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Numerical Methods for Evaluation of Double integrals with Continuous Integrands

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Abstract:
The main objective of this research is to introduce new numerical methods for calculating double integrations with continuous integrands. We combined between the best methods of finding approximation values for single integration, based on accelerating Romberg with a Trapezoidal rule (RT), the Mid- point rule (RM), the Simpson rule (RS) on the outer dimension Y with the Suggested method (RSu) on the inner dimension X. We call the obtained methods by RT(RSu), RM(RSu) and RS(RSu) respectively. Applying these methods on double integrals gives high accuracy results in short time and with few partial periods.

1. Introduction:

Numerical integral is one of the mathematics branches that connects between analytical mathematics and computer, it is very important in physical engineering applications. Moreover, it has been used in dentistry where dentists used some numerical methods to find the surface area of the cross section of the root canal.

Because of the importance of double integrals in calculating surface area, middle centres, the intrinsic limitations of flat surfaces and finding the volume under the surface of double integral, so many researchers worked in the field of double integrals. Furthermore, they improved the results using different accelerating methods. Usually, by these methods, we get an approximation solution which means that there is an error, so here is the role of numerical integrals.

In 1984, Muhammad [7] applied complex methods such as Romberg (Romberg) method, Gauss (Gauss) method, Romberg (Gauss) method, and Gauss (Romberg) method on many examples of double integrals with continuous integrands and he compared between all these complex methods to find that Gauss method (Gauss) is the best one in terms of accuracy and velocity of approach to the values of analytic integrals as well as the number of partial periods. However, Al-Taey [2] used accelerating Romberg with the base point on the outer dimension Y beside the base of Simpson on the internal dimension X to give good results in terms of accuracy and a few partial periods. In the other side, RMM method, Romberg acceleration method on the values obtained by Mid-point method on the inner and outer dimensions X, Y when the number of partial periods on the two dimensions is equal was presented by Egghead [4]. For more information about this ,see [1] and[9] .

In this work, we present three new numerical methods for calculating double integrals with continuous integrands using single integral methods, RT, RM,RS, [7] and R(Su), [8] when the number of divisions on the outer dimension is not equal to the number of divisions on the internal dimension and we get good results in terms of accuracy and velocity of approach to the values of analytic integrals as well as the number of partial periods.

2 . Newton-Cotes Formulas

The Newton-Coats formulas are the most important methods of numerical integrals, such as Trapezoidal rules, Mid-point, Simpson. In this section we present these methods and their correction limits. Let's assume the integral J is defined to be :

$$J = \int_s^t g(x)dx = \alpha(k) + \delta_\alpha(k) + R_\alpha \qquad \ldots(1)$$

[5].

Such that $\alpha(k)$ is the numerical base to calculate the value of integral J, α represents the type of the rule, $\delta_\alpha(k)$ is the correction terms for the rule $\alpha(k)$ and R_α is the remainder after truncation some terms from $\delta_\alpha(k)$.

The general formulas for the above three methods are:

$$T(k) = \frac{k}{2}\left[g(s) + g(t) + 2\sum_{p=1}^{w-1} g(s+pk) \right] \qquad \text{Trapezoidal Rule}$$

$$M(k) = \sum_{p=1}^{w} g(s+(p-0.5)k) \qquad \text{Mid-point Rule}$$

$$S(k) = \frac{k}{3}\left[g(s) + g(t) + 2\sum_{p=1}^{\frac{w}{2}-1} g(s+2pk) + 4\sum_{p=1}^{\frac{w}{2}} g(s+(2p-1)k) \right] \qquad \text{Simpson's Rule}$$

A great importance for the correction limits is to improve the value of integral and accelerate the numerical value of integral to the analytical or exact one. Therefore, Fox [5] has found the series of correction terms for each rule of Newton-Coates rules for continuous integrals. Thus the series of correction terms for $T(k)$, $M(k)$ and $S(k)$ respectively are:

$$\delta_T(k) = \varpi_T k^2 + \theta_T k^4 + \vartheta_T k^6 + \cdots$$

$$\delta_M(k) = \varpi_M k^2 + \theta_M k^4 + \vartheta_M k^6 + \cdots$$

$$\delta_S(k) = \varpi_S k^4 + \theta_S k^6 + \vartheta_S k^8 + \cdots$$

Such that $\varpi_S, \theta_S, \vartheta_S, \cdots, \varpi_M, \theta_M, \vartheta_M, \cdots, \varpi_T, \theta_T, \vartheta_T, \cdots$ are constants.

3. Suggested method (Su)

The Suggested method that represented by [8] and others, is one of the single integral methods that derived from the Trapezoidal and Mid-point rules. This method gave good results in terms of accuracy and velocity of approach to the values of analytic integrals better than the mid-trapezoid method. The general formula of the Suggested method is :

$$Su(k)=\frac{k}{4}\left[\mathrm{g}(s)+\mathrm{g}(t)+2\,\mathrm{g}(s+(w-0.5)k)+2\sum_{p=1}^{w-1}(\mathrm{g}(s+(p-0.5)k)+g(s+pk))\right]$$

Where its correction terms are:

$\delta_{Su}(k)=\lambda_{Su}k^2+\xi_{Su}k^4+\psi_{Su}k^6+\cdots$, such that $\lambda_{Su},\xi_{Su},\psi_{Su},\cdots$ are constants.

4. Romberg Accelerating

Werner Werner (1909-2003) represented this method in 1955, in which a triangular arrangement consist from numerical approximations for the definite integral by applying Richardson's external adjustment repeatedly to Trapezoidal rule, Mid-point or Simpson's rule which represents some of the Newton Coates formulas. The general formula for this accelerating is :

$$\alpha=\frac{\left(2^L\alpha\left(\frac{k}{2}\right)-\alpha(k)\right)}{\left(2^L-1\right)}$$

Such that $L=4,6,8,10,...$(in Simpson's rule) and $L=2,4,6,8,10,...$ (in Trapezoidal rule and Mid-point rule), α is the value in that we written it in a new column in tables whereas $\alpha(k)$ and $\alpha\left(\frac{k}{2}\right)$ are values in its previous column. Moreover, the first column from the table represents one of Newton Coates values [3].

5. Numerical Methods for Calculating double Integrals with Numerically continuous integrands:

In this section, we will apply the compound method RT(RSu) to calculate the approximated value for the integral :

$$J=\int_u^v\int_s^t g(x,y)dxdy \qquad \ldots(2)$$

We can write it as:

$$J=\int_u^v G(y)dy \qquad \ldots(3)$$

Such that: $G(y)=\int_s^t g(x,y)dx$ …(4)

For example, the approximated value for (3) on outer dimension y by Trapezoidal rule is :

$$J=\frac{k}{2}\left(G(u)+G(v)+2\sum_{p=1}^{w_1-1}G(y_p)\right)+\delta_T(k) \qquad \ldots(5)$$

where $y_p = u + pk$, $p = 1, 2, ..., w_1 - 1$, w_1 is the number of the interval $[u, v]$ divisions, $k = \frac{u - v}{w_1}$ and the correction terms on outer dimension Y is $\delta_T(k)$. We shall mention that in Trapezoidal and Midpoint case, $w_1 = 1, 2, 4, 8, ...$ while in Simpson case, $w_1 = 2, 4, 8, ...$.

To calculate $G(y_p), G(v), G(u)$ approximately using formula (4), we write:

$$G(u) = \int_s^t g(x, u)dx \quad \dots(6)$$

$$G(v) = \int_s^t g(x, v)dx \quad \dots(7)$$

$$G(y_p) = \int_s^t g(x, y_p)dx , \quad \dots(8)$$

Applying the suggested method (on inner dimension X) on (6),(7),(8), we get the following formulas:

$$G(u) = \frac{\bar{k}}{4}\left[g(s,u) + g(t,u) + 2g(s + (w_2 - 0.5)k, u) + 2\sum_{r=1}^{w_2 - 1} (g(s + (r - 0.5)k, u) + g(s + rk, u)) \right] + \bar{\delta}_1(\bar{k}) \quad ...(9)$$

$$G(v) = \frac{\bar{k}}{4}\left[g(s,v) + g(t,v) + 2g(s + (w_2 - 0.5)k, v) + 2\sum_{r=1}^{w_2 - 1} (g(s + (r - 0.5)k, v) + g(s + rk, v)) \right] + \bar{\delta}_2(\bar{k}) \quad ...(10)$$

$$G(y_p) = \frac{\bar{k}}{4}\left[g(s,y_p) + g(t,y_p) + 2g(s + (w_2 - 0.5)k, y_p) + 2\sum_{r=1}^{w_2 - 1} (g(s + (r - 0.5)k, y_p) + g(s + rk, y_p)) \right] + \bar{\delta}_3(\bar{k}) \quad ...(11)$$

where w_2 is the number of the interval $[s, t]$ divisions, $\bar{k} = \frac{t - s}{w_2}$ and the correction terms on outer dimension X for (9),(10) and (11) are $\bar{\delta}_3(\bar{k}), \bar{\delta}_2(\bar{k}), \bar{\delta}_1(\bar{k})$ respectively.

To improve the results, we applied Romberg accelerating on (9),(10) and (11), then substituted the new results in (5) to get the approximated value for the integral J in the formula. After substitute the correction terms $\delta_T(k)$ in (5) and applying Romberg accelerating again, we will calculate the approximated value for the integral J in (3) which means that we got the approximated value for the original integral J in (2).

Remark: With the same above process, we can calculate the double integrals using RS(RSu) and RM(RSu).

6. Examples:

6.1 :

The integrand of $\int_0^1 \int_0^1 y e^{2.5x + 0.6y} dxdy$ that defined for $(x, y) \in [0,1] \times [0,1]$ which analytic value is 3.36906774253669 (approximates to fourteen decimal) had been shown in Fig.:1.

From (4) we have $G(y)=\frac{2}{5}y\left[e^{2.5+0.6y}-e^{0.6y}\right]$, applying the three methods $RT(RSu), RM(RSu), RS(RSu)$ with $\overline{Eps}=10^{-12}$ for the outer dimension y and $\overline{\overline{Eps}}=10^{-14}$ for the inner dimension x as follows:

Applying $RT(RSu), RS(RSu)$ with $w_1=w_2=32$, we got a value that equal to the exact value on partial interval 2^{12}, 2^{11} respectively. Moreover by applying $RM(RSu)$ with $w_1=16$, $w_2=32$, we got a value that approximates to thirteen decimal) on partial interval 2^{11}. All these details shown in tables (1),(2) and (3).

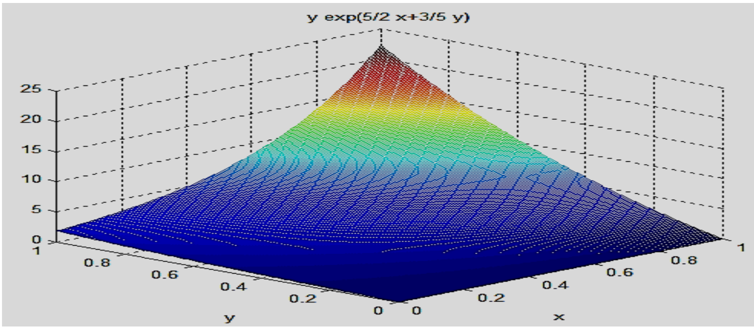

Fig.1

6.2 :

The integrand of $\int_2^3\int_1^2 \sqrt{y}\ln(x+y)dxdy$ that defined for $(x,y)\in[1,2]\times[2,3]$ which analytic value is 2.18658641209591 (approximates to fourteen decimal) had been shown in Fig.:2.

From (4) we have $G(y)=\sqrt{y}\left[(2+y)\ln(2+y)-(1+y)\ln(1+y)-1\right]$, applying the three methods $RT(RSu), RM(RSu), RS(RSu)$ with $\overline{\overline{Eps}}=\overline{Eps}=10^{-14}$ as follows:

Applying $RT(RSu), RS(RSu)$ with $w_1=32$, $w_2=16$, we got a value that equal to the exact value on partial interval 2^{11}, 2^{10} respectively. Moreover by applying $RM(RSu)$ with $w_1=w_2=16$, we got the same value on partial interval 2^{10}. All these details shown in tables (4),(5) and (6).

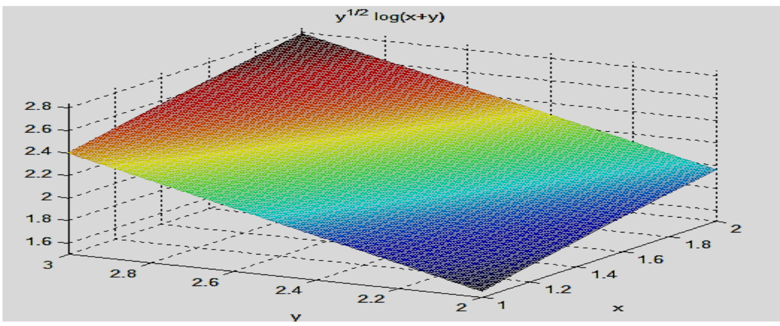

Fig.:2

6.3:

The integrand of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cosh(0.4x+0.6y)\,dxdy$ that defined for $(x,y)\in[\frac{\pi}{4},\frac{\pi}{2}]\times[\frac{\pi}{4},\frac{\pi}{2}]$ which analytic value is 1.11150994188443 (approximates to fourteen decimal) had been shown in Fig.:3.

From (4) we have $G(y)=2.5[\sinh(0.2\pi+0.6y)-\sinh(0.1\pi+0.6y)]$, applying the three methods $RT(RSu), RM(RSu), RS(RSu)$ with $\overline{\overline{Eps}}=10^{-14}$ and $\overline{Eps}=10^{-12}$ with $w_1=w_2=16$, we got a value that equal to the exact value on partial interval 2^{10}, 2^{9} respectively. All these details shown in tables (7),(8) and (9).

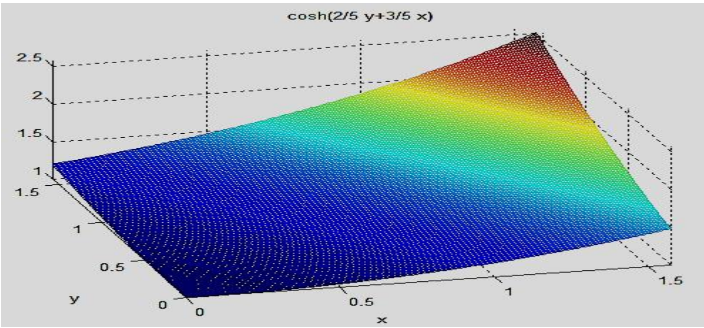

Fig.:3

6.4

The integral $\int_{1.5}^{2}\int_{1.5}^{2} \frac{\cos(\frac{\pi}{4}x)+\sin(\frac{\pi}{4}y)}{\sqrt[3]{x^3+y^3}}\,dxdy$ that defined on $(x,y)\in[1.5,2]\times[1.5,2]$, but its analytic value is unknown had been shown in Fig.:4.

From (4) we have $\int_{1.5}^{2}\int_{1.5}^{2} \frac{\cos(\frac{\pi}{4}x)+\sin(\frac{\pi}{4}y)}{\sqrt[3]{x^3+y^3}}\,dxdy$,applying the three methods $RT(RSu), RM(RSu)$, $RS(RSu)$ with $\overline{\overline{Eps}}=10^{-14}$ and $\overline{Eps}=10^{-12}$ we got that although, this integral with unknown analytic value, but applying $RT(RSu), RS(RSu)$ with $(w_1=w_2=16)$ and $(w_1=w_2=32)$ for RS(RSu), its value will be 0.13289635209694 which fixed. Thus we can say that this fixed value is the correct value for the integral approximates to fourteen decimal on partial interval 2^{10} and 2^{9} respectively. All these details shown in tables (10),(11) and (12).

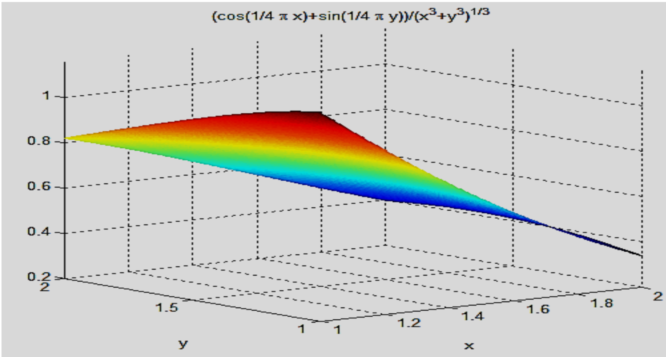

Fig.:4

6.5

The integral $\int_2^3\int_2^3 (xy)^{\frac{1}{y}}\,dxdy$ that defined on $(x,y)\in[2,3]\times[2,3]$, but its analytic value is unknown had been shown in Fig.:5.

From (4) we have $\int_2^3\int_2^3 (xy)^{\frac{1}{y}}\,dxdy$, applying the three methods $RS(RSu), RM(RSu), RT(RSu)$ with $\overline{\overline{Eps}} = 10^{-14}$ and $\overline{Eps} = 10^{-12}$ we got that although, this integral with unknown analytic value, but applying $RT(RSu), RS(RSu)$ with $(w_1 = 32, w_2 = 16)$ and for RM(RSu), $(w_1 = 64, w_2 = 16)$, its value will be 2.08319749522837 which fixed. Thus we can say that this fixed value is the correct value for the integral approximates to fourteen decimal on partial interval 2^{11} and 2^{10} respectively. All these details shown in tables (13),(14) and (15).

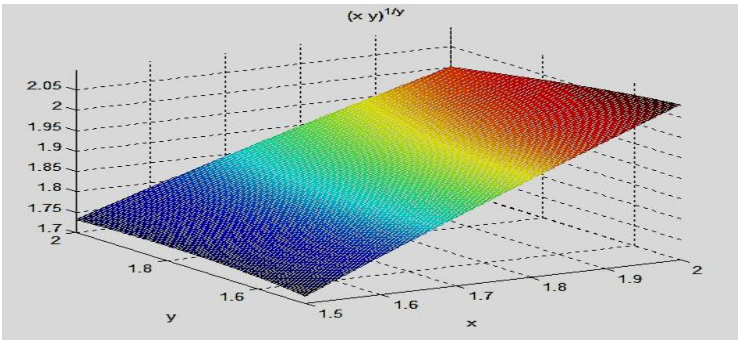

Fig:5

7. Conclusion :

In this work, we calculate the approximated value for the double integrals with continuous integrands using the compound methods which consist of RSu (on inner dimension) and RT, RM, RS (on outer dimension), and we called them as $RT(RSu), RM(RSu)$ and $RS(RSu)$

respectively. All tables show that, these compound methods give correct value (for many decimals) on numbers of partial periods compared with the exact values of the integrals. In more accuracy, we got exact values for the integrals approximated to (13-14) decimal on (16-64) partial period using these compound methods. Moreover, using these three methods allow us to calculate the value for unknown analytic value integrals when the approximated value is fixed like was in Example 6.4 and 6.5. Therefore, the compound methods give results with higher accuracy in calculating the double integrals with continuous integrands on less partial periods and less time.

8. Tables:

w_1	RT(RSu)	w_2
1	4.07516649621022	32
2	3.37102722720621	32
4	3.36906927731316	32
8	3.36906774282088	32
16	3.36906774253671	32
32	3.36906774253669	32

Table (1) Calculate Double integral $\int_0^1\int_0^1 ye^{2.5x+0.6}\,dx\,dy$ by Method RT(RSu)

w_1	RS(RSu)	w_2
2	3.37102722720621	32
4	3.36906927731315	32
8	3.36906774282088	32
16	3.36906774253671	32
32	3.36906774253669	32

Table (3) Calculate Double integral $\int_0^1\int_0^1 ye^{2.5x+0.6}\,dx\,dy$ by Method RS(RSu)

w_1	RM(RSu)	w_2
1	3.01895759270421	32
2	3.36735607115673	32
4	3.36906625628148	32
8	3.36906774225475	32
16	3.36906774253668	32

Table (2) Calculate Double integral $\int_0^1\int_0^1 ye^{2.5x+0.6}\,dx\,dy$ by Method RM(RSu)

w_1	RT(RSu)	w_2
1	2.18419784576378	16
2	2.18659051479755	16
4	2.18658642711805	16
8	2.18658641211480	16
16	2.18658641209592	16
32	2.18658641209591	16

Table (4) Calculate Double integral $\int_2^2\int_1^2 \sqrt{y}\ln(x+y)\,dx\,dy$ by Method RT(RSu)

w_1	RM(RSu)	w_2
1	2.18778684931444	16
2	2.18658285039849	16
4	2.18658639758039	16
8	2.18658641207719	16
16	2.18658641209591	16

Table (5) Calculate Double integral $\int_2^3\int_1^2 \sqrt{y}\ln(x+y)\,dx\,dy$ by Method RM(RSu)

w_1	RS(RSu)	w_2
2	2.18659051479755	16
4	2.18658642711805	16
8	2.18658641211479	16
16	2.18658641209592	16
32	2.18658641209591	16

Table (6) Calculate Double integral $\int_2^3\int_1^2 \sqrt{y}\ln(x+y)\,dx\,dy$ by Method RS(RSu)

w_1	RT(RSu)	w_2
1	1.12063674231684	16
2	1.11151369028585	16
4	1.11150994243478	16
8	1.11150994188444	16
16	1.11150994188442	16

Table (7) Calculate Double integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cosh(0.4x+0.6y)\,dx\,dy$ by Method RT(RSu)

w_1	RM(RSu)	w_2
1	1.10695216427036	16
2	1.11150666306509	16
4	1.11150994135130	16
8	1.11150994188440	16
16	1.11150994188442	16

Table (8) Calculate Double integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cosh(0.4x+0.6y)\,dx\,dy$ by Method RM(RSu)

w_1	RS(RSu)	w_2
2	1.11151369028585	16
4	1.11150994243478	16
8	1.11150994188444	16
16	1.11150994188442	16

Table (9) Calculate Double integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cosh(0.4x+0.6y)\,dx\,dy$ by Method RS(RSu)

w_1	RT(RSu)	w_2
1	0.13129250128304	16
2	0.13289643420586	16
4	0.13289635946722	16
8	0.13289635208948	16
16	0.13289635209694	16
32	0.13289635209694	16

Table (10) Calculate Double integral $\int_{1.5}^{2}\int_{1.5}^{2} \frac{\cos(\frac{\pi}{4}x)+\sin(\frac{\pi}{4}y)}{\sqrt[3]{x^2+y^2}}\,dx\,dy$ by Method RT(RSu)

w_1	RM(RSu)	w_2
1	0.13369840066727	16
2	0.13289629407091	16
4	0.13289634494230	16
8	0.13289635210433	16
16	0.13289635209694	16
32	0.13289635209694	16

Table (11) Calculate Double integral $\int_{1.5}^{2}\int_{1.5}^{2} \frac{\cos(\frac{\pi}{4}x)+\sin(\frac{\pi}{4}y)}{\sqrt[3]{x^2+y^2}}\,dx\,dy$ by Method RM(RSu)

w_1	RS(RSu)	w_2
2	0.13289643420586	16
4	0.13289635946722	16
8	0.13289635208948	16
16	0.13289635209694	16
32	0.13289635209694	16

Table (12) Calculate Double integral $\int_{1.5}^{2}\int_{1.5}^{2} \frac{\cos(\frac{\pi}{4}x)+\sin(\frac{\pi}{4}y)}{\sqrt[3]{x^2+y^2}}\,dx\,dy$ by Method RS(RSu)

w_1	RT(RS u)	w_2
1	2.09341026575913	16
2	2.08315319000490	16
4	2.08319635360170	16
8	2.08319748886946	16
16	2.08319749521717	16
32	2.08319749522837	16
64	2.08319749522837	16

Table (13) Calculate Double integral $\int_2^3\int_2^3 (xy)^{\frac{1}{y}}\,dx\,dy$ by Method RT(RS u)

w_1	RM(RS u)	w_2
1	2.07802465212778	16
2	2.08323412174891	16
4	2.08319858866010	16
8	2.08319750151529	16
16	2.08319749523954	16
32	2.08319749522838	16
64	2.08319749522837	16

Table (14) Calculate Double integral $\int_2^3\int_2^3 (xy)^{\frac{1}{y}}\,dx\,dy$ by Method RM(RS u)

w_2	RS(RS u)	w_2
2	2.08315319000490	16
4	2.08319635360170	16
8	2.08319748886946	16
16	2.08319749521717	16
32	2.08319749522837	16
64	2.08319749522837	16

Table (15) Calculate Double integral $\int_2^3\int_2^3 (xy)^{\frac{1}{y}}\,dx\,dy$ by Method RS(RS u)

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