NUMERICAL METHODS

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An ISO 9001:2008 Certified Company
VAYU EDUCATION OF INDIA
2/25, Ansari Road, Darya Ganj-110 002

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ISBN: 978-93-83137-43-5

First Edition: 2013

Rs. ₹ 160/-

Laser Typesetting :
Upasana Graphics, New Delhi

Published by:

VAYU EDUCATION OF INDIA
2/25, Ansari Road, Darya Ganj, New Delhi-110 002
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PREFACE

The application of Numerical Methods has become an integral part of the life for all the modern software professionals. With the advent of powerful small computers with highly increased speed, power and flexibility of numerical computing; almost all the universities offer courses in numerical methods in their curriculum. It is a fact that the students who can better understand and apply the numerical method to solve the practical problems successfully make a better software professionals. The author has made their extreme effort for the same with great care so that the students who are beginner in the field of numerical methods can understand the implementation of the numerical methods in computer.

The contents of the book are derived from latest revised syllabus for B.C.A. students of all Universities. The primary aim of the book is to provide students with a sound background of numerical methods as well as it's implementation in computer using C language.

I have exerted a conscious effort to make the book students-friendly. I hope the students would find not only useful but also interesting. I have made our best effort to bring the book free from errors. However, the author will be grateful, if the readers communicate any omissions or errors or any valuable suggestions for the improvement of the quality of the book for the next edition.

—***Author***

TABLE OF CONTENTS

CHAPTER 1

NUMERICAL SOLUTION OF ALGEBRAIC & TRANSCEDENTAL EQUATIONS

1.1 ANALYTICAL METHODS VS NUMERICAL METHODS

Let us consider an equation $ax^2 + bx + c = 0$. The solution of this equation can be obtained either by the technique of algebra or by using **shridhara charya** method. These methods come under the category of Analytical method. There are many algebraic and transcendental equation that can not be solved by the analytical method *e.g.* consider the following example

$$5^x + x - 8 = 0$$

It seems very simple but cannot be solved by analytical method. In such situation where analytical method is helpless to provide any solution, in that situation numerical method play an important role in obtaining the approximate solution up to the desired level of accuracy. This is the advantage of numerical method over the analytical methods.

The determination of solution of algebraic and transcendental equations is of great interest not only to mathematician but also to scientists, engineers and software developers in many situations.

1.2 ALGEBRIC AND TRANSCENDENTAL EQUATION

An expression of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$, where, $a_0, a_1, ..., a_n$ and $a_n \neq 0$, n is ppositive integer, is called a polynomial in x of degree n. When the polynomial is equated to zero *i.e.* $f(x) = 0$, is called an algebraic equation of degree n. The equations $2x^3 + 3x^2 + 5x + 6 = 0$ is an algebraic equation of degree 3.

If $f(x)$ contains some other functions such as logarithmic, exponential, trignometric e.t.c., then $f(x) = 0$ is called a transcendental equation. The equations $3x^2 + 2x + \sin x = 0$, $e^x + 2x + 3 = 0$, $3x^3 + 2x^2 + \log_{10} x + \cos x + 6 = 0$ etc are known as transcendental equations because these equations are containing either **trignometric** or logarithmic or exponential functions or in combination of these functions.

1.3 ZEROS OF ALGEBRAIC AND TRANSCENDENTAL EQUATION

In general, an algebraic and transcendental equation in x is written as $f(x) = 0$. The roots of this equation *i.e.* those values of x for which the equation is satisfied, is also known as zero(*s*) of algebraic and transcendental equation.

1.4 ITERATIVE METHOD

An iterative method is a method in which certain number of steps required to solve the problems are repeated until the desired level of accuracy in the solution is obtained. The iterative method, also known as trial and error method are based on the idea of successive approximations. Most of the numerical methods are of iterative type. They start with one or more initial approximations by repeating a fixed sequence of steps till the solution with reasonable accuracy is obtained. ***Now the question arises when to stop the iterative process*** i.e. when the iteration should terminate. The answer of this question will be clear from the following topic.

1.5 WHEN TO STOP THE ITERATIVE PROCESS

If a number is correct to **n decimal place** then the error in number is $\frac{1}{2}10^{-n}$.

i.e.,

If x is correct to **one decimal place** then the allowed error $= \frac{1}{2}10^{-1} = 0.05$.

If x is correct to **two decimal place** then the allowed error $= \frac{1}{2}10^{-2} = 0.005$.

If x is correct to **three decimal place** then the allowed error $= \frac{1}{2}10^{-3} = 0.0005$.

If x is correct to **four decimal place** then the allowed error $= \frac{1}{2}10^{-4} = 0.00005$.

Now consider the following successive approximations by iterative method

First approximation, $x_1 = 2.53789$
Second approximation, $x_2 = 2.54645$
Third approximation, $x_3 = 2.56145$
Fourth approximation, $x_4 = 2.\mathbf{563}67$
Fifth approximation, $x_5 = 2.\mathbf{563}98$

From above approximation, consider the last two approximations x_4 and x_5. Now we compute the error as follows

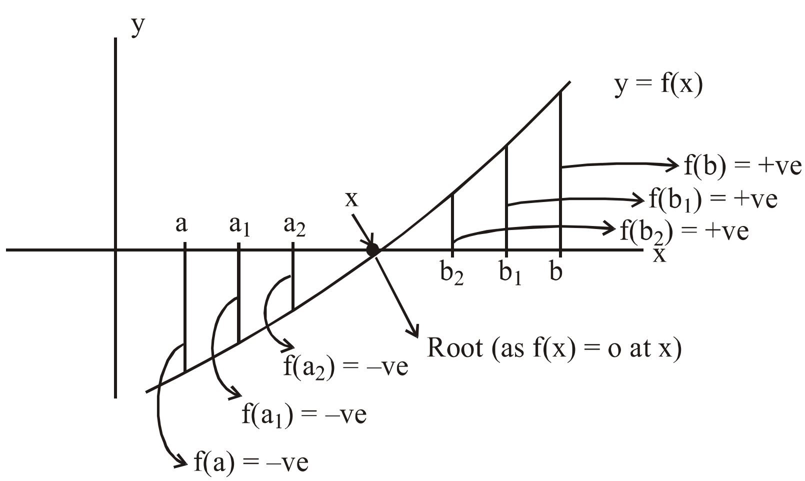
$$e = |x_5 - x_4| = |2.56398 - 2.56367| = 0.00031 \le 0.0005$$

Thus from above example it is clear that if the solution is required correct to three decimal place then the iterative process should be stopped because at this stage the allowed error in the solution is less than 0.0005. We can also observed from the above that when the three decimal place in successive approximations are same and the difference of fourth decimal digit is ≤ 5 then we can stop the iterative process because in such situation the allowed error in the solution will be always ≤ 0.0005. Similarly, for four decimal accuracy in the solution, the iterative process should stop when the four decimal place in two successive approximations are same and the difference between fifth decimal digit is ≤ 5 in such situation the allowed error in the solution will be always ≤ 0.00005.

In general, *when solution is required correct to n decimal place then we observe the successive approximations and when the two consecutive approximations are same up to n decimal place and the difference of $(n + 1)^{th}$ decimal digit is <5 then we stop the iterative process and we get the solution correct to n decimal place.*

Selection of Appropriate Initial Approximation (*i.e.* Interval)

Let us consider the equation $f(x) = 0$. The solution of this equation will be those values of x for which $f(x) = 0$. Now to start the iterative method we need an appropriate initial approximation enclosing the roots as shown in the following figure.



From the above figure it is clear that if at the point a and b, $f(a)$ and $f(b)$ be of opposite sign then it must enclose the root. Therefore the interval (a, b) will be appropriate. Instead of this interval (a, b) there may be so many other interval which encloses the same root *e.g.* (a_1, b_1), (a_2, b_2) etc. Since, the no of iteration directly depends on the interval, therefore we choose the small interval enclosing the roots to avoid the unneccessary iteration while solving the problem.

From the figure it is easy to choose the interval, but in practice we do not plot the graph and use trial and error method as given below.

Starting any values of x, we observe the values of $f(x)$. If on increasing the value of x, $f(x)$ is decreasing then a situation will come when the function will change the sign. Now the consecutive values of x for which the sign of $f(x)$ changes will be the required interval within which the root will lie. For example consider the following equation.

$$x^3 - 4x - 9 = 0$$

Here, $f(x) = x^3 - 4x - 9$

Now by trial and error, we can start the values of x from 0 onwards and observe the sign of the function as follows.

For, $x = 0$, $f(0) = -9$

$x = 1$, $f(1) = -12$

$x = 2$, $f(2) = -9$

$x = 3$, $f(3) = +6$

Thus, for $x = 2$ and $x = 3$ the sign of the function are of opposite sign. Therefore the required interval for this equation will be $a = 2$ and $b = 3$, which will enclose the roots.

Now from the first guess about the interval we can search another small interval to avoid an neccessary iteration. For this we take different values within 2 and 3 as follows.

$x = 2.5$, $f(2.5) = -3.375$

$x = 2.6$, $f(2.6) = -1.824$

$x = 2.7$, $f(2.7) = -0.117$

$x = 2.8$, $f(2.8) = +1.752$

Thus, from the above it is clear that for $x = 2.7$ and 2.8 the function has opposite sign. Therefore the root must lie in between 2.7 and 2.8. Hence, a = 2.7 and b = 2.8 will also be the appropriate initial approximation. The difference between the interval (2, 3) and (2.7, 2.8) is this that in case of first interval (2, 3) the number of iteration will be large but in case of second interval (2.7, 2.8) it will be less. So, the interval (2.7, 2.8) will be better than previous interval and consume less time to compute the root.

Again, we can choose another small interval. For this we take diffferent values within 2.7 and 2.8. Since $f(2.7) = -0.117$ which is small and indicates that the root is near 2.7. Therefore, For $x = 2.7, \quad f(2.7) = -0.117$

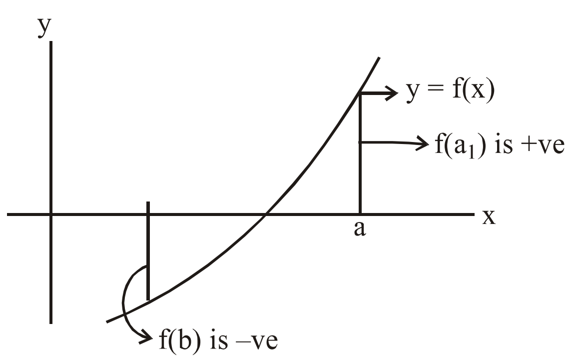
$x = 2.71, \quad f(2.71) = +0.062511$

Now, $a = 2.70$, and $b = 2.71$ will be the best initial approximation covering the roots.

The purpose of above discussion is this that when the interval will be small then the number of iteration will be small that save the time and effort in finding the roots of equation. There are several method some of them are slow and some one is fast. Therefore the appropriate selection plays an important role to save our effort. Here we have discussed some interval (2, 3), (2.7, 2.8) and (2.70, 2.71). All these initial approximations are correct and give the same solution upto desired level of accuracy but they differ in their number of iteration.

1.6 BISECTION METHOD (BOLZANO METHOD)

Bisection method is an iterative method which is used to find out the root of an algebraic and transcedental equation $f(x) = 0$. In this method first we find an interval within which the root lies and for assurance of this the sign of the function must be of oppostie sign *i.e.* if $f(a)$ is +ve then $f(b)$ is –ve then we say that (a, b) is the correct interval that enclose the root of equation as shown in following figure.



Thus, from above figure it is clear that when $f(a)$ and $f(b)$ will have opposite sign then the root must be enclosed by a and b.

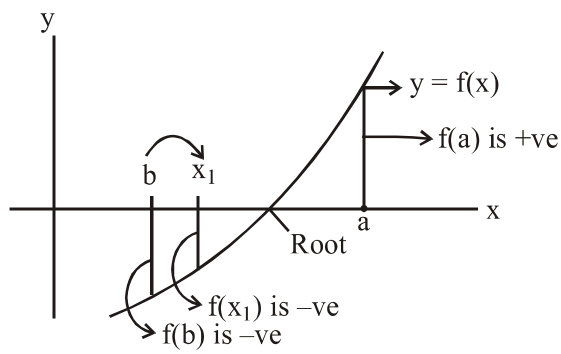
Now, we find the new approximation to root as follows,

$$x_1 = \frac{a+b}{2}, \text{ and then find } f(x_1)$$

Now, three case may arise, which are as follows.

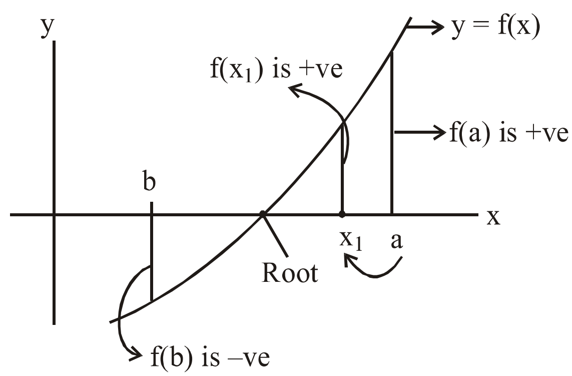
Case I: When $f(x_1) = 0$ then x_1 is the root and the iterative process will stop. The possibility of this case is very few and normally does not occure.

Case II: When $f(x_1) \neq 0$ *i.e.* $f(x_1) < 0$ *i.e.* $f(x_1)$ is $-ve$ as shown in following figure.



From the above figure it is clear that the new interval enclosing the root will be x_1 and a *i.e.*, b has to be changed by x_1, *i.e.* when $f(x_1)$ and $f(b)$ have the same sign. Thus the new interval will be (a, $b = x_1$,).

Case III: When $f(x_1) > 0$ *i.e.* $f(x_1) = +ve$ as shown in following figure.



From the above figure it is clear that the new interval enclosing the root will be x_1 and b *i.e.* **'a' has to be changed by 'x_1'** *i.e.* when $f(x_1)$ and $f(a)$ have the same sign. Thus the new interval will be $(a = x_1, b)$

Thus, from the above it is clear that

- If $f(x_1)$ and $f(a)$ have same sign then **a** will be changed by x_1 *i.e.* $a = x_1$ and ***b* will remain as it is.**
- If $f(x_1)$ and $f(b)$ have same sign then ***b*** will be changed by x_1 *i.e.* $b = x_1$ and ***a* will remain as it is.**

Now with this new interval (*a*, *b*) we again find the second approximation to the root $x_2 = \frac{a+b}{2}$. Again either *a* or *b* will be changed by x_2 depending upon the sign of $f(x_2)$. In this way we find the successive approximations to the root x_3, x_4, ...so on. Now the iteration process will stop when two consecutive approximations are same **upto *n* decimal for *n* decimal accuracy.** *e.g.* ***for three decimal accuracy any two consecutive approximation will be same upto three decimal place and the difference of fourth decimal digit will be ≤5.***

Now the question arise, how many decimal digit should be retained during the intermediate calculations. The answer of this question depends on the accuracy required in the solution. If it is required to find root correct to four decimal place then we retain one or two more digit *i.e.* five or six decimal digit either by round off or by discarded throughout the calculation.

Since the Bisection method is slower than the other method. Therefore, appropriate interval should be choosen to avoid unneccessary calculation.

EXAMPLE 1.1: Find a real root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to four decimal places.

SOLUTION: Here $f(x) = x^3 - 4x - 9$

(We have discussed about the selection of appropriate interval for this equation under the topic 'selection of appropriate initial approximation').

∵ $f(2.70) = -0.117$ and $f(2.71) = +0.062511$ which are of opposite sign.

∴ The root lies between 2.70 and 2.71 *i.e.* $a = 2.70$ and $b = 2.71$.

∴ First approximation to the root is

$$x_1 = \frac{a+b}{2} = \frac{2.70 + 2.71}{2} = 2.705$$

i.e. $x_1 = 2.705$ and $f(2.705) = -0.02745$ i.e. –ve

Now either *a* or *b* will change

∵ $f(a) = f(2.7)$ is – ve and $f(x_1) = f(2.705)$ is – ve

⇒ *a* will be changed by x_1 *i.e.* $a = 2.705$ and *b* will remain as it is.

∴ The new interval become ($a = 2.705$, $b = 2.71$)

Now **second approximation** to the root is given by

$$x_2 = \frac{a+b}{2} = \frac{2.705 + 2.71}{2} = 2.7075$$

i.e. $x_2 = 2.7075$ and $f(x_2) = f(2.7075) = 0.01748$ *i.e.* + ve

Since $f(b) = f(2.71)$ is – ve and $f(x_2)$ is + ve

$\Rightarrow$ ***b*** will be changed by x_2 *i.e.* $b = 2.7075$ and **a** will be unchanged.

$\therefore$ The new interval become (a = 2.705, b = 2.7075)

Third approximation to the root is given by

$$x_3 = \frac{a+b}{2} = \frac{2.705 + 2.7075}{2} = 2.70625$$

i.e. $x_3 = 2.70625$ and $f(x_3) = f(2.70625)$ is – ve

Since $f(a) = f(2.705)$ is – ve and $f(x_3) = f(2.70625)$ is – ve

$\Rightarrow$ ***a*** will be changed by x_3 *i.e.* $a = 2.70625$ and ***b*** will be unchanged.

$\therefore$ The new interval become (a = 2.70625, b = 2.7075).

Fourth approximation to root is given by

$$x_4 = \frac{a+b}{2} = \frac{2.70625 + 2.7075}{2} = 2.706875$$

i.e. $x_4 = 2.70687$ and $f(x_4) = f(2.70687)$ is + *ve*

$\because$ $f(b) = f(2.7075)$ is + ve and $f(x_4)$ is + ve

$\Rightarrow$ ***b*** will be changed by x_4 *i.e.* $b = 2.70687$ and ***a*** will be unchanged.

$\therefore$ The new interval become (a = 2.70625, b = 2.70687).

Fifth approximation to root is given by

$$x_5 = \frac{a+b}{2} = \frac{2.70625 + 2.70687}{2} = 2.70656$$

i.e. $x_5 = 2.70656$ and $f(x_5) = f(2.70656)$ is + ve

$\because$ $f(b) = f(2.70687)$ is + ve and $f(x_5)$ is + ve

$\Rightarrow$ Again ***b*** will be changed by x_5 *i.e.* $b = 2.70656$ and ***a*** will be unchanged.

$\therefore$ The new interval become (a = 2.70625, b = 2.70656).

Sixth approximation to root is given by

$$x_6 = \frac{a+b}{2} = \frac{2.70625 + 2.70656}{2} = 2.70656$$

i.e. $x_6 = 2.70656$

Since, in the last two approximations x_6 and x_5, there is no change upto four decimal place (even upto five decimal place). Therefore, the iteration process will stop.

Therefore, the root of given equation correct of four decimal place is ***x* = 2.7065** (after discarding fifth decimal digit).

Note: Since the bisection method is very slow therefore we should choose very small interval enclosing the root otherwise the number of iteration will be large. For the above problem the other intervals are (2, 3), (2.7, 2.8). If we choose first interval *i.e.* (2, 3) then the number of iteration (*i.e.* approximation to the root) will be large as compared to the interval (2.7, 2.8). Similarly if we choose second interval *i.e.* (2.7, 2.8) then the number of iteration will be greater than the interval (2.70, 2.71). To observe this fact students are advised to solve the above problem by taking these two intervals separately and the solution will be same correct to four decimal place yielding larger number of iteration.

EXAMPLE 1.2: Find a real root of the equation $x\log_{10}x = 1.2$ using bisection method correct to four decimal places.

SOLUTION

Given that $x\log_{10}x = 1.2$ *i.e.* $x\log_{10}x - 1.2 = 0$

Here, $f(x) = x\log_{10}x - 1.2$

∵ $f(2.74)$ is –ve and $f(2.75)$ is +ve which are of opposite sign.

∴ The root lies between 2.74 and 2.75 *i.e.* $a = 2.74$ and $b = 2.75$.

∴ The **First approximation** to the root using bisection method is given by

$$x_1 = \frac{a+b}{2} = \frac{2.74+2.75}{2} = 2.745$$

i.e. $x_1 = 2.745$ and $f(x_1) = f(2.745)$ is $+$ve

∵ $f(b) = f(2.75)$ is $+$ve and $f(x_1) = f(2.745)$ is $+$ve

⇒ ***b*** will be changed by $\boldsymbol{x_1}$ *i.e.* $b = 2.745$ and ***a*** will be unchanged.

∴ The new interval enclosing the root become ($a = 2.74$, $b = 2.745$)

∴ The **second approximation** to the root is given by

$$x_2 = \frac{a+b}{2} = \frac{2.74+2.745}{2} = 2.7425$$

i.e. $x_2 = 2.7425$ and $f(x_2) = f(2.7425)$ is $+$ve

∵ $f(b) = f(2.745)$ is $+$ve and $f(x_2)$ is $+$ve

⇒ ***b*** will be again changed by $\boldsymbol{x_2}$ *i.e.* $b = 2.7425$ and **a** will be unchanged.

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