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ORIGINAL ARTICLE

Numerical approximation of an electro-elastic frictional contact problem modeled by hemivariational inequality

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Abstract
In this paper, an electro-elastic frictional contact problem is studied numerically as a hemivariational inequality. Convergence of the Galerkin approximation for the hemivariational inequality is proved, and Céa's type inequalities are derived for error estimation. The results are applied to the electro-elastic contact problem, and an optimal order error estimate is deduced for linear element approximation. Finally, two numerical examples are reported, providing numerical evidence of the optimal convergence order theoretically predicted.

Keywords Hemivariational inequality · Galerkin approximation · Optimal order error estimate · Electro-elastic material · Frictional contact

Mathematics Subject Classification 65N30 · 65N15 · 74M10 · 74M15

1 Introduction

This paper is devoted to numerical analysis of a static problem for contact between an electro-elastic body and an electrically conductive foundation. Such contact phenomena arise in engineering devices, e.g., switches in radiotronics and measuring equipment. The main feature of the electro-elastic contact problem is the coupling between the mechanical and electrical properties of the materials. In the coupled system, the forces acting on the body affect the appearance of electric charge, and in turn, the action of the electric field generates stress or strain in the body. Recently, various contact problems for piezoelectric materials have attracted much interest. For existence and uniqueness of a weak solution of the contact problems, the reader is referred to Lerguet et al. (2007), Migórski (2006), Migórski et al. (2010), Migórski et al. (2011). On numerical approximation of piezoelectric contact problems, only

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a few references can be found in the literature. In Barboteu et al. (2008a), two frictionless contact models for electro-elastic materials are studied in the framework of a coupled system consisting of a variational inequality and an elliptic equation. Linear finite element is used to solve the problem numerically, and some error estimates are derived. In Barboteu and Sofonea (2009), an algorithm based on the finite element method and the backward Euler scheme is applied to solve a quasistatic contact model for electro-viscoelastic materials. An augmented Lagrangian method is described, yet no error estimates or convergence results are provided. In Sofonea et al. (2012), a quasistatic contact model for electro-elastic–visco-plastic materials is investigated. The weak formulation is a coupled system involving an evolutionary variational inequality and an elliptic equation. An error estimate is provided for a fully discrete scheme. More references on piezoelastic contact problems can be found in Barboteu et al. (2008b), Barboteu et al. (2008c), Han et al. (2007), Hüeber et al. (2013).

So far, the existing handful of papers on numerical analysis of piezoelectric contact problems are for mathematical formulations in the form of variational inequalities. For more complicated engineering applications, piezoelectric contact problems in the form of hemivariational inequalities arise naturally. While variational inequalities are non-smooth problems with convex structures, hemivariational inequalities are non-smooth problems allowing non-convex terms. Recent years have witnessed the development of numerical analysis of hemivariational inequalities. In Han et al. (2014), a static varaitional–hemivariational inequality is analyzed theoretically and numerically, and an optimal order error estimate is presented for the first time in the literature on finite element solutions of hemivariational inequalities. In Barboteu et al. (2015), a hyperbolic hemivariational inequality arising in dynamic contact is studied and an optimal order error estimate is derived for the numerical method based on the linear finite elements for the spatial discretization and the backward Euler finite difference for the time derivative. More recent results on numerical analysis of various types of hemivariational inequalities, including history-dependent hemivariational inequalities, can be found in Han (2018), Han et al. (2019), Han et al. (2017), Han et al. (2018), Xu et al. (2019a), Xu et al. (2019b) and the lengthy paper Han and Sofonea (2019).

Recently, a new contact model for piezoelectric materials is studied in Gamorski and Migórski (2018). In this model, a unilateral Signorini-type contact condition with multivalued normal compliance is adopted. Strong coupling between the displacement and electric potential appears in constitutive relation, contact condition, friction law and electric contact. The weak formulation of the contact problem is a hemivariational inequality and existence of a unique weak solution is proved. The purpose of this paper is to numerically study and simulate the contact model in Gamorski and Migórski (2018). Galerkin approximation of the hemivariational inequality is considered with convergence shown and Céa's type inequalities derived. For the new contact model, an optimal order error estimate is derived for linear finite element solutions. This is the first paper containing an optimal order error bound for the numerical solution of a hemivariational inequality modeling piezoelectric contact problems.

The rest of this paper is organized as follows. In Sect. 2, we recall some preliminaries in nonlinear analysis. In Sect. 3, we present an abstract hemivariational inequality and consider its Galerkin approximation. In Sect. 4, we state the new contact model for piezoelectric materials. Galerkin approximation to the problem is analyzed, and an optimal order error estimate is derived for linear finite element solutions. Finally in Sect. 5, we report numerical results providing numerical evidence for the theoretically predicted optimal convergence orders.

2 Preliminaries

In this section, we recall some preliminary materials in nonlinear analysis. Let X be a normed space and X^* be its dual space, the duality pairing of X^* and X is denoted by $\langle\cdot,\cdot\rangle_{X^*\times X}$. The norms in X and X^* are denoted by $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$, respectively. The notions of generalized directional derivative and subdifferential will be needed (Clarke 1983).

Definition 1 Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative (in the sense of Clarke) of ϕ at a point $u \in X$ along a direction $v \in X$ is defined by

$$\phi^0(u; v) = \limsup_{y\to u,\, \lambda\downarrow 0} \frac{\phi(y+\lambda v)-\phi(y)}{\lambda}.$$

Definition 2 Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function. The subdifferential or the generalized gradient (in the sense of Clarke) of ϕ at the point $u \in X$ is the following subset of the dual space X^*:

$$\partial\phi(u) = \{\xi \in X^* \mid \phi^0(u; v) \geq \langle\xi, v\rangle_{X^*\times X} \quad \forall\, v \in X\}.$$

The Clarke directional derivative is subadditive, i.e., for every $u \in X$,

$$\phi^0(u; v_1 + v_2) \leq \phi^0(u; v_1) + \phi^0(u; v_2) \quad \forall\, v_1, v_2 \in X.$$

More properties concerning the Clarke directional derivative and subdifferential can be found in the books Clarke (1983); Denkowski et al. (2003); Migórski et al. (2013).

We also recall the following definitions, useful in the analysis of hemivariational inequality problems (Han et al. 2014; Xu et al. 2019a, b).

Definition 3 An operator $A : X \to X^*$ is called

(a) monotone, if $\langle Au - Av, u - v\rangle_{X^*\times X} \geq 0$ for any $u, v \in X$;
(b) strongly monotone, if $\langle Au - Av, u - v\rangle_{X^*\times X} \geq c\,\|u - v\|_X^2$ for any $u, v \in X$ with some constant $c > 0$;
(c) pseudomonotone, if it is bounded and for any sequence u_n converging to u weakly in X such that $\limsup\langle Au_n, u_n - u\rangle_{X^*\times X} \leq 0$, then $\langle Au, u - v\rangle_{X^*\times X} \leq \liminf\langle Au_n, u_n - v\rangle_{X^*\times X}$ for any $v \in X$.

3 An abstract hemivariational inequality and its Galerkin approximation

Based on the problem introduced by Gamorski and Migórski (2018), an abstract hemivariational inequality is presented in this section. We use the Galerkin method to approximate the hemivariational inequality, derive Céa's type inequality and show convergence of the numerical solution. Below, C represents a generic positive constant which does not depend on the mesh size h, but may depend on the data of the problem and the solution.

Let X be a reflexive Banach space with norm $\|\cdot\|_X$ and let X_1 be a Banach space with norm $\|\cdot\|_{X_1}$. Their dual spaces are denoted by X^* and X_1^*. Let there be given a set $K \subset X$ and two operators $A : X \to X^*$, $\gamma : X \to X_1$. Denote $K_1 = \gamma(K)$; thus $K_1 \subset X_1$. Let $J : K_1 \times X_1 \to \mathbb{R}$ be a function, Lipschitz continuous with respect to its second argument, and let J^0 be the Clarke directional derivative of J with respect to its second argument. Given $f \in X^*$, consider the following abstract hemivariational inequality.

Problem 4 *Find an element $u \in K$ such that*

$$\langle Au - f, v - u\rangle_{X^*\times X} + J^0(\gamma u, \gamma u; \gamma v - \gamma u) \geq 0 \quad \forall v \in K. \tag{1}$$

The form of the problem is slightly different from that presented in Gamorski and Migórski (2018), to accommodate the need to derive optimal order error estimates.

Remark 5 In applications to contact mechanics, γ represents a trace operator, and X_1 can be chosen as the space of vector-valued or scalar-valued square integrable functions over the contact boundary. Generally, the function J is defined as an integral over the contact boundary, such as $J(\gamma u)$ in Han (2018), Han et al. (2017), Han et al. (2018). In dealing with multi-physics coupling problems, for example, problems with electro-elastic coupling, it is useful to express the integral in the form $J(u, u)$ (Gamorski and Migórski 2018).

The following conditions are needed in the well-posedness and numerical analysis (Gamorski and Migórski 2018; Han et al. 2017):

$$K \text{ is a closed and convex subset of } X \text{ with } 0_X \in K. \tag{2}$$

$$\gamma \in \mathcal{L}(X; X_1), \text{ for some constant } c_\gamma > 0, \ \|\gamma v\|_{X_1} \leq c_\gamma \|v\|_X \quad v \in X. \tag{3}$$

$$\begin{cases} A : X \to X^* \text{ is} \\ \quad (a) \text{ pseudomonotone;} \\ \quad (b) \text{ strongly monotone, i.e., there exists } m_A > 0 \text{ such that} \\ \qquad \langle Av_1 - Av_2, v_1 - v_2\rangle_{X^*\times X} \geq m_A\|v_1 - v_2\|_X^2 \quad \forall v_1, v_2 \in X. \end{cases} \tag{4}$$

$$\begin{cases} J : K_1 \times X_1 \to \mathbb{R} \text{ is a function such that} \\ \quad \text{(a) } J(z, \cdot) \text{ is locally Lipschitz on } X_1 \quad \forall z \in K_1; \\ \quad \text{(b) there exists } c_0, c_1, c_2 \geq 0 \text{ such that } \forall z \in K_1, \ \forall w \in X_1 \\ \qquad\qquad \|\partial J(z, w)\|_{X_1^*} \leq c_0 + c_1\|z\|_{X_1} + c_2\|w\|_{X_1}; \\ \quad \text{(c) there exist } \alpha_J, \beta_J > 0 \text{ such that } \forall z_1, z_2 \in K_1, \ \forall w_1, w_2 \in X_1. \\ \qquad J^0(z_1, w_1; w_2 - w_1) + J^0(z_2, w_2; w_1 - w_2) \\ \qquad\qquad \leq \alpha_J\|w_1 - w_2\|_{X_1}^2 + \beta_J\|z_1 - z_2\|_{X_1}\|w_1 - w_2\|_{X_1}. \end{cases} \tag{5}$$

$$\alpha_J c_\gamma^2 + \max\{\beta_J, c_1\}c_\gamma^2 < m_A. \tag{6}$$

$$f \in X^*. \tag{7}$$

The assumption $0_X \in K$ is valid for various contact problems, which is used to simplify the calculation (Han et al. 2017). For instance, we can derive the following two inequalities from that

$$\langle Av, v\rangle_{X^*\times X} = \langle Av - A0, v\rangle_{X^*\times X} + \langle A0, v\rangle_{X^*\times X},$$
$$\langle \partial J(z, w), w\rangle_{X_1^*\times X_1} = \langle \partial J(z, w) - \partial J(z, 0), w\rangle_{X_1^*\times X_1} + \langle \partial J(z, 0), w\rangle_{X_1^*\times X_1}.$$

Combining with (4)(b), (5)(b) and (5)(c), for any $v \in X$, $z \in K_1$ and $w \in X_1$, we have

$$\langle Av, v\rangle_{X^*\times X} \geq m_A\|v\|_X^2 - \|A0\|_{X^*}\|v\|_X, \tag{8}$$

$$\langle \partial J(z, w), w\rangle_{X_1^*\times X_1} \geq -\alpha_J\|w\|_{X_1}^2 - (c_0 + c_1\|z\|_{X_1})\|w\|_{X_1}. \tag{9}$$

The condition (4)(c) is equivalent to the inequality

$$\langle \partial J(z_1, w_1) - \partial J(z_2, w_2), w_1 - w_2\rangle_{X_1^*\times X_1} \geq -\alpha_J\|w_1 - w_2\|_{X_1}^2$$
$$- \beta_J\|z_1 - z_2\|_{X_1}\|w_1 - w_2\|_{X_1}$$

for any $z_1, z_2 \in K_1$, for any $w_1, w_2 \in X_1$ (Migórski et al. 2013). If $\beta_J = 0$, then the above inequality reduces to the so-called relaxed monotonicity condition which is widely used in the literature (Han 2018; Han et al. 2014, 2017, 2019).

The function J is defined on $K_1 \times X_1$; thus, the condition (5) has differences with the one presented in the paper Gamorski and Migórski (2018). The relation (6) is a modified smallness condition. By slightly modifying the proof in Gamorski and Migórski (2018), we have the existence of a unique solution to Problem 4.

Theorem 6 *Assume* (2)–(7)*. Then Problem 4 has a unique solution* $u \in K$.

Now, we give a discrete scheme of Problem 4. Let $X^h \subset X$ be a finite dimensional subspace where $h > 0$ denotes a spatial discretization parameter. Assume $K^h \subset X^h \cap K$, which is used to approximate the set K. As for K, we assume $0 \in K^h$, an assumption valid in our applications. Automatically, we have $\gamma(K^h) \subset \gamma(K) = K_1$. Then, the Galerkin approximation of Problem 4 is the following.

Problem 7 *Find* $u^h \in K^h$ *such that*

$$\langle Au^h - f, v^h - u^h\rangle_{X^*\times X} + J^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u^h) \geq 0 \quad \forall\, v^h \in K^h. \tag{10}$$

By a discrete analog of Theorem 6, we know that under assumptions (2)–(7), Problem 7 has a unique solution $u^h \in K^h$. The following uniform boundedness result on the discrete solution will be useful in convergence analysis and error estimation.

Proposition 8 *The solution* $u^h \in K^h$ *of Problem 7 is uniformly bounded in X, independent of* h.

Proof We let $v^h = 0$ in Problem 7 to get

$$\langle Au^h, u^h\rangle_{X^*\times X} \leq J^0(\gamma u^h, \gamma u^h; -\gamma u^h) + \langle f, u^h\rangle_{X^*\times X}. \tag{11}$$

We take $z_1 = z_2 = \gamma u^h$, $w_1 = \gamma u^h$ and $w_2 = 0$ in (5)(c) to get

$$J^0(\gamma u^h, \gamma u^h; -\gamma u^h) + J^0(\gamma u^h, 0; \gamma u^h) \leq \alpha_J \|\gamma u^h\|_{X_1}^2.$$

Thus,

$$\begin{aligned} J^0(\gamma u^h, \gamma u^h; -\gamma u^h) &\leq \alpha_J \|\gamma u^h\|_{X_1}^2 - J^0(\gamma u^h, \gamma 0; \gamma u^h) \\ &\leq \alpha_J \|\gamma u^h\|_{X_1}^2 + (c_0 + c_1\|\gamma u^h\|_{X_1})\|\gamma u^h\|_{X_1} \\ &\leq (\alpha_J + c_1)c_\gamma^2 \|u^h\|_X^2 + c_0 c_\gamma \|u^h\|_X. \end{aligned} \tag{12}$$

Obviously,

$$\langle f, u^h\rangle_{X^*\times X} \leq \|f\|_{X^*}\|u^h\|_X. \tag{13}$$

Taking $v = u^h$ in (8), we have

$$m_A\|u^h\|_X^2 - \|A0\|_{X^*}\|u^h\|_X \leq \langle Au^h, u^h\rangle_{X^*\times X}. \tag{14}$$

Combining (11), (12), (13) and (14), we get

$$\begin{aligned} m_A\|u^h\|_X^2 &\leq (\alpha_J + c_1)c_\gamma^2\|u^h\|_X^2 + (c_0 c_\gamma + \|A0\|_{X^*})\|u^h\|_X \\ &\leq (\alpha_J + c_1)c_\gamma^2\|u^h\|_X^2 + c_3\|u^h\|_X, \end{aligned}$$

where $c_3 = c_0 c_\gamma + \|A0\|_{X^*}$. Recalling the condition (6), we find that

$$\|u^h\|_X \le \frac{c_3}{m_A - (\alpha_J + c_1)c_\gamma^2}.$$

Thus, $\|u^h\|_X$ is uniformly bounded with respect to h. □

To proceed further in convergence analysis and error estimation, we assume $A : X \to X^*$ is Lipschitz continuous with a Lipschitz constant $L_A > 0$, i.e.,

$$\|Au - Av\|_{X^*} \le L_A \|u - v\|_X \quad \forall\, u, v \in X. \tag{15}$$

We also assume $\{K^h\}$ approaches K in the sense that

$$\forall\, v \in K,\ \exists\, v^h \in K^h \text{ such that } v^h \to v \text{ in } X \text{ as } h \to 0. \tag{16}$$

Theorem 9 *Assume (2)–(7) and (15). Then, the following error bound holds for Problem 7:*

$$\|u - u^h\|_X^2 \le C\left(\|u - v^h\|_X^2 + \|\gamma u - \gamma v^h\|_{X_1} + R(u, v^h)\right) \quad \forall\, v^h \in K^h, \tag{17}$$

where $R(u, v^h)$ is the residual term defined by

$$R(u, v^h) = \langle Au, v^h - u\rangle_{X^*\times X} + J^0(\gamma u, \gamma u; \gamma v^h - \gamma u) - \langle f, v^h - u\rangle_{X^*\times X}. \tag{18}$$

Proof Apply (4) (b) with $v_1 = u$ and $v_2 = u^h$,

$$m_A \|u - u^h\|_X^2 \le \langle Au - Au^h, u - u^h\rangle_{X^*\times X}$$

which is rewritten as

$$m_A \|u - u^h\|_X^2 \le \langle Au - Au^h, u - v^h\rangle_{X^*\times X} + \langle Au, v^h - u\rangle_{X^*\times X} \tag{19}$$

$$+ \langle Au, u - u^h\rangle_{X^*\times X} - \langle Au^h, v^h - u^h\rangle_{X^*\times X}. \tag{20}$$

Take $v = u^h$ in (1) to get

$$\langle Au, u - u^h\rangle_{X^*\times X} \le J^0(\gamma u, \gamma u; \gamma u^h - \gamma u) + \langle f, u - u^h\rangle_{X^*\times X}. \tag{21}$$

Using (10),

$$-\langle Au^h, v^h - u^h\rangle_{X^*\times X} \le J^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u^h) + \langle f, u^h - v^h\rangle_{X^*\times X}. \tag{22}$$

Combining (18)–(22), we have

$$m_A \|u - u^h\|_X^2 \le \langle Au - Au^h, u - v^h\rangle_{X^*\times X} + R(u, v^h) + R_J(u, u^h, v^h), \tag{23}$$

where

$$R_J(u, u^h, v^h) = J^0(\gamma u, \gamma u; \gamma u^h - \gamma u) + J^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u^h) - J^0(\gamma u, \gamma u; \gamma v^h - \gamma u).$$

Use the subadditive property of the generalized directional derivative,

$$J^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u^h) \le J^0(\gamma u^h, \gamma u^h; \gamma u - \gamma u^h) + J^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u).$$

Then,

$$R_J(u,u^h,v^h) \le \Big[J^0(\gamma u,\gamma u;\gamma u^h-\gamma u)+J^0(\gamma u^h,\gamma u^h;\gamma u-\gamma u^h)\Big] \\ + J^0(\gamma u^h,\gamma u^h;\gamma v^h-\gamma u)-J^0(\gamma u,\gamma u;\gamma v^h-\gamma u).$$

Apply (5) (b) and (c),

$$R_J(u,u^h,v^h) \le \alpha_J\|\gamma u-\gamma u^h\|_{X_1}^2+\beta_J\|\gamma u-\gamma u^h\|_{X_1}^2 \\ +\Big(2c_0+(c_1+c_2)(\|\gamma u\|_{X_1}+\|\gamma u^h\|_{X_1})\Big)\|\gamma u-\gamma v^h\|_{X_1}.$$

By Proposition 8, $\|\gamma u^h\|_{X_1}$ is bounded independent of h. So

$$R_J(u,u^h,v^h) \le (\alpha_J+\beta_J)\,c_\gamma^2\|u-u^h\|_X^2+C\,\|\gamma u-\gamma v^h\|_{X_1}.$$

Hence, from (23), we have

$$m_A\|u-u^h\|_X^2 \le L_A\|u-u^h\|_X\|u-v^h\|_X+R(u,v^h) \\ +(\alpha_J+\beta_J)c_\gamma^2\|u-u^h\|_X^2+C\|\gamma u-\gamma v^h\|_{X_1}.$$

Using the smallness condition (7), and applying the modified Cauchy–Schwarz inequality in the form

$$L_A\|u-u^h\|_X\|u-v^h\|_X \le \epsilon\,\|u-u^h\|_X^2+\frac{L_A^2}{4\,\epsilon}\,\|u-v^h\|_X^2$$

with a sufficiently small $\epsilon>0$, we have the error bound (17). □

In the case $K^h\subset K$, the residual term $R(u,v^h)$ is nonnegative. Additionally, in the case $K^h\not\subset K$, $R(u,v^h)$ may be negative. Theorem 9 is the starting point for convergence analysis and error estimation. Now, we give the convergence result for the numerical solution.

Theorem 10 *Assume* (2)–(7), (15) *and* (16). *Then, the numerical solution of Problem 7 converges strongly to the solution of Problem 4.*

Proof The residual term $R(u,v^h)$ can be bounded as follows:

$$R(u,v^h) \le \left(\|Au\|_{X^*}+\Big(c_0+(c_1+c_2)c_\gamma\|u\|_X\Big)+\|f\|_{X^*}\right)\|u-v^h\|_X. \tag{24}$$

Together with (17), we have

$$\|u-u^h\|_X \le C\left(\|u-v^h\|_X+\|u-v^h\|_X^{\frac{1}{2}}\right) \quad \forall\, v^h\in K^h.$$

Choose $v^h\in K^h$ such that $v^h\to u$ as $h\to 0$ to obtain $\|u-u^h\|_X\to 0$ as $h\to 0$. □

Now we consider the particular case where $K=X$, i.e., the original hemivariational inequality is not constrained and is posed over the entire space X. In this case, Problem 4 takes the following form.

Problem 11 *Find an element $u\in X$ such that*

$$\langle Au-f,v-u\rangle_{X^*\times X}+J^0(\gamma u,\gamma u;\gamma v-\gamma u)\ge 0 \quad \forall\, v\in X. \tag{25}$$

The discrete scheme for Problem 11 is as follows.

Problem 12 *Find an discrete solution $u^h \subset X^h$ such that*

$$\langle Au^h - f, v^h - u^h \rangle_{X^* \times X} + J^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u^h) \geq 0 \quad \forall\, v^h \in X^h. \tag{26}$$

Let us show that in this special case, the Céa's type inequality takes a simpler form. We replace v by $2u - v$ in (25) to get

$$\langle Au - f, u - v \rangle_{X^* \times X} + J^0(\gamma u, \gamma u; \gamma u - \gamma v) \geq 0 \quad \forall\, v \in X. \tag{27}$$

Taking $v = v^h$ in (27), we get

$$\langle Au, v^h - u \rangle_{X^* \times X} \leq J^0(\gamma u, \gamma u; \gamma u - \gamma v^h) - \langle f, u - v^h \rangle_{X^* \times X}. \tag{28}$$

Combining (20), (21), (22) and (28), we have

$$m_A \|u - u^h\|_X^2 \leq \langle Au - Au^h, u - v^h \rangle_{X^* \times X} + \widetilde{R}_J(u, u^h, v^h), \tag{29}$$

where

$$\widetilde{R}_J(u, u^h, v^h) = J^0(\gamma u, \gamma u; \gamma u - \gamma v^h) + J^0(\gamma u, \gamma u; \gamma u^h - \gamma u) \\ + J^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u^h).$$

By the subadditive property of the generalized directional derivative,

$$J^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u^h) \leq J^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u) + J^0(\gamma u^h, \gamma u^h; \gamma u - \gamma u^h).$$

Thus,

$$\widetilde{R}_J(u, u^h, v^h) \leq \Big[J^0(\gamma u, \gamma u; \gamma u^h - \gamma u) + J^0(\gamma u^h, \gamma u^h; \gamma u - \gamma u^h) \Big] \\ + J^0(\gamma u, \gamma u; \gamma u - \gamma v^h) + J^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u).$$

Apply (6) (b), (c),

$$\widetilde{R}_J(u, u^h, v^h) \leq (\alpha_J + \beta_J) c_\gamma^2 \|u - u^h\|_X^2 \\ + \Big(2c_0 + (c_1 + c_2) c_\gamma (\|u\|_X + \|u^h\|_X)\Big) \|\gamma u - \gamma v^h\|_{X_1}.$$

Note that u^h is uniformly bounded; we have

$$\widetilde{R}_J(u, u^h, v^h) \leq (\alpha_J + \beta_J) c_\gamma^2 \|u - u^h\|_X^2 + C\, \|\gamma u - \gamma v^h\|_{X_1}. \tag{30}$$

Combine (29) and (30),

$$m_A \|u - u^h\|_X^2 \leq L_A \|u - u^h\|_X \|u - v^h\|_X + (\alpha_J + \beta_J) c_\gamma^2 \|u - u^h\|_X^2 \\ + C \|\gamma u - \gamma v^h\|_{X_1}. \tag{31}$$

Using the smallness condition (7), we can derive the following Céa's type inequality from (31):

$$\|u - u^h\|_X \leq C \left(\|u - v^h\|_X + \|\gamma u - \gamma v^h\|_{X_1}^{\frac{1}{2}} \right) \quad \forall\, v^h \in X^h. \tag{32}$$

Comparing (32) with (17), we observe that the residual term $R(u, v^h)$ in (17) results from the constraint $u \in K$, and it is absent in (32) since Problem 11 and Problem 12 are inequality problems without this constraint.

4 Numerical approximation of an electro-elastic contact problem

In this section, we perform numerical analysis for the static electro-elastic contact model studied in Gamorski and Migórski (2018). The weak formulation of the contact problem is a coupled system involving two hemivariational inequalities. Existence and uniqueness of a weak solution of the problem is proved in Gamorski and Migórski (2018). We use the theoretical results in Sect. 3 for the discussion of the numerical solution of the electro-elastic contact problem. Galerkin approximation of the problem is analyzed, and an optimal order error estimate is derived for linear finite element solutions.

We start with a description of the physical setting. Let $d = 2$ or 3 be the spatial dimension and symbols i, j, k, l be the subscripts satisfying $1 \le i, j, k, l \le d$. An electro-elastic body is in contact with an electrically conductive foundation, and the process is static. The body occupies an open, bounded and connected domain $\Omega \subset \mathbb{R}^d$ and is subject to mechanical and electrical constraints. Let Γ be the boundary of Ω, which is assumed to be Lipschitz continuous. The outward unit normal vector over Γ is denoted by $\boldsymbol{\nu}$. For the mechanical boundary conditions, we assume Γ is divided into three parts Γ_D, Γ_N and Γ_C with disjoint relative interiors; for the electrical boundary conditions, we assume $\Gamma_D \cup \Gamma_N$ is partitioned into two parts Γ_a and Γ_b with disjoint relative interiors. Moreover, we assume $\text{meas}(\Gamma_D) > 0$, $\text{meas}(\Gamma_a) > 0$ and $\text{meas}(\Gamma_C) > 0$.

As usual, $\mathbb{R}^d$ denotes the d-dimensional Euclidean vector space, and $\mathbb{S}^d$ denotes the space of second-order symmetric tensor field on $\mathbb{R}^d$. The inner products and corresponding norms in $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{u}\| = (\boldsymbol{u} \cdot \boldsymbol{u})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{u} = (u_i), \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively. Note that summation convention over repeated indices is implied. For a displacement $\boldsymbol{u} \in \mathbb{R}^d$, the symbols u_ν and $\boldsymbol{u}_{\boldsymbol{\tau}}$ represent its normal and tangential components on the boundary, respectively, defined by

$$u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu} \ \text{ and } \ \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{u} - u_\nu \boldsymbol{\nu}.$$

The linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

For a stress tensor $\boldsymbol{\sigma} \in \mathbb{S}^d$, the notation σ_ν and $\boldsymbol{\sigma}_\tau$ represent its normal and tangential components on Γ, respectively, denoted by

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu} \ \text{ and } \ \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}.$$

$\text{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$ is the divergence of $\boldsymbol{\sigma}$. Notation $\mathcal{A} = (a_{ijkl})$ denotes an elasticity tensor of fourth order.

Let $\boldsymbol{D} \in \mathbb{R}^d$ be an electric displacement field, and $\text{div}\boldsymbol{D} = (D_{i,i})$ be its divergence. Let φ be an electric potential. The notation $\boldsymbol{\beta} = (\beta_{ij})$ denotes a linear electric permittivity tensor of second order. The symbol $\mathbb{P} = (p_{ijk})$ represents a linear piezoelectric tensor of third order. We have

$$\mathcal{P}\boldsymbol{\tau} \cdot \boldsymbol{v} = \boldsymbol{\tau} \cdot \mathcal{P}^{\mathsf{T}}\boldsymbol{v} \quad \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \boldsymbol{v} \in \mathbb{R}^d,$$

where $\mathcal{P}^{\mathsf{T}}$ is the transpose of the tensor $\mathcal{P}$. We adopt the linear constitutive law from Batra and Yang (1995) for the coupling of the mechanical and electrical responses of the piezoelectric materials. The pointwise formulation of the electro-elastic contact problem is the following.

Problem 13 *Find a displacement $\boldsymbol{u} : \Omega \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \to \mathbb{R}$ and an electric displacement field $\boldsymbol{D} : \Omega \to \mathbb{R}^d$ that satisfy the differential equations:*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{P}^T\nabla\varphi \quad \text{in } \Omega, \tag{33}$$

$$\boldsymbol{D} = \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\beta}\nabla\varphi \quad \text{in } \Omega, \tag{34}$$

$$\mathrm{Div}\boldsymbol{\sigma} + \boldsymbol{f_0} = \boldsymbol{0} \quad \text{in } \Omega, \tag{35}$$

$$\mathrm{div}\boldsymbol{D} - q_0 = 0 \quad \text{in } \Omega, \tag{36}$$

with the standard boundary conditions:

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{37}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_N \quad \text{on } \Gamma_N, \tag{38}$$

$$\varphi = 0 \quad \text{on } \Gamma_a, \tag{39}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} = q_b \quad \text{on } \Gamma_b, \tag{40}$$

and the inequality boundary conditions:

$$\left.\begin{array}{l} u_\nu \le g_0,\ \sigma_\nu + \xi \le 0, \\ (\sigma_\nu + \xi)(u_\nu - g_0) = 0, \\ \xi \in h_\nu(\varphi - \varphi_0)\partial j_\nu(u_\nu - g_0) \end{array}\right\} \quad \text{on } \Gamma_C, \tag{41}$$

$$-\boldsymbol{\sigma_\tau} \in h_{\boldsymbol{\tau}}(\varphi - \varphi_0, u_\nu - g_0)\mu(\|\boldsymbol{u_\tau}\|)\partial j_\tau(\boldsymbol{u_\tau}) \quad \text{on } \Gamma_C, \tag{42}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} \in h_e(u_\nu - g_0)\partial j_e(\varphi - \varphi_0) \quad \text{on } \Gamma_C. \tag{43}$$

Next, we give a short explanation for the relations (33)–(43). Equations (33) and (34) represent the constitutive law for electro-elastic material, where $\mathcal{A}$, $\mathcal{P}$ and $\boldsymbol{\beta}$ are elasticity, piezoelectric and permittivity tensors, respectively. The relations (35) and (36) are the balance equations for stress and electric displacement, respectively, where $\boldsymbol{f_0}$ is the density of volume force and q_0 is the density of the volume electric charge.

The displacement boundary condition is given by (37), i.e., the body is clamped on Γ_D, and the surface traction of density on Γ_N is (38). The relation (39) means the electric potential vanishes on Γ_a, and in (40) a surface electric charge of density q_b acts on Γ_b.

The condition (41) describes the contact on Γ_C, which is modeled by unilateral constraint and multivalued normal compliance condition coupled with electric potential. Here, g_0 denotes the thickness of the elastic material laid on the rigid foundation, and φ_0 denotes the electric potential on the electrically conductive foundation, h_ν is the stiffness coefficient.

The inclusion (42) is a coupled relation for friction on Γ_C, in which ∂j_τ is the Clarke subdifferential, $h_{\boldsymbol{\tau}}$ is a function for the effect of electric potential and normal displacement, and μ is the frictional coefficient.

The condition (43) describes a regularized condition for the electrical contact. The electrical conductivity coefficient h_e is a nonnegative function, and ∂j_e is the Clarke subdifferential which is multivalued and nonmonotone. More interpretations about this electro-elastic contact model can be found in Gamorski and Migórski (2018).

The following hypotheses for Problem 13 are also imposed in Gamorski and Migórski (2018). On the elasticity tensor $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$,

$$\begin{cases} (a)\ \mathcal{A}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ (b)\ \mathcal{A}(\boldsymbol{x}, \cdot) \text{ is continuous on } \mathbb{S}^d \text{ and } \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0} \text{ for a.e. } \boldsymbol{x} \in \Omega; \\ (c)\ \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})\|_{\mathbb{S}^d} \le a_0(\boldsymbol{x}) + a_1 \|\boldsymbol{\varepsilon}\|_{\mathbb{S}^d} \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \text{ where} \\ \quad a_0 \in L^2(\Omega), a_0 \ge 0 \text{ and } a_1 \ge 0; \\ (d)\ (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2_{\mathbb{S}^d} \text{ for a.e. } \boldsymbol{x} \in \Omega, \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ with } m_{\mathcal{A}} > 0. \end{cases} \tag{44}$$

On the electric permittivity tensor $\boldsymbol{\beta} = (\beta_{ij})$,

$$\begin{cases} (a)\ \beta_{ij} = \beta_{ji} \in L^\infty(\Omega); \\ (b) \text{ there exists constant } m > 0 \text{ such that} \\ \quad \beta_{ij}(\boldsymbol{x})\xi_i\xi_j \ge m\|\boldsymbol{\xi}\|^2_{\mathbb{R}^d} \ \text{ for all } \boldsymbol{\xi} = (\xi_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{45}$$

On the piezoelectric tensor $\mathcal{P} = (\mathcal{P}_{ijk})$,

$$\begin{cases} (a)\ \mathcal{P}_{ijk} \in L^\infty(\Omega); \\ (b)\ \mathcal{P}\boldsymbol{\tau} \cdot \boldsymbol{v} = \boldsymbol{\tau} \cdot \mathcal{P}^{\mathsf{T}}\boldsymbol{v} \text{ for } \boldsymbol{\tau} \in \mathbb{S}^d, \boldsymbol{v} \in \mathbb{R}^d. \end{cases} \tag{46}$$

The electro-elastic constitutive law is described by the above three tensors, which reduces to the decoupled state in the case $\mathcal{P} = \boldsymbol{0}$.

The densities of volume force, surface contraction, volume electric charge and surface electric charge satisfy the following regularities

$$\boldsymbol{f}_0 \in L^2(\Omega; \mathbb{R}^d), \ \boldsymbol{f}_N \in L^2(\Gamma_N; \mathbb{R}^d), \ q_0 \in L^2(\Omega), \ q_b \in L^2(\Gamma_b). \tag{47}$$

The gap function g_0 and the electric potential φ_0 satisfy

$$g_0 \ge 0 \text{ a.e. on } \Gamma_C \ \text{ and } \ g_0, \varphi_0 \in L^\infty(\Omega). \tag{48}$$

The stiffness coefficient $h_\nu : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a)\ \forall\, r \in \mathbb{R}, \ h_\nu(\cdot, r) \text{ is measurable on } \Gamma_C; \\ (b)\ h_\nu(\boldsymbol{x}, \cdot) \text{ is continuous on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_C; \\ (c)\ 0 \le h_\nu(\boldsymbol{x}, r) \le \overline{h}_\nu \text{ for all } r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } \overline{h}_\nu > 0. \end{cases} \tag{49}$$

The functions $j_\nu : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a)\ \forall\, r \in \mathbb{R}, \ j_\nu(\cdot, r) \text{ is measurable on } \Gamma_C, \text{ and there exists } e_1 \in L^2(\Gamma_C) \\ \quad \text{such that } j_\nu(\cdot, e_1(\cdot)) \in L^1(\Gamma_C); \\ (b)\ j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_C; \\ (c)\ |\partial j_\nu(\boldsymbol{x}, r)| \le c_{0\nu} + c_{1\nu}|r| \text{ for all } r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } c_{0\nu}, c_{1\nu} \ge 0; \\ (d)\ j^0_\nu(\boldsymbol{x}, r_1; r_2 - r_1) + j^0_\nu(\boldsymbol{x}, r_2; r_1 - r_2) \le \alpha_{j_\nu}|r_1 - r_2|^2 \text{ for all } r_1, r_2 \in \mathbb{R}, \\ \quad \text{a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } \alpha_{j_\nu} \ge 0. \end{cases} \tag{50}$$

The functions $h_\tau : \Gamma_C \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a)\ \forall\, r_1, r_2 \in \mathbb{R}, \ h_\tau(\cdot, r_1, r_2) \text{ is measurable on } \Gamma_C; \\ (b)\ h_\tau(\boldsymbol{x}, \cdot, \cdot) \text{ is continuous on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_C; \\ (c)\ 0 \le h_\tau(\boldsymbol{x}, r_1, r_2) \le \overline{h}_\nu \text{ for all } r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } \overline{h}_\tau > 0. \end{cases} \tag{51}$$

The coefficient of friction $\mu : \Gamma_C \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\begin{cases} (a)\ \forall\, r \in \mathbb{R}_+,\ \ \mu(\cdot, r) \text{ is measurable on } \Gamma_C; \\ (b)\ |\mu(\cdot, r_1) - \mu(\cdot, r_2)| \le L_\mu |r_1 - r_2| \text{ for all } r_1, r_2 \in \mathbb{R}_+,\ \text{ a.e. } \boldsymbol{x} \in \Gamma_C \\ \quad \text{with } L_\mu > 0; \\ (c)\ \mu(\boldsymbol{x}, r) \le \mu_0 \text{ for all } r \in \mathbb{R}_+,\ \text{ a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } \mu_0 > 0. \end{cases} \tag{52}$$

The function $j_\tau : \Gamma_C \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\begin{cases} (a)\ \forall\, \boldsymbol{\xi} \in \mathbb{R}^d,\ \ j_\tau(\cdot, \boldsymbol{\xi}) \text{ is measurable on } \Gamma_C,\ \text{ and there exists } \boldsymbol{e}_2 \in L^2(\Gamma_C; \mathbb{R}^d) \\ \quad \text{such that } j_\tau(\cdot, \boldsymbol{e}_2(\cdot)) \in L^1(\Gamma_C); \\ (b)\ j_\tau(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } \boldsymbol{x} \in \Gamma_C; \\ (c)\ |\partial j_\tau(\boldsymbol{x}, \boldsymbol{\xi})| \le c_{0\tau} + c_{1\tau} \|\boldsymbol{\xi}\|_{\mathbb{R}^d} \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d,\ \text{ a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } c_{0\tau}, c_{1\tau} \ge 0; \\ (d)\ j_\tau^0(\boldsymbol{x}, \boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + j_\tau^0(\boldsymbol{x}, \boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \le \alpha_{j_\tau} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_{\mathbb{R}^d}^2 \text{ for all } \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d, \\ \quad \text{a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } \alpha_{j_\tau} \ge 0. \end{cases} \tag{53}$$

Finally, for the condition (43), the function $h_e : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a)\ \forall\, r \in \mathbb{R},\ \ h_e(\cdot, r) \text{ is measurable on } \Gamma_C; \\ (b)\ h_e(\boldsymbol{x}, \cdot) \text{ is continuous on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_C; \\ (c)\ 0 \le h_e(\boldsymbol{x}, r) \le \overline{h}_e \text{ for all } r \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } \overline{h}_e > 0. \end{cases} \tag{54}$$

The function $j_e : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a)\ \forall\, r \in \mathbb{R},\ \ j_e(\cdot, r) \text{ is measurable on } \Gamma_C,\ \text{ and there exists } e_3 \in L^2(\Gamma_C) \\ \quad \text{such that } j_e(\cdot, e_3(\cdot)) \in L^1(\Gamma_C); \\ (b)\ j_e(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_C; \\ (c)\ |\partial j_\nu(\boldsymbol{x}, r)| \le c_{0e} + c_{1e}|r| \text{ for all } r \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } c_{0e}, c_{1e} \ge 0; \\ (d)\ j_e^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \le \alpha_{j_e} |r_1 - r_2|^2 \text{ for all } r_1, r_2 \in \mathbb{R}, \\ \quad \text{a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } \alpha_{j_e} \ge 0. \end{cases} \tag{55}$$

The function spaces for mechanical and electrical unknowns in Problem 13 are as follows:

$$H = L^2(\Omega; \mathbb{R}^d),\ \mathcal{H} = L^2(\Omega; \mathbb{S}^d),\ \text{ and } \mathcal{H}_1 = \{\boldsymbol{\tau} \in \mathcal{H} \mid \mathrm{Div}\boldsymbol{\tau} \in H\}$$

with the corresponding inner products

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} d\boldsymbol{x},$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} d\boldsymbol{x},$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\boldsymbol{\sigma}, \mathrm{Div}\boldsymbol{\tau})_H.$$

The space

$$V = \{\boldsymbol{v} = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_D\}$$

is a closed subspace of $H^1(\Omega; \mathbb{R}^d)$ and the set

$$K = \{\boldsymbol{v} \in V \mid v_\nu \le g_0 \text{ a.e. on } \Gamma_C\}$$

is used for the admissible displacement field. Obviously, K is a closed convex subset of V with $0_V \in K$. Since $\mathrm{meas}(\Gamma_D) > 0$, it follows from Korn's inequality that V is a Hilbert space with the inner product $(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$ and norm $\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}}$ for $\boldsymbol{u}, \boldsymbol{v} \in V$. Let W and Φ be the function spaces defined by

$$W = \{\boldsymbol{D} \in H \mid \mathrm{div}\boldsymbol{D} \in L^2(\Omega)\}, \quad \Phi = \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ a.e. on } \Gamma_a\}.$$

On the space Φ, the inner product $(\varphi, \psi)_\Phi = (\nabla\varphi, \nabla\psi)_H$ and norm $\|\psi\|_\Phi = \|\nabla\psi\|_H$ for all $\varphi, \psi \in \Phi$ are considered. Since meas$(\Gamma_a) > 0$, by Poincaré's inequality, we can conclude that Φ is a Hilbert space. Let $Y = V \times \Phi$, the inner product and norm are

$$(\boldsymbol{y}, \boldsymbol{z})_Y = (\boldsymbol{u}, \boldsymbol{v})_V + (\varphi, \psi)_\Phi, \ \|\boldsymbol{y}\|_Y = (\boldsymbol{y}, \boldsymbol{y})_Y^{\frac{1}{2}} \quad \forall\, \boldsymbol{y} = (\boldsymbol{u}, \varphi), \boldsymbol{z} = (\boldsymbol{v}, \psi) \in Y.$$

Denote $U = K \times \Phi$, which is a closed and convex subset of Y with $0_Y \in U$.

For the trace operators $\gamma_1 : V \to L^2(\Gamma_C; \mathbb{R}^d)$ and $\gamma_2 : \Phi \to L^2(\Gamma_C)$, we have $\|\gamma_1 \boldsymbol{v}\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le \|\gamma_1\| \|\boldsymbol{v}\|_V$ for all $\boldsymbol{v} \in V$ and $\|\gamma_2 w\|_{L^2(\Gamma_C)} \le \|\gamma_2\| \|w\|_\Phi$ for all $w \in \Phi$ by Sobolev trace theorem. Let $Y_1 = L^2(\Gamma_C; \mathbb{R}^d) \times L^2(\Gamma_C)$, and $\gamma : Y \to Y_1$ be an operator defined by $\gamma \boldsymbol{y} = (\gamma_1 \boldsymbol{u}, \gamma_2 \varphi)$ for all $\boldsymbol{y} = (\boldsymbol{u}, \varphi) \in Y$. Denote $c_\gamma = \max\{\|\gamma_1\|, \|\gamma_2\|\}$. Then, $\|\gamma \boldsymbol{y}\|_{Y_1} \le c_\gamma \|\boldsymbol{y}\|_Y$. Let $U_1 = \gamma_1(K) \times L^2(\Gamma_C)$. Then U_1 is a convex, closed subset of Y_1 with $0_{Y_1} \in U_1$. Below, we omit the operators γ_1 and γ_2 for simplicity where no ambiguity may arise.

The functionals $\boldsymbol{f} \in V^*$ and $q \in \Phi^*$ are defined by

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V^*\times V} = (\boldsymbol{f}_0, \boldsymbol{v})_H + (\boldsymbol{f}_N, \boldsymbol{v})_{L^2(\Gamma_N;\mathbb{R}^d)} \quad \forall\, \boldsymbol{v} \in V,$$
$$\langle q, \psi\rangle_{\Phi^*\times\Phi} = (q_0, \psi)_{L^2(\Omega)} - (q_b, \psi)_{L^2(\Gamma_b)} \quad \forall\, \psi \in \Phi,$$

respectively. The weak formulation of Problem 13 has been derived in Gamorski and Migórski (2018).

Problem 14 *Find a displacement $\boldsymbol{u} \in K$ and an electric potential $\varphi \in \Phi$ such that*

$$\begin{aligned} &\Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{P}^T\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\Big)_{\mathcal{H}} + \int_{\Gamma_C} \Big(h_\nu(\varphi - \varphi_0) j_\nu^0(u_\nu - g_0; v_\nu - u_\nu) \\ &\quad + h_{\boldsymbol{\tau}}(\varphi - \varphi_0, u_\nu - g_0)\mu(\|\boldsymbol{u}_{\boldsymbol{\tau}}\|) j_\tau^0(\boldsymbol{u}_{\boldsymbol{\tau}}; \boldsymbol{v}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}})\Big)\mathrm{d}\Gamma \\ &\ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle_{V^*\times V} \quad \forall\, \boldsymbol{v} \in K, \end{aligned} \tag{56}$$

$$\begin{aligned} &\Big(\boldsymbol{\beta}\nabla\varphi - \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla(\psi - \varphi)\Big)_H + \int_{\Gamma_C} h_e(u_\nu - g_0) j_e^0(\varphi - \varphi_0; \psi - \varphi)\mathrm{d}\Gamma \\ &\quad \ge \langle q, \psi - \varphi\rangle_{\Phi^*\times\Phi} \quad \forall\, \psi \in \Phi. \end{aligned} \tag{57}$$

Note that Problem 14 is equivalent to the following system.

Problem 15 *Find $(\boldsymbol{u}, \varphi) \in K \times \Phi$ such that*

$$\begin{aligned} &\Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{P}^T\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\Big)_{\mathcal{H}} + \Big(\boldsymbol{\beta}\nabla\varphi - \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla(\psi - \varphi)\Big)_H \\ &\quad + \int_{\Gamma_C} \Big(h_\nu(\varphi - \varphi_0) j_\nu^0(u_\nu - g_0; v_\nu - u_\nu) \\ &\quad + h_{\boldsymbol{\tau}}(\varphi - \varphi_0, u_\nu - g_0)\mu(\|\boldsymbol{u}_{\boldsymbol{\tau}}\|) j_\tau^0(\boldsymbol{u}_{\boldsymbol{\tau}}; \boldsymbol{v}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}}) \\ &\quad + h_e(u_\nu - g_0) j_e^0(\varphi - \varphi_0; \psi - \varphi)\Big)\mathrm{d}\Gamma \\ &\ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle_{V^*\times V} + \langle q, \psi - \varphi\rangle_{\Phi^*\times\Phi} \quad \forall\, (\boldsymbol{v}, \psi) \in K \times \Phi. \end{aligned} \tag{58}$$

It is shown in Gamorski and Migórski (2018) that under assumptions (44)–(55) and a smallness condition

$$\max\{\overline{h}_\nu\alpha_{j_\nu}, \overline{h}_\tau\mu_0\alpha_{j_\tau}, \overline{h}_e\alpha_{j_e}\}c_\gamma^2 < \min\{m_{\mathcal{A}}, m_\beta\}, \tag{59}$$

Problem 15 has a unique solution $(\boldsymbol{u}, \varphi) \in K \times \Phi$.

Next, we introduce the Galerkin approximation for Problem 15. Assume Ω is a polygonal/tetrahedtron and $\{\mathcal{T}^h\}$ is a regular family of triangular/tetrahedral partitions of $\overline{\Omega}$. The partitions are compatible with the boundary splittings $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$ and $\Gamma_D \cup \Gamma_N = \Gamma_a \cup \Gamma_b$, i.e., if the intersection between the side/face of an element and the boundary is nonempty and its $d-1$ dimensional measurement is positive, then the associated side/face lies entirely in one of the sets $\overline{\Gamma}_D$, $\overline{\Gamma}_N$, $\overline{\Gamma}_C$ or $\overline{\Gamma}_a$, $\overline{\Gamma}_b$. Corresponding to each partition $\{\mathcal{T}^h\}$, define

$$V^h = \{\boldsymbol{v}^h \in C(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in \mathbb{P}_1(T)^d \text{ for } T \in \mathcal{T}^h,\ \boldsymbol{v}^h = \mathbf{0} \text{ on } \Gamma_D\},$$
$$\Phi^h = \{\varphi^h \in C(\overline{\Omega}) \mid \varphi^h|_T \in \mathbb{P}_1(T) \text{ for } T \in \mathcal{T}^h,\ \varphi^h = 0 \text{ on } \Gamma_a\}.$$

Define $K^h = \{\boldsymbol{v}^h \in V^h \mid v_\nu^h \le g_0$ at node points on $\Gamma_C\}$. We only consider the specific case $K^h \subset K$ for simplicity, similar results can be generalized to the case of $K^h \not\subseteq K$ (Han (2018); Xu et al. (2019b)). The discrete scheme for Problem 15 is the following.

Problem 16 *Find* $(\boldsymbol{u}^h,\ \varphi^h) \in K^h \times \Phi^h$ *such that*

$$\begin{aligned}
&\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}^h) + \mathcal{P}^\mathsf{T}\nabla\varphi^h, \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}^h)\right)_{\mathcal{H}} + \left(\boldsymbol{\beta}\nabla\varphi^h - \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}^h), \nabla(\psi^h - \varphi^h)\right)_H \\
&\quad + \int_{\Gamma_C}\Big(h_\nu(\varphi^h - \varphi_0)j_\nu^0(u_\nu^h - g_0; v_\nu^h - u_\nu^h) \\
&\quad + h_{\boldsymbol{\tau}}(\varphi^h - \varphi_0, u_\nu^h - g_0)\mu(\|\boldsymbol{u}_{\boldsymbol{\tau}}^h\|)j_\tau^0(\boldsymbol{u}_{\boldsymbol{\tau}}^h; \boldsymbol{v}_{\boldsymbol{\tau}}^h - \boldsymbol{u}_{\boldsymbol{\tau}}^h) \\
&\quad + h_e(u_\nu^h - g_0)j_e^0(\varphi^h - \varphi_0; \psi^h - \varphi^h)\Big)\mathrm{d}\Gamma \\
&\ge \langle \boldsymbol{f}, \boldsymbol{v^h} - \boldsymbol{u}^h\rangle_{V^*\times V} + \langle q, \psi^h - \varphi^h\rangle_{\Phi^*\times\Phi} \quad \forall(\boldsymbol{v}^h,\ \varphi^h) \in K^h \times \Phi^h.
\end{aligned} \tag{60}$$

Similar to Problem 15, there is a unique solution to Problem 16. Now we use Theorem 9 to bound the error. For this purpose, we need to assume solution regularities:

$$(\boldsymbol{u}, \varphi) \in H^2(\Omega; \mathbb{R}^d) \times H^2(\Omega) \tag{61}$$

and the trace of $(\boldsymbol{u}, \varphi)$ satisfies

$$u_\nu \in H^2(\Gamma_C), \quad \boldsymbol{u}_{\boldsymbol{\tau}} \in H^2(\Gamma_C; \mathbb{R}^d), \quad \varphi \in H^2(\Gamma_C). \tag{62}$$

The operator $A : Y \to Y^*$ is defined by

$$\langle A\boldsymbol{y}, \boldsymbol{z}\rangle_{Y^*\times Y} = \left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{P}^\mathsf{T}\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}} + \left(\boldsymbol{\beta}\nabla\varphi - \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\psi\right)_H \tag{63}$$

for all $\boldsymbol{y} = (\boldsymbol{u}, \varphi) \in Y$, $\boldsymbol{z} = (\boldsymbol{v}, \psi) \in Y$. The functional $J : U_1 \times Y_1 \to \mathbb{R}$ is defined by

$$\begin{aligned}
J(\gamma\boldsymbol{z}, \gamma\boldsymbol{y}) = \int_{\Gamma_C}\Big(&h_\nu(\psi - \varphi_0)j_\nu(u_\nu - g_0) \\
&+ h_{\boldsymbol{\tau}}(\psi - \varphi_0, v_\nu - g_0)\mu(\|\boldsymbol{v}_{\boldsymbol{\tau}}\|)j_\tau(\boldsymbol{u}_{\boldsymbol{\tau}}) \\
&+ h_e(v_\nu - g_0)j_e(\varphi - \varphi_0)\Big)\mathrm{d}\Gamma.
\end{aligned} \tag{64}$$

for all $\boldsymbol{y} = (\boldsymbol{u}, \varphi) \in Y,\ \boldsymbol{z} = (\boldsymbol{v}, \psi) \in U$. The functional $\widetilde{\boldsymbol{f}} \in Y^*$ is defined by

$$\langle \widetilde{\boldsymbol{f}}, \boldsymbol{z} \rangle_{Y^* \times Y} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^* \times V} + \langle q, \psi \rangle_{\Phi^* \times \Phi} \quad \forall \boldsymbol{z} = (\boldsymbol{v}, \psi) \in Y.$$

Let $Y^h = V^h \times \Phi^h$, $U^h = K^h \times \Phi^h$, $\boldsymbol{y}^h = (\boldsymbol{u}^h, \varphi^h) \in U^h$ and $\boldsymbol{s}^h = (\boldsymbol{v}^h,\ \psi^h) \in U^h$, the conditions in Theorem 9 can be verified straightforward (see Gamorski and Migórski (2018)). Using the theoretical result (17), we get

$$\|\boldsymbol{y} - \boldsymbol{y}^h\|_Y^2 \le C\Big(\|\boldsymbol{y} - \boldsymbol{s}^h\|_Y^2 + \|\boldsymbol{y} - \boldsymbol{s}^h\|_{Y_1} + R(\boldsymbol{y}, \boldsymbol{s}^h)\Big), \tag{65}$$

where

$$R(\boldsymbol{y}, \boldsymbol{s}^h) = \langle A\boldsymbol{y}, \boldsymbol{s}^h - \boldsymbol{y} \rangle_{Y^* \times Y} + J^0(\gamma \boldsymbol{z}, \gamma \boldsymbol{y}; \gamma \boldsymbol{s}^h - \gamma \boldsymbol{y}) - \langle \widetilde{\boldsymbol{f}}, \boldsymbol{s}^h - \boldsymbol{y} \rangle_{Y^* \times Y}. \tag{66}$$

J^0 can be bounded by (Gamorski and Migórski (2018))

$$\begin{aligned} J^0(\gamma \boldsymbol{z}, \gamma \boldsymbol{y}; \gamma \boldsymbol{s}^h - \gamma \boldsymbol{y}) \le \int_{\Gamma_C} \Big(& h_\nu(\psi - \varphi_0) j_\nu^0(u_\nu - g_0; v_\nu^h - u_\nu) \\ & + h_{\boldsymbol{\tau}}(\psi - \varphi_0, v_\nu - g_0)\mu(\|\boldsymbol{u}_{\boldsymbol{\tau}}\|) j_\tau^0(\boldsymbol{u}_{\boldsymbol{\tau}}; \boldsymbol{v}_{\boldsymbol{\tau}}^h - \boldsymbol{u}_{\boldsymbol{\tau}}) \\ & + h_e(v_\nu - g_0) j_e^0(\varphi - \varphi_0; \psi^h - \varphi) \Big) \mathrm{d}\Gamma, \end{aligned} \tag{67}$$

then we have

$$R(\boldsymbol{y}, \boldsymbol{s}^h) \le R_1(\boldsymbol{y}, \boldsymbol{s}^h) + R_2(\boldsymbol{y}, \boldsymbol{s}^h),$$

where

$$\begin{aligned} R_1(\boldsymbol{y}, \boldsymbol{s}^h) = & \Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{P}^\top \nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u})\Big)_{\mathcal{H}} \\ & + \int_{\Gamma_C} \Big(h_\nu(\varphi - \varphi_0) j_\nu^0(u_\nu - g_0; v_\nu^h - u_\nu) \\ & + h_{\boldsymbol{\tau}}(\varphi - \varphi_0, u_\nu - g_0)\mu(\|\boldsymbol{u}_{\boldsymbol{\tau}}\|) j_\tau^0(\boldsymbol{u}_{\boldsymbol{\tau}}; \boldsymbol{v}_{\boldsymbol{\tau}}^h - \boldsymbol{u}_{\boldsymbol{\tau}}) \Big) \mathrm{d}\Gamma \\ & - (\boldsymbol{f_0}, \boldsymbol{v}^h - \boldsymbol{u})_H - (\boldsymbol{f_N}, \boldsymbol{v}^h - \boldsymbol{u})_{L^2(\Gamma_N;\ \mathbb{R}^d)}, \end{aligned} \tag{68}$$

and

$$\begin{aligned} R_2(\boldsymbol{y}, \boldsymbol{s}^h) = & \Big(\boldsymbol{\beta}\nabla\varphi - \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla(\psi^h - \varphi)\Big)_H \\ & + \int_{\Gamma_C} \Big(h_e(u_\nu - g_0) j_e^0(\varphi - \varphi_0; \psi^h - \varphi) \Big) \mathrm{d}\Gamma \\ & - (q_0, \psi^h - \varphi)_{L^2(\Omega)} + (q_b, \psi^h - \varphi)_{L^2(\Gamma_b)}. \end{aligned} \tag{69}$$

We use (58) to bound the residual terms $R_1(\boldsymbol{y}, \boldsymbol{s}^h)$ and $R_2(\boldsymbol{y}, \boldsymbol{s}^h)$. Define

$$\begin{aligned} \boldsymbol{\sigma} &= \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{P}^\top \nabla\varphi && \text{in } \Omega, \\ \boldsymbol{D} &= \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\beta}\nabla\varphi && \text{in } \Omega. \end{aligned}$$

In addition to the solution regularity assumptions (61), we further assume

$$\boldsymbol{\sigma} \in H^1(\Omega; \mathbb{S}^d), \quad \boldsymbol{D} \in H^1(\Omega; \mathbb{R}^d). \tag{70}$$

Note that (70) follows from (61) if $\mathcal{P}$ and $\boldsymbol{\beta}$ are continuously differentiable on $\overline{\Omega}$, and $\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})$ is a smooth function of $\boldsymbol{x}$ and $\boldsymbol{\varepsilon}$. The solution regularity (70) further implies

$$\boldsymbol{\sigma}\boldsymbol{\nu} \in L^2(\Gamma_C; \mathbb{R}^d), \quad \boldsymbol{D}\cdot\boldsymbol{\nu} \in L^2(\Gamma_C).$$

Define spaces

$$\widetilde{V} = \{\widetilde{\boldsymbol{v}} \in C^\infty(\overline{\Omega}; \mathbb{R}^d) \mid \widetilde{\boldsymbol{v}} = \boldsymbol{0} \ \text{ on } \Gamma_D \cup \Gamma_C\},$$
$$\widetilde{\Phi} = \{\widetilde{\varphi} \in C^\infty(\overline{\Omega}) \mid \widetilde{\varphi} = 0 \text{ on } \Gamma_a \cup \Gamma_C\}.$$

On one hand for any $\widetilde{\boldsymbol{v}} \in \widetilde{V}$, we take $(\boldsymbol{u} \pm \widetilde{\boldsymbol{v}}, \varphi) \in \widetilde{V} \times \Phi$ in (58) to get

$$\Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{P}^{\mathsf{T}}\nabla\varphi, \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{v}})\Big)_{\mathcal{H}} = \langle \boldsymbol{f}, \widetilde{\boldsymbol{v}}\rangle_{V^*\times V} \quad \forall \widetilde{\boldsymbol{v}} \in \widetilde{V}. \tag{71}$$

Using Green's formula, we get

$$-(\mathrm{Div}\boldsymbol{\sigma}, \widetilde{\boldsymbol{v}})_H + \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\widetilde{\boldsymbol{v}}\mathrm{d}\Gamma = \int_\Omega \boldsymbol{f}_0\cdot\widetilde{\boldsymbol{v}}\mathrm{d}x + \int_{\Gamma_N} \boldsymbol{f}_N\cdot\widetilde{\boldsymbol{v}}\mathrm{d}\Gamma. \tag{72}$$

Since $\widetilde{\boldsymbol{v}} \in \widetilde{V}$ is arbitrary, we can deduce the following pointwise relations (cf. Han and Sofonea 2002, Section 8.1) for derivation details on a similar problem) from (71),

$$\mathrm{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{a.e. in } \Omega, \tag{73}$$
$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_N \quad \text{a.e. on } \Gamma_N. \tag{74}$$

Multiplying (73) with $\boldsymbol{v}^h - \boldsymbol{u}$ and integrating over Ω, we use Green's formula and (74) to get

$$\int_\Omega \boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}))d\boldsymbol{x} - \int_\Omega \boldsymbol{f}_0\cdot(\boldsymbol{v}^h - \boldsymbol{u})d\boldsymbol{x} - \int_{\Gamma_N} \boldsymbol{f}_N\cdot(\boldsymbol{v}^h - \boldsymbol{u})\mathrm{d}\Gamma$$
$$= \int_{\Gamma_C} \boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v}^h - \boldsymbol{u})\mathrm{d}\Gamma.$$

Then,

$$R_1(\boldsymbol{y}, \boldsymbol{s}^h) = \int_{\Gamma_C} \Big(\boldsymbol{\sigma}\boldsymbol{\nu}\cdot(\boldsymbol{v}^h - \boldsymbol{u}) + h_\nu(\varphi - \varphi_0)j_\nu^0(u_\nu - g_0; v_\nu^h - u_\nu)$$
$$+ h_{\boldsymbol{\tau}}(\varphi - \varphi_0, u_\nu - g_0)\mu(\|\boldsymbol{u}_{\boldsymbol{\tau}}\|)j_\tau^0(\boldsymbol{u}_{\boldsymbol{\tau}}; \boldsymbol{v}_{\boldsymbol{\tau}}^h - \boldsymbol{u}_{\boldsymbol{\tau}})\Big)\mathrm{d}\Gamma.$$

Together with (49)–(53) and regularity $\boldsymbol{\sigma}\boldsymbol{\nu} \in L^2(\Gamma_C; \mathbb{R}^d)$, we have

$$R_1(\boldsymbol{y}, \boldsymbol{s}^h) \le C\|\boldsymbol{u} - \boldsymbol{v}^h\|_{L^2(\Gamma_C;\mathbb{R}^d)}. \tag{75}$$

On the other hand for any $\widetilde{\varphi} \in \widetilde{\Phi}$, we take $(\boldsymbol{u}, \varphi \pm \widetilde{\varphi}) \in K \times \widetilde{\Phi}$ in (58) to get

$$\Big(\boldsymbol{\beta}\nabla\varphi - \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\widetilde{\varphi}\Big)_H = \langle q, \widetilde{\varphi}\rangle_{\Phi^*\times\Phi} \quad \forall \widetilde{\varphi} \in \widetilde{\Phi}. \tag{76}$$

Using Green's formula, we get

$$(\mathrm{div}\boldsymbol{D}, \widetilde{\varphi})_{L^2(\Omega)} - \int_\Gamma \boldsymbol{D}\cdot\boldsymbol{\nu}\widetilde{\varphi}\mathrm{d}\Gamma = (q_0, \widetilde{\varphi})_{L^2(\Omega)} - (q_b, \widetilde{\varphi})_{L^2(\Gamma_b)}. \tag{77}$$

Since $\widetilde{\varphi} \in \widetilde{\Phi}$ is arbitrary, we obtain

$$\operatorname{div} \boldsymbol{D} - q_0 = 0 \text{ a.e. in } \Omega, \tag{78}$$

$$\boldsymbol{D} \cdot \boldsymbol{\nu} = q_b \text{ a.e. on } \Gamma_b. \tag{79}$$

Multiplying (78) with $\psi^h - \varphi$ and integrating over Ω, we obtain the following equation by Green's formula and (79),

$$\int_\Omega -\boldsymbol{D} \cdot \nabla(\psi^h - \varphi) d\boldsymbol{x} - \int_\Omega q_0(\psi^h - \varphi) d\boldsymbol{x} + \int_{\Gamma_b} q_b(\psi^h - \varphi) \mathrm{d}\Gamma$$
$$= \int_{\Gamma_C} -\boldsymbol{D} \cdot \boldsymbol{\nu}(\psi^h - \varphi) \mathrm{d}\Gamma.$$

Thus

$$R_2(\boldsymbol{y}, \boldsymbol{s}^h) = \int_{\Gamma_C} \Big(-\boldsymbol{D} \cdot \boldsymbol{\nu}(\psi^h - \varphi) + h_e(u_\nu - g_0) j_e^0(\varphi - \varphi_0; \psi^h - \varphi) \Big) \mathrm{d}\Gamma.$$

Combining (54), (55) and $\boldsymbol{D} \cdot \boldsymbol{\nu} \in L^2(\Gamma_C)$, we have

$$R_2(\boldsymbol{y}, \boldsymbol{s}^h) \le C \|\varphi - \psi^h\|_{L^2(\Gamma_C)}. \tag{80}$$

Therefore

$$R(\boldsymbol{y}, \boldsymbol{s}^h) \le C \Big(\|\boldsymbol{u} - \boldsymbol{v}^h\|_{L^2(\Gamma_C;\mathbb{R}^d)} + \|\varphi - \psi^h\|_{L^2(\Gamma_C)} \Big). \tag{81}$$

Together with (65), we have

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 + \|\varphi - \varphi^h\|_\Phi^2 \le C \inf_{v^h \in K^h} \Big(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V^2 + \|\boldsymbol{u} - \boldsymbol{v}^h\|_{L^2(\Gamma_C;\mathbb{R}^d)} \Big) \\ + C \inf_{\psi^h \in \Phi^h} \Big(\|\varphi - \psi^h\|_\Phi^2 + \|\varphi - \psi^h\|_{L^2(\Gamma_C)} \Big). \end{aligned} \tag{82}$$

Using standard interpolation error estimation Atkinson and Han (2009); Ciarlet (1978), we summarize the optimal error estimate for linear element method in the form of a theorem.

Theorem 17 *Assume* (44)–(55), (59) *and solution regularities* (61)–(62), (70). *Then the following error bound holds for the numerical solution:*

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 + \|\varphi - \varphi^h\|_\Phi^2 \le C\, h^2. \tag{83}$$

5 Numerical experiments

In this section, we report results from two numerical experiments on two-dimensional models for Problem 13. The emphasis is on numerical evidence for convergence behaviors predicted in the theoretical analysis.

We first provide the configuration for the numerical examples. Let $\Omega = (0, L_1) \times (0, L_2)$ be a rectangle. The divisions for the boundary Γ satisfy that $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$, $\Gamma_D \cup \Gamma_N = \Gamma_a \cup \Gamma_b$. For the numerical simulation, we let $\Gamma_a = \Gamma_D$ and $\Gamma_b = \Gamma_N$; see Figure 1.

The elasticity tensor $\mathcal{A}$ satisfies

$$(\mathcal{A}\boldsymbol{\varepsilon})_{ij} = \frac{E\kappa}{1-\kappa^2}(\varepsilon_{11} + \varepsilon_{22})\delta_{ij} + \frac{E}{1+\kappa}\varepsilon_{ij}, \quad 1 \le i, j \le 2.$$

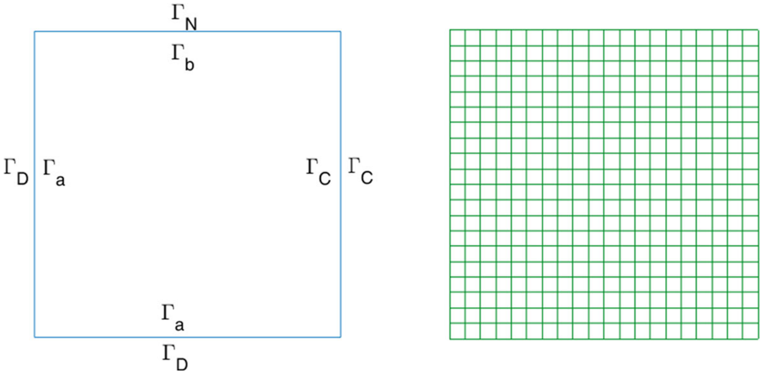


Fig. 1 The body Ω (left) and the mesh $T^h(\Omega)$ (right)

E is Young's modulus, κ is the Poisson ratio of the material and δ_{ij} denotes the Kronecker symbol. The electric permitivity tensor $\boldsymbol{\beta}$ satisfies $\boldsymbol{\beta}_{ij} = 5\delta_{ij}$, and the piezoelectric tensor $\mathcal{P}$ satisfies

$$\mathcal{P}\boldsymbol{\varepsilon} = \left(\frac{1}{4}(\varepsilon_{11} + \varepsilon_{22}), \frac{1}{8}(\varepsilon_{12} + \varepsilon_{21})\right)^T. \tag{84}$$

To facilitate numerical implementation, simplified contact conditions are considered and presented as follows. We set $h_\nu = 0$, then $\xi = 0$, and the condition (41) changes to the classical Signorini contact condition

$$u_\nu \leq g_0, \ \sigma_\nu \leq 0, \ \sigma_\nu(u_\nu - g_0) = 0 \quad \text{on } \Gamma_C. \tag{85}$$

The condition (42) is simplified by

$$-\boldsymbol{\sigma_\tau} \in S\partial j_\tau(\boldsymbol{u_\tau}) \text{ on } \Gamma_C, \tag{86}$$

where $S \geq 0$ is a given value, and the function $j_\tau(\boldsymbol{u_\tau})$ is defined by

$$j_\tau(\boldsymbol{u_\tau}) = \int_0^{\|\boldsymbol{u_\tau}\|} \eta(t)dt, \quad \text{and} \quad \eta(t) = (a-b)e^{-\alpha t} + b \tag{87}$$

with $a \geq b > 0$ and $\alpha > 0$ (Han et al. (2017)). Additionally, the relation (86) is equivalent to the frictional condition

$$\|\boldsymbol{\sigma}_\tau\| \leq \eta(0)S \text{ if } \|\boldsymbol{u}_\tau\| = 0, \ -\boldsymbol{\sigma}_\tau = \eta(\|\boldsymbol{u}_\tau\|)S\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \neq 0 \quad \text{on } \Gamma_C. \tag{88}$$

For the condition (43), two different versions are used in the numerical examples below.

For the domain Ω, a uniform rectangular finite element mesh $T^h(\Omega)$ is used, in which the intervals $[0, L_1]$ and $[0, L_2]$ are divided into $1/h$ equal parts; see Figure 1. In the numerical experiments, continuous linear element spaces are used. The following parameters are used in the numerical experiments

$$\begin{aligned} &L_1 = 1\,\text{m}, \quad L_2 = 1\,\text{m}, \quad E = 1\,\text{N/m}^2, \quad \kappa = 0.3, \\ &g_0 = 0.025\,\text{m}, \quad \alpha = 100, \quad a = 0.04, \quad b = 0.02, \quad S = 1\,\text{N}, \\ &\boldsymbol{f}_0 = \boldsymbol{0}, \quad \boldsymbol{f}_N = (0, -0.25\,x)^T\ \text{N/m}, \quad q_0 = 1, \quad q_b = 0. \end{aligned}$$

Numerical example 1

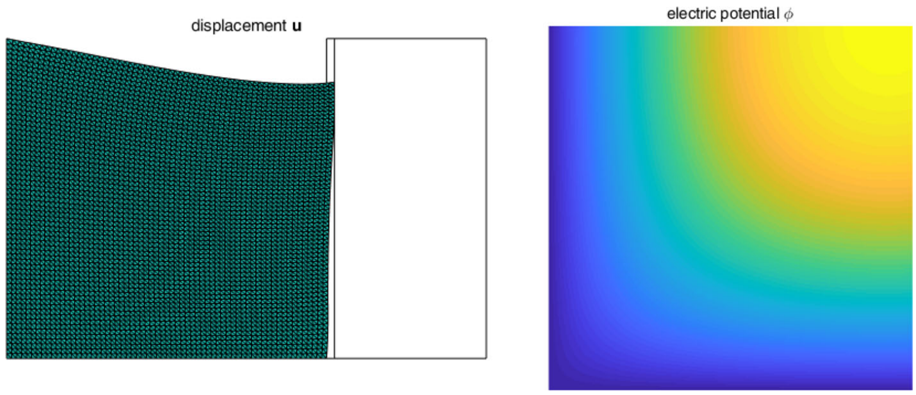


Fig. 2 The displacement of Ω (left) and the electric potential of Ω (right)

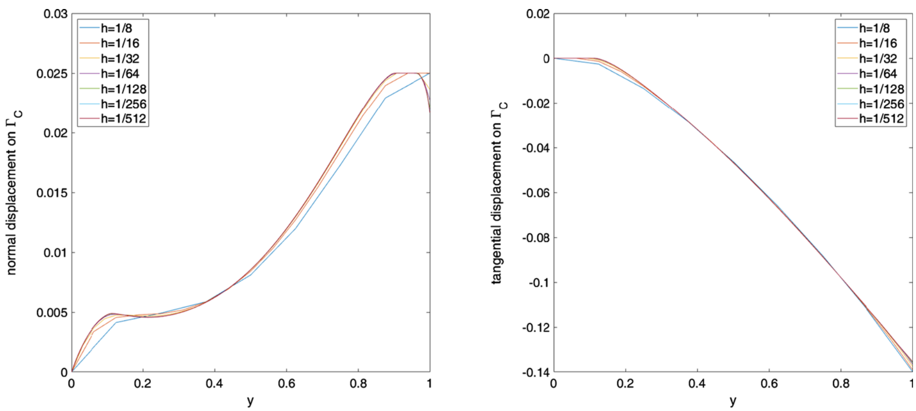


Fig. 3 The normal displacement on Γ_C (left) and the tangential displacement on Γ_C (right)

In this example, we consider a single-valued electric contact condition. Thus, the relation (43) is

$$\boldsymbol{D} \cdot \boldsymbol{\nu} = q_C \quad \text{on } \Gamma_C, \tag{89}$$

where $q_C : \mathbb{R} \to \mathbb{R}_+$ is continuous. Define

$$j_e(t) = \int_0^t q_C(s)ds, \quad t \in \mathbb{R},$$

we know that $\partial j_e(t) = q_C(t)$ for all $t \in \mathbb{R}$ by (Migórski et al. 2013, Lemma 3.50). Thus, the Eq. (89) is of the form (43).

In computation we take $q_C = 0$, i.e., there is no electric charge on the contact surface Γ_C. The numerical solution with $h = 1/64$ is presented in Fig. 2, and the normal and tangential displacements on Γ_C with different mesh sizes are depicted in Fig. 3.

To calculate the errors and convergence rates of the displacement and the electric potential, we take the numerical solution on a fine mesh with $h = 1/512$ as the "reference" solution, and report the numerical results in Table 1 and Fig. 4. From the numerical results, we can see that the numerical convergence orders of $\|\boldsymbol{u} - \boldsymbol{u}^h\|_1$ and $\|\varphi - \varphi^h\|_1$ in H^1-norm are both near 1, which matches the theoretical error estimate (83).

Table 1 Errors and numerical convergence orders for Example 1

h	1/8	1/16	1/32	1/64	1/128
$\|\boldsymbol{u}-\boldsymbol{u}^h\|_1$	9.4929e–3	5.6033e–3	2.9126e–3	1.5235e–3	7.3063e–4
Order	–	0.7606	0.9440	0.9349	1.0602
$\|\varphi-\varphi^h\|_1$	2.9941e–4	8.2339e–5	2.8658e–5	1.3003e–5	6.4240e–6
Order	–	1.8625	1.5226	1.1401	1.0173
$\|\boldsymbol{u}-\boldsymbol{u}^h\|_0$	1.0682e–3	3.8120e–4	1.2311e–4	4.2323e–5	1.2260e–5
Order	–	1.4866	1.6306	1.5404	1.7875
$\|\varphi-\varphi^h\|_0$	9.8648e–5	2.2000e–5	4.3434e–6	8.5122e–7	2.4684e–7
Order	–	2.1648	2.3406	2.3512	1.7860

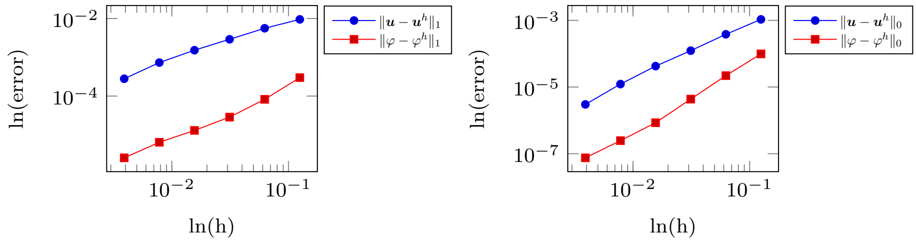


Fig. 4 The log–log plot of H^1 errors (left) and the log–log plot of L^2 errors (right)

Numerical example 2

In the second numerical example, we use the original subdifferential form (43), i.e.,

$$\boldsymbol{D}\cdot\boldsymbol{\nu} \in h_e(u_\nu - g_0)\partial j_e(\varphi-\varphi_0) \text{ on } \Gamma_C.$$

Here we use h_e as in Gamorski and Migórski (2018):

$$h_e(r) = \begin{cases} 0 & \text{if } \ r < -\delta_e, \\ k_e\frac{r+\delta_e}{\delta_e} & \text{if } \ -\delta_e \le r \le 0, \\ k_e & \text{if } \ r > 0, \end{cases} \tag{90}$$

where $k_e \ge 0$ is the electrical conductivity coefficient, and $\delta_e > 0$ is small parameter. j_e is defined by

$$j_e(t) = \begin{cases} 0 & \text{if } \ t < 0, \\ -e^{-ct} + dt + 1 & \text{if } \ t \ge 0, \end{cases} \tag{91}$$

where c and d are two positive parameters. Its generalized subdifferential is

$$\partial j_e(t) = \begin{cases} 0 & \text{if } \ t < 0, \\ [0, c+d] & \text{if } \ t = 0, \\ ce^{-ct} + d & \text{if } \ t > 0. \end{cases} \tag{92}$$

The following parameters are also needed in computation

$$\delta_e = 0.005, \quad k_e = 1, \quad \varphi_0 = 0.03, \quad c = 0.01, \quad d = 0.01.$$

Numerical solution with $h = 1/64$ is illustrated in Fig. 5. The normal displacement, the tangential displacement and the electric potential on Γ_C obtained on different mesh sizes are

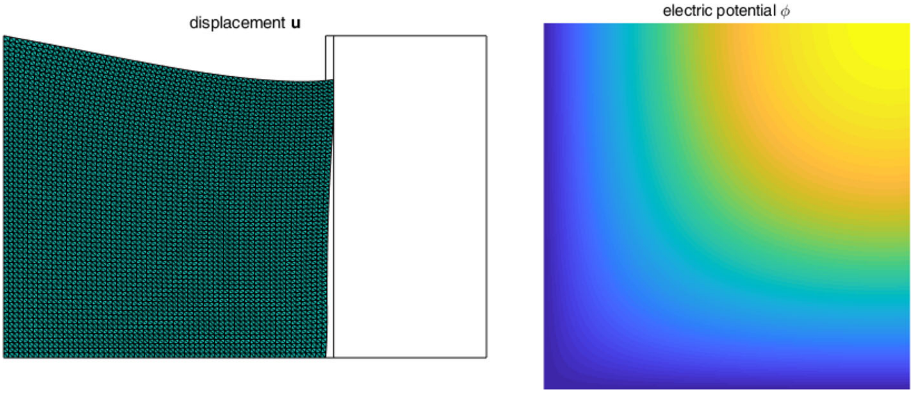


Fig. 5 The displacement of Ω (left) and the electric potential of Ω (right)

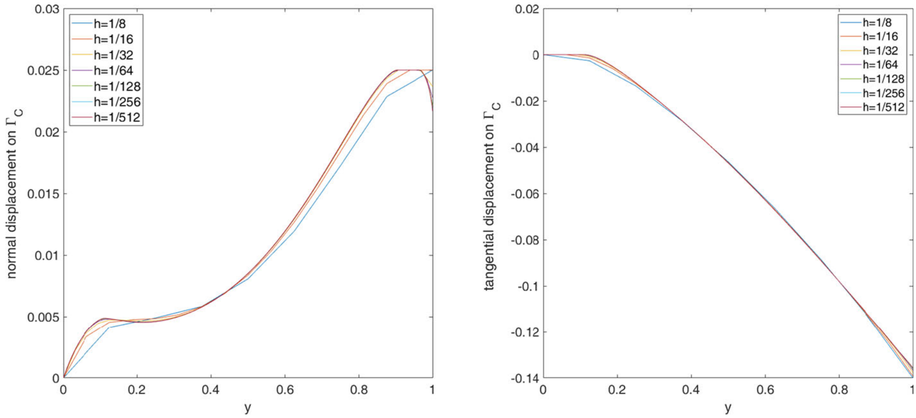
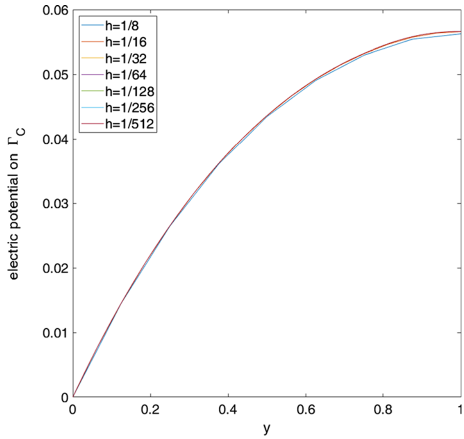


Fig. 6 The normal displacement on Γ_C (left) and the tangential displacement on Γ_C (right)

Fig. 7 The electric potential on Γ_C



depicted in Fig. 6 and Fig. 7. Using the numerical solution with mesh size $h = 1/512$ as the "reference" solution, we present the numerical errors and numerical convergence orders of displacement and electric potential in Table 2 and Fig. 8. Again, we observe that the numerical

Table 2 Errors and numerical convergence orders for Example 2

h	1/8	1/16	1/32	1/64	1/128
$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1$	9.4991e–3	5.6053e–3	2.9174e–3	1.5238e–3	7.3226e–4
Order	–	0.7610	0.9421	0.9370	1.0572
$\|\varphi - \varphi_h\|_1$	4.1270e–4	1.0951e–4	3.3055e–5	1.5291e–5	6.7134e–6
Order	–	1.9140	1.7281	1.1122	1.1876
$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0$	1.0735e–3	3.8166e–4	1.2264e–4	4.2366e–5	1.2442e–5
Order	–	1.4920	1.6379	1.5335	1.7677
$\|\varphi - \varphi_h\|_0$	1.4790e–4	3.0958e–5	4.1839e–6	9.5847e–7	2.5814e–7
Order	–	2.2562	2.8874	2.1260	1.8926

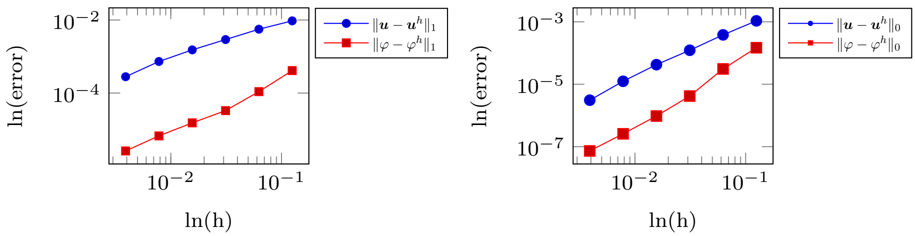


Fig. 8 The log–log plot of H^1 errors (left) and the log–log plot of L^2 errors (right)

convergence orders of $\|\boldsymbol{u} - \boldsymbol{u}_h\|_1$ and $\|\varphi - \varphi_h\|_1$ are both close to the theoretically predicted value of 1.

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