in Other Bases - Convert to Base 10

There are two types of questions involve converting repeating decimals in other bases. The first asks you to convert them into a base-10 fraction while the second asks you to keep the base the same as the repeating decimal. We'll tackle the base-10 conversion here and in the next section we'll look at keeping the bases the same. We'll start off with a simple type of conversion where you only need to use the sum of an infinite geometric series (Section 2.2.1) in order to solve. Here is an example of that type:

Change $.555\ldots_8$ to a base 10 fraction.

For these types of problems, you can apply the change of base (as explained in Section 3.2.2) to produce an infinite geometric series which you can then sum using the well-known formula:

$$.555\ldots_8 = \frac{5}{8} + \frac{5}{64} + \frac{5}{512}\ldots = \frac{\frac{5}{8}}{\left(1 - \frac{1}{8}\right)} = \frac{5}{8} \times \frac{8}{7} = \mathbf{\frac{5}{7}}$$

Although these problems look pretty intimidating they are pretty straightforward to solve. Now there is a more complicated form of the repeated decimal problem that uses a general variant of all the procedures outlined in Section 3.3.2 and Section 3.3.3 (because of the complexity, I have a hard time imagining they'd do something like Section 3.3.4, but you can certainly extend these methods to come up with a procedure).

For instance, for a repeating fraction in the form $.xyxyxy_b$, with base b, the procedure for converting to a base-10 fraction is:

1. For the numerator, convert the two-digit number xy into base-10.
2. The denomminator is $(b^2 - 1)$
3. Reduce the fraction if necessary.

Problem: Change $.353535\ldots_8$ to a base-10 fraction.

Solution: Numerator: $35_8 = 29_{10}$; Denominator: $8^2 - 1 = 63$. So your answer is: $\mathbf{\dfrac{29}{63}}$

For a repeating fraction in the form $.xyyyy_b$, the procedure is:

1. For the numerator, convert the two-digit number xy into base-10 and subtract x from it.
2. For the denominator, it is $b(b-1)$
3. Reduce the fraction if necessary.

Problem: Change $.3555\ldots_8$ to a base-10 fraction.

Solution: Numerator: $35_8 - 3_8 = 26_{10}$; Denominator: $8(8-1) = 56$. So your answer is: $\mathbf{\dfrac{26}{56}} \Rightarrow \mathbf{\dfrac{13}{28}}$

Here are some more practice problems to familiarize you with the procedures:

Problem Set 4.5.3

1. Change $0.444\ldots_8$ to a base-10 fraction. ______
2. Change $0.444\ldots_9$ to a base-10 fraction. ______
3. Change $0.333\ldots_8$ to a base-10 fraction. ______
4. Change $0.444\ldots_7$ to a base-10 fraction. ______
5. Change $0.777\ldots_9$ to a base-10 fraction. ______
6. $0.3131\ldots_5 =$ ____________________ $_{10}$ (fraction)
7. Change $0.3444\ldots_6$ to a base-10 fraction. ______
8. Change $0.4777\ldots_8$ to a base-10 fraction. ______
9. Change $0.3222\ldots_7$ to a base-10 fraction. ______
10. Convert $0.1666\ldots_8$ to a base-10 fraction. ______
11. Convert $0.1333\ldots_4$ to a base-10 fraction. ______
12. Change $0.3444\ldots_7$ to a base-10 fraction. ______
13. $0.232323\ldots_5 =$ ____________________ $_{10}$ (fraction)
14. Change $0.6333\ldots_7$ to a base-10 fraction. ______
15. Change $0.4666\ldots_8$ to a base-10 fraction. ______
16. $0.1313\ldots_5 =$ ____________________ $_1$0 (fraction)
17. Change $0.1444\ldots_6$ to a base-10 fraction. ______

4.5.4 Repeating Decimals in Other Bases - Keeping Same Base

If the problem asks you to keep the same base, it's easiest (at least for me) to do the following procedure:

1. Follow the previous Section (4.5.3) and convert the repeating decimal into a base-10.
2. Typically, you'll see some reduction in the fraction.
3. Convert both the numerator and the denominator back into the original base.

Let's do most of the same practice problems from the last section but we'll keep it in the same base this time:

Problem Set 4.5.4

1. Change $0.444\ldots_8$ to a base-8 fraction. ______
2. Change $0.444\ldots_9$ to a base-9 fraction. ______
3. Change $0.333\ldots_8$ to a base-8 fraction. ______
4. Change $0.444\ldots_7$ to a base-7 fraction. ______
5. Change $0.777\ldots_9$ to a base-9 fraction. ______
6. $0.3131\ldots_5 =$ ________________ $_5$ (fraction)
7. Change $0.3444\ldots_6$ to a base-6 fraction. ______
8. Change $0.4777\ldots_8$ to a base-8 fraction. ______
9. Change $0.3222\ldots_7$ to a base-7 fraction. ______
10. Convert $0.1666\ldots_8$ to a base-8 fraction. ______
11. Change $0.3444\ldots_7$ to a base-7 fraction. ______
12. $0.232323\ldots_5 =$ ________________ $_5$ (fraction)
13. $0.1313\ldots_5 =$ ________________ $_5$ (fraction)
14. Change $0.1444\ldots_6$ to a base-6 fraction. ______

4.5.5 Remainders with $\frac{a}{p}$, $\frac{b}{p}$, and $\frac{ab}{p}$

It's best just to show a practice problem first so you know the type of question I'm talking about:

Problem: If $\dfrac{6x}{7}$ has a remainder of 2 and $\dfrac{2y}{7}$ has a remainder of 3 what is the remainder of $\dfrac{4xy}{7}$?

As explained in Section 1.4.5, "the remainders after algebra is equal to the algebra of the remainders," you can do the multiplication of the first two expression which translates to the multiplication of the remainders. From there, you can divide the calculated expression (and their equivalent calculated remainders) to get what the question is asking for:

$$\frac{6x}{7} \times \frac{2y}{7} = \frac{12xy}{7} \div 3 = \frac{4xy}{7}$$

Same operations with equivalent remainders:

$$2 \times 3 = 6 \div 3 = \mathbf{2}$$

Now sometimes when you do the algebra on the remainders, you wind up with a fractional answer. You can either figure out small, individual values for x and y and use them in the problem expression to calculate the remainder, or you can think about what xy fits the expression. Here is an example to show both methods:

Problem: If $\dfrac{2x}{7}$ has a remainder of 1 and $\dfrac{y}{7}$ has a remainder of 3 what is the remainder of $\dfrac{xy}{7}$?

Solution 1: Multiplying the first two expressions and dividing by 2 yields: $\dfrac{xy}{7} \cong 1.5$ which is a road-block. From the first expression, run through small values of x to find $x = 4$; do the same for the second expression to find the value $y = 3$; therefore $\dfrac{xy}{7} = \dfrac{12}{7}$ which has a remainder of **5**.

Solution 2: Again, multiplying the first two expressions and dividing by 2 yields: $\frac{xy}{7} \cong 1.5$. Multiplying by 2 gives $\frac{2xy}{7} \cong 3$. Treating xy together and running through small values yields $xy = 5$, therefore $\frac{xy}{7} = \frac{5}{7}$ which has a remainder of **5**.

Both methods work OK, so it's up to you to find which way you are more comfortable with. Here are some more practice problems for you:

Problem Set 4.5.5

1. If $\frac{a}{9}$ has a remainder of 7 and $\frac{b}{9}$ has a remainder of 5 then $\frac{ab}{9}$ has a remainder of ________
2. If $\frac{a}{8}$ has a remainder of 2, and $\frac{b}{8}$ has a remainder of 7, then $\frac{ab}{8}$ has a remainder of ________
3. If $\frac{3x}{5}$ has a remainder of 4 and $\frac{3y}{5}$ has a remainder of 1 then $\frac{xy}{5}$ has a remainder of ____
4. If $\frac{2x}{7}$ has a remainder of 3 and $\frac{2y}{7}$ has a remainder of 4, then $\frac{xy}{7}$ has a remainder of ____
5. If $\frac{2x}{7}$ has a remainder of 5 and $\frac{3y}{7}$ has a remainder of 4, then $\frac{xy}{7}$ has a remainder of ____

4.5.6 Minimal and Maximum Value of Expressions

This is a problem where knowing too much Calculus actually winds up slowing you down. Here is an example problem:

What is the minimum value of $2(x-1)^2 + 3$?

People who have done some basic Calculus will want to take the derivative of the expression and set it equal to zero, solve for x, then substitute that value into the original expression – a very time consuming process! Oftentimes, these minimum value problems can be easily solved by taking a step back and noticing that the minimum value of a square is 0! Applying that above, the minimum value $(x-1)^2$ can be is 0 – this makes the minimum value of the expression: $2(0) + 3 = \mathbf{3}$.

Sometimes the expression isn't factored, so you'll have to complete the square in order to get a more discernible expression where you can apply the above "trick." Sometimes, the expression might have a trigonometric function, such as sin and cos where you just need to know that those minimal values are -1. I'd only revert to taking derivatives if you absolutely can't figure our the simplified form.
For expressions involving trigonometric functions, you might be asked to calculate maximum values as well. Just remember that for sin and cos, those achieve a max value of 1. Here are a few practice problems concerning minimum and maximum values of expressions:

Problem Set 4.5.6

1. The minimum value of $f(x) = (x+2)^2 + 2$ is _
2. The minimum value of $y = 2x^2 + 3$ is ________
3. The minimum value of $f(x) = 2x^2 + 4x + 2$ is .
4. The minimum value of $f(x) = x^2 - 2$ is ______
5. The minimum value of $\sin(2x) - 3$ is ________

6. The minimum value of $y = x^2 + 4x$ is at $y =$ _

7. The minimum value of $\sin(3x) - 5$ is ________

8. The minimum value of $\sin(2x) - 3$ is ________

9. The minimum value of $y = x^2 + 2x - 3$ is ____

10. The horizontal asymptote of $y = 4^x + 2$ is ____

11. The maximum value of $\cos(3x) - 2$ is ________

12. The maximum value of $5 - \cos(3x)$ is ________

13. The minimum value of $y = 3x^2 + 4x$ is _______

14. The range of the function $y = -x^4 + 4$ is $y \leq$ _

5 Solutions

The following are solutions to the practice problems proposed in the previous sections.

Problem Set 1.1:

2850	7020	561	4233
3392	5265	6992	1150
8648	1728	918	847
3828	5418	4968	5644
4680	7776	5073	2450
1088	6688	242	2484
1820	1320	2976	2240
4836	3819	1232	5680
3111	1215	896	1739
7663	8613	2646	1943
3724	3525	2618	4606
2059	5610	845	704
930	2795	5328	2009
1610	5400	3127	7462
238	1809	2125	2475
4384	31672	46812	29430
17277	36950	21489	48312
42658	3564	5994	69030
22270	27664	11022	13545
294849	128472	211554	124890
13431	397946	185364	283251
190005	66789	293007	176712

Problem Set 1.2.1:

1. 594
2. 792
3. 418
4. 5082
5. 814
6. 726
7. 2.53
8. 572
9. 2706
10. 50616
11. 18
12. 3927
13. 25
14. 35631
15. .275
16. 4884
17. 34
18. 9657
19. 26
20. 5883
21. 27
22. 203
23. 2178
24. 27
25. 4551
26. 3885
27. 38295
28. 222333
29. 1155
30. 14641
31. 36663
32. 704
33. 333
34. 22.077
35. 2.42%
36. 1573
37. 252
38. 22066
39. 14641
40. 1452
41. 858
42. 2662
43. 2420
44. 2310
45. 23
46. 13794
47. 12100
48. 1815
49. (*) $648 - 717$
50. 182
51. 6776
52. (*) $31181 - 34465$
53. 14641
54. 23
55. 6006
56. 136653
57. 2310
58. 15004
59. (*) $75897 - 83853$

Problem Set 1.2.2:

1. 124634
2. 2363.4
3. 37269
4. 345
5. 222
6. 24846
7. $15.15
8. 448844
9. (*) $14488 - 16014$
10. (*) $2398 - 2652$

Problem Set 1.2.3:

1. 6000
2. 10800
3. 6.5
4. 3700
5. 825
6. 2.56
7. 3675
8. 10450
9. 24.64
10. 101
11. .44
12. 80800
13. (*) 376 – 417
14. $\frac{6}{25}$
15. 5225
16. 850
17. (*) 1265 – 1400
18. 85.6%
19. 16.16
20. 7575
21. (*) 376 – 417
22. $\frac{7}{25}$
23. 80.24
24. 7675
25. 1280
26. (*) 185 – 205
27. 50075
28. 4125
29. 6600
30. 4950
31. (*) 14842800 – 16405200

Problem Set 1.2.4:

1. 3600
2. 4800
3. .88
4. 6300
5. (*) 560 – 620
6. 2100
7. 1800
8. (*) 10504127 – 11609825
9. .64
10. (*) 719 – 796
11. 1.28
12. 19800
13. 64
14. 54000
15. 72.6

Problem Set 1.2.5:

1. 40000
2. (*) 189992 – 209992
3. 1.104
4. (*) 425 – 471
5. (*) 2102 – 2324
6. (*) 1054 – 1166
7. 200000
8. (*) 6628 – 7326
9. (*) 192850 – 213150
10. 6000
11. (*) 139 – 154
12. (*) 4620 – 5107
13. (*) 3917 – 4330
14. (*) 384750 – 425250
15. 153000

16. (*) 307 – 341
17. 121
18. (*) 597668 – 660582
19. (*) 8957133 – 9899991
20. (*) 114 – 126
21. 183000
22. (*) 7440353 – 8223549
23. (*) 1261 – 1395
24. (*) 646 – 714
25. (*) 22757 – 25153
26. 210000
27. (*) 3360 – 3715
28. 9300
29. (*) 347699652 – 384299616
30. .02
31. (*) 5842616 – 6457630
32. (*) 2020 – 2233
33. (*) 3528 – 3900
34. (*) 321 – 356
35. (*) 474999 – 525000
36. (*) 1030 – 1140
37. (*) 326 – 362
38. (*) 1576 – 1743
39. (*) 461428 – 510000
40. (*) 38240 – 42267
41. (*) 182076 – 201242
42. 60.25
43. (*) 593749 – 656250
44. (*) 652 – 721
45. (*) 775848 – 857518
46. (*) 1056 – 1168
47. (*) 2253 – 2492
48. (*) 93755 – 103625
49. (*) 4303 – 4757
50. (*) 450570 – 498000
51. (*) 84142 – 93000
52. (*) 583 – 646
53. (*) 58163 – 64286
54. (*) 7546054 – 8340376
55. (*) 664694 – 734662
56. (*) 1644 – 1818
57. 40625
58. (*) 99071 – 109500
59. (*) 232071 – 256500
60. (*) 113491195 – 125437637
61. (*) 18457124 – 20399980
62. (*) 484306 – 535286
63. (*) 6641817 – 7340957
64. (*) 24 – 28
65. (*) 35624 – 39375
66. (*) 47362 – 52348
67. (*) 5277 – 5834
68. (*) 118 – 132
69. (*) 8130 – 8986
70. (*) 6332 – 7000
71. (*) 54204 – 59910
72. (*) 237 – 263
73. (*) 50805 – 56154
74. (*) 14776 – 16332
75. (*) 12324 – 13622
76. (*) 200163 – 221233
77. (*) 577 – 639
78. (*) 21855 – 24157
79. (*) 632 – 700
80. (*) 605 – 670
81. (*) 1159 – 1283
82. (*) 3167 – 3502
83. (*) 139 – 155
84. (*) 117040 – 129362

Problem Set 1.2.6:

1. 7.8
2. 72
3. 96
4. 720
5. 2842
6. 4368
7. 840
8. 4368
9. 3.6
10. 2016
11. 378
12. 4410
13. 22.5
14. 4140
15. 1.5
16. 10.56
17. 700

Problem Set 1.2.7:

1. 8633
2. 9312
3. 11227
4. 9021
5. 11021
6. 8277
7. 11016
8. 11663
9. 8544
10. 8924
11. 10712
12. 10506
13. 8556
14. 11342
15. 8633
16. 9212
17. 8554
18. 8918
19. 987042
20. 9888
21. 9579
22. 980099
23. 1013036
24. 10379
25. 9672
26. 9888
27. 982081
28. 10088
29. 1011024
30. (*) 18062 − 19964
31. 12996

Problem Set 1.2.8:

1. 6.25
2. 1.225
3. 625
4. 13225
5. 3025
6. −24.75
7. (*) 19699 − 21773
8. 255025
9. 14

Problem Set 1.2.9:

1. 3364
2. 260100
3. 2209
4. 2809
5. (*) 111720 − 123480
6. 3136
7. 3481
8. 1681

Problem Set 1.2.10:

1. 7224
2. 3021
3. 2496
4. 3596
5. 48.96
6. 7216
7. 864
8. 63.84
9. 24.91
10. 3025
11. 9984
12. 7225
13. 1225
14. 5625
15. 4225
16. 1225
17. 441
18. 4225
19. 2025
20. 7225
21. 3025
22. 1064
23. 2000
24. 5625
25. 4842
26. 900
27. 5625
28. 5625
29. 3025
30. 9975
31. 4200
32. 936
33. 7200
34. 625
35. 1073
36. −7200
37. 9856
38. (*) 4745 − 5245
39. (*) 4016 − 4440
40. (*) 9305 − 10285
41. (*) 3035 − 3355
42. (*) 26270 − 29036
43. (*) 101076 111716
44. 14400
45. (*) 62132 − 68673
46. (*) 1423267 − 1573085

Problem Set 1.2.11:

1. 1462
2. 736
3. 403
4. 252
5. 1944
6. 976
7. 765
8. 574
9. 1458
10. 2268
11. 1008
12. 1612

Problem Set 1.3.1:

1. 40804
2. 164836
3. 253009
4. 368449
5. 43264
6. 93636
7. 259081
8. 646416
9. 495616
10. 166464
11. 362404
12. 91809
13. 826281
14. 161604
15. 499849
16. 34013
17. 644809
18. 163216
19. 262144
20. 37942
21. 374544
22. 96942
23. 509796
24. 49374
25. 23632
26. 67196
27. 24969
28. 49731
29. 46144
30. 204020
31. 35143
32. 15004
33. 842724
34. 38688
35. 37942
36. 274576
37. 41363
38. 19881
39. 108332
40. 25864
41. 144288144
42. 444889

Problem Set 1.3.2:

1. 640
2. 810
3. 450
4. 0
5. 1210
6. −660
7. 16.9
8. 10020
9. 1280
10. 380
11. 12030
12. 0
13. 441
14. 960
15. 5100
16. 660
17. 490
18. −10030
19. 384
20. 14.4
21. 196
22. 1024
23. 330
24. 484
25. 2450
26. 870
27. 256
28. 3540

29. -196
30. 289
31. -289
32. 1080
33. 4830
34. 2002
35. 1210
36. 2160
37. 6320
38. 1188
39. 363
40. 14280
41. 1560
42. -324
43. 3300
44. 9900
45. 0
46. -1210
47. 2775
48. 540
49. 576
50. 16770
51. 910
52. (*) $1825 - 2019$
53. 3300
54. 720
55. (*) 12108
13384
56. (*) $9076 - 10032$
57. 1056
58. 11990
59. (*) $8015 - 8859$
60. 672
61. (*) $2050 - 2266$
62. 1584
63. (*) $4698 - 5194$
64. 2250
65. 4662
66. -588
67. (*) $9516 - 10518$
68. 2100
69. 3774
70. (*) $3659 - 4045$

Problem Set 1.3.3:

1. 2521
2. 313
3. 481
4. 1301
5. 3281
6. 12961

Problem Set 1.3.4:

1. 462
2. 1920
3. 380
4. 128
5. 160
6. 124
7. 1920
8. 12960
9. 550
10. 3760
11. 6380
12. 120
13. 2280
14. 3360
15. 128
16. 1680
17. 880
18. 450
19. 550
20. 128

21. 5300
22. 2300

Problem Set 1.3.5:

1. 9090
2. 505
3. 5353
4. 6868
5. 4141
6. 4545
7. 6161
8. 5858

Problem Set 1.3.6:

1. 145
2. 140
3. −115
4. 133
5. 272
6. −109
7. 264
8. 175
9. −97
10. 193
11. 153
12. 107
13. −172
14. 11.2
15. −5.07
16. 254
17. 540
18. 218
19. 300
20. −30
21. −94
22. 525
23. 326
24. 321
25. 170
26. 438
27. −363
28. 302
29. 720
30. 218
31. −1560
32. −70
33. −170
34. 288
35. 105
36. 18
37. 78
38. −238
39. −900
40. 168
41. 1014
42. −1540
43. −616
44. 715
45. −272
46. 672
47. 894

Problem Set 1.3.7:

1. 1575
2. 4275
3. 2275
4. 4675
5. 2925
6. 2975
7. 6175
8. 5225

Problem Set 1.3.8:

1. $35\frac{1}{16}$
2. $72\frac{2}{9}$
3. $12\frac{4}{25}$
4. $29\frac{1}{6}$
5. $101\frac{1}{16}$
6. $139\frac{1}{36}$
7. $75\frac{1}{36}$
8. $245\frac{1}{121}$
9. $137\frac{4}{25}$
10. 53.04
11. $40\frac{4}{9}$
12. $101\frac{1}{49}$
13. $53\frac{1}{25}$
14. $131\frac{1}{25}$
15. $29\frac{4}{25}$
16. $131\frac{1}{64}$
17. $138\frac{2}{3}$
18. 131
19. $64\frac{4}{9}$
20. $160\frac{4}{9}$
21. $351\frac{1}{49}$
22. 9
23. 9.03
24. 41
25. $44\frac{4}{9}$
26. 79.04
27. $21\frac{7}{15}$
28. 5
29. 5.7
30. $12\frac{24}{25}$

Problem Set 1.3.9:

1. $8\frac{9}{14}$
2. $19\frac{9}{25}$
3. $15\frac{16}{23}$
4. $22\frac{25}{32}$
5. $13\frac{9}{19}$
6. $24\frac{25}{34}$
7. $28\frac{9}{34}$
8. $8\frac{9}{17}$
9. $11\frac{9}{14}$
10. $23\frac{9}{16}$
11. $13\frac{16}{17}$
12. $-\frac{13}{14}$
13. $-\frac{17}{18}$
14. $-2\frac{16}{25}$
15. $-2\frac{8}{17}$
16. $30\frac{16}{21}$
17. $-2\frac{7}{16}$
18. $-\frac{11}{12}$
19. $-3\frac{11}{15}$
20. $-2\frac{8}{17}$
21. $-1\frac{13}{17}$
22. $67\frac{9}{38}$
23. $-1\frac{11}{15}$

Problem Set 1.3.10:

1. (*) 2553 – 2823
2. (*) 25356 – 28026
3. (*) 3149 – 3481
4. (*) 5958 – 6586
5. (*) 106799 – 118041
6. (*) 34108 – 37700
7. (*) 39398 – 43545
8. (*) 126445 – 139755
9. (*) 14630 – 16170
10. (*) 624255 – 689967
11. (*) 97917 – 108225
12. (*) 760958 – 841060
13. (*) 31005 – 34269
14. (*) 9771 – 10801
15. (*) 80548 – 89028
16. (*) 65555 – 72457
17. (*) 60693 – 67083
18. (*) 60762 – 67158
19. (*) 2048 – 2265
20. (*) 86184 – 95256
21. (*) 157586 – 174174
22. (*) 7524 – 8316
23. (*) 34108 – 37700
24. (*) 523488 – 578592
25. (*) 25536 – 28224
26. (*) 298452 – 329868
27. (*) 260646 – 288084
28. (*) 1740 – 1924
29. (*) 8257 – 9127
30. (*) 5728 – 6332
31. (*) 28260 – 31236
32. (*) 3513 – 3883
33. (*) 53437500 – 59062500
34. (*) 25650 – 28350
35. (*) 95000 – 105000
36. (*) 475089 – 525099
37. (*) 3910 – 4322
38. (*) 150292 – 166114
39. 2592
40. (*) 3406 – 3766

Problem Set 1.4.1:

1. 0
2. 2
3. 3
4. 3
5. 0
6. 5
7. 4

Problem Set 1.4.2:

1. 2
2. 5
3. 0
4. 7
5. 2
6. 8

Problem Set 1.4.3:

1. 8
2. 5
3. 9
4. 9
5. 0
6. 8
7. 5
8. 7
9. 4
10. 4

Problem Set 1.4.4:

1. 4
2. 2
3. 2
4. 6
5. 0
6. 3
7. 0
8. 6
9. 7
10. 0
11. 6
12. 7
13. 0
14. 2
15. 6

Problem Set 1.4.5:

1. 1
2. 3
3. 0
4. 2
5. 2
6. 2
7. 4
8. 0
9. 4
10. 1
11. 2
12. 5
13. 2
14. 2
15. 2
16. 3
17. 3
18. 0
19. 5
20. 2
21. 4
22. 3
23. 6
24. 2
25. 3
26. 2
27. 2
28. 2

Problem Set 1.4.6:

1. $39\frac{1}{3}$
2. $55\frac{8}{9}$
3. $222\frac{5}{9}$
4. $35\frac{2}{3}$
5. $50\frac{2}{3}$
6. $137\frac{1}{9}$
7. $1371\frac{2}{3}$
8. 55

Problem Set 1.4.7:

1. 2.5%
2. 7.5%
3. 17.5%
4. 52.5%
5. 1.075
6. .075
7. $1\frac{1}{4}\%$
8. 20%
9. 17.5%
10. 40
11. 27.5%
12. .045
13. 18
14. .025
15. 435%
16. 6.25%
17. $\frac{11}{40}$
18. .32
19. 8%
20. .0081

Problem Set 1.5.1:

1. 198
2. -396
3. 1998
4. 495
5. 99
6. 2997
7. -3996
8. -999
9. -198
10. -4995

Problem Set 1.5.2:

1. $-1\frac{1}{6}$
2. $-1\frac{14}{15}$
3. -2
4. $-1\frac{17}{20}$
5. $-1\frac{4}{7}$
6. $-\frac{7}{8}$
7. $-4\frac{1}{8}$
8. $-5\frac{1}{10}$
9. $-1\frac{8}{9}$
10. $-7\frac{1}{14}$
11. $-3\frac{1}{6}$
12. $-1\frac{5}{6}$
13. $-8\frac{1}{12}$
14. $-6\frac{1}{12}$
15. $-4\frac{1}{2}$
16. $-1\frac{3}{5}$

Problem Set 1.5.3:

1. $\frac{4}{21}$
2. $\frac{1}{24}$
3. $\frac{3}{40}$
4. $1\frac{1}{6}$

Problem Set 1.5.4:

1. $2\frac{1}{156}$
2. $2\frac{1}{30}$
3. $2\frac{16}{285}$
4. $\frac{4}{15}$
5. $1\frac{4}{35}$
6. $1\frac{4}{143}$
7. $1\frac{36}{91}$
8. $\frac{1}{30}$
9. $1\frac{4}{195}$
10. 2
11. 3
12. $-\frac{31}{35}$
13. $1\frac{4}{255}$
14. $1\frac{16}{165}$
15. $1\frac{4}{143}$
16. $1\frac{1}{210}$
17. $3\frac{1}{156}$
18. $1\frac{2}{35}$
19. $1\frac{1}{132}$
20. $\frac{49}{330}$
21. $-\frac{145}{154}$

Problem Set 1.5.5:

1. $\frac{13}{252}$
2. $\frac{9}{203}$
3. $\frac{17}{520}$
4. $\frac{22}{915}$
5. $\frac{19}{495}$
6. $\frac{19}{1342}$
7. $\frac{11}{584}$
8. $\frac{9}{430}$
9. $-\frac{11}{42}$
10. $\frac{17}{900}$
11. $-\frac{37}{1620}$
12. $\frac{11}{328}$
13. $-\frac{22}{435}$
14. $\frac{13}{328}$
15. $\frac{17}{333}$
16. $\frac{7}{165}$
17. $\frac{27}{784}$
18. $\frac{19}{1342}$
19. $\frac{11}{448}$
20. $-\frac{11}{414}$
21. $\frac{11}{328}$
22. $\frac{18}{979}$

Problem Set 2.1.1:

1. 784
2. 10.24
3. 841
4. 256
5. 961
6. 4.84
7. 1156
8. 289
9. 529
10. 361
11. 324
12. 5.76
13. 529
14. 1024
15. 484
16. 196
17. 441
18. 576
19. 9.61
20. 7.29

21. 784
22. 1156
23. 676
24. 289
25. 1089
26. -27
27. (*) $972 - 1075$
28. (*) $372 - 412$
29. -224
30. .324
31. (*) $14546 - 16078$
32. (*) $7553 - 8349$
33. (*) $36495 - 40337$
34. (*) $379 - 420$
35. (*) $28227 - 31200$
36. (*) $27132 - 29990$
37. (*) $9098 - 10057$
38. (*) $13166 - 14553$
39. (*) $79344 - 87698$
40. (*) $241 - 267$
41. (*) $496 - 549$
42. (*) $975 - 1078$
43. (*) $184756 - 204206$

Problem Set 2.1.2:

1. 12
2. 1331
3. 2744
4. -7
5. 1728
6. 4096
7. 2
8. 1331
9. -1728
10. 13
11. -9
12. 512
13. 3375
14. 1728
15. $\frac{5}{4}$
16. 2197
17. 343
18. -11
19. 216
20. 3375
21. (*) $1653 - 1828$
22. $\frac{1}{2}$
23. 370
24. -2
25. 1.2
26. 64000
27. 1331
28. 1.1
29. (*) $692464 - 765356$
30. .9
31. (*) $1682982 - 1860140$
32. -54
33. $\frac{1}{2}$
34. 225
35. -217
36. (*) $169059 - 186855$
37. 343000
38. (*) $1641486 - 1814374$
39. (*) $2669363 - 2950349$
40. 4096

Problem Set 2.1.3:

1. 160
2. -83
3. 32
4. 243
5. -61
6. 3
7. 160
8. 98
9. 40
10. -26
11. 81
12. 3200000
13. 729
14. .04
15. 4000
16. 2560000
17. (*) $3242-3584$
18. 40000
19. 7
20. -144
21. .081
22. 288
23. 29
24. (*) $61-69$
25. 2.5
26. 64000
27. 512000
28. 648000
29. 98000
30. 14400
31. 21600
32. 2025000
33. 2500
34. 64800
35. 8100000
36. 144000

Problem Set 2.1.4:

1. $\frac{1}{8}$
2. 220%
3. .56
4. $\frac{5}{8}$
5. 2.125
6. 1
7. 60%
8. .125
9. .81
10. 6.25%
11. $-.875$
12. 275%
13. $\frac{5}{9}$
14. $\frac{2}{3}$
15. $-\frac{16}{17}$
16. .46
17. $-\frac{8}{9}$
18. $\frac{3}{8}$
19. .8
20. $42\frac{6}{7}\%$
21. $77\frac{7}{9}\%$
22. $\frac{1}{2}$
23. .56
24. $\frac{9}{11}$
25. $\frac{4}{3}$
26. 43.75%
27. 176
28. $28\frac{4}{7}\%$
29. $-.375$
30. $-\frac{4}{3}$
31. $7\frac{1}{7}\%$
32. $\frac{7}{16}$
33. 15
34. 3
35. $-.27$
36. 121
37. $-\frac{7}{18}$
38. 31.25%

39. 1331

40. $\frac{3}{14}$

41. 8

42. $10\frac{4}{5}$

43. $21\frac{3}{7}\%$

44. $\frac{5}{14}$

45. 6

46. $\frac{11}{14}$

47. 10021

48. $8\frac{1}{3}\%$

49. $78\frac{4}{7}\%$

50. 800

51. 1331

52. $80\frac{1}{3}$

53. 135

54. $\frac{9}{14}$

55. 13.31

56. $187\frac{1}{2}\%$

57. $\frac{1}{16}$

58. $121\frac{3}{7}\%$

59. $\frac{3}{7}$

60. $\frac{3}{80}$

61. $\frac{11}{1000}$

62. $\frac{13}{14}$

63. $\frac{1}{14}$

64. 2400

65. .06875

66. $92\frac{6}{7}\%$

67. $\frac{2}{65}$

68. $107\frac{1}{7}\%$

69. $\frac{3}{14}$

Problem Set 2.1.5:

1. 12012
2. 54
3. 505.05
4. 25025
5. 70707
6. 37
7. 20020
8. 15015
9. 27027
10. 29
11. 60
12. 36036
13. 7
14. 1073
15. 30030
16. 70070
17. 999
18. 55055
19. 75075
20. 153153
21. 10010
22. 18018
23. 7070.7
24. 121.121
25. 35035
26. 909.09
27. 505505
28. 9009
29. 1111.11
30. 303303
31. 5005
32. $28\frac{7}{9}$
33. 7007
34. 36
35. (*) $712 - 788$
36. $\frac{1}{3}$
37. 6006
38. 49
39. 13
40. 48
41. 10010
42. 96
43. 256
44. 11011

45. 9009
46. 36036
47. 7007
48. 60
49. $8\frac{2}{9}$
50. 9009
51. 90
52. $16\frac{4}{9}$
53. 11011
54. 96
55. $24\frac{2}{3}$
56. 147
57. $32\frac{8}{9}$
58. 81
59. 74
60. $789\frac{1}{3}$
61. 37
62. 13013
63. 324
64. 185
65. 175
66. 9009
67. 15015

Problem Set 2.1.6:

1. 2042
2. 44
3. 2003
4. 199
5. 1666
6. 444
7. 277
8. 1459
9. 999
10. 2222
11. -89
12. 2100
13. 999
14. 534
15. 2017
16. 2007
17. 1664
18. 1666
19. 1364
20. 2006
21. 556
22. 505
23. 1530
24. $66\frac{8}{9}$
25. 34
26. 2005
27. 20
28. 401
29. 2997
30. 11011
31. 50175
32. 84
33. 22066
34. 10.1

Problem Set 2.1.7:

1. 20
2. 20
3. 16
4. 4
5. 12
6. 6
7. 20
8. 6
9. 12
10. 20

Problem Set 2.1.8:

1. (*) $185 - 205$
2. (*) $683 - 756$
3. (*) $51 - 58$
4. (*) $290 - 322$
5. (*) $5052 - 5585$
6. (*) $342 - 379$
7. (*) $1608 - 1778$
8. (*) $7495 - 8285$
9. (*) $995 - 1100$
10. (*) $664 - 734$
11. (*) $15384 - 17005$
12. (*) $1221 - 1350$
13. (*) $5052 - 5585$
14. (*) $493 - 546$
15. (*) $46339 - 51218$
16. (*) $524 - 581$

Problem Set 2.1.9:

1. 22
2. 126
3. 4.5
4. 240
5. 132
6. 220
7. 440
8. 1760
9. 81
10. 3520
11. 30
12. 10
13. 44
14. .5
15. 2160
16. 8
17. 66
18. 22.5
19. 11

Problem Set 2.1.10:

1. 81
2. 1728
3. 81
4. 3
5. 27
6. 9
7. 5184
8. 2.5
9. 10000
10. $1\frac{1}{3}$
11. 1.5
12. 1
13. 36
14. 3456
15. 500

Problem Set 2.1.11:

1. 4
2. 32
3. 48
4. 693
5. 154
6. 308

7. 96
8. 16
9. $\frac{3}{8}$
10. 6
11. 2
12. 50%
13. 400%
14. 37.5%
15. 225%
16. 400%
17. 11
18. 600%
19. 37.5%
20. 1155
21. 2
22. 1.75
23. 5
24. 3
25. 2.5
26. 112
27. $\frac{1}{8}$
28. 231
29. 63
30. 12
31. 147
32. 128
33. 320

Problem Set 2.1.12:

1. 77
2. -40
3. 37

Problem Set 2.2.1.:

1. 132
2. 231
3. 169
4. 123
5. 18
6. 240
7. 100
8. $\frac{2}{5}$
9. 96
10. 132
11. 81
12. 506
13. 1.5
14. 441
15. 396
16. 255
17. 4
18. $-1\frac{1}{8}$
19. 143
20. $\frac{4}{5}$
21. 117
22. $5\frac{1}{3}$
23. 2.5
24. 108
25. 462
26. -13
27. 264
28. 528
29. $4\frac{1}{6}$
30. 98
31. 126
32. 207
33. 91
34. $6\frac{1}{4}$
35. $2\frac{2}{3}$
36. 255

37. 147
38. 98
39. 1150
40. 16
41. 264
42. (*) $418 - 464$
43. 396
44. 242
45. 81
46. 6
47. 294
48. 726
49. 161
50. 273
51. 168
52. 528
53. 37
54. 3.2
55. (*) $179763 - 198687$
56. (*) $4138 - 4574$
57. 11
58. 135
59. 1.5
60. $9\frac{1}{3}$
61. 80
62. 4.8
63. 77
64. (*) $7866 - 8696$
65. 3
66. 16
67. (*) $25863 - 28587$
68. (*) $1231 - 1361$

Problem Set 2.2.2:

1. 750
2. 372
3. 514
4. 660
5. 804
6. 610
7. 893
8. 534
9. 284
10. 304
11. 372
12. 114
13. 88
14. 6
15. 196
16. 143
17. 319
18. 693
19. 748

Problem Set 2.2.3:

1. 3
2. 9
3. 96
4. 4
5. 10
6. 10
7. 8
8. 12
9. 9
10. 124
11. 36
12. 56
13. 20
14. 42
15. 5
16. 15
17. 192
18. 78
19. 42
20. 8

21. 56
22. 70
23. 7
24. 55
25. 385
26. 24
27. 24
28. 39
29. 54
30. 35
31. 160
32. 7
33. 24
34. 7
35. 35
36. 124

Problem Set 2.2.4:

1. 5
2. 9
3. 5
4. 27
5. 20
6. 35
7. 2
8. 14

Problem Set 2.2.5:

1. 140
2. 108
3. 45
4. 1080
5. 120
6. 1440
7. 133
8. 540

Problem Set 2.2.6:

1. 70
2. 40
3. 35
4. 176
5. 276
6. 112
7. 35
8. 51
9. 45
10. 66
11. 78
12. 66
13. 36
14. 18

Problem Set 2.2.7:

1. 9
2. 40
3. 26
4. 15
5. 6
6. 8
7. 33
8. 5

9. 15
10. 6
11. 7
12. 6
13. 9
14. 8
15. 84
16. 84

Problem Set 2.2.8:

1. 3
2. 6
3. $2\sqrt{3}$
4. 12
5. 4
6. 18
7. 3
8. 9

Problem Set 2.2.9:

1. 726
2. 144π
3. 27
4. 96
5. 64
6. 216
7. 512
8. 1728
9. 224

Problem Set 2.2.10:

1. 60
2. 10
3. 20
4. 35
5. 840
6. 30
7. 28
8. 10
9. 336
10. 56
11. 36
12. 2
13. 6
14. 24
15. $\frac{1}{120}$
16. 6
17. 4
18. 12
19. 720
20. $\frac{1}{6}$
21. 10
22. 200

Problem Set 2.2.11:

1. $-\frac{1}{2}$
2. $\frac{8}{3}$
3. 0
4. 1
5. $\frac{1}{2}$

6. -10
7. $-\frac{1}{2}$
8. -1
9. 10
10. $\frac{1}{3}$
11. 112.5
12. 36
13. 0
14. -1
15. 0
16. $\frac{1}{2}$
17. 1
18. -2
19. -2
20. -1
21. 108
22. -1
23. -1
24. 45
25. $\frac{14}{9}$
26. 1
27. 12
28. -1
29. 2
30. 15
31. -4
32. 3
33. 225
34. $\frac{1}{3}$
35. $\frac{\sqrt{2}}{2}$
36. -1
37. -1
38. $\frac{3}{2}$
39. -1
40. $\frac{1}{2}$
41. 0
42. $-\frac{1}{3}$
43. 1
44. $-\frac{1}{3}$
45. $-\frac{3}{4}$
46. $-\frac{1}{2}$
47. 45
48. $\frac{1}{2}$
49. $-\frac{3}{4}$
50. $\frac{1}{4}$
51. $\frac{6}{5}$
52. $\frac{1}{4}$
53. $\frac{1}{3}$
54. $\frac{1}{4}$
55. 4
56. $\frac{1}{2}$
57. $-\frac{3}{4}$
58. $-\frac{1}{4}$
59. $3\frac{1}{2}$

Problem Set 2.2.12:

1. 1
2. $\frac{1}{2}$
3. $\frac{1}{2}$
4. $-\frac{1}{2}$
5. $\frac{3}{4}$
6. 68
7. $\frac{3}{4}$
8. $-\frac{1}{2}$
9. 3
10. $\frac{1}{2}$
11. $-\frac{1}{4}$
12. 308
13. $-\frac{1}{2}$
14. $\frac{1}{2}$
15. $\frac{1}{4}$
16. $\frac{1}{2}$
17. $\frac{1}{2}$

18. $\frac{7}{25}$
19. $\frac{1}{2}$
20. $\frac{1}{4}$
21. $-\frac{1}{2}$
22. 1
23. $-\frac{1}{2}$
24. 1
25. $\frac{1}{2}$
26. $-\frac{1}{2}$
27. -2

Problem Set 2.2.13:

1. 4
2. 5
3. 2
4. 3
5. 8
6. -3
7. 5
8. -2
9. $\frac{1}{6}$
10. $\frac{1}{2}$
11. 10π
12. 2

Problem Set 2.2.14:

1. -9
2. -5
3. $1\frac{1}{4}$

Problem Set 2.2.15:

1. -2
2. $-\frac{1}{24}$
3. $\frac{2}{3}$
4. $\sqrt{17}$
5. $\frac{4}{3}$
6. $-\frac{1}{3}$

Problem Set 3.1.1:

1. 7
2. 320
3. 108
4. 24
5. 9
6. 315
7. 12
8. 285
9. 13
10. 364
11. 6
12. 18
13. 108
14. 324
15. 4
16. 102
17. 216
18. 84
19. 432
20. 420

21. 11
22. 144
23. 17
24. -260
25. 160
26. 420
27. 180
28. 42
29. 288
30. 108
31. 22
32. 693
33. 96
34. 201
35. 336
36. 63
37. 144
38. -76
39. 14
40. 720
41. 360
42. 168

Problem Set 3.1.3:

1. -64
2. 1728
3. 0
4. 8
5. 64
6. 4
7. 32
8. 0
9. 16
10. 96
11. 128
12. 9
13. 96
14. 1458
15. 2500
16. -2
17. -3456
18. 16
19. 16000

Problem Set 3.1.4:

1. $\frac{3}{2}$
2. 9
3. $\frac{2}{3}$
4. -7
5. 0
6. $-\frac{3}{4}$
7. 3
8. $-\frac{3}{5}$
9. $\frac{1}{4}$
10. $-\frac{2}{3}$
11. 0
12. $\frac{5}{2}$
13. $\frac{3}{5}$
14. $\frac{1}{6}$
15. -3
16. -2
17. $-\frac{1}{4}$
18. 2
19. 7
20. -36
21. 3
22. $-\frac{3}{4}$
23. $-\frac{1}{4}$
24. 0
25. -4

Problem Set 3.1.5:

1. 9
2. 9
3. 6
4. 3
5. 3
6. 7
7. -1
8. -1
9. 6

Problem Set 3.1.6:

1. 1224
2. 630.9
3. $\frac{1}{8}$
4. 2
5. $\frac{2}{7}$
6. $\frac{1}{64}$
7. 289
8. 29.2
9. 2.5
10. 324
11. 216
12. $2\frac{2}{3}$
13. $\frac{4}{3}$
14. 0
15. -6
16. 343
17. 13
18. 4
19. 144
20. 2
21. 0
22. 25

Problem Set 3.1.7:

1. $\frac{1}{9}$
2. 2
3. 2
4. 6
5. -3
6. $\frac{4}{3}$
7. 9
8. 7
9. -3
10. 1
11. 3
12. 1
13. 3
14. 1
15. $\frac{8}{3}$
16. 1
17. 6
18. 2
19. 5
20. 3
21. 3
22. $1\frac{1}{2}$
23. 2
24. 1
25. $-.5$
26. 0
27. $\frac{3}{4}$
28. 243
29. 1
30. 4
31. 2

32. 0

33. $\frac{3}{2}$

34. 0

35. -1.5

36. 22

37. 0

38. 8

39. 5

40. (*) $791 - 876$

41. .5

42. 2

43. 8

44. $\frac{1}{16}$

45. 9

46. 12

47. 8

48. 1

49. 16

50. $\frac{1}{3}$

51. $\frac{1}{2}$

Problem Set 3.1.8:

1. 45
2. 60
3. 66
4. 78
5. 36
6. 28
7. 22
8. 48
9. 36
10. $\frac{3}{4}$
11. $\frac{2}{3}$
12. 40

Problem Set 3.1.9:

1. (*) $117 - 131$
2. 94
3. (*) $145 - 161$
4. (*) $172 - 191$
5. (*) $2430 - 2686$
6. 87
7. (*) $2368 - 2618$
8. (*) $395 - 438$
9. (*) $2407 - 2661$
10. (*) $887 - 981$
11. (*) $496 - 549$
12. (*) $186 - 207$
13. (*) $170 - 189$
14. (*) $128 - 142$
15. (*) $150 - 167$
16. (*) $489 - 541$
17. (*) $271 - 301$
18. (*) $486 - 539$
19. (*) $831 - 919$
20. (*) $270 - 299$
21. (*) $296 - 328$
22. (*) $279 - 309$
23. (*) $7276 - 8043$
24. (*) $200220 - 221297$
25. (*) $26596 - 29397$
26. (*) $197162 - 217917$
27. (*) $217 - 241$
28. (*) $62366 - 68932$
29. (*) $1258 - 1392$

Problem Set 3.1.10:

1. 15
2. 61
3. -6
4. 24
5. 54
6. 48
7. $\frac{3}{2}$
8. 25
9. 1600
10. 1
11. -44
12. 31
13. 9
14. 15
15. 53
16. -7
17. -7
18. 50
19. -41
20. 7
21. 0
22. 41
23. 3721
24. -243
25. 4
26. 0
27. $16 + 16i$
28. 1
29. $\frac{12}{13}$
30. -64
31. $\frac{1}{5}$
32. 625
33. $\frac{12}{13}$
34. 169

Problem Set 3.1.11:

1. $-\frac{4}{3}$
2. 2.5
3. $\frac{1}{3}$
4. 3
5. $-\frac{7}{3}$
6. $\frac{2}{3}$
7. 1
8. $3\frac{1}{2}$
9. -3
10. -4
11. 1
12. -1
13. 1
14. -5
15. $-\frac{2}{3}$
16. 1
17. $\frac{7}{3}$
18. -2
19. 0
20. 2
21. -1
22. 7
23. 1
24. 7

Problem Set 3.1.12:

1. 17
2. 120
3. 65
4. 91
5. 217
6. 720
7. 26
8. 256

9. 110
10. 398
11. 101
12. 46
13. 513
14. 511
15. 45
16. .25

Problem Set 3.1.13:

1. $\frac{1}{12}$
2. $\frac{1}{18}$
3. $\frac{1}{6}$
4. $\frac{1}{7}$
5. $\frac{9}{13}$
6. $\frac{4}{5}$
7. $\frac{13}{20}$
8. $\frac{1}{216}$
9. $\frac{3}{8}$
10. $\frac{9}{8}$
11. $\frac{1}{3}$
12. $\frac{7}{29}$
13. $\frac{5}{4}$
14. $\frac{5}{8}$
15. $\frac{3}{5}$
16. $\frac{1}{25}$
17. $\frac{5}{4}$
18. $\frac{3}{5}$
19. $\frac{3}{2}$
20. $\frac{1}{18}$
21. $\frac{1}{6}$
22. $\frac{1}{5}$
23. $\frac{5}{13}$
24. $\frac{1}{4}$
25. $\frac{7}{36}$
26. $\frac{1}{2}$
27. $\frac{1}{3}$
28. $\frac{5}{6}$
29. $\frac{3}{4}$
30. $\frac{1}{5}$

Problem Set 3.1.14:

1. 4
2. 32
3. 16
4. 5
5. 1
6. 3
7. 4
8. 5
9. 4
10. 5
11. 4
12. 2
13. 8
14. 7
15. 5
16. 6
17. 4
18. 254
19. 15
20. 15
21. 1

Problem Set 3.2.1:

1. 57
2. 1230
3. 254
4. 103
5. 102
6. 312
7. 1010
8. 11000
9. 1010
10. 1210
11. 110
12. 21
13. 404
14. 234
15. 1414
16. 27
17. 202
18. 3210
19. 2300
20. 333
21. 50
22. 72
23. 10101
24. 1101
25. 1355
26. 9
27. 443
28. 102
29. 1101
30. 1011
31. 2220
32. 140
33. 104
34. 69
35. 10101
36. 1323
37. 1331
38. 1414
39. 1234
40. 2332
41. 5
42. 32
43. 100
44. 38
45. 25

Problem Set 3.2.2:

1. $\frac{17}{25}$
2. $\frac{19}{25}$
3. $\frac{57}{343}$
4. $\frac{15}{16}$
5. $\frac{69}{125}$
6. $\frac{9}{16}$
7. $\frac{7}{12}$
8. $\frac{13}{24}$
9. $\frac{52}{125}$
10. $\frac{35}{36}$
11. $\frac{124}{125}$
12. $\frac{24}{25}$
13. $\frac{9}{25}$
14. .21
15. .55
16. .33
17. .74
18. .21
19. .42

Problem Set 3.2.3:

1. 120
2. 340
3. 10
4. 341
5. 4
6. 115
7. 181
8. 12
9. 35
10. 22
11. -44
12. 606
13. 33
14. 31
15. 1102
16. 210
17. 121
18. 143
19. 143
20. 220
21. 24
22. 30
23. 22
24. 142
25. 21
26. 104
27. 1221
28. 44
29. 1331
30. 31
31. 64
32. 1221
33. 121
34. 231
35. 330
36. 222
37. 124
38. 1331

Problem Set 3.2.4:

1. 1120
2. 1122
3. 133
4. 11011
5. 78
6. 11100101
7. 11011
8. 223
9. 101101
10. 100011010
11. 110110
12. 33
13. 23
14. 11011
15. 123

Problem Set 3.2.6:

1. $\frac{5}{6}$
2. 3
3. $\frac{6}{7}$
4. $\frac{7}{8}$
5. $\frac{1}{4}$

Problem Set 3.3.2:

1. $\frac{3}{11}$
2. $\frac{41}{99}$
3. $\frac{7}{33}$
4. $\frac{9}{11}$
5. $\frac{4}{11}$
6. $\frac{2}{99}$
7. $\frac{8}{11}$
8. $\frac{5}{33}$
9. $\frac{308}{999}$
10. $\frac{77}{333}$
11. $\frac{101}{333}$
12. $\frac{11}{111}$

Problem Set 3.3.3:

1. $\frac{7}{30}$
2. $\frac{29}{90}$
3. $\frac{19}{90}$
4. $\frac{29}{90}$
5. $\frac{11}{900}$

Problem Set 3.3.4:

1. $\frac{211}{990}$
2. $\frac{61}{495}$
3. $\frac{229}{990}$
4. $\frac{151}{494}$
5. $\frac{203}{990}$
6. $\frac{311}{990}$
7. $\frac{269}{990}$
8. $\frac{233}{990}$
9. $\frac{47}{990}$
10. $\frac{106}{495}$
11. $\frac{61}{495}$

Problem Set 3.4:

1. 3
2. 4
3. 1
4. 4
5. 2
6. 3
7. 1
8. 5
9. 2
10. 4
11. 2
12. 1
13. 3
14. 9

Problem Set 3.5.1:

1. 719
2. 5039
3. 40319
4. -117
5. 152

Problem Set 3.5.2:

1. $8\frac{1}{7}$
2. $10\frac{1}{9}$
3. $6\frac{5}{6}$
4. $10\frac{9}{10}$
5. -100
6. 600
7. $\frac{1}{15}$
8. -12
9. -113
10. 120
11. -718
12. 32
13. 4
14. 15
15. 25
16. 35
17. -80
18. 48
19. -9
20. 60

Problem Set 3.5.3:

1. 0
2. 6
3. 1
4. 6
5. 3
6. 0

Problem Set 3.6.1:

1. $\frac{3}{7}$
2. 2
3. 4
4. 3
5. 3
6. 27
7. 0
8. $\frac{9}{2}$

Problem Set 3.6.2:

1. -11
2. 2
3. 5
4. 78
5. -8
6. 24
7. 252
8. -32
9. -48
10. 18
11. 2
12. -15
13. 14
14. 4
15. 172
16. 19
17. 6
18. 18
19. -7
20. 1

21. 24
22. 54
23. -68
24. 54
25. 24
26. -224
27. 84
28. -2

Problem Set 3.6.3:

1. $8\frac{2}{3}$
2. 6
3. 15
4. 18
5. 4
6. $\frac{4}{7}$
7. $4\frac{2}{3}$
8. 12
9. 2
10. 0
11. $\frac{3}{2}$
12. $8\frac{2}{3}$
13. 6
14. $\frac{2}{3}$
15. $\frac{1}{3}$
16. $\frac{2}{3}$
17. $5\frac{1}{3}$
18. 6
19. 9
20. 2
21. 4
22. 4
23. -3
24. 12
25. $\frac{3}{5}$
26. 84
27. 2
28. $\frac{2}{3}$
29. $\frac{3}{4}$
30. 9
31. $\frac{18}{5}$
32. $3\frac{3}{4}$
33. 4
34. 6
35. 2
36. 3
37. 4
38. $1\frac{1}{2}$
39. 6

Problem Set 4.1.1:

1. 5338
2. 13013
3. 6868
4. 4316
5. 10706
6. 1456
7. 5083
8. 5535
9. 7161
10. 15352
11. 6099
12. 4270
13. 12054
14. 10665
15. 4913
16. 9801
17. 7296
18. 13518
19. 3456
20. 23115

21. 32895

Problem Set 4.1.2:

1. 65932
2. 43264
3. 163216
4. 40382
5. 499849
6. 29213
7. 161604
8. 646416
9. 49374
10. 826281
11. 39483
12. 93636
13. 45066
14. 77665
15. 41616
16. 166464
17. 179568
18. 49731
19. 31161
20. 27772
21. 38688
22. 37942
23. 46144
24. 37023
25. 108332
26. 17365
27. 70503
28. 16074
29. 36693
30. 32680
31. 28248
32. 20520
33. 179568
34. 26964
35. 129480
36. 69015
37. 35482
38. 134550

Problem Set 4.1.3:

1. 5609
2. 13216
3. 2024
4. 4224
5. 38016
6. 13209
7. 112221
8. 50609

Problem Set 4.1.5:

1. $10\frac{1}{4}$
2. $12\frac{2}{15}$
3. $17\frac{1}{10}$
4. $11\frac{1}{7}$
5. $25\frac{1}{14}$
6. $10\frac{2}{13}$
7. $9\frac{1}{17}$
8. $-\frac{13}{14}$
9. $-\frac{17}{18}$
10. $-2\frac{8}{17}$
11. $-2\frac{7}{16}$
12. $-1\frac{13}{17}$

13. $-2\frac{8}{17}$

Problem Set 4.1.6:

1. 2450
2. 6320
3. 3540
4. 4830
5. 600
6. 9900

Problem Set 4.2.1:

1. 3.75
2. 480
3. 176
4. .25
5. 40
6. 120
7. (*) 3147-3477

Problem Set 4.2.3:

1. 3.5
2. 16
3. 5.4
4. 0
5. 5
6. 16
7. 0
8. 14
9. 18

Problem Set 4.2.4:

1. (*) $176 - 194$
2. 7

Problem Set 4.3.1:

1. 122
2. 133
3. 548
4. 318
5. 480
6. 909

Problem Set 4.3.2:

1. 207
2. 304
3. 195
4. 284
5. 304
6. 514
7. 550
8. 372
9. 114
10. 460
11. 88
12. 748
13. 693
14. 162
15. 660
16. 750
17. 517
18. 407
19. 608

Problem Set 4.3.4:

1. 79
2. 104
3. 271
4. 273
5. 200

Problem Set 4.4.1:

1. $\frac{2}{91}$
2. $-\frac{2}{105}$
3. $\frac{4}{957}$
4. $\frac{2}{495}$
5. $-\frac{5}{497}$
6. $-\frac{1}{210}$
7. $\frac{1}{392}$
8. $\frac{2}{6225}$

Problem Set 4.4.2:

1. -2
2. -10
3. -8
4. 38
5. 44
6. 10

Problem Set 4.4.3:

1. $\frac{2}{3}$
2. $\frac{3}{5}$
3. $\frac{5}{7}$
4. $\frac{8}{21}$
5. $\frac{1}{3}$

Problem Set 4.4.4:

1. $5\frac{5}{6}$
2. $3\frac{3}{19}$
3. $-1\frac{1}{2}$
4. $-1\frac{8}{13}$
5. $2\frac{10}{13}$
6. $3\frac{3}{5}$
7. -4
8. $1\frac{1}{7}$
9. 12

Problem Set 4.4.5:

1. 1
2. $1\frac{2}{5}$
3. $1\frac{4}{5}$
4. $\frac{2}{13}$
5. $\frac{1}{5}$
6. 1
7. $\frac{1}{5}$
8. $\frac{2}{5}$

Problem Set 4.4.6:

1. 1
2. $\frac{7}{5}$
3. $\frac{7}{13}$
4. 2
5. $\frac{3}{5}$
6. 1
7. $\frac{7}{4}$
8. 7

Problem Set 4.5.1:

1. 3
2. 10
3. 4
4. 15
5. 6
6. 10
7. 10
8. 15
9. 15
10. 70
11. 35

Problem Set 4.5.2:

1. 1777
2. 393
3. 3777
4. 1888
5. 4777
6. 2444
7. 3444
8. 21515
9. 424
10. 2888

Problem Set 4.5.3:

1. $\frac{4}{7}$
2. $\frac{1}{2}$
3. $\frac{3}{7}$
4. $\frac{2}{3}$
5. $\frac{7}{8}$
6. $\frac{2}{3}$
7. $\frac{19}{30}$
8. $\frac{5}{8}$
9. $\frac{10}{21}$
10. $\frac{13}{56}$
11. $\frac{1}{2}$
12. $\frac{11}{21}$
13. $\frac{13}{24}$
14. $\frac{13}{14}$
15. $\frac{17}{28}$
16. $\frac{1}{3}$
17. $\frac{3}{10}$

Problem Set 4.5.4:

1. $\frac{4}{7}$
2. $\frac{1}{2}$
3. $\frac{3}{7}$
4. $\frac{2}{3}$
5. $\frac{7}{8}$
6. $\frac{2}{3}$
7. $\frac{31}{50}$
8. $\frac{43}{70}$
9. $\frac{13}{30}$
10. $\frac{15}{70}$
11. $\frac{31}{60}$
12. $\frac{23}{44}$
13. $\frac{1}{3}$
14. $\frac{3}{14}$

Problem Set 4.5.5:

1. 8
2. 6
3. 1
4. 3
5. 1

Problem Set 4.5.6:

1. 2
2. 3
3. 0
4. -2
5. -4
6. -4
7. -6
8. -4
9. -4
10. 2
11. -1
12. 6
13. $-\frac{4}{3}$
14. 4