NUMERICAL METHODS

VI SEMESTER

CORE COURSE

B Sc MATHEMATICS

(2011 Admission)

UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

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UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

STUDY MATERIAL

Core Course

B Sc Mathematics

VI Semester

NUMERICAL METHODS

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Layout: *Computer Section, SDE*

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B.Sc. DEGREE PROGRAMME

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M M 6 B 1 1 **: NUMERICAL METHODS**

4 credits 30 weightage

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S.S. Sastry : Introductory Methods of Numerical Analysis, Fourth Edition, PHI.

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References

1. S. Sankara Rao : Numerical Methods of Scientists and Engineer, 3rd ed., PHI.

2. F.B. Hidebrand : Introduction to Numerical Analysis, TMH.

3. J.B. Scarborough : Numerical Mathematical Analysis, Oxford and IBH.

1

FIXED POINT ITERATION METHOD

Nature of numerical problems

I`]gZ_X ^ReYV^ReZTR] VbfReZ`_d Zd R_ Z^a`ceR_e cVbfZcV^V_e W`c gRcZ`fd ScR_TYVd `W dTZV_TV& JYV WZV]U `W _f^VcZTR] R_R]jdZd Via]`cVd eYV eVTY_ZbfVd eYRe XZgV Raac`iZ^ReV d`]feZ`_d e` dfTY ac`S]V^d hZeY eYV UVdZcVU RTTfcRTj&

Computer based solutions

JYV ^R[`c deVad Z_g`]gVU e` d`]gV R XZgV_ ac`S]V^ fdZ_X R T`^afeVc RcV2

1. D`UV]Z_X2 IVeeZ_X fa R ^ReYV^ReZTR] ^`UV]$ ZǷV&$ W`c^f]ReZ_X eYV ac`S]V^ Z_ ^ReYV^ReZTR] eVc^d$ eR\Z_X Z_e` RTT`f_e eYV ejaV `W T`^afeVc `_V hR_ed e` fdV&
2. :Y``dZ_X R_ Raac`acZReV _f^VcZTR] ^VeY`U R]X`cZeY^! e`XVeYVc hZeY R acV]Z^Z_Rcj Vcc`c R_R]jdZd VdeZ^ReZ`_ `W Vcc`c$ UVeVc^Z_ReZ`_ `W deVad$ dZkV VeT&!
3. Gc`XcR^^Z_X$ fdfR]]j deRceZ_X hZeY R W]`hTYRce dY`hZ_X R S]`T\ UZRXcR^ `W eYV ac`TVUfcVd e` SV aVcW`c^VU Sj eYV T`^afeVc R_U eYV_ hcZeZ_X$ dRj$ R :## ac`XcR^&
4. FaVcReZ`_ `c T`^afeVc ViVTfeZ`_&
5. @_eVcacVeReZ`_ `W cVdf]ed$ hYZTY ^Rj Z_T]fUV UVTZdZ`_d e` cVcf_ ZW WfceYVc UReR RcV _VVUVU&

Errors

Ef^VcZTR]]j T`^afeVU d`]feZ`_d RcV dfS[VTe e` TVceRZ_ Vcc`cd& DRZ_]j eYVcV RcV eYcVV ejaVd `W Vcc`cd& JYVj RcV Z_YVcV_e Vcc`cd$ ecf_TReZ`_ Vcc`cd R_U Vcc`cd UfV e` c`f_UZ_X&

1. ***Inherent errors or experimental errors*** RcZdV UfV e` eYV Rddf^aeZ`_d ^RUV Z_ eYV ^ReYV^ReZTR] ^`UV]Z_X `W ac`S]V^& @e TR_ R]d` RcZdV hYV_ eYV UReR Zd `SeRZ_VU Wc`^ TVceRZ_ aYjdZTR] ^VRdfcV^V_ed `W eYV aRcR^VeVcd `W eYV ac`S]V^& ZǷV&$ Vcc`cd RcZdZ_X Wc`^ ^VRdfcV^V_ed&
2. ***Truncation errors*** RcV eY`dV Vcc`cd T`ccVda`_UZ_X e` eYV WRTe eYRe R WZ_ZeV `c Z_WZ_ZeV! dVbfV_TV `W T`^afeReZ`_R] deVad _VTVddRcj e` ac`UfTV R_ ViRTe cVdf]e Zd recf_TReVUs acV^ReFcV]j RWeVc R TVceRZ_ _f^SVc `W deVad&
3. ***Round of errors*** RcV Vcc`cd RcZdZ_X Wc`^ eYV ac`TVdd `W c`f_UZ_X `WW UfcZ_X T`^afeReZ`_& JYVdV RcV R]d` TR]]VU @ELMMFKD" ZǷV& UZdTRcUZ_X R]] UVTZ^R]d Wc`^ d`^V UVTZ^R]d `_&

Error in Numerical Computation

Due to errors that we have just discussed, it can be seen that our numerical result is an approximate value of the (sometimes unknown) exact result, except for the rare case where the exact answer is sufficiently simple rational number.

If $\tilde{a}$ is an approximate value of a quantity whose exact value is a, then the difference $\varepsilon = \tilde{a} - a$ is called the absolute error of $\tilde{a}$ or, briefly, the error of $\tilde{a}$. Hence, $\tilde{a} = a + \varepsilon$, i.e.

Approximate value = True value + Error.

For example, if $\tilde{a} = 10.52$ is an approximation to $a = 10.5$, then the error is $\varepsilon = 0.02$. The relative error, ε_r, of $\tilde{a}$ is defined by

$$|\varepsilon_r| = \frac{|\varepsilon|}{|a|} = \frac{|\text{Error}|}{|\text{Truevalue}|}$$

For example, consider the value of $\sqrt{2}\,(=1.414213...)$ up to four decimal places, then

$$\sqrt{2} = 1.4142 + \text{Error}.$$

$$|\text{Error}| = |1.4142 - 1.41421| = .00001,$$

taking 1.41421 as true or exact value. Hence, the relative error is

$$\varepsilon_r = \frac{0.00001}{1.4142}.$$

We note that

$\varepsilon_r \approx \dfrac{\varepsilon}{\tilde{a}}$ if $|\varepsilon|$ is much less than $|\tilde{a}|$.

We may also introduce the quantity $\gamma = a - \tilde{a} = -\varepsilon$ and call it the correction, thus, $a = \tilde{a} + \gamma$, i.e.

True value = Approximate value + Correction.

Error bound for $\tilde{a}$ is a number β such that $|\tilde{a} - a| \le \beta$, i.e., $|\varepsilon| \le \beta$.

Number representations

Integer representation

Floating point representation

D `de UZXZeR] T` ^ afeVcd YRgV eh` hRjd `W cVacVdV_eZ_X _f ^ SVcd$ TR]]VU **fixed point** R_U **floating point**& @_ R WZiVU a`Z_e djdeV^ eYV _f ^ SVcd RcV cVacVdV_eVU Sj R WZiVU _f ^ SVc `W UVTZ^R] a]RTVd V&X& .*&+-O$ (&()+$)&(((&

@_ R W]`ReZ_X a`Z_e djdeV^ eYV _f ^ SVcd RcV cVacVdV_eVU hZeY R WZiVU _f ^ SVc `W dZX_ZWZTR_e UZXZed$ W`c ViR^a]V

(&.*+0 ×)(+ (&)/), ×)(⁻)+ –(&*(((×)()

R]d` hcZeeV_ Rd (&.*+0 <(+ (&)/), < –)+ –(&*(((<()

`c ^`cV dZ^a]j (&.*+0 #(+ (&)/), –)+ –(&*(((#()

Significant digits

Significant digit `W R _f ^ SVc @ Zd R_j XZgV_ UZXZe `W @$ ViTVae a`ddZS]j W`c kVc`d e` eYV]VWe `W eYV WZcde _`_kVc` UZXZe eYRe dVcgV `_]j e` WZi eYV a`dZeZ`_ `W eYV UVTZ^R] a`Z_e& JYfd$ R_j `eYVc kVc` Zd R dZX_ZWZTR_e UZXZe `W @!& =`c ViR^a]V$ VRTY `W eYV _f ^ SVc)+.($)&+.($ (&()+.(YRd , dZX_ZWZTR_e UZXZed&

Round off rule to discard the k + 1th and all subsequent decimals

(*a*) ***Rounding down*** @W eYV _f ^ SVc Re H#)!eY UVTZ^R] e` SV UZdTRcUVU Zd ]Vdd eYR_ YR]W R f_Ze Z_ eYV HeY a]RTV$ ]VRgV eYV HeY UVTZ^R] f_TYR_XVU& =`c ViR^a]V$ c`f_UZ_X `W O&,+ e` ) UVTZ^R] XZgVd O&, R_U c`f_UZ_X `W .&*O) e` * UVTZ^R] a]RTVd XZgVd .&*O&

(*b*) ***Rounding up*** @W eYV _f ^ SVc Re H#)!eY UVTZ^R] e` SV UZdTRcUVU Zd XcVReVc eYR_ YR]W R f_Ze Z_ eYV HeY a]RTV$ RUU ) e` eYV HeY UVTZ^R]& =`c ViR^a]V$ c`f_UZ_X `W O&,O e`) UVTZ^R] XZgVd O&- R_U c`f_UZ_X `W .&*// e` * UVTZ^R]d XZgVd .&*O&

(*c*) @W Ze Zd ViRTe]j YR]W R f_Ze$ c`f_U `WW e` eYV _VRcVde VgV_ UVTZ^R]& =`c ViR^a]V$ c`f_UZ_X `WW O&,- R_U O&-- e` ) UVTZ^R] XZgVd O&, R_U O&. cVdaVTeZgV]j& H`f_UZ_X `WW .&*.- R_U .&*/- e` * UVTZ^R]d XZgVd .&*. R_U .&*O cVdaVTeZgV]j&

Example =Z_U eYV c``ed `W eYV W`]]`hZ_X VbfReZ`_d fdZ_X , dZX_ZWZTR_e WZXfcVd Z_ eYV TR]Tf]ReZ`_&

>! U* – ,U # * 5 (R_U ?! U* – ,(U # * 5 (&

9LIRQFLK

8 W`c ^ f]R W`c eYV c``ed U)$ U* `W R bfRUcReZT Vbf ReZ`_ >U* # ?U # @ 5 (Zd

F! $x_1 = \frac{1}{2a}(-b + \sqrt{b^2 - 4ac})$ R_U $x_2 = \frac{1}{2a}(-b - \sqrt{b^2 - 4ac})$ &

= f ceYVc ^ `cV$ dZ\_TV U)U* 5 @%>" R\_`eYVc W`c ^ f]R W`c eYVdV c``ed Zd

FF! $x_1 = \frac{1}{2a}(-b + \sqrt{b^2 - 4ac})$ $ R_U $x_2 = \frac{c}{ax_1}$

=`c eYV Vb f ReZ`_ Z_ >!$ W`c ^ f]R F! XZgVd$

U) 5 * # $\sqrt{2}$ 5 * #)&,) , 5 +&,) ,$

U* 5 * – $\sqrt{2}$ 5 * –)&,) , 5 (&-0.

R_U W`c ^ f]R FF! XZgVd$

U) 5 * # $\sqrt{2}$ 5 * #)&,) , 5 +&,) ,$

U* 5 *&(((' +&,) , 5 (&-0-0&

=`c eYV Vb f ReZ`_ Z_ ?!$ W`c ^ f]R F! XZgVd$

U) 5 *(# $\sqrt{398}$ 5 *(#)1&1- 5 +1&1-$

U* 5 *(– $\sqrt{398}$ 5 *(–)1&1- 5 (&(-

R_U W`c ^ f]R FF! XZgVd$

U) 5 *(# $\sqrt{398}$ 5 *(#)1&1- 5 +1&1-$

U* 5 *(&(((' +1&1- 5 (&(-((.&

Example : `\_gVce eYV UVTZ^R] \_f ^SVc hYZTY Zd Z\_ eYV SRdV )(! 0)&- e` Zed SZ_Rc j W`c^ `W SRdV *!&

9LIRQFLK E`eV eYRe 0)&-!)(50·)()#)·)((#-·)(%)

E`h 0)&- 5 .,#).#)#(&-5*· #*, #*(# *%)5)()((()&)!*&

			HV^RZ_UVc	Gc`UfTe	@_eVXVc aRce	
*	0)		(&- m *	)&(	)	↓
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Example : `_gVce eYV SZ_Rcj _f^SVc )()(&)() e` Zed UVTZ^R] W`c^&

9LIRQFLK

)()(&)()!* 5)· *+ #)· *) #)· *%) #)· *%+

5 0 # * # (&- # (&)*-5)(&.*-!)(

Numerical Iteration Method

8 **numerical iteration method** `c dZ^a]j **iteration method** Zd R ^ReYV^ReZTR] ac`TVUfcV eYRe XV_VcReVd R dVbfV_TV `W Z^ac`gZ_X Raac`iZ^ReV d`]feZ`_d W`c R T]Rdd `W ac`S]V^d& 8 daVTZWZT hRj `W Z^a]V^V_eReZ`_ `W R_ ZeVcReZ`_ ^VeY`U$ Z_T]fUZ_X eYV eVc^Z_ReZ`_ TcZeVcZR$ Zd TR]]VU R_ R]X`cZeY^ `W eYV ZeVcReZ`_ ^VeY`U& @_ eYV ac`S]V^d `W WZ_UZ_X eYV d`]feZ`_ `W R_ VbfReZ`_ R_ ZeVcReZ`_ ^VeY`U fdVd R_ Z_ZeZR] XfVdd e` XV_VcReV dfTTVddZgV Raac`iZ^ReZ`_d e` eYV d`]feZ`_&

IZ_TV eYV ZeVcReZ`_ ^VeY`Ud Z_g`]gV cVaVeZeZ`_ `W eYV dR^V ac`TVdd ^R_j eZ^Vd$ T`^afeVcd TR_ RTe hV]] W`c WZ_UZ_X d`]feZ`_d `W VbfReZ`_ _f^VcZTR]]j& I`^V `W eYV ZeVcReZ`_ ^VeY`Ud W`c WZ_UZ_X d`]feZ`_ `W VbfReZ`_d Z_g`]gVd)! 9ZdVTeZ`_ ^VeY`U$ *! DVeY`U `W WR]dV a`dZeZ`_ HVXf]R%WR]dZ DVeY`U!$ +! EVh e`_%HRaYd`_ ^VeY`U&

8 _f^VcZTR] ^VeY`U e` d`]gV VbfReZ`_d ^Rj SV R]`_X ac`TVdd Z_ d`^V TRdVd& @W eYV ^VeY`U]VRUd e` gR]fV T]`dV e` eYV ViRTe d`]feZ`_$ eYV_ hV dRj eYRe eYV ^VeY`U Zd T`_gVcXV_e& FeYVchZdV$ eYV ^VeY`U Zd dRZU e` SV UZgVcXV_e&

Solution of Algebraic and Transcendental Equations

F_V `W eYV ^`de T`^^`_ ac`S]V^ V_T`f_eVcVU Z_ V_XZ_VVcZ_X R_R]jdZd Zd eYRe XZgV_ R Wf_TeZ`_ W i!$ WZ_U eYV gR]fVd `W i W`c hYZTY W i! 5 (& JYV d`]feZ`_ gR]fVd `W i! RcV _`h_

Rd eYV **roots** `W eYV VbfReZ`_ W i! 5 ($ `c eYV **zeroes** `W eYV Wf_TeZ`_ W i!& JYV c``ed `W VbfReZ`_d ^Rj SV cVR] `c T`^a]Vi&

@_ XV_VcR]$ R_ VbfReZ`_ ^Rj YRgV R_j _f^SVc `W cVR]! c``ed$ `c _` c``ed Re R]]& =`c ViR^a]V$ dZ_ i o i 5 (YRd R dZ_X]V c``e$ _R^V]j$ i 5 ($ hYVcVRd eR_ i o i 5 ( YRd Z_WZ_ZeV _f^SVc `W c``ed i 5 ($ l ,&,1+$ l /&/*-$ t !&

Algebraic and Transcendental Equations

C U! - (Zd TR]]VU R_ **algebraic equation** ZW eYV T`ccVda`_UZ_X $f(x)$ Zd R a`]j_`^ZR]& 8_ ViR^a]V Zd /U* # U% 0 5 (& $f(x)=0$ Zd TR]]VU **transcendental equation** ZW eYV $f(x)$ T`_eRZ_d ecZX`_`^VecZT$ `c Via`_V_eZR] `c]`XRcZeY^ZT Wf_TeZ`_d& <iR^a]Vd `W ecR_dTV_UV_eR] VbfReZ`_d RcV dZ_ i o i 5 ($ $\tan x - x = 0$ R_U $7x^3 + \log(3x-6) + 3e^x \cos x + \tan x = 0.$

JYVcV RcV eh` ejaVd `W ^VeY`Ud RgRZ]RS]V e` WZ_U eYV c``ed `W R]XVScRZT R_U ecR_dTV_UV_eR] VbfReZ`_d `W eYV W`c^ W i! 5 (&

1. Direct Methods: ; ZcVTe ^VeY`Ud XZgV eYV ViRTe gR]fV `W eYV c``ed Z_ R WZ_ZeV _f^SVc `W deVad& MV Rddf^V YVcV eYRe eYVcV RcV _` c`f_U `WW Vcc`cd& ; ZcVTe ^VeY`Ud UVeVc^Z_V R]] eYV c``ed Re eYV dR^V eZ^V&

2. Indirect or Iterative Methods: @_UZcVTe `c ZeVcReZgV ^VeY`Ud RcV SRdVU `_ eYV T`_TVae `W dfTTVddZgV Raac`iZ^ReZ`_d& JYV XV_VcR] ac`TVUfcV Zd e` deRce hZeY `_V `c ^`cV Z_ZeZR] Raac`iZ^ReZ`_ e` eYV c``e R_U `SeRZ_ R dVbfV_TV `W ZeVcReVd $x_k$ hYZTY Z_ eYV ]Z^Ze T`_gVcXVd e` eYV RTefR] `c ecfV d`]feZ`_ e` eYV c``e& @_UZcVTe `c ZeVcReZgV ^VeY`Ud UVeVc^Z_V `_V `c eh` c``ed Re R eZ^V& JYV Z_UZcVTe `c ZeVcReZgV ^VeY`Ud RcV WfceYVc UZgZUVU Z_e` eh` TReVX`cZVd2 ScRT\VeZ_X R_U `aV_ ^VeY`Ud& JYV ScRT\VeZ_X ^VeY`Ud cVbfZcV eYV ]Z^Zed SVehVV_ hYZTY eYV c``e]ZVd$ hYVcVRd eYV `aV_ ^VeY`Ud cVbfZcV eYV Z_ZeZR] VdeZ^ReZ`_ `W eYV d`]feZ`_& 9ZdVTeZ`_ R_U =R]dV a`dZeZ`_ ^VeY`Ud RcV eh` \_`h_ ViR^a]Vd `W eYV ScRT\VeZ_X ^VeY`Ud& 8^`_X eYV `aV_ ^VeY`Ud$ eYV EVhe`_%HRaYd`_ Zd ^`de T`^^`_]j fdVU& JYV ^`de a`af]Rc ^VeY`U W`c d`]gZ_X R _`_%]Z_VRc VbfReZ`_ Zd eYV

EVhe`_%HRaYd`_ ^VeY`U R_U eYZd ^VeY`U YRd R YZXY cReV `W T`_gVcXV_TV e` R d`]feZ`_&

@_ eYZd TYRaeVc R_U Z_ eYV T`^Z_X TYRaeVcd$ hV acVdV_e eYV W`]]`hZ_X Z_UZcVTe `c ZeVcReZgV ^VeY`Ud hZeY Z]]fdecReZgV ViR^a]Vd2

)& =ZiVU G`Z_e @eVcReZ`_ DVeY`U

*& 9ZdVTeZ`_ DVeY`U

+& DVeY`U `W =R]dV G`dZeZ`_ HVXf]R =R]dZ DVeY`U!

,& EVhe`_%HRaYd`_ DVeY`U EVhe`_qd ^VeY`U!

Fixed Point Iteration Method

: `_dZUVc

$$f(x)=0 \qquad\qquad t \quad)!$$

JcR_dW`c^ )! e` eYV W`c^$

$$x=¤(x). \qquad\qquad t \quad *!$$

JR\V R_ RcSZecRcj U(R_U eYV_ T`^afeV R dVbfV_TV U)$ U*$ U+$ & & & cVTfcdZgV]j Wc`^ R cV]ReZ`_ `W eYV W`c^

$$x_{n+1}=\phi(x_n) \quad (n=0,1,\ldots) \qquad t \quad +!$$

8 **solution** `W *! Zd TR]]VU **fixed point** `W¤& J` R XZgV_ Vbf ReZ`_)! eYVcV ^Rj T`ccVda`_U dVgVcR] Vbf ReZ`_d *! R_U eYV SVYRgZ`fc$ VdaVTZR]]j$ Rd cVXRcUd daVVU `W T`_gVcXV_TV `W ZeVcReZgV dVbf V_TVd U($ U)$ U*$ U+$ & & & ^Rj UZWWVc RTT`cUZ_X]j&

Example l`]gV $f(x)=x^2-3x+1=0,$ Sj WZiVU a`Z_e ZeVcReZ`_ ^VeY`U&

9LIRQFLK

McZeV eYV XZgV_ Vbf ReZ`_ Rd

$x^2=3x-1$ `c $x=3-1/x$&

:Y``dV $¤(x)=3-\frac{1}{x}$$ JYV_ $¤'(x)=\frac{1}{x^2}$ and $|¤'(x)|<1$ `_ eYV Z_eVcgR] $(1, 2)$.

?V_TV eYV ZeVcReZ`_ ^VeY`U TR_ SV Raa]ZVU e` eYV <b& +!&

JYV ZeVcReZgV W`c^ f]R Zd XZgV_ Sj

$$x_{n+1}=3-\frac{1}{x_n} \qquad K - ($)$ *$ & & & !$$

leRceZ_X hZeY$ $x_0=1$$ hV `SeRZ_ eYV dVbf V_TV

U(5)&((($ U) 5*&((($ U* 5*&-(($ U+ 5 *&.(($ U, 5*&.)-$ & & &

Question : K_UVc hYRe Rddf ^aeZ`_d `_ ¤ R_U x_0, U`Vd 8]X`cZeY^) T`_gVcXV 7 MYV_ U`Vd eYV dVbf V_TV (x_n) `SeRZ_VU Wc`^ eYV ZeVcReZgV ac`TVdd +! T`_gVcXV 7

MV R_dhVc eYZd Z_ eYV W`]]`hZ_X eYV`cV^$ eYRe Zd R df WWZTZV_e T`_UZeZ`_ W`c T`_gVcXV_TV `W ZeVcReZ`_ ac`TVdd

Theorem CVe $x = ¶$ SV R c``e `W $f(x) = 0$ R_U ]Ve 2 SV R_ Z_eVcgR] T`_eRZ_Z_X eYV a`Z_e $x = ¶$. CVe $¤(x)$ SV T`_eZ_f`fd Z_ 2" hYVcV $¤(x)$ Zd UVWZ_VU Sj eYV VbfReZ`_ $x = ¤(x)$ hYZTY Zd VbfZgR]V_e e` $f(x) = 0$. JYV_ ZW $|¤'(x)| < 1$ W`c R]] U Z_ 2" eYV dVbfV_TV `W Raac`iZ^ReZ`_d $x_0, x_1, x_2, \cdots, x_n$ UVWZ_VU Sj

$$x_{n+1} = \phi(x_n) \quad (n = 0, 1, \ldots)$$

T`_gVcXVd e` eYV c``e ¶, ac`gZUVU eYRe eYV Z_ZeZR] Raac`iZ^ReZ`_ $x_0$ Zd TY`dV_ Z_ 2&

Example =Z_U R cVR] c``e `W eYV VbfReZ`_ $x^3 + x^2 - 1 = 0$ `_ eYV Z_eVcgR] $[0, 1]$ hZeY R_ RTTfcRTj `W 10^{-4}.

J` WZ_U eYZd c``e$ hV cVhcZeV eYV XZgV_ VbfReZ`_ Z_ eYV W`c^

$$x = \frac{1}{\sqrt{x+1}}$$

JR\V

$$¤(x) = \frac{1}{\sqrt{x+1}}. \text{ JYV\_ } ¤(x) = -\frac{1}{2}\frac{1}{(x+1)^{\frac{3}{2}}}$$

$$\max_{[0,1]} |¤'(x)| = \left|\frac{1}{2\sqrt{8}}\right| = k = 0.17678 < 0.2.$$

: Y``dV $¤(x) = 3 - \frac{1}{x}$$ JYV_ $¤'(x) = \frac{1}{x^2}$ and $|¤'(x)| < 1$ `_ eYV Z_eVcgR] $(1, 2)$.

?V_TV eYV ZeVcReZ`_ ^VeY`U XZgVd2

n	x_n	$\sqrt{x_n + 1}$	$x_{n+1} = 1/\sqrt{x_n + 1}$
0	0.75	1.3228756	0.7559289
1	0.7559289	1.3251146	0.7546517
2	0.7546617	1.3246326	0.7549263

8e eYZd deRXV$

$$|x_{n+1} - x_n| = 0.7549263 - 0.7546517 = 0.0002746,$$

hYZTY Zd]Vdd eYR_ (&(((,& JYV ZeVcReZ`_ Zd eYVcVW`cV eVc^Z_ReVU R_U eYV c``e e` eYV cVbfZcVU RTTfcRTj Zd (&/-,1&

Example KdV eYV ^VeY`U `W ZeVcReZ`_ e` WZ_U R a`dZeZgV c``e$ SVehVV_ (R_U)$ `W eYV VbfReZ`_ $xe^x = 1$.

McZeZ_X eYV VbfReZ`_ Z_ eYV W`c^

$$x = e^{-x}$$

MV WZ_U eYRe $¤(x) = e^{-x}$ R_U d` $¤'(x) = -e^{-x}$ &

?V_TV $|¤'(x)| < 1$ W`c $x < 1$, hYZTY Rdd f cVd eYRe eYV ZeVcReZgV ac`TVdd UVWZ_VU Sj eYV Vb f ReZ`_ $x_{n+1} = ¤(x_n)$ hZ]] SV T`_gVcXV_e$ hYV_ $x < 1$.

JYV ZeVcReZgV W`c^ f]R Zd

$$x_{n+1} = \frac{1}{e^{x_n}} \qquad (n = 0, 1, \ldots)$$

IeRceZ_X hZeY $x_0 = 1$, hV WZ_U eYRe eYV d f TTVddZgV ZeVcReVd RcV XZgV_ Sj

$$x_1 = 1/e = 0.3678794, \quad x_2 = \frac{1}{ex_1} = 0.6922006,$$

$$x_3 = 0.5004735, \qquad x_4 = 0.6062435,$$

$$x_5 = 0.5453957, \qquad x_6 = 0.5796123,$$

MV RTTVae .&-,-+1-/ Rd R_ Raac`iZ^ReV c``e&

Example =Z_U eYV c``e `W eYV Vb f ReZ`_ $2x = \cos x + 3$ T`ccVTe e` eYcVV UVTZ^R] a]RTVd&

MV cVhcZeV eYV Vb f ReZ`_ Z_ eYV W`c^

$$x = \frac{1}{2}(\cos x + 3)$$

d` eYRe

$$¤ = \frac{1}{2}(\cos x + 3),$$

R_U

$$|¤'(x)| = \left|\frac{\sin x}{2}\right| < 1.$$

?V_TV eYV ZeVcReZ`_ ^VeY`U TR_ SV Raa]ZVU e` eYV Vb& +! R_U hV deRce hZeY $x_0 = ®/2$. JYV d f TTVddZgV ZeVcReVd RcV

$$x_1 = 1.5, \quad x_2 = 1.535, \quad x_3 = 1.518,$$
$$x_4 = 1.526, \quad x_5 = 1.522, \quad x_6 = 1.524,$$
$$x_7 = 1.523, \quad x_8 = 1.524.$$

MV RTTVae eYV d`] f eZ`_ Rd)&-*, T`ccVTe e` eYcVV UVTZ^R] a]RTVd&

Example =Z_U R d`] f eZ`_ `W $f(x) = x^3 + x - 1 = 0$, Sj WZiVU a`Z_e ZeVcReZ`_&

U+ # U o) 5 (TR_ SV hcZeeV_ Rd $x\left(x^2+1\right)=1$ $ `c $x=\frac{1}{x^2+1}$&

E `eV eYRe

$$|¤'(x)|=\frac{2|x|}{\left(1+x^2\right)^2}<1$$ W`c R_j cVR] U$

d` Sj eYV JYV`cV^ hV TR_ ViaVTe R d`]feZ`_ W`c R_j cVR] _f^SVc U( Rd eYV deRceZ_X a`Z_e&

: Y``dZ_X U(5)$ R_U f_UVcX`Z_X TR]Tf]ReZ`_d Z_ eYV ZeVcReZgV W`c^f]R

$$x_{n+1}=¤(x_n)=\frac{1}{1+x_n^2}$$ K 5 ($)$&&&!$ t ,!

hV XVe eYV dVbfV_TV

U(5)&((($ U) 5(&-(($ U* 5(&0(($ U+ 5(&.)($

U, 5 (&/*1$ U- 5(&.-+$ U. 5(&/()$ &&&

R_U hV TY``dV (&/() Rd R_ Raac`iZ^ReV! d`]feZ`_ e` eYV XZgV_ VbfReZ`_&

Example I`]gV eYV VbfReZ`_ $x^3=\sin x$. : `_dZUVcZ_X gRcZ`fd $¤(x)$, UZdTfdd eYV T`_gVcXV_TV `W eYV d`]feZ`_&

? `h U` eYV Wf_TeZ`_d hV T`_dZUVcVU W`c $¤(x)$ T`^aRcV7 JRS]V dY`hd eYV cVdf]ed `W dVgVcR]

ZeVcReZ`_d fdZ_X Z_ZeZR] gR]fV $x_0=1$ R_U W`fc UZffVcV_e Wf_TeZ`_d W`c $¤(x)$& ?VcV x_n Zd eYV gR]fV `W i

`_ eYV KeY ZeVcReZ`_ &

8_dhVc2

MYV_ $¤(x)=\sqrt[3]{\sin x}$, hV YRgV2

$x_1=$ (&1,,(01*,)*,+(.3 $x_2=0.93215560685805$

$x_3=0.92944074461587$ 3 $x_4=0.92881472066057$

MYV_ $¤(x)=\frac{\sin x}{x^2}$, hV YRgV2

$x_1=$ (&0,),/(10,0(/1(3 $x_2=1.05303224555943$

$x_3=0.78361086350974$ 3 $x_4=1.14949345383611$

HVWVccZ_X e` JYV`cV^$ hV TR_ dRj eYRe W`c $¤(x)=\frac{\sin x}{x^2}$, eYV ZeVcReZ`_ U`Vd_qe T`_gVcXV&

MYV_ $¤(x)=x+\sin x-x^3$, hV YRgV2

$x_1 =$ 0.841470984807903 $x_2 = 0.99127188988250$

$x_3 = 0.85395152069647$ 3 $x_4 = 0.98510419085185$

MYV_ $¤(x)=x-\frac{\sin x - x^3}{\cos x - 3x^2}$, hV YRgV2

$x_1 =$ (&1+--,1+1(.-,./3 $x_2 = 0.92989141894368$

$x_3 = 0.92886679103170$ 3 $x_4 = 0.92867234089417$

Example >ZgV R]] a`ddZS]V ecR_da`dZeZ`_d e` $x=¤(x)$, R_U d`]gV $f(x)=x^3+4x^2-10=0.$

G`ddZS]V JcR_da`dZeZ`_d e` $x=¤(x)$, RcV

$$x=¤_1(x)=x-x^3-4x^2+10,$$

$$x=¤_2(x)=\sqrt{\frac{10}{x}-4x},$$

$$x=¤_3(x)=\frac{1}{2}\sqrt{10-x^3}$$

$$x=¤_4(x)=\sqrt{\frac{10}{4+x}}$$

$$x=¤_5(x)=x-\frac{x^3+4x^2-10}{3x^2+8x}$$

=`c $x=¤_1(x)=x-x^3-4x^2+10$, _f^VcZTR] cVdf]ed RcV2

$x_0 = 1.5;$ $x_2 = -0.875$

$x_3 = 6.732;$ $x_4 = -469.7$ 3

?V_TV U`Vd_qe T`_gVcXV&

=`c $x=¤_2(x)=\sqrt{\frac{10}{x}-4x}$, _f^VcZTR] cVdf]ed RcV2

$x_0 = 1.5;$ $x_2 = 0.8165$

$x_3 = 2.9969;$ $x_4 = (-8.65)^{1/2}$ 3

=`c $x = ¤_3(x) = \frac{1}{2}\sqrt{10 - x^3}$, _f ^VcZTR] cVdf]ed RcV2

$x_0 = 1.5;$ $x_2 = 1.2869$
$x_3 = 1.4025;$ $x_4 = 1.3454$ 3

Exercises

I`]gV eYV W`]]` hZ_X Vbf ReZ`_d Sj ZeVcReZ`_ ^VeY`U2

- $\sin x = \frac{x+1}{x-1}$
- U, 5 U ! (&) -
- $3x - \cos x - 2 = 0$
- $x^3 - 5x + 3 = 0,$
- $x^3 + x + 1 = 0$
- $x = \frac{1}{6}\left(x^3 + 3\right)$
- $3x = 6 + \log_{10} x$
- $x = \frac{1}{5}\left(x^3 + 3\right)$
- $2x - \log_{10} x = 7$
- $x^3 = 2x^2 + 10x = 20$
- $2\sin x = x$
- $\cos x = 3x - 1$
- $x^3 + x^2 = 100$
- $3x + \sin x = e^x$

2

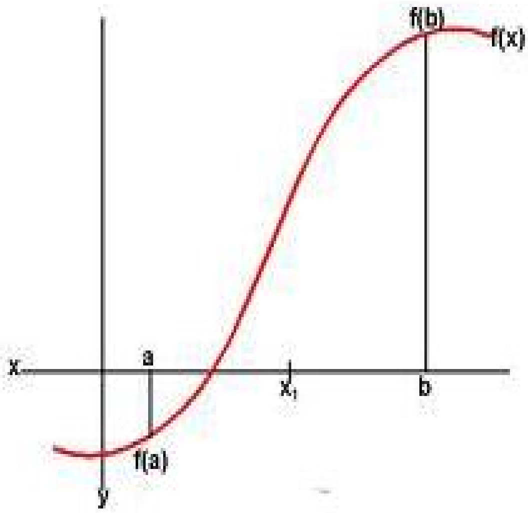
BISECTION AND REGULA FALSI METHODS

Bisection Method

JYV SZdVTeZ`_ ^VeY`U Zd `_V `W eYV ScRT\VeZ_X ^VeY`Ud W`c WZ_UZ_X c``ed `W R_ VbfReZ`_& =`c R XZgV_ R Wf_TeZ`_ C U" XfVdd R_ Z_eVcgR] hYZTY ^ZXYe T`_eRZ_ R c``e R_U aVcW`c^ R _f^SVc `W ZeVcReZ`_d$ hYVcV$Z_ VRTY ZeVcReZ`_ eYV Z_eVcgR] T`_eRZ_Z_X eYV c``e Zd XVe YR]gVU&

JYV **bisection method** Zd SRdVU `_ eYV Z_eVc^VUZReV gR]fV eYV`cV^ W`c T`_eZ_f`fd Wf_TeZ`_d&

Intermediate value theorem for continuous functions: @W f Zd R T`_eZ_f`fd Wf_TeZ`_ R_U $f(a)$ R_U $f(b)$ YRgV `aa`dZeV dZX_d$ eYV_ Re ]VRde `_V c``e ]ZVd Z_ SVehVV_ $a$ R_U $b.$ @W eYV Z_eVcgR] $(a, b)$ Zd d^R]] V_`fXY$ Ze Zd ]Z\V]j e` T`_eRZ_ R dZ_X]V c``e&



Z&V&$ R_ Z_eVcgR] P>" ?Q ^fde T`_eRZ_ R kVc` `W R T`_eZ_f`fd Wf_TeZ`_ C ZW eYV ac`UfTe $f(a)f(b)<0.$ >V`^VecZTR]]j$ eYZd ^VR_d eYRe ZW $f(a)f(b)<0,$ eYV_ eYV TfcgV f YRd e` Tc`dd eYV U%RiZd Re d`^V a`Z_e Z_ SVehVV_ > R_U ?&

Algorithm : Bisection Method

I faa`dV hV hR_e e` WZ_U eYV d`]feZ`_ e` eYV VbfReZ`_ $f(x)=0$ $ hYVcV C Zd T`_eZ_f`fd&

>ZgV_ R Wf_TeZ`_ $f(x)$ T`_eZ_f`fd `_ R_ Z_eVcgR] P>($?(Q R_U dReZdWjZ_X $f(a_0)f(b_0)<0.$

=`c K 5 ($ )$ *$ t f_eZ] eVc^Z_ReZ`_ U`2

: `^afeV $x_n=\frac{1}{2}(a_n+b_n)$ &

@W $f(x_n)=0$ " RTTVae UK Rd R d`]feZ`_ R_U de`a&

<]dV T`_eZ_fV&

@W $f(a_n)\, f(x_n) < 0$ $ R c``e]ZVd Z_ eYV Z_eVcgR] $(a_n,\, x_n)$ &

IVe $a_{n+1} = a_n,\, b_{n+1} = x_n$ &

@W $f(a_n)\, f(x_n) > 0,$ R c``e]ZVd Z_ eYV Z_eVcgR] $(x_n,\, b_n)$ &

IVe $a_{n+1} = x_n,\, b_{n+1} = b_n$ &

JYV_ $f(x) = 0$ W`c d`^V U Z_ $[a_{n+1},\, b_{n+1}]$ &

JVde W`c eVc^Z_ReZ`_&

Criterion for termination

A convenient criterion is to compute the percentage error $£_r$ defined by

$$£_r = \left| \frac{x_r' - x_r}{x_r'} \right| \times 100\%.$$

hYVcV x_r' Zd eYV _Vh gR]fV `W $x_r$ & JYV T`^afeReZ`_d TR_ SV eVc^Z_ReVU hYV_ $£_r$ SVT`^Vd]Vdd eYR_ R acVdTcZSVU e`]VcR_TV$ dRj $£_p$. @_ RUUZeZ`_$ eYV ^RiZ^f^ _f^SVc `W ZeVcReZ`_d ^Rj R]d` SV daVTZWZVU Z_ RUgR_TV&

I`^V `eYVc eVc^Z_ReZ`_ TcZeVcZR RcV Rd W`]]`hd2

- JVc^Z_ReZ`_ RWeVc 6 deVad 6 XZgV_$ WZiVU!
- JVc^Z_ReZ`_ ZW | UK#) – UK | ≤ ε ε 6 (XZgV_!
- JVc^Z_ReZ`_ ZW |C UK! | ≤ α α 6(XZgV_!&

@_ eYZd TYRaeVc `fc TcZeVcZ`_ W`c eVc^Z_ReZ`_ Zd eVc^Z_ReV eYV ZeVcReZ`_ ac`TVdd RWeVc d`^V WZ_ZeV deVad& ?`hVgVc$ hV _`eV eYRe eYZd Zd XV_VcR]]j _`e RUgZdRS]V$ Rd eYV deVad ^Rj _`e SV dfWWZTZV_e e` XVe R_ Raac`iZ^ReV d`]feZ`_&

Example I`]gV U+ o 1U!) 5 ( W`c eYV c``e SVehVV_ U 5 * R_U U 5 ,$ Sj SZdVTeZ`_ ^VeY`U&

>ZgV_ $f(x) = x^3 - 9x + 1$ & E`h $f(2) = -9,\; f(4) = 29$ d` eYRe $f(2)\,f(4) < 0$ R_U YV_TV R c``e]ZVd SVehVV_ * R_U ,&

IVe >(5 * R_U ?(5 ,& JYV_

$$x_0 = \frac{(a_0 + b_0)}{2} = \frac{2+4}{2} = 3 \quad \text{R\_U} \quad f(x_0) = f(3) = 1\&$$

lZ_TV $f(2)f(3) < 0$$ R c``e]ZVd SVe hVV_ * R_U +$ YV_TV hV dVe >) 5 >(5 * R_U $b_1 = x_0 = 3$& JYV_

$$x_1 = \frac{(a_1 + b_1)}{2} = \frac{2+3}{2} = 2.5 \quad \text{R\_U} \quad f(x_1) = f(2.5) = -5.875$$

lZ_TV $f(2)f(2.5) > 0,$ R c``e]ZVd SVe hVV_ *&- R_U +$ YV_TV hV dVe $a_2 = x_1 = 2.5$ R_U $b_2 = b_1 = 3$$

JYV_ $x_2 = \frac{(a_2 + b_2)}{2} = \frac{2.5+3}{2} = 2.75$ R_U $f(x_2) = f(2.75) = -2.9531.$

JYV deVad RcV Z]] fdecReVU Z_ eYV W`]]` hZ_X eRS]V&

K	x_n	$f(x_n)$
(	+	)&((((
)	*&-	– -&0/-
*	*&/-	– *&1-+)
+	*&0/-	–)&)))+
,	*&1+/-	– (&(1()

Example =Z_U R cVR] c``e `W eYV Vb fReZ`_ $f(x) = x^3 - x - 1 = 0.$

lZ_TV $f(1)$ Zd _VXReZgV R_U $f(2)$ a`dZeZgV$ R c``e]ZVd SVe hVV_) R_U * R_U eYVcVW`cV hV eR\V $x_0 = 3/2 = 1.5.$ JYV_

$f(x_0) = \frac{27}{8} - \frac{3}{2} = \frac{15}{8}$ Zd a`dZeZgV R_U YV_TV $f(1)\ f(1.5) < 0$ R_U ?V_TV eYV c``e]ZVd SVe hVV_) R_U)&- R_U hV `SeRZ_

$$x_1 = \frac{1+1.5}{2} = 1.25$$

$f(x_1) = -19/64,$ hYZTY Zd _VXReZgV R_U YV_TV $f(1)\ f(1.25) > 0$ R_U YV_TV R c``e]ZVd SVe hVV_)&*- R_U)&-& 8]d`$

$$x_2 = \frac{1.25+1.5}{2} = 1.375$$

The procedure is repeated and the successive approximations are

$$x_3 = 1.3125, \quad x_4 = 1.34375, \quad x_5 = 1.328125, \text{ etc.}$$

Example Find a positive root of the equation $xe^x = 1$, which lies between 0 and 1.

Let $f(x) = xe^x - 1$. Since $f(0) = -1$ and $f(1) = 1.718$, it follows that a root lies between 0 and 1. Thus,

$$x_0 = \frac{0+1}{2} = 0.5.$$

Since $f(0.5)$ is negative, it follows that a root lies between 0.5 and 1. Hence the new root is 0.75, i.e.,

$$x_1 = \frac{.5+1}{2} = 0.75.$$

Since $f(x_1)$ is positive, a root lies between 0.5 and 0.75. Hence

$$x_2 = \frac{.5+.75}{2} = 0.625$$

Since $f(x_2)$ is positive, a root lies between 0.5 and 0.625. Hence

$$x_3 = \frac{.5+.625}{2} = 0.5625.$$

We accept 0.5625 as an approximate root.

Merits of bisection method

a) The iteration using bisection method always produces a root, since the method brackets the root between two values.

b) As iterations are conducted, the length of the interval gets halved. So one can guarantee the convergence in case of the solution of the equation.

c) the Bisection Method is simple to program in a computer.

Demerits of bisection method

R! JYV T`_gVcXV_TV `W eYV SZdVTeZ`_ ^VeY`U Zd d]`h Rd Ze Zd dZ^a]j SRdVU `_ YR]gZ_X eYV Z_eVcgR]&

S! 9ZdVTeZ`_ ^VeY`U TR__`e SV Raa]ZVU `gVc R_ Z_eVcgR] hYVcV eYVcV Zd R UZdT`_eZ_fZej&

T! 9ZdVTeZ`_ ^VeY`U TR__`e SV Raa]ZVU `gVc R_ Z_eVcgR] hYVcV eYV Wf_TeZ`_ eR\Vd R]hRjd gR]fVd `W eYV dR^V dZX_&

U! JYV ^VeY`U WRZ]d e` UVeVc^Z_V T`^a]Vi c``ed&

V! @W `_V `W eYV Z_ZeZR] XfVddVd a_0 `c $b_0$ Zd T]`dVc e` eYV ViRTe d`]feZ`_$ Ze hZ]] eR\V ]RcXVc _f^SVc `W ZeVcReZ`_d e` cVRTY eYV c``e&

Exercises

=Z_U R cVR] c``e `W eYV W`]]`hZ_X Vbf ReZ`_d Sj SZdVTeZ`_ ^VeY`U&

)& $3x=\sqrt{1+\sin x}$ *& $x^3+1.2x^2-=4x+48$

+& $e^x=3x$,& $x^3-4x-9=0$

-& $x^3+3x-1=0$.& $3x=\cos x+1$

/& $x^3+x^2-1=0$ 0& $2x=3+\cos x$

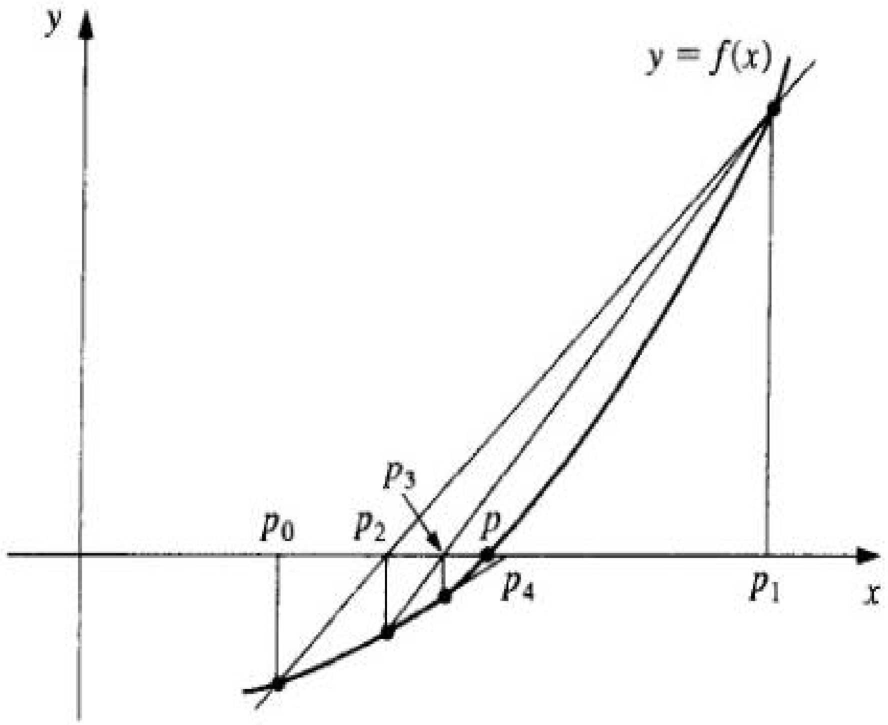
1& $x^4=3$)(& U+– -U 5 .

))& $\cos x=\sqrt{x}$)*& $x^3-x^2-x-3=0,$

)+& U, 5 U ! (&)- _VRc U 5 (&

Regula Falsi method or Method of False Position

JYZd ^VeY`U Zd R]d` SRdVU `_ eYV Z_eVc^VUZReV gR]fV eYV`cV^& @_ eYZd ^VeY`U R]d`$ Rd Z_ SZdVTeZ`_ ^VeY`U$ hV TY``dV eh` a`Z_ed >K R_U ?K dfTY eYRe $f(a_n)$ R_U $f(b_n)$ RcV `W `aa`dZeV dZX_d Z&V&$ $f(a_n)f(b_n)<0)$& JYV_$ Z_eVc^VUZReV gR]fV eYV`cV^ dfXXVded eYRe R kVc` `W C]ZVd Z_ SVehVV_ >K R_U ?K$ ZW C Zd R T`_eZ_f`fd Wf_TeZ`_&



Algorithm: >ZgV_ R Wf_TeZ`_ $f(x)$ T`_eZ_f`fd `_ R_ Z_eVcgR] P>(\$?(Q R_U dReZdWjZ_X $f(a_0)f(b_0)<0.$

=`c K 5 (\$ )\$ *\$ t f_eZ] eVc^Z_ReZ`_ U`2

: `^afeV

$$x_n = \frac{\begin{vmatrix} a_n & b_n \\ f(a_n) & f(b_n) \end{vmatrix}}{f(b_n) - f(a_n)}$$ &

@W $f(x_n)=0$ \$ RTTVae x_n Rd R d`]feZ`_ R_U de`a&

<]dV T`_eZ_fV&

@W $f(a_n)f(x_n)<0,$ dVe $a_{n+1}=a_n,\ b_{n+1}=x_n$ & <]dV dVe $a_{n+1}=x_n,\ b_{n+1}=b_n$ &

JYV_ $f(x)=0$ W`c d`^V U Z_ $[a_{n+1}, b_{n+1}]$ &

Example KdZ_X cVXf]R%WR]dZ ^VeY`U\$ WZ_U R cVR] c``e `W eYV Vbf ReZ`_\$

$f(x)=x^3+x-1=0,$ _VRc U 5)&

?VcV _`eV eYRe C (! 5 %) R_U $f(0)=-1$&   ?V_TV $f(0)f(1)<0$$ d` Sj Z_eVc^VUZReV gR]fV eYV`cV^ R c``e ]ZVd Z_ SVehVV_ ( R_U )& MV dVRcTY W`c eYRe c``e Sj cVXf]R WR]dZ ^VeY`U R_U hV hZ]] XVe R_ Raac`iZ^ReV c``e&

IVe >(5 (R_U ?(5)& JYV_

$$x_0=\frac{\begin{vmatrix} a_0 & b_0 \\ f(a_0) & f(b_0)\end{vmatrix}}{f(b_0)-f(a_0)}=\frac{\begin{vmatrix} 0 & 1 \\ -1 & 1\end{vmatrix}}{1-(-1)}=0.5$$

R_U $f(x_0)=f(0.5)=-0.375$&

IZ_TV $f(0)f(0.5)>0$$ R c``e]ZVd SVehVV_ (&- R_U)& IVe $a_1=x_0=0.5$ R_U $b_1=b_0=1$&

JYV_

$$x_1=\frac{\begin{vmatrix} a_1 & b_1 \\ f(a_1) & f(b_1)\end{vmatrix}}{f(b_1)-f(a_1)}=\frac{\begin{vmatrix} 0.5 & 1 \\ -0.375 & 1\end{vmatrix}}{1-(-0.375)}=0.6364$$

R_U $f(x_1)=f(0.6364)=-0.1058.$

IZ_TV $f(0.6364)f(x_1)>0$$ R c``e]ZVd SVehVV_ x_1 R_U) R_U YV_TV hV dVe $a_2=x_1=0.6364$ R_U $b_2=b_1=1.$ JYV_

$$x_2=\frac{\begin{vmatrix} a_2 & b_2 \\ f(a_2) & f(b_2)\end{vmatrix}}{f(b_2)-f(a_2)}=\frac{\begin{vmatrix} 0.6364 & 1 \\ -0.1058 & 1\end{vmatrix}}{1-(-0.1058)}=0.6712$$

R_U $f(x_2)=f(0.6712)=-0.0264$

IZ_TV $f(0.6712)f(0.6364)>0,$ R c``e]ZVd SVehVV_ x_2 R_U)$ R_U YV_TV hV dVe $a_3=x_2=0.6364$ R_U $b_3=b_1=1$&

JYV_ $$x_3=\frac{\begin{vmatrix} a_3 & b_3 \\ f(a_3) & f(b_3)\end{vmatrix}}{f(b_3)-f(a_3)}=\frac{\begin{vmatrix} 0.6712 & 1 \\ -0.0264 & 1\end{vmatrix}}{1-(-0.0264)}=0.6796$$

R_U $f(x_3)=f(0.6796)=-0.0063\approx 0$&

IZ_TV $f(0.6796)\approx 0.0000$ hV RTTVae (&./1. Rd R_ Raac`iZ^ReV! d`]feZ`_ `W $x^3-x-1=0$&

Example >ZgV_ eYRe eYV Vbf ReZ`_ $x^{2.2}=69$ YRd R c``e SVe hVV_ - R_U 0& KdV eYV ^VeY`U `W cVXf]R%WR]dZ e` UVeVc^Z_V Ze&

CVe $f(x)=x^{2.2}-69$. MV WZ_U

$$f(5)=-3450675846 \text{ R\_U } f(8)=-28.00586026.$$

$$x_1=\frac{\begin{vmatrix}5 & 8\\ f(5) & f(8)\end{vmatrix}}{f(8)-f(5)}=\frac{5(28.00586026)-8(-34.50675846)}{28.00586026+34.50675846)} \qquad = 6.655990062\,\&$$

E`h$ $f(x_1)=-4.275625415$ R_U eYVcVW`cV$ $f(5)\ f(x_1)>0$ R_U YV_TV eYV c``e]ZVd SVe hVV_ 6.655990062 R_U 0&(& Gc`TVVUZ_X dZ^Z]Rc]j$

$$x_2=6.83400179,\quad x_3=6.850669653,$$

JYV T`ccVTe c``e Zd $x_3=6.8523651\cdots$, d` eYRe $x_3$ Zd T`ccVTe e` eYVdV dZX_ZWZTR_e WZXfcVd& MV RTTVae $6.850669653$ Rd R_ Raac`iZ^ReV c``e&

Theoretical Exercises with Answers:

)& MYRe Zd eYV UZWWVcV_TV SVe hVV_ R]XVScRZT R_U ecR_dTV_UV_eR] Vbf ReZ`_d7

8_d2 8_ Vbf ReZ`_ $f(x)=0$ Zd TR]]VU R_ R]XVScRZT Vbf ReZ`_ ZW eYV T`ccVda`_UZ_X $f(x)$ Zd R a`]j_`^ZR]$ hYZ]V$ $f(x)=0$ Zd TR]]VU ecR_dTV_UV_eR] Vbf ReZ`_ ZW eYV $f(x)$ T`_eRZ_d ecZX`_`^VecZT$ `c Via`_V_eZR] `c ]`XRcZeY^ZT Wf_TeZ`_d&

*& MYj hV RcV fdZ_X _f^VcZTR] ZeVcReZgV ^VeY`Ud W`c d`]gZ_X Vbf ReZ`_d7

8_d2 8d R_R]jeZT d`]feZ`_d RcV `WeV_ VZeYVc e`` eZcVd`^V `c dZ^a]j U`_`e ViZde$ hV _VVU e` WZ_U R_ Raac`iZ^ReV ^VeY`U `W d`]feZ`_& JYZd Zd hYVcV _f^VcZTR] R_R]jdZd T`^Vd Z_e` eYV aZTefcV&

+& 9RdVU `_ hYZTY acZ_TZa]V$ eYV SZdVTeZ`_ R_U cVXf]R%WR]dZ ^VeY`U Zd UVgV]`aVU7

8_d2 JYVdV ^VeY`Ud RcV SRdVU `_ eYV FKQBOJBAF>QB S>IRB QEBLOBJ CLO @LKQFKRLRP CRK@QFLKP2 deReVU Rd $ r@W f Zd R T`_eZ_f`fd Wf_TeZ`_ R_U $f(a)$ R_U $f(b)$ YRgV `aa`dZeV dZX_d$ eYV_ Re ]VRde `_V c``e ]ZVd Z_ SVe hVV_ $a$ R_U $b$. @W eYV Z_eVcgR] $(a,b)$ Zd d^R]] V_`fXY$ Ze Zd ]Z\V]j e` T`_eRZ_ R dZ_X]V c``e& s

,& MYRe RcV eYV RUgR_eRXVd R_U UZdRUgR_eRXVd `W eYV ScRT\VeZ_X ^VeY`Ud]Z\V SZdVTeZ`_ R_U cVXf]R%WR]dZ7

Ans: i) The bisection and regula-falsi method is always convergent. Since the method brackets the root, the method is guaranteed to converge. The main disadvantage is, if it is not possible to bracket the roots, the methods cannot applicable. For example, if $f(x)$ is such that it always takes the values with same sign, say, always positive or always negative, then we cannot work with bisection method. Some examples of such functions are

- $f(x)=x^2$ which take only non-negative values and
- $f(x)=-x^2$, which take only non-positive values.

Exercises

Find a real root of the following equations by false position method:

1. $x^3-5x=6$	2. $4x=e^x$
3. $x\log_{10}x=1.2$	4. $\tan x+\tanh x=0$
5. $e^{-x}=\sin x$	6. $x^3-5x-7=0$
7. $x^3+2x^2+10x-20=0$	8. $2x-\log_{10}x=7$
9. $xe^x=\cos x$	10. $x^3-5x+1=0$
11. $e^x=3x$	12. $x^2-\log_e x=12$
13. $3x-\cos x=1$	14. $2x-3\sin x=5$
15. $2x=\cos x+3$	16. $xe^x=3$
17. $\cos x=\sqrt{x}$	18. $x^3-5x+3=0$

Ramanujan's Method

We need the following Theorem:

Binomial Theorem: If n is any rational number and $|x|<1$, then

$$(1+x)^n=1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\ .\ .\ .\ +\frac{n(n-1)\ .\ .\ .\ (n-(r-1))}{1\cdot 2\cdot\ .\ .\ .\ .\ \cdot r}x^r+\ .\ .\ .$$

@_ aRceZT f]Rc$

$$(1+x)^{-1}=1-x+x^2-x^3+\ . \ . \ . \ +(-1)^n x^n+\ . \ . \ .$$

R_U $$(1-x)^{-1}=1+x+x^2+x^3+\ . \ . \ . \ +x^n+\ . \ . \ .$$

@_UZR_ DReYV^ReZTZR_ IcZ_ZgRdR HR^R_f[R_)00/%)1*(! UVdTcZSVU R_ ZeVcReZgV ^VeY`U hYZTY TR_ SV fdVU e` UVeVc^Z_V eYV d^R]]Vde c``e `W eYV Vbf ReZ`_

$$f(x)=0,$$

hYVcV $f(x)$ Zd `W eYV W`c^

$$f(x)=1-(a_1x+a_2x^2+a_3x^2+a_4x^4+\cdots).$$

=`c d^R]]Vc gR] fVd `W U" hV TR_ hcZeV

$$[1-(a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots)]^{-1}=b_1+b_2x+b_3x^2+\cdots$$

< i aR_UZ_X eYV]VWe%YR_U dZUV fdZ_X SZ_`^ZR] eYV`cV^ $ hV `SeRZ_

$$1+(a_1x+a_2x^2+a_3x^3+\cdots)+(a_1x+a_2x^2+a_3x^3+\cdots)^2+\cdots$$

$$=b_1+b_2x+b_3x^2+\cdots$$

: `^aRcZ_X eYV T`VWWZTZV_ed `W ]Z\V a`hVcd `W U `_ S`eY dZUVd `W hV `SeRZ_

$$\left.\begin{aligned}&b_1=1,\\&b_2=a_1=a_1b_1,\\&b_3=a_1^2+a_2=a_1b_2+a_2b_1,\\&\vdots\\&b_n=a_1b_{n-1}+a_2b_{n-2}+\cdots+a_{n-1}b_1 \quad n=2,3,\cdots\end{aligned}\right\}$$

JYV_ b_n/b_{n+1} Raac`RTY R c``e `W eYV Vbf ReZ`_ $f(x)=0$ &

Example =Z_U eYV d^R]]Vde c``e `W eYV Vbf ReZ`_

$$f(x)=x^3-6x^2+11x-6=0.$$

9LIRQFLK

JYV XZgV_ Vbf ReZ`_ TR_ SV hcZeeV_ Rd $f(x)$

$$f(x)=1-\frac{1}{6}(11x-6x^2+x^3)$$

: ` ^ aRcZ_X$

$$a_1 = \frac{11}{6}, \quad a_2 = -1, \quad a_3 = \frac{1}{6}, \quad a_4 = a_5 = \cdots = 0$$

J` Raa]j HR^R_ f[R_qd ^VeY`U hV hcZeV

$$1 - \left(\frac{11x - 6x^2 + x^3}{6} \right)^{-1} = b_1 + b_2 x + b_3 x^2 + \cdots$$

?V_TV$

$$b_1 = 1;$$

$$b_2 = a_1 = \frac{11}{6};$$

$$b_3 = a_1 b_2 + a_2 b_1 = \frac{121}{36} - 1 = \frac{85}{36};$$

$$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = \frac{575}{216};$$

$$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 = \frac{3661}{1296};$$

$$b_6 = a_1 b_5 + a_2 b_4 + a_3 b_3 + a_4 b_2 + a_5 b_1 = \frac{22631}{7776};$$

JYVcVW`cV$

$$\frac{b_1}{b_2} = \frac{6}{11} = 0.54545\ 3 \qquad \frac{b_2}{b_3} = \frac{66}{85} = 0.7764705$$

$$\frac{b_3}{b_4} = \frac{102}{115} = 0.8869565\ 3 \qquad \frac{b_4}{b_5} = \frac{3450}{3661} = 0.9423654$$

$$\frac{b_5}{b_6} = \frac{3138}{3233} = 0.9706155$$

9j Z_daVTeZ`_$ R c``e `W eYV XZgV_ Vbf ReZ`_ Zd f_Ze j R_U Ze TR_ SV dVV_ eYRe eYV dfTTVddZgV T`_gVcXV_ed $\frac{b_n}{b_{n+1}}$ Raac`RTY eYZd c``e&

Example =Z_U R c``e `W eYV Vbf ReZ`_ $xe^x = 1$.

CVe $xe^x = 1$

HVTR]] $\qquad e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

?V_TV$

$$f(x) = 1 - \left(x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \cdots \right) = 0$$

$$a_1 = 1, \qquad a_2 = 1, \qquad a_3 = \frac{1}{2}, \qquad a_4 = \frac{1}{6}, \qquad a_5 = \frac{1}{24}, \cdots$$

MV eYV_ YRgV

$$b_1 = 1;$$

$$b_2 = a_2 = 1;$$

$$b_3 = a_1 b_2 + a_2 b_1 = 1 + 1 = 2;$$

$$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = 2 + 1 + \frac{1}{2} = \frac{7}{2};$$

$$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 = \frac{7}{2} + 2 + \frac{1}{2} + \frac{1}{6} = \frac{37}{6};$$

$$b_6 = a_1 b_5 + a_2 b_4 + a_3 b_3 + a_4 b_2 + a_5 b_1 = \frac{37}{6}; + \frac{7}{2} + 1 + \frac{1}{6} + \frac{1}{24} = \frac{261}{24};$$

JYVcVW`cV$

$$\frac{b_2}{b_3} = \frac{1}{2} = 0.5 \; 3 \qquad \frac{b_3}{b_4} = \frac{4}{7} = 0.5714 \; 3$$

$$\frac{b_4}{b_5} = \frac{21}{37} = 0.56756756 \; 3 \qquad \frac{b_5}{b_6} = \frac{148}{261} = 0.56704980 \; \&$$

Example KdZ_X HR^R_f[R_qd ^VeY`U$ WZ_U R cVR] c``e `W eYV Vbf ReZ`_

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \cdots = 0.$$

9LIRQFLK

CVe $\qquad f(x) = 1 - \left[x - \dfrac{x^2}{(2!)^2} + \dfrac{x^3}{(3!)^2} - \dfrac{x^4}{(4!)^2} + \cdots \right] = 0.$

?VcV

$$a_1 = 1, \qquad a_2 = -\frac{1}{(2!)^2}, \qquad a_3 = \frac{1}{(3!)^2}, \qquad a_4 = -\frac{1}{(4!)^2},$$

$$a_5 = \frac{1}{(5!)^2}, \quad a_6 = -\frac{1}{(6!)^2}, \cdots$$

McZeZ_X

$$\left\{1 - \left[x - \frac{x^2}{(2!)} + \frac{x^3}{(3!)^2} - \frac{x^4}{(4!)^2} + \cdots\right]\right\}^{-1} = b_1 + b_2 x + b_3 x^2 + \cdots \$$$

hV `SeRZ_

$$b_1 = 1,$$

$$b_2 = a_1 = 1,$$

$$b_3 = a_1 b_2 + a_2 b_1 = 1 - \frac{1}{(2!)^2} = \frac{3}{4};$$

$$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = \frac{3}{4} - \frac{1}{(2!)^2} + \frac{1}{(3!)^2} = \frac{3}{4} - \frac{1}{4} + \frac{1}{36} = \frac{19}{36},$$

$$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1$$

$$= \frac{19}{36} - \frac{1}{4} \times \frac{3}{4} + \frac{1}{36} \times 1 - \frac{1}{576} = \frac{211}{576}.$$

@e W`]]` hd

$$\frac{b_1}{b_2} = 1; \qquad \frac{b_2}{b_3} = \frac{4}{3} = 1.333\cdots;$$

$$\frac{b_3}{b_4} = \frac{3}{4} \times \frac{36}{19} = \frac{27}{19} = 1.4210\cdots, \qquad \frac{b_4}{b_5} = \frac{19}{36} \times \frac{576}{211} = 1.4408\cdots,$$

hYVcV eYV]Rde cVd f]e Zd T`ccVTe e` eYcVV dZX_ZWZTR_e WZX f cVd&

Example =Z_U R c``e `W eYV Vbf ReZ`_ $\sin x = 1 - x.$

KdZ_X eYV V i aR_dZ`_ `W $\sin x,$ eYV XZgV_ Vbf ReZ`_ ^R j SV hcZeeV_ Rd

$$f(x) = 1 - \left(x + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = 0.$$

?VcV

$$a_1 = 2, \quad a_2 = 0, \quad a_3 = \frac{1}{6}, \quad a_4 = 0,$$

$$a_5 = \frac{1}{120}, \quad a_6 = 0, \quad a_7 = -\frac{1}{5040}, \cdots$$

hV hcZeV

$$\left[1 - \left(2x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)\right]^{-1} = b_1 + b_2 x + b_3 x^2 + \cdots$$

MV eYV_ `SeRZ_

$$b_1 = 1;$$

$$b_2 = a_1 = 2;$$

$$b_3 = a_1 b_2 + a_2 b_1 = 4;$$

$$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = 8 - \frac{1}{6} = \frac{47}{6};$$

$$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 = \frac{46}{3};$$

$$b_6 = a_1 b_5 + a_2 b_4 + a_3 b_3 + a_4 b_2 + a_5 b_1 = \frac{3601}{120};$$

JYVcVW`cV$

$$\frac{b_1}{b_2} = \frac{1}{2}; \qquad \frac{b_2}{b_3} = \frac{1}{2};$$

$$\frac{b_3}{b_4} = \frac{24}{27} = 0.5106382 \qquad \frac{b_4}{b_5} = \frac{47}{92} = 0.5108695$$

$$\frac{b_5}{b_6} = \frac{1840}{3601} = 0.5109691$$&

JYV c``e$ T`ccVTe e` W` f c UVTZ^R] a]RTVd Zd (&-))(

Exercises

)& KdZ_X HR^R_f[R_qd ^VeY`U$ `SeRZ_ eYV WZcde%VZXYe T`_gVcXV_ed `W eYV Vbf ReZ`_

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \cdots = 0$$

*& KdZ_X HR^R_f[R_qd ^VeY`U$ WZ_U eYV cVR] c``e `W eYV Vbf ReZ`_ $x + x^3 = 1.$

3

NEWTON RAPHSON ETC..

JYV EVhe`_%HRaYd`_ ^VeY`U$ `c EVhe`_ DVeY`U$ Zd R a`hVcWf] eVTY_ZbfV W`c d`]gZ_X VbfReZ`_d _f^VcZTR]]j& CZ\V d` ^fTY `W eYV UZWWVcV_eZR] TR]Tf]fd$ Ze Zd SRdVU `_ eYV dZ^a]V ZUVR `W]Z_VRc Raac`iZ^ReZ`_&

Newton – Raphson Method

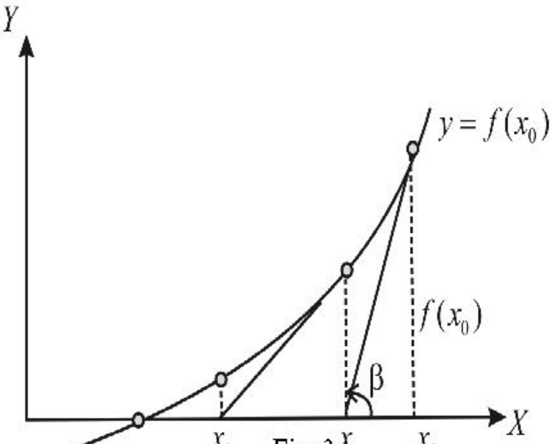


Fig. 3

: `_dZUVc $f(x)=0$$ hYVcV $f$ YRd T`_eZ_f`fd UVcZgReZgV $f'$& =c`^ eYV WZXfcV hV TR_ dRj eYRe Re $x=a,\ y=f(a)=0$3 hYZTY ^VR_d eYRe > Zd R d`]feZ`_ e` eYV VbfReZ`_ $f(x)=0$& @_ `cUVc e` WZ_U eYV gR]fV `W >$ hV deRce hZeY R_j RcSZecRcj a`Z_e U(& =c`^ WZXfcV hV TR_ dVV eYRe$ eYV eR_XV_e e` eYV TfcgV C Re $(x_0, f(x_0))$ hZeY d]`aV $f'(x_0)$! e`fTYVd eYV U%RiZd Re U)&

E`h$ $\tan \quad = f'(x_0) = \dfrac{f(x_0)-f(x_1)}{x_0-x_1}$$

8d $f(x_1)=0,$ eYV RS`gV dZ^a]ZWZVd e`

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

@_ eYV dVT`_U deVa$ hV T`^afeV

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}\$$$

Z_ eYV eYZcU deVa hV T`^afeV

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

R_U d` `_& D`cV XV_VcR]]j$ hV hcZeV $x_{n+1}$ Z_ eVc^d `W $x_n,\ f(x_n)$ R_U $f'(x_n)$ W`c $n=1, 2, \ldots$ Sj ^VR_d `W eYV **Newton-Raphson** W`c^f]R

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The refinement on the value of the root x_n is terminated by any of the following conditions.

(i) Termination after a pre-fixed number of steps

(ii) After *n* iterations where, $\left|x_{n+1} - x_n\right| \leq \varepsilon \left(\textit{for a given } \varepsilon > 0 \right)$, or

(iii) After *n* iterations, where $f(x_n) \leq \alpha \ \left(\textit{for a given } \alpha > 0 \right)$.

Termination after a fixed number of steps is not advisable, because a fine approximation cannot be ensured by a fixed number of steps.

Algorithm: The steps of the Newton-Raphson method to find the root of an equation $f(x) = 0$ are

1. Evaluate $f'(x)$
2. Use an initial guess of the root, x_i, to estimate the new value of the root, x_{i+1}, as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

3. Find the absolute relative approximate error $\left|\in_a\right|$ as

$$\left|\in_a\right| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

4. Compare the absolute relative approximate error with the pre-specified relative error tolerance, $\in_s$. If $\left|\in_a\right| > \in_s$ then go to Step 2, else stop the algorithm. Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

The method can be used for both algebraic and transcendental equations, and it also works when coefficients or roots are complex. It should be noted, however, that in the case of an algebraic equation with real coefficients, a complex root cannot be reached with a real starting value.

Example lVe fa R EVhe`_ ZeVcReZ`_ W`c T`^afeZ_X eYV dbfRcV c``e `W R XZgV_ a`dZeZgV _f^SVc& KdZ_X eYV dR^V WZ_U eYV dbfRcV c``e `W * ViRTe e` dZi UVTZ^R] a]RTVd&

CVe @ SV R XZgV_ a`dZeZgV _f^SVc R_U ]Ve U SV Zed a`dZeZgV dbfRcV c``e$ d` eYRe $x = \sqrt{c}$ & JYV_ $x^2 = c$ `c

$$f(x) = x^2 - c = 0$$

$$f'(x) = 2x$$

Using the Newton's iteration formula we have

$$x_{n+1} = x_n - \frac{x_n^2 - c}{2x_n}$$

or $$x_{n+1} = \frac{x_n}{2} + \frac{c}{2x_n}$$

or $$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right), \; n = 0.\, 1,\, 2, \cdots,$$

Now to find the square root of 2, **let** $c = 2$, so that

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), \; n = 0,\, 1,\, 2, \cdots$$

Choose $x_0 = 1$. Then

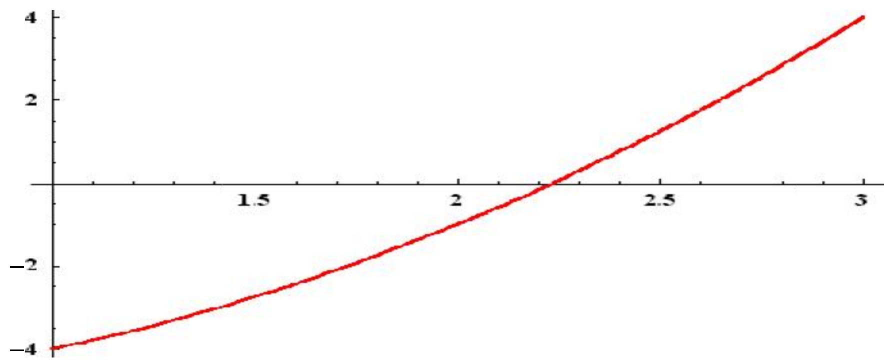
$x_1 = 1.500000$, $x_2 = 1.416667$, $x_3 = 1.414216$, $x_4 = 1.414214$, …

and accept 1.414214 as the square root of 2 exact to 6 D.

Historical Note: Heron of Alexandria (60 CE) used a pre-algebra version of the above recurrence. It is still at the heart of computer algorithms for finding square roots.

Example. Let us find an approximation to $\sqrt{5}$ to ten decimal places.

Note that $\sqrt{5}$ is an irrational number. Therefore the sequence of decimals which defines $\sqrt{5}$ will not stop. Clearly $\sqrt{5}$ is the only zero of $f(x) = x^2 - 5$ on the interval $[1, 3]$. See the Figure.



CVe (x_n) SV eYV dfTTVddZgV Raac`iZ^ReZ`_d `SeRZ_VU eYc`fXY EVhe`_d ^VeY`U& MV YRgV

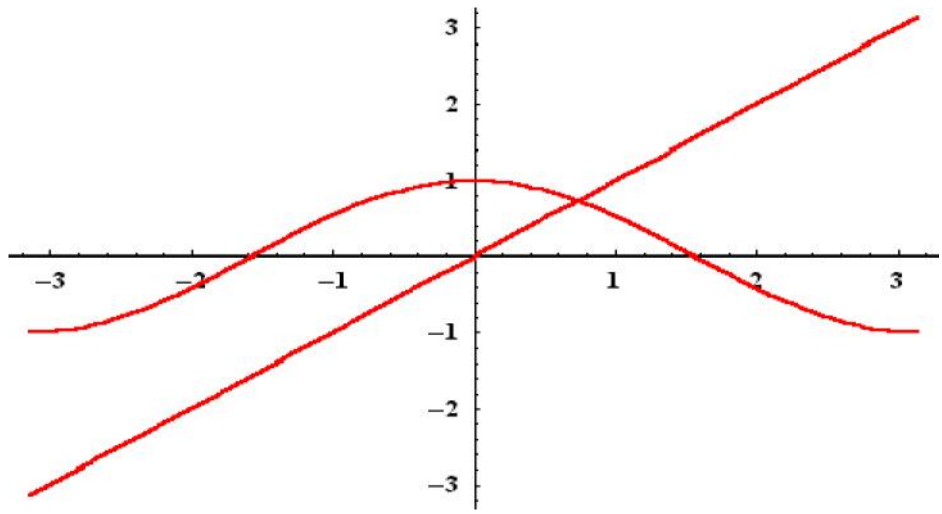
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 5}{2x_n}.$$

CVe fd deRce eYZd ac`TVdd Sj eR\Z_X U) 5 *&

$$\begin{aligned}
x_1 &= 2 \\
x_2 &= 2.25 \\
x_3 &= 2.23611111111111111111111111111111 \\
x_4 &= 2.23606797791580400276052449965 4934 \\
x_5 &= 2.23606797749978969644787282832 7110 \\
x_6 &= 2.23606797749978969640917366873 1276
\end{aligned}$$

Example. CVe fd Raac`iZ^ReV eYV `_]j d`]feZ`_ e` eYV Vbf ReZ`_ $x = \cos x$

@_ WRTe$]``\Z_X Re eYV XcRaYd hV TR_ dVV eYRe eYZd Vbf ReZ`_ YRd `_V d`]feZ`_&



JYZd d`]feZ`_ Zd R]d` eYV `_]j kVc` `W eYV Wf_TeZ`_ $f(x)=x-\cos x$ & I` _`h hV dVV Y`h EVhe`_d ^VeY`U ^Rj SV fdVU e` Raac`iZ^ReV O& IZ_TV O Zd SVehVV_ (R_U ®/2 $ hV dVe U) 5)& JYV cVde `W eYV dVbfV_TV Zd XV_VcReVU eYc`fXY eYV W`c^f]R

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n-\cos(x_n)}{1+\sin(x_n)}.$$

MV YRgV

$$
\begin{aligned}
x_1 &= 1.\\
x_2 &= 0.750363867840243893034942306682177\\
x_3 &= 0.739112890911361670360585290904890\\
x_4 &= 0.739085133385283969760125120856804\\
x_5 &= 0.739085133215160641661702625685026\\
x_6 &= 0.739085133215160641655312087673873\\
x_7 &= 0.739085133215160641655312087673873\\
x_8 &= 0.739085133215160641655312087673873
\end{aligned}
$$

Example 8aa]j EVhe`_qd ^VeY`U e` d`]gV eYV R]XVScRZT Vbf ReZ`_ $f(x)=x^3+x-1=0$ T`ccVTe e` . UVTZ^R] a]RTVd& IeRce hZeY U(5)!

$f(x)=x^3+x-1$$

$f'(x)=3x^2+1$

R_U dfSdeZefeZ_X eYVdV Z_ EVhe`_qd ZeVcReZgV W`c^f]R$ hV YRgV

$x_{n+1}=x_n-\dfrac{x_n^3+x_n-1}{3x_n^2+1}$ `c $x_{n+1}=\dfrac{2x_n^3+1}{3x_n^2+1}$ $K- ($)$*$ t &

IeRceZ_X Wc`^ U(5)&(((((($

$x_1+0.750000,\ x_2=0.686047,\quad x_3=0.682340,\ x_4=0.682328,\ \cdots$ R_U hV RTTVae (&.0*+*0 Rd R_ Raac`iZ^ReV d`]feZ`_ `W $f(x)=x^3+x-1=0$ T`ccVTe e` . UVTZ^R] a]RTVd&

Example IVe fa EVhe`_%HRaYd`_ ZeVcReZgV W`c^f]R W`c eYV Vbf ReZ`_

$$x\log_{10}x-1.2=0.$$

9LIRQFLK

JR\V $f(x)=x\log_{10}x-1.2.$

E`eZ_X eYRe $\log_{10} x = \log_e x \cdot \log_{10} e \approx 0.4343 \log_e x,$

hV `SeRZ_ $f(x) = 0.4343 x \log_e x - 1.2.$

$$f'(x) = 0.4343 \log_e x + 0.4343 x \times \frac{1}{x} = \log_{10} x + 0.4343$$

R_U YV_TV eYV EVhe`_qd ZeVcReZgV W`c^ f]R W`c eYV XZgV_ VbfReZ`_ Zd

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{0.4343 x \log_e x_n - 1.2}{\log_{10} x + 0.4343}$$ &

Example =Z_U eYV a`dZeZgV d`]feZ`_ `W eYV ecR_dTV_UV_eR] VbfReZ`_

$$2 \sin x = x$$ &

?VcV $f(x) = x - 2\sin x$ \$

d` eYRe $f'(x) = 1 - 2\cos x$

I fSdeZefeZ_X Z_ EVhe`_qd ZeVcReZgV W`c^ f]R\$ hV YRgV

$$x_{n+1} = x_n - \frac{x_n - 2\sin x_n}{1 - 2\cos x_n}$$ \$ $n = 0.1, 2, \cdots$ `c

$$x_{n+1} = \frac{2(\sin x_n - x_n \cos x_n)}{1 - 2\cos x_n} = \frac{N_n}{D_n}$$ \$ $n = 0.1, 2, \cdots$

hYVcV hV eR\V $N_n = 2(\sin x_n - x_n \cos x_n)$ R_U $D_n = 1 - 2\cos x_n$ \$ e` VRdj `fc TR]Tf]ReZ`_& LR]fVd TR]Tf]ReVU Re VRTY deVa RcV Z_UZTReVU Z_ eYV W`]]`hZ_X eRS]V IeRceZ_X hZeY $x_0 = 2$!&

K	UK	6K	/K	UK!)
(	*&(((	+&,0+	)&0+*	)&1()
)	)&1()	+&)*-	)&.,0	)&01.
*	)&01.	+&)(/	)&.+1	)&01.

)&01. Zd R_ Raac`iZ^ReV d`]feZ`_ e` $2\sin x = x$ &

Example KdV EVhe`_%HRaYd`_ ^VeY`U e` WZ_U R c``e `W eYV VbfReZ`_ $x^3 - 2x - 5 = 0.$

?VcV $f(x) = x^3 - 2x - 5$ R_U $f'(x) = 3x^2 - 2.$?V_TV EVhe`_qd ZeVcReZgV W`c^ f]R SVT`^Vd

$$x_{n+1} = x_n - \frac{x_n^{3`} - 2x_n - 5}{3x_n^2 - 2}$$

: Y``dZ_X $x_0 = 2,$ hV `SeRZ_ $f(x_0) = -1$ R_U $f'(x_0) = 10.$

$$x_1 = 2 - \left(-\frac{1}{10}\right) = 2.1$$

$$f(x_1) = (2.1)^3 - 2(2.1) - 5 = 0.06,$$

R_U $\quad f'(x_1) = 3(2.1)^2 - 2 = 11.23.$

$$x_2 = 2.1 - \frac{0.061}{11.23} = 2.094568.$$

*&(1,-.0 Zd R_ Raac` iZ^ReV c``e&

Example =Z_U R c``e `W eYV Vb f ReZ`_ $x\sin x + \cos x = 0.$

MV YRgV

$$f(x) = x\sin x + \cos x \;\text{ R\_U }\; f'(x) = x\cos x.$$

?V_TV eYV ZeVcReZ`_ W`c^ f]R Zd

$$x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$$

MZeY $x_0 = ®,$ eYV d f TTVddZgV ZeVcReVd RcV XZgV_ SV]` h2

n	x_n	$f(x_n)$	x_{n+1}
0	3.1416	–1.0	2.8233
1	2.8233	–0.0662	2.7986
2	2.7986	–0.0006	2.7984
3	2.7984	0.0	2.7984

Example =Z_U R cVR] c``e `W eYV Vb f ReZ`_ $x = e^{-x},$ f dZ_X eYV EVhe`_ o HRaYd`_ ^VeY`U&

$$f(x) = xe^x - 1 = 0$$

CVe $x_0 = 1.$ JYV_

$$x_1 = 1 - \frac{e-1}{2e} = \frac{1}{2}\left(1 + \frac{1}{e}\right) = 0.6839397$$

E`h $f(x_1) = 0.3553424$, R_U $f'(x_1) = 3.337012$,

$$x_2 = 0.6839397 - \frac{0.3553424}{3.337012} = 0.5774545.$$

$x_3 = 0.5672297$ R_U $x_4 = 0.5671433.$

Example W i! 5 iu*#]_i YRd R c``e _VRc i 5)&-& KdV eYV EVhe`_%HRaYd`_ W`c^f]R e` `SeRZ_ R SVeeVc VdeZ^ReV&

?VcV i(5)&-$W)&-!5 u(&- #]_)&-!5 u(&(1,-

$$f'(x) = 1 + \frac{1}{x};\ f'(1.5) = \frac{5}{3};\ x_1 = 1.5 - \frac{(-0.0945)}{1.6667} = 1.5567$$

JYV EVhe`_%HRaYd`_ W`c^f]R TR_ SV fdVU RXRZ_2 eYZd eZ^V SVXZ__Z_X hZeY )&--./ Rd `fc Z_ZeZR]

$$x_2 = 1.5567 - \frac{(-0.0007)}{1.6424} = 1.5571$$

JYZd Zd Z_ WRTe eYV T`ccVTe gR]fV `W eYV c``e e` , U&a&

Generalized Newton's Method

@W ¶ Zd R c``e `W $f(x) = 0$ hZeY ^f]eZa]ZTZej M$ eYV_ eYV XV_VcR]ZkVU EVhe`_qd W`c^f]R Zd

$$x_{n+1} = x_n - p\frac{f(x_n)}{f'(x_n)},$$

IZ_TV ¶ Zd R c``e `W $f(x) = 0$ hZeY ^f]eZa]ZTZej M$ Ze W`]]`hd eYRe ¶ Zd R c``e `W $f'(x) = 0$ hZeY ^f]eZa]ZTZej $(p-1)$, `W $f''(x) = 0$ hZeY ^f]eZa]ZTZej $(p-2)$, R_U d` `_& ?V_TV eYV Vi acVddZ`_d

$$x_0 - p\frac{f(x_0)}{f'(x_0)},\ x_0 - (p-1)\frac{f'(x_0)}{f''(x_0)},\ x_0 - (p-2)\frac{f'''(x_0)}{f''''(x_0)}$$

^fde YRgV eYV dR^V gR]fV ZW eYVcV Zd R c``e hZeY ^f]eZa]ZTZej M" ac`gZUVU eYRe eYV Z_ZeZR] Raac`iZ^ReZ`_ $x_0$ Zd TY`dV_ df WWZTZV_e]j T]`dV e` eYV c``e&

Example =Z_U R U`f S]V c``e `W eYV VbfReZ`_

$$f(x) = x^3 - x^2 - x + 1 = 0.$$

?VcV $f'(x) = 3x^2 - 2x - 1$, R_U $f''(x) = 6x - 2$. MZeY $x_0 = 0.8$, hV `SeRZ_

$$x_0 - 2\frac{f(x_0)}{f'(x_0)} = 0.8 - 2\frac{0.072}{-(0.68)} = 1.012,$$

R_U

$$x_0 - \frac{f'(x_0)}{f''(x_0)} = 0.8 - \frac{-(0.68)}{2.8} = 1.043,$$

JYV T]`dV_Vdd `W eYVdV gR]fVd Z_UZTReVd eYRe eYVcV Zd R U`fS]V] c``e _VRc e` f_Zej& =`c eYV _Vie Raac`iZ^ReZ`_$ hV TY``dV $x_1 = 1.01$ R_U `SeRZ_

$$x_1 - 2\frac{f(x_1)}{f'(x_1)} = 1.01 - 0.0099 = 1.0001,$$

R_U
$$x_1 - \frac{f'(x_1)}{f''(x_1)} = 1.01 - 0.0099 = 1.0001,$$

?V_TV hV T`_T]fUV eYRe eYVcV Zd R U`fS]V c``e Re $x = 1.0001$ hYZTY Zd dfWWZTZV_e]j T]`dV e` eYV RTefR] c``e f_Zej&

F_ eYV `eYVc YR_U$ ZW hV Raa]j EVhe`_%HRaYd`_ ^VeY`U hZeY $x_0 = 0.8,$ hV `SeRZ_ $x_1 = 0.8 + 0.106 \approx 0.91,$ R_U $x_2 = 0.91 + 0.046 \approx 0.96.$

Exercises

1. 8aac`iZ^ReV eYV cVR] c``e e` e h` W`fc UVTZ^R] a]RTVd `W $x^3 + 5x - 3 = 0$
2. 8aac`iZ^ReV e` W`fc UVTZ^R] a]RTVd $\sqrt[3]{3}$
3. =Z_U R a`dZeZgV c``e `W eYV VbfReZ`_ $x^4 + 2x + 1 = 0$ T`ccVTe e` , a]RTVd `W UVTZ^R]d& :Y``dV U(5)&+!
4. <ia]RZ_ Y`h e` UVeVc^Z_V eYV dbfRcV c``e `W R cVR] _f^SVc Sj $N - R$ ^VeY`U R_U fdZ_X Ze UVeVc^Z_V $\sqrt{3}$ T`ccVTe e` eYcVV UVTZ^R] a]RTVd&
5. =Z_U eYV gR]fV `W $\sqrt{2}$ T`ccVTe e` W`fc UVTZ^R]d a]RTVd fdZ_X EVhe`_ HRaYd`_ ^VeY`U&
6. KdV eYV EVhe`_%HRaYd`_ ^VeY`U$ hZeY + Rd deRceZ_X a`Z_e$ e` WZ_U R WcRTeZ`_ eYRe Zd hZeYZ_ 10^{-8} `W $\sqrt{10}$ &
7. ; VdZX_ EVhe`_ ZeVcReZ`_ W`c eYV TfSV c``e& :R]Tf]ReV $\sqrt[3]{7}$ $ deRceZ_X Wc`^ U( 5 * R_U aVcW`c^Z_X + deVad&
8. :R]Tf]ReV $\sqrt{7}$ Sj EVhe`_qd ZeVcReZ`_$ deRceZ_X Wc`^ U( 5 * R_U TR]Tf]ReZ_X U)$ U*$ U+& : `^aRcV eYV cVdf]ed hZeY eYV gR]fV $\sqrt{7} = 2.645751$

9. ; VdZX_ R EVhe`_qd ZeVcReZ`_ W`c T`^afeZ_X HeY c``e `W R a`dZeZgV _f^SVc @&

10. =Z_U R]] cVR] d`]feZ`_d `W eYV W`]]`hZ_X VbfReZ`_d Sj EVhe`_qd ZeVcReZ`_ ^VeY`U&

>! dZ_ U 5 $\frac{x}{2}$. ?!]_ U 5) o *U @! $\cos x = \sqrt{x}$

11. KdZ_X EVhe`_%HRaYd`_ ^VeY`U$ WZ_U eYV c``e `W eYV VbfReZ`_ $x^3 - x^2 - x - 3 = 0,$ T`ccVTe e` eYcVV UVTZ^R] a]RTVd

12. 8aa]j EVhe`_qd ^VeY`U e` eYV VbfReZ`_

$$x^3 - 5x + 3 = 0$$

deRceZ_X Wc`^ eYV XZgV_ $x_0 = 2$ R_U aVcW`c^Z_X + deVad&

13. 8aa]j EVhe`_qd ^VeY`U e` eYV VbfReZ`_

$$x^4 - x^3 - 2x - 34 = 0$$

deRceZ_X Wc`^ eYV XZgV_ $x_0 = 3$ R_U aVcW`c^Z_X + deVad&

14. 8aa]j EVhe`_qd ^VeY`U e` eYV VbfReZ`_

$$x^3 - 3.9x^2 + 4.79x - 1.881 = 0$$

deRceZ_X Wc`^ eYV XZgV_ $x_0 = 1$ R_U aVcW`c^Z_X + deVad&

Ramanujan's Method

MV _VVU eYV W`]]`hZ_X JYV`cV^2

9Z_`^ZR] JYV`cV^2 @W K Zd R_j cReZ`_R] _f^SVc R_U $|x| < 1$ $ eYV_

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \ . \ . \ . \ + \frac{n(n-1) \ . \ . \ . \ (n-(r-1))}{1\cdot 2\cdot \ . \ . \ . \ \cdot r}x^r + \ . \ . \ .$$

@_ aRceZT f]Rc$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ . \ . \ . + (-1)^n x^n + \ . \ . \ .$$

R_U $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ . \ . \ . + x^n + \ . \ . \ .$

@_UZR_ DReYV^ReZTZR_ I cZ_ZgRdR HR^R_f[R_)00/%)1*(! UVdTcZSVU R_ ZeVcReZgV ^VeY`U hYZTY TR_ SV fdVU e` UVeVc^Z_V eYV d^R]]Vde c``e `W eYV VbfReZ`_

$$f(x) = 0,$$

hYVcV $f(x)$ Zd `W eYV W`c ^

$$f(x) = 1 - (a_1 x + a_2 x^2 + a_3 x^2 + a_4 x^4 + \cdots).$$

=`c d ^ R]]Vc gR] f Vd `W U" hV TR_ hcZeV

$$[1 - (a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)]^{-1} = b_1 + b_2 x + b_3 x^2 + \cdots$$

< i aR_UZ_X eYV]VWe%YR_U dZUV fdZ_X SZ_` ^ZR] eYV`cV^ $ hV `SeRZ_

$$1 + (a_1 x + a_2 x^2 + a_3 x^3 + \cdots) + (a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2 + \cdots$$

$$= b_1 + b_2 x + b_3 x^2 + \cdots$$

: ` ^ aRcZ_X eYV T`VWWZTZV_ed `W ]Z\V a` hVcd `W U `_ S`eY dZUVd `W hV `SeRZ_

$$\left.\begin{aligned} b_1 &= 1, \\ b_2 &= a_1 = a_1 b_1, \\ b_3 &= a_1^2 + a_2 = a_1 b_2 + a_2 b_1, \\ &\vdots \\ b_n &= a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 \quad n = 2, 3, \cdots \end{aligned}\right\}$$

JYV_ b_n / b_{n+1} Raac`RTY R c``e `W eYV Vbf ReZ`_ $f(x) = 0$ &

Example =Z_U eYV d ^ R]]Vde c``e `W eYV Vbf ReZ`_

$$f(x) = x^3 - 6x^2 + 11x - 6 = 0.$$

9LIRQFLK

JYV XZgV_ Vbf ReZ`_ TR_ SV hcZeeV_ Rd $f(x)$

$$f(x) = 1 - \frac{1}{6}(11x - 6x^2 + x^3)$$

: ` ^ aRcZ_X$

$$a_1 = \frac{11}{6}, \quad a_2 = -1, \quad a_3 = \frac{1}{6}, \quad a_4 = a_5 = \cdots = 0$$

J` Raa]j HR^R_f[R_qd ^VeY`U hV hcZeV

$$1 - \left(\frac{11x - 6x^2 + x^3}{6} \right)^{-1} = b_1 + b_2 x + b_3 x^2 + \cdots$$

?V_TV$

$$b_1 = 1;$$

$$b_2 = a_1 = \frac{11}{6};$$

$$b_3 = a_1 b_2 + a_2 b_1 = \frac{121}{36} - 1 = \frac{85}{36};$$

$$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = \frac{575}{216};$$

$$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 = \frac{3661}{1296};$$

$$b_6 = a_1 b_5 + a_2 b_4 + a_3 b_3 + a_4 b_2 + a_5 b_1 = \frac{22631}{7776};$$

JYVcVW`cV$

$$\frac{b_1}{b_2} = \frac{6}{11} = 0.54545\,3 \qquad \frac{b_2}{b_3} = \frac{66}{85} = 0.7764705$$

$$\frac{b_3}{b_4} = \frac{102}{115} = 0.8869565\,3 \qquad \frac{b_4}{b_5} = \frac{3450}{3661} = 0.9423654$$

$$\frac{b_5}{b_6} = \frac{3138}{3233} = 0.9706155$$

9j Z_daVTeZ`_$ R c``e `W eYV XZgV_ Vbf ReZ`_ Zd f_Zej R_U Ze TR_ SV dVV_ eYRe eYV dfTTVddZgV T`_gVcXV_ed $\frac{b_n}{b_{n+1}}$ Raac`RTY eYZd c``e&

Example =Z_U R c``e `W eYV Vbf ReZ`_ $xe^x = 1$.

CVe $xe^x = 1$

HVTR]] $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

?V_TV$

$$f(x) = 1 - \left(x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \cdots \right) = 0$$

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = \frac{1}{2}, \quad a_4 = \frac{1}{6}, \quad a_5 = \frac{1}{24}, \cdots$$

MV eYV_ YRgV

$b_1 = 1;$

$b_2 = a_2 = 1;$

$b_3 = a_1 b_2 + a_2 b_1 = 1 + 1 = 2;$

$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = 2 + 1 + \frac{1}{2} = \frac{7}{2};$

$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 = \frac{7}{2} + 2 + \frac{1}{2} + \frac{1}{6} = \frac{37}{6};$

$b_6 = a_1 b_5 + a_2 b_4 + a_3 b_3 + a_4 b_2 + a_5 b_1 = \frac{37}{6}; + \frac{7}{2} + 1 + \frac{1}{6} + \frac{1}{24} = \frac{261}{24};$

JYVcVW`cV$

$\frac{b_2}{b_3} = \frac{1}{2} = 0.5$ 3 $\qquad \frac{b_3}{b_4} = \frac{4}{7} = 0.5714$ 3

$\frac{b_4}{b_5} = \frac{21}{37} = 0.56756756$ 3 $\qquad \frac{b_5}{b_6} = \frac{148}{261} = 0.56704980$ &

Example KdZ_X HR^R_f[R_qd ^VeY`U$WZ_U R cVR] c``e `W eYV Vbf ReZ`_

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \cdots = 0.$$

9LIRQFLK

CVe $\qquad f(x) = 1 - \left[x - \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} - \frac{x^4}{(4!)^2} + \cdots \right] = 0.$

?VcV

$$a_1 = 1, \quad a_2 = -\frac{1}{(2!)^2}, \quad a_3 = \frac{1}{(3!)^2}, \quad a_4 = -\frac{1}{(4!)^2},$$

$$a_5 = \frac{1}{(5!)^2}, \quad a_6 = -\frac{1}{(6!)^2}, \cdots$$

McZeZ_X

$$\left\{1-\left[x-\frac{x^2}{(2!)}+\frac{x^3}{(3!)^2}-\frac{x^4}{(4!)^2}+\cdots\right]\right\}^{-1}=b_1+b_2x+b_3x^2+\cdots\$$$

hV `SeRZ_

$b_1 = 1,$

$b_2 = a_1 = 1,$

$$b_3 = a_1b_2 + a_2b_1 = 1 - \frac{1}{(2!)^2} = \frac{3}{4};$$

$$b_4 = a_1b_3 + a_2b_2 + a_3b_1 = \frac{3}{4} - \frac{1}{(2!)^2} + \frac{1}{(3!)^2} = \frac{3}{4} - \frac{1}{4} + \frac{1}{36} = \frac{19}{36},$$

$$b_5 = a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1$$

$$= \frac{19}{36} - \frac{1}{4}\times\frac{3}{4} + \frac{1}{36}\times 1 - \frac{1}{576} = \frac{211}{576}.$$

@e W`]]` hd

$\frac{b_1}{b_2} = 1;$ $\qquad$ $\frac{b_2}{b_3} = \frac{4}{3} = 1.333\cdots;$

$\frac{b_3}{b_4} = \frac{3}{4}\times\frac{36}{19} = \frac{27}{19} = 1.4210\cdots,$ $\qquad$ $\frac{b_4}{b_5} = \frac{19}{36}\times\frac{576}{211} = 1.4408\cdots,$

hYVcV eYV]Rde cVd f]e Zd T`ccVTe e` eYcVV dZX_ZWZTR_e WZX fcVd&

Example =Z_U R c``e `W eYV Vb fReZ`_ $\sin x = 1 - x.$

KdZ_X eYV ViaR_dZ`\_ `W $\sin x$, eYV XZgV_ Vb fReZ`_ ^R j SV hcZeeV_ Rd

$$f(x) = 1 - \left(x + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = 0.$$

?VcV

$a_1 = 2, \quad a_2 = 0, \quad a_3 = \frac{1}{6}, \quad a_4 = 0,$

$a_5 = \frac{1}{120}, \quad a_6 = 0, \quad a_7 = -\frac{1}{5040}, \cdots$

hV hcZeV

$$\left[1-\left(2x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\cdots\right)\right]^{-1}=b_1+b_2x+b_3x^2+\cdots$$

MV eYV_ `SeRZ_

$$b_1=1;$$

$$b_2=a_1=2;$$

$$b_3=a_1b_2+a_2b_1=4;$$

$$b_4=a_1b_3+a_2b_2+a_3b_1=8-\frac{1}{6}=\frac{47}{6};$$

$$b_5=a_1b_4+a_2b_3+a_3b_2+a_4b_1=\frac{46}{3};$$

$$b_6=a_1b_5+a_2b_4+a_3b_3+a_4b_2+a_5b_1=\frac{3601}{120};$$

JYVcVW`cV$

$$\frac{b_1}{b_2}=\frac{1}{2}; \qquad \frac{b_2}{b_3}=\frac{1}{2};$$

$$\frac{b_3}{b_4}=\frac{24}{27}=0.5106382 \qquad \frac{b_4}{b_5}=\frac{47}{92}=0.5108695$$

$$\frac{b_5}{b_6}=\frac{1840}{3601}=0.5109691\&$$

JYV c``e$ T`ccVTe e` W` fc UVTZ^R] a]RTVd Zd (&-))(

Exercises

)& KdZ_X HR^R_f[R_qd ^VeY`U$ `SeRZ_ eYV WZcde%VZXYe T`_gVcXV_ed `W eYV Vbf ReZ`_

$$1-x+\frac{x^2}{(2!)^2}-\frac{x^3}{(3!)^2}+\frac{x^4}{(4!)^2}-\cdots=0$$

*& KdZ_X HR^R_f[R_qd ^VeY`U$ WZ_U eYV cVR] c``e `W eYV Vbf ReZ`_ $x+x^3=1.$

The Secant Method

MV YRgV dVV_ eYRe eYV EVhe`_%HRaYd`_ ^VeY`U cVbfZcVd eYV VgR] fReZ`_ `W UVcZgReZgVd `W eYV Wf_TeZ`_ R_U eYZd Zd _`e R] hRjd a`ddZS]V$ aRceZT f]Rc] j Z_ eYV TRdV `W Wf_TeZ`_d RcZdZ_X Z_ acRTeZTR] ac`S]V^d& @_ eYV dVTR_e ^VeY`U$ eYV UVcZgReZgV Re $x_n$ Zd Raac`iZ^ReVU Sj eYV W`c^ f]R

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}},$$

hYZTY TR_ SV hcZeeV_ Rd

$$f_n' = \frac{f_n - f_{n-1}}{x_n - x_{n-1}},$$

hYVcV $f_n = f(x_n)$. ?V_TV$ eYV EVhe`_%HRaYd`_ W`c^ f]R SVT`^Vd

$$x_{n+1} = x_n - \frac{f_n(x_n - x_{n-1})}{f_n = f_{n-1}} = \frac{x_{n+1} f_n - x_n f_{n-1}}{f_n = f_{n-1}}.$$

@e dY` f]U SV _`eVU eYRe eYZd W`c^ f]R cVbfZcVd eh` Z_ZeZR] Raac`iZ^ReZ`_d e` eYV c``e&

Example =Z_U R cVR] c``e `W eYV Vbf ReZ`_ $x^3 - 2x - 5 = 0$ f dZ_X dVTR_e ^VeY`U&

CVe eYV eh` Z_ZeZR] Raac`iZ^ReZ`_d SV XZgV_ Sj $x_{-1} = 2$ R_U $x_0 = 3$.

MV YRgV

$$f(x_{-1}) = f_1 = 8 - 9 = -1, \quad \text{R\_U} \quad f(x_0) = f_0 = 27 - 11 = 16.$$

$$x_1 = \frac{2(16) - 3(-1)}{17} = \frac{35}{17} = 2.058823529.$$

8]d`$

$$f(x_1) = f_1 = -0.390799923.$$

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{3(-0.390799923) - 2.058823529(16)}{-16.390799923} = 2.08126366.$$

8XRZ_

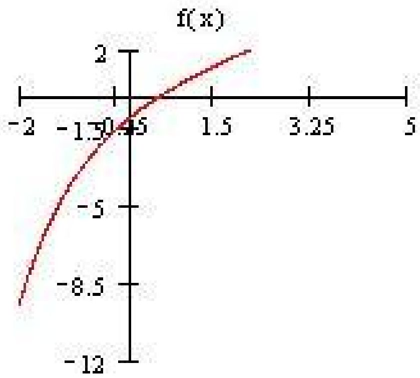
$$f(x_2) = f_2 = -0.147204057.$$

$$x_3 = 2.094824145.$$

Example: =Z_U R cVR] c``e `W eYV Vbf ReZ`_ $x - e^{-x} = 0$ f dZ_X dVTR_e ^VeY`U&

Solution

JYV XcRaY `W $f(x) = x - e^{-x}$ Zd Rd dY`h_ YVcV&



CVe fd Rddf ^V eYV Z_ZeZR] Raac`iZ^ReZ`_ e` eYV c``ed Rd) R_U *& JYRe Zd T`_dZUVc $x_{-1}=1$ R_U $x_0=2$

$$f(x_{-1}) = f_{-1} = 1 - e^{-1} = 1 - 0.367879441 = 0.632120559 \quad \text{R\_U}$$

$$f(x_0) = f_0 = 2 - e^{-2} = 2 - 0.135335283 = 1.864664717.$$

IeVa)2 GfeeZ_X $n = 0$ $ hV `SeRZ_ $x_1 = \dfrac{x_{-1} f_0 - x_0 f_{-1}}{f_0 - f_{-1}}$

?VcV$ $x_1 = \dfrac{1(1.864664717) - 2(0.632120559)}{1.864664717 - 0.632120559} = \dfrac{0.600423599}{1.232544158} = 0.487142.$

8]d`$

$$f(x_1) = f_1 = 0.487142 - e^{-0.487142} = \text{-}0.12724.$$

IeVa *2 GfeeZ_X $n = 1$ $ hV `SeRZ_

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{2(\text{-}0.12724) - 0.487142(1.864664717)}{\text{-}0.12724 - 1.864664717} = \frac{\text{-}1.16284}{\text{-}1.99190} = 0.58378$$

8XRZ_

$$f(x_2) = f_2 = 0.58378 - e^{-0.58378} = 0.02599.$$

IeVa +2 IVeeZ_X $n = 2$ $

$$x_3 = \frac{x_1 f_2 - x_2 f_1}{f_2 - f_1} = \frac{0.487142(0.02599) - 0.58378(\text{-}0.12724)}{0.02599 - (\text{-}0.12724)} = \frac{0.08694}{0.15323} = 0.56738$$

$$f(x_3) = f_3 = 0.56738 - e^{-0.56738} = 0.00037.$$

IeVa ,2 IVeeZ_X $n = 3$ Z_ "!$

$$x_4 = \frac{x_2 f_3 - x_3 f_2}{f_3 - f_2} = \frac{0.58378(0.00037) - 0.56738(0.02599)}{0.00037 - 0.02599} = \frac{-0.01453}{-0.02562} = 0.5671$$

8aac`iZ^ReZ_X e` eYcVV UZXZed$ eYV c``e TR_ SV T`_dZUVcVU Rd (&-./&

Exercises

)& ;VeVc^Z_V eYV cVR] c``e `W eYV Vbf ReZ`_ $xe^x = 1$ fdZ_X eYV dVTR_e ^VeY`U& :`^aRcV j`fc cVdf]e hZeY eYV ecfV gR]fV `W $x = 0.567143\cdots$&

*& KdV eYV dVTR_e ^VeY`U e` UVeVc^Z_V eYV c``e$]jZ_X SVehVV_ - R_U 0$ `W eYV Vbf ReZ`_ $x^{2.2} = 69.$

Objective Type Questions

R! JYV EVhe`_%HRaYd`_ ^VeY`U W`c^f]R W`c WZ_UZ_X eYV dbfRcV c``e `W R cVR] _f^SVc : Wc`^ eYV Vbf ReZ`_ $x^2 - C = 0$ Zd$

Z! $x_{n+1} = \frac{x_n}{2}$ ZZ! $x_{n+1} = \frac{3x_n}{2}$ ZZZ! $x_{n+1} = \frac{1}{2}\left(x_n + \frac{C}{x_n}\right)$ Zg! E`_V `W eYVdV

S! JYV _Vie ZeVcReZgV gR]fV `W eYV c``e `W $2x^2 - 3 = 0$ fdZ_X eYV EVhe`_%HRaYd`_ ^VeY`U$ ZW eYV Z_ZeZR] XfVdd Zd *$ Zd

Z!)&*/- ZZ!)&+/- ZZZ!)&,/- Zg! E`_V `W eYVdV

T! JYV _Vie ZeVcReZgV gR]fV `W eYV c``e `W $2x^2 - 3 = 0$ fdZ_X eYV dVTR_e ^VeY`U$ ZW eYV Z_ZeZR] XfVddVd RcV * R_U +$ Zd

Z!) ZZ!)&*- ZZZ!)&- Zg! E`_V `W eYVdV

U! @_ dVTR_e ^VeY`U$

Z! $x_{n+1} = \frac{x_{n-1} f_n - x_n f_{n-1}}{f_n - f_{n-1}}$ ZZ! $x_{n+1} = \frac{x_n f_n - x_{n-1} f_{n-1}}{f_n - f_{n-1}}$ ZZZ! $x_{n+1} = \frac{x_{n-1} f_{n-1} - x_n f_n}{f_{n-1} - f_n}$

Zg! E`_V `W eYVdV

Answers

(a) ZZZ! $x_{n+1} = \frac{1}{2}\left(x_n + \frac{C}{x_n}\right)$

S! ZZ!)&+/-

T! ZZZ!)&-

U! Z! $x_{n+1} = \dfrac{x_{n-1} f_n - x_n f_{n-1}}{f_n - f_{n-1}}$

Theoretical Questions with Answers:

)& MYRe Zd eYV UZWWVcV_TV SVehVV_ ScRT\VeZ_X R_U `aV_ ^VeY`U7

8_d2 =`c WZ_UZ_X c``ed `W R _`_]Z_VRc Vbf ReZ`_ $f(x)=0$$ ScRT\VeZ_X ^VeY`U cVbfZcVd eh` XfVddVd hYZTY T`_eRZ_ eYV ViRTe c``e& 9fe Z_ `aV_ ^VeY`U Z_ZeZR] XfVdd `W eYV c``e Zd _VVUVU hZeY`fe R_j T`_UZeZ`_ `W ScRT\VeZ_X W`c deRceZ_X eYV ZeVcReZgV ac`TVdd e` WZ_U eYV d`]feZ`_ `W R_ VbfReZ`_&

*& MYV_ eYV >V_VcR]ZkVU EVhe`_qd ^VeY`Ud W`c d`]gZ_X VbfReZ`_d Zd YV]aWf]7

8_d2 J` d`]gV eYV WZ_U eYV ``e `W $f(x)=0$ hZeY ^f]eZa]ZTZej M$ eYV XV_VcR]ZkVU EVhe`_qd W`c^f]R Zd cVbfZcVU&

+& MYRe Zd eYV Z^a`ceR_TV `W IVTR_e ^VeY`U `gVc EVhe`_%HRaYd`_ ^VeY`U7

8_d2 EVhe`_%HRaYd`_ ^VeY`U cVbfZcVd eYV VgR]fReZ`_ `W UVcZgReZgVd `W eYV Wf_TeZ`_ R_U eYZd Zd _`e R]hRjd a`ddZS]V$ aRceZTf]Rc]j Z_ eYV TRdV `W Wf_TeZ`_d RcZdZ_X Z_ acRTeZTR] ac`S]V^d& @_ dfTY dZefReZ`_d IVTR_e ^VeY`U YV]ad e` d`]gV eYV VbfReZ`_ hZeY R_ Raac`iZ^ReZ`_ e` eYV UVcZgReZgV&

''''''''''''''''''''''''

4

FINITE DIFFERENCES OPERATORS

=`c R Wf_TeZ`_ j5W i!$ Ze Zd XZgV_ eYRe $y_0, y_1, ..., y_n$ RcV eYV gR]fVd `W eYV gRcZRS]V j T`ccVda`_UZ_X e` eYV Vbf ZUZdeR_e RcX f ^V_ed$ $x_0, x_1, ..., x_n$ $ hYVcV $x_1 = x_0 + h, x_2 = x_0 + 2h, \; x_3 = x_0 + 3h, ..., x_n = x_0 + nh$ & @_ eYZd TRdV$ VgV_ eY`fXY CRXcR_XV R_U UZgZUVU UZWWVcV_TV Z_eVca`]ReZ`_ a`]j_`^ZR]d TR_ SV fdVU W`c Z_eVca`]ReZ`_$ d`^V dZ^a]Vc Z_eVca`]ReZ`_ W`c^f]Rd TR_ SV UVcZgVU& =`c eYZd$ hV YRgV e` SV WR^Z]ZRc hZeY d`^V WZ_ZeV UZWWVcV_TV `aVcRe`cd R_U WZ_ZeV UZWWVcV_TVd$ hYZTY hVcV Z_ec`UfTVU Sj lZc @dRRT EVhe`_& =Z_ZeV UZWWVcV_TVd UVR] hZeY eYV TYR_XVd eYRe eR\V a]RTV Z_ eYV gR]fV `W R Wf_TeZ`_ W i! UfV e` WZ_ZeV TYR_XVd Z_ i& =Z_ZeV UZWWVcV_TV `aVcRe`cd Z_T]fUV$ W`chRcU UZWWVcV_TV `aVcRe`c$ SRT\hRcU UZWWVcV_TV `aVcRe`c$ dYZWe `aVcRe`c$ TV_ecR] UZWWVcV_TV `aVcRe`c R_U ^VR_ `aVcRe`c&

- **Forward difference operator (Δ) :**

=`c eYV gR]fVd $y_0, y_1, ..., y_n$ `W R Wf_TeZ`_ j5W i!$ W`c eYV Vbf ZUZdeR_e gR]fVd $x_0, x_1, x_2, ..., x_n$ $ hYVcV $x_1 = x_0 + h, x_2 = x_0 + 2h, \; x_3 = x_0 + 3h, ..., x_n = x_0 + nh$ $ eYV W`chRcU UZWWVcV_TV `aVcRe`c $\Delta$ Zd UVWZ_VU `_ eYV Wf_TeZ`_ W i! Rd$

$$\Delta f(x_i) = f(x_i + h) - f(x_i) = f(x_{i+1}) - f(x_i)$$

JYRe Zd$

$$\Delta y_i = y_{i+1} - y_i$$

JYV_$ Z_ aRceZTf]Rc

$$\Delta f(x_0) = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0)$$

$$\Rightarrow \quad \Delta y_0 = y_1 - y_0$$

$$\Delta f(x_1) = f(x_1 + h) - f(x_1) = f(x_2) - f(x_1)$$

$$\Rightarrow \quad \Delta y_1 = y_2 - y_1$$

VeT&$

$\Delta y_0, \Delta y_1, ..., \Delta y_i, ...$ RcV _`h_ Rd eYV **first forward differences.**

JYV dVT`_U W`chRcU UZWWVcV_TVd RcV UVWZ_VU Rd$

$$\Delta^2 f(x_i) = \Delta[\Delta f(x_i)] = \Delta[f(x_i + h) - f(x_i)]$$
$$= \Delta f(x_i + h) - \Delta f(x_i)$$
$$= f(x_i + 2h) - f(x_i + h) - [f(x_i + h) - f(x_i)]$$
$$= f(x_i + 2h) - 2f(x_i + h) + f(x_i)$$
$$= y_{i+2} - 2y_{i+1} + y_i$$

@_ aRceZT f]Rc$

$$\Delta^2 f(x_0) = y_2 - 2y_1 + y_0 \quad or \quad \Delta^2 y_0 = y_2 - 2y_1 + y_0$$

JYV eYZcU W`c h RcU UZWWVcV_TVd RcV$

$$\Delta^3 f(x_i) = \Delta[\Delta^2 f(x_i)]$$
$$= \Delta[f(x_i + 2h) - 2f(x_i + h) + f(x_i)]$$
$$= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$$

@_ aRceZT f]Rc$

$$\Delta^3 f(x_0) = y_3 - 3y_2 + 3y_1 - y_0 \quad or \quad \Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

@_ XV_VcR] eYV _eY W`c h RcU UZWWVcV_TV$

$$\Delta^n f(x_i) = \Delta^{n-1} f(x_i + h) - \Delta^{n-1} f(x_i)$$

JYV UZWWVcV_TVd $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 ....$ RcV TR]]VU eYV **leading differences.**

=`c h RcU UZWWVcV_TVd TR_ SV h cZeeV_ Z_ R eRS f]Rc W`c ^ Rd W`]]` h d2

i	j	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	$y_0 = f(x_o)$			
		$\Delta y_0 = y_1 - y_0$		
x_1	$y_1 = f(x_1)$		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_2	$y_2 = f(x_2)$		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
x_3	$y_3 = f(x_3)$			

Example : `_decfTe eYV W`chRcU UZWWVcV_TV eRS]V W`c eYV W`]]`hZ_X U gR]fVd R_U Zed T`ccVda`_UZ_X C gR]fVd&

U	(&)	(&+	(&-	(&/	(&1	)&)	)&+
C	(&((+	(&(./	(&),0	(&*,0	(&+/(	(&-)0	(&.1/

U	C	∆C	∆*C	∆+C	∆,C	∆-C
(&)	(&((+					
		(&(.,				
(&+	(&(./		(&()/			
		(&(0)		(&((*		
(&-	(&),0		(&()1		(&(()	
		(&)((		(&((+		(&(((
(&/	(&*,0		(&(**		(&(()	
		(&)**		(&((,		(&(((
(&1	(&+/(		(&(*.		(&(()	
		(&),0		(&((-		
)&)	(&-)0		(&(+)			
		(&)/1				
)&+	(&.1/					

Example : `_decfTe eYV W`chRcU UZWWVcV_TV eRS]V$ hYVcV $f(x)=\frac{1}{x}$" U -) (&*!*$, ;&

U	$f(x)=\frac{1}{x}$	∆C WZcde UZWWVcV_TV	∆*C dVT`_U UZWWVcV_TV	∆+C	∆,C	∆-C
)&(	)&(((					
		%(&)../				
)&*	(&0+++		(&(,//			
		%(&))1(		%(&()0(		
)&,	(&/),+		(&(*1/		(&((0*	%(&((,-
		%(&(01+		%(&((10		
)&.	(&.*-(		(&()11		(&((+/	
		%(&(.1,		%(&((.)		
)&0	(&---.		(&()+0			
		%(&(--.				
*&(	(&-(((					

Example : `_dec f Te eYV W`c h RcU UZWWVcV_TV eRS]V W`c eYV UReR

$$x: -2 \quad 0 \quad 2 \quad 4$$
$$y = f(x): \quad 4 \quad 9 \quad 17 \quad 22$$

JYV W`c h RcU UZWWVcV_TV eRS]V Zd Rd W`]]` hd2

i	j 5W i !	Δy	$\Delta^2 y$	$\Delta^3 y$
%*	,			
		Δy_0 5-		
(	1		$\Delta^2 y_0$ 5+	
		Δy_1 50		$\Delta^3 y_0$ 5%.
*	) /		$\Delta^2 y_1$ 5%+	
		Δy_2 5-		
,	**			

Properties of Forward difference operator (Δ):

(i) =`c h RcU UZWWVcV_TV `W R T`_deR_e W f_TeZ`_ Zd kVc`&

Gc``W2 : `_dZUVc eYV T`_deR_e W f_TeZ`_ $f(x) = k$

JYV_$ $\Delta f(x) = f(x+h) - f(x) = k - k = 0$

(ii) =`c eYV W f_TeZ`_d $f(x)$ *and* $g(x)$ 3 $\Delta\big(f(x)+g(x)\big) = \Delta f(x) + \Delta g(x)$

Gc``W2 9 j UVWZ_ZeZ`_$

$$\begin{aligned}\Delta\big(f(x)+g(x)\big) &= \Delta\big((f+g)(x)\big)\\ &= (f+g)(x+h) - (f+g)(x)\\ &= f(x+h) + g(x+h) - \big(f(x)+g(x)\big)\\ &= f(x+h) - f(x) + g(x+h) - g(x)\\ &= \Delta f(x) + \Delta g(x)\end{aligned}$$

(iii)Gc`TVVUZ_X Rd Z_ ZZ!$ W`c eYV T`_deR_ed > R_U ?$

$$\Delta\big(af(x) + bg(x)\big) = a\Delta f(x) + b\Delta g(x)\&$$

(iv)=`c h RcU UZWWVcV_TV `W eYV ac`U f Te `W e h` W f_TeZ`_d Zd XZgV_ S j$

$$\Delta\big(f(x)g(x)\big) = f(x+h)\Delta g(x) + g(x)\Delta f(x)$$

Gc``W2

$$\Delta\big(f(x)g(x)\big)=\Delta\big((fg)(x)\big)$$
$$=(fg)(x+h)-(fg)(x)$$
$$=f(x+h)g(x+h)-f(x)g(x)$$

8UUZ_X R_U dfSecRTeZ_X $f(x+h)g(x)$ $ eYV RS`gV XZgVd

$$\Delta\big(f(x)g(x)\big)=f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)$$

$$=f(x+h)\big[g(x+h)-g(x)\big]+g(x)\big[f(x+h)-f(x)\big]$$
$$=f(x+h)\Delta g(x)+g(x)\Delta f(x)$$

E`eV 2 8UUZ_X R_U dfSecRTeZ_X $g(x+h)f(x)$ Z_deVRU `W $f(x+h)g(x)$ $ Ze TR_ R]d` SV ac`gVU eYRe

$$\Delta\big(f(x)g(x)\big)=g(x+h)\Delta f(x)+f(x)\Delta g(x)$$

(v) =`chRcU UZWWVcV_TV `W eYV bf`eZV_e `W eh` Wf_TeZ`_d Zd XZgV_ Sj

$$\Delta\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x+h)g(x)}$$

Gc``W2

$$\Delta\left(\frac{f(x)}{g(x)}\right)=\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}$$
$$=\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}$$
$$=\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{g(x+h)g(x)}$$

$$=\frac{g(x)\big[f(x+h)-f(x)\big]-f(x)\big[g(x+h)-g(x)\big]}{g(x+h)g(x)}$$

$$=\frac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x+h)g(x)}$$

Following are some results on forward differences:

HVdf]e)2 JYV KeY W`chRcU UZWWVcV_TV `W R a`]j_`^ZR] `W UVXcVV K Zd T`_deR_e hYV_ eYV gR]fVd `W eYV Z_UVaV_UV_e gRcZRS]V RcV Re VbfR] Z_eVcgR]d&

HVd f]e *2 @W K Zd R_ Z_eVXVc$

$$f(a+nh) = f(a) + {}^nC_1 \Delta f(a) + {}^nC_2 \Delta^2 f(a) + \cdots + \Delta^n f(a)$$

W`c eYV a`] j_` ^ZR] W i! Z_ i&

Forward Difference Table

U	C	ΔC	Δ*C	Δ+C	Δ,C	Δ-C	Δ.C
U(	C(						
U)	C)	ΔC(	Δ*C(				
				Δ+C(			
U*	C*	ΔC)	Δ*C*		Δ,C(		
				Δ+C)		Δ-C(	
U+	C+	ΔC*	Δ*C*		Δ,C)		Δ.C(
				Δ+C*		Δ-C)	
U,	C,	ΔC+	Δ*C+		Δ,C*		
				Δ+C+			
U-	C-	ΔC,	Δ*C,				
		ΔC-					
U.	C.						

Example <i acVdd $\Delta^2 f_0$ R_U $\Delta^3 f_0$ Z_ eVc ^d `W eYV gR] f Vd `W eYV W f_TeZ`_ C&

$$\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = f_2 - f_1 - (f_1 - f_0) = f_2 - 2f_1 + f_0$$

$$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0 = \Delta f_2 - \Delta f_1 - (\Delta f_1 - \Delta f_0)$$

$$= (f_3 - f_2) - (f_2 - f_1) - (f_2 - f_1) + (f_1 - f_0)$$

$$= f_3 - 3f_2 + 3f_1 - f_0$$

@_ XV_VcR]$

$$\Delta^n f_0 = f_n - {}^nC_1 f_{n-1} + {}^nC_2 f_{n-2} - {}^nC_3 f_{n-3} + \ldots + (-1)^n f_0$$ &

@W hV hcZeV VK e` UV_`eV CK eYV RS`gV cVd f]ed eR\Vd eYV W`]]` hZ_X W`c ^d2

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^n y_0 = y_n - {}^nC_1\, y_{n-1} + {}^nC_2\, y_{n-2} - {}^nC_3\, y_{n-3} + \ldots + (-1)^n y_0$$

Example IY`h eYRe eYV gR]fV `W VK TR_ SV ViacVddVU Z_ eVc^d `W eYV]VRUZ_X gR]fV V(R_U eYV]VRUZ_X UZWWVcV_TVd $\Delta y_0, \Delta^2 y_0, \ldots, \Delta^n y_0$.

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=`c \_`eReZ`\_R] T`_gV_ZV_TV$ hV ecVRe VK Rd CK R_U d` `_&!

=c`^ eYV W`ch RcU UZWWVcV_TV eRS]V hV YRgV

$$\left.\begin{aligned} \Delta f_0 &= f_1 - f_0 \quad \text{or} \quad f_1 = f_0 + \Delta f_0 \\ \Delta f_1 &= f_2 - f_1 \quad \text{or} \quad f_2 = f_1 + \Delta f_1 \\ \Delta f_2 &= f_3 - f_2 \quad \text{or} \quad f_3 = f_2 + \Delta f_2 \end{aligned}\right\}$$

R_U d` `_& IZ^Z]Rc]j$

$$\left.\begin{aligned} \Delta^2 f_0 &= \Delta f_1 - \Delta f_0 \text{ or } \Delta f_1 = \Delta f_0 + \Delta^2 f_0 \\ \Delta^2 f_1 &= \Delta f_2 - \Delta f_1 \text{ or } \Delta f_2 = \Delta f_1 + \Delta^2 f_1 \end{aligned}\right\}$$

R_U d` `_& IZ^Z]Rc]j$ hV TR_ hcZeV

$$\left.\begin{aligned} \Delta^3 f_0 &= \Delta^2 f_1 - \Delta^2 f_0 \text{ or } \Delta^2 f_1 = \Delta^2 f_0 + \Delta^3 f_0 \\ \Delta^3 f_1 &= \Delta^2 f_2 - \Delta^2 f_1 \text{ or } \Delta^2 f_2 = \Delta^2 f_1 + \Delta^3 f_1 \end{aligned}\right\}$$

R_U d` `_& 8]d`$ hV TR_ hcZeV f_2 Rd

$$\begin{aligned} f_2 &= (f_0 + \Delta f_0) + (\Delta f_0 + \Delta^2 f_0) \\ &= f_0 + 2\Delta f_0 + \Delta^2 f_0 \\ &= (1+\Delta)^2 f_0 \end{aligned}$$

?V_TV

$$\begin{aligned} f_3 &= f_2 + \Delta f_2 \\ &= (f_1 + \Delta f_1) + \Delta f_0 + 2\Delta^2 f_0 + \Delta^3 f_0 \\ &= f_0 + 3\Delta f_0 + 3\Delta^2 f_0 + \Delta^3 f_0 \\ &= (1+\Delta)^3 f_0 \end{aligned}$$

JYRe Zd$ hV TR_ dj^S`]ZTR]]j hcZeV

$$f_1 = (1+\Delta) f_0 \; , \; f_2 = (1+\Delta)^2 f_0 \; , \; f_3 = (1+\Delta)^3 f_0 \, .$$

:`\_eZ\_fZ\_X eYZd ac`TVUfcV$ hV TR_ dY`h$ Z_ XV_VcR]

$$f_n = (1+\Delta)^n f_0 \, .$$

KdZ_X SZ_`^ZR] ViaR\_dZ`_$ eYV RS`gV Zd

$$f_n = f_0 + {}^nC_1\Delta f_0 + {}^nC_2\Delta^2 f_0 + \ldots + \Delta^n f_0$$

JYfd

$$f_n = \sum_{i=0}^{n} {}^nC_i \, \Delta^i f_0 .$$

Backward Difference Operator

For the values $y_0, y_1, \ldots, y_n$ of a function y=f(x), for the equidistant values $x_0, x_1, \ldots, x_n$, where $x_1 = x_0 + h, x_2 = x_0 + 2h,\ x_3 = x_0 + 3h, \ldots, x_n = x_0 + nh$, the **backward difference operator** ∇ is defined on the function f(x) as,

$$\nabla f\left(x_i\right) = f\left(x_i\right) - f\left(x_i - h\right) = y_i - y_{i-1},$$

which is the **first backward difference.**

In particular, we have the first backward differences,

$$\nabla f\left(x_1\right) = y_1 - y_0;\ \nabla f\left(x_2\right) = y_2 - y_1 \ etc$$

The second backward difference is given by

$$\begin{aligned} \nabla^2 f\left(x_i\right) &= \nabla\left(\nabla f\left(x_i\right)\right) = \nabla\left[f\left(x_i\right) - f\left(x_i - h\right)\right] = \nabla f\left(x_i\right) - \nabla f\left(x_i - h\right) \\ &= \left[f\left(x_i\right) - f\left(x_i - h\right)\right] - \left[f\left(x_i - h\right) - f\left(x_i - 2h\right)\right] \\ &= \left(y_i - y_{i-1}\right) - \left(y_{i-1} - y_{i-2}\right) \\ &= y_i - 2y_{i-1} + y_{i-2} \end{aligned}$$

Similarly, the third backward difference, $\nabla^3 f\left(x_i\right) = y_i - 3y_{i-1} + 3y_{i-2} - y_{i-3}$ and so on.

Backward differences can be written in a tabular form as follows:

x	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_o	$y_0 = f(x_o)$			
		$\nabla y_1 = y_1 - y_0$		
x_1	$y_1 = f(x_1)$		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_2	$y_2 = f(x_2)$		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		
x_3	$y_3 = f(x_3)$			

Relation between backward difference and other differences:

1. $\Delta y_0 = y_1 - y_0 = \nabla y_1$; $\Delta^2 y_0 = y_2 - 2y_1 + y_0 = \nabla^2 y_2$ etc.

2. $\Delta - \nabla = \Delta\nabla$

Proof: Consider the function f(x).

$$\Delta f(x) = f(x+h) - f(x)$$
$$\nabla f(x) = f(x) - f(x-h)$$

$$\begin{aligned}(\Delta - \nabla)(f(x)) &= \Delta f(x) - \nabla f(x)\\ &= [f(x+h) - f(x)] - [f(x) - f(x-h)]\\ &= \Delta f(x) - \Delta f(x-h)\\ &= \Delta[f(x) - f(x-h)]\\ &= \Delta[\nabla f(x)]\end{aligned}$$

$$\Rightarrow \quad \Delta - \nabla = \Delta\nabla$$

3. $\nabla = \Delta E^{-1}$

Proof: Consider the function f(x).

$$\nabla f(x) = f(x) - f(x-h) = \Delta f(x-h) = \Delta E^{-1} f(x) \quad \Rightarrow \quad \nabla = \Delta E^{-1}$$

4. $\nabla = 1 - E^{-1}$

Proof: Consider the function f(x).

$$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1} f(x) = \left(1 - E^{-1}\right) f(x) \quad \Rightarrow \quad \nabla = 1 - E^{-1}$$

Problem: Construct the backward difference table for the data

$$x : -2 \quad 0 \quad 2 \quad 4$$
$$y = f(x) : -8 \quad 3 \quad 1 \quad 12$$

Solution: The backward difference table is as follows:

x	Y=f(x)	∇y	$\nabla^2 y$	$\nabla^3 y$
-2	-8			
		∇y_1 =3-(-8)=11		
0	3		$\nabla^2 y_2$ =-2-11= -13	
		∇y_2 =1-3=-2		$\nabla^3 y_3$ =13-(-13)=26
2	1		$\nabla^2 y_3$ =11-(2)=13	
		∇y_3 =12-1=11		
4	12			

Backward Difference Table

U	C	∇C	∇*C	∇+C	∇,C	∇-C	∇.C
U(	C(						
U)	C)	∇C)	∇*C*				
U*	C*	∇C*	∇*C+	∇+C+	∇,C,	∇-C-	
U+	C+	∇C+	∇*C,	∇+C,	∇,C-		∇.C.
U,	C,	∇C,	∇*C-	∇+C-	∇,C.	∇-C.	
U-	C-	∇C-	∇*C.	∇+C.			
U.	C.	∇C.					

Example lY`h eYRe R_j gR]fV `W C `c V! TR_ SV Viac VddVU Z_ eVc^d `W CK `c VK ! R_U Zed SRT\ hRcU UZWWVcV_TVd&

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$\nabla f_n = f_n - f_{n-1}$ Z^a]ZVd $f_{n-1} = f_n - \nabla f_n$

R_U $\nabla f_{n-1} = f_{n-1} - f_{n-2}$ Z^a]ZVd $f_{n-2} = f_{n-1} - \nabla f_{n-1}$

$\nabla^2 f_n = \nabla f_n - \nabla f_{n-1}$ Z^a]ZVd $\nabla f_{n-1} = \nabla f_n - \nabla^2 f_n$

=c`^ VbfReZ`_d)!e` +!$ hV `SeRZ_

$$f_{n-2} = f_n - 2\nabla f_n + \nabla^2 f_n \&$$

lZ^Z]Rc]j$ hV TR_ dY`h eYRe

$$f_{n-3} = f_n - 3\nabla f_n + 3\nabla^2 f_n - \nabla^3 f_n \&$$

lj^S`]ZTR]]j$ eYVdV cVdf]ed TR_ SV cVhcZeeV_ Rd W`]]`hd2

$$f_{n-1} = (1-\nabla) f_n \ , \ f_{n-2} = (1-\nabla)^2 f_n \ , \ f_{n-3} = (1-\nabla)^3 f_n .$$

JYfd$ Z_ XV_VcR]$ hV TR_ hcZeV

$$f_{n-r} = (1-\nabla)^r f_n \&$$

Z&V&$ $f_{n-r} = f_n - {}^rC_1 \nabla f_n + {}^rC_2 \nabla^2 f_n - \ . \ . \ . + \ (-1)^r \nabla^r f_n$

@W hV hcZeV VK e` UV_`eV CK eYV RS`gV cVdf]e Zd2

$$y_{n-r} = y_n - {}^rC_1 \nabla y_n + {}^rC_2 \nabla^2 y_n - \ . \ . \ . + \ (-1)^r \nabla^r y_n$$

Central Differences

Central difference operator ¢ for a function f(x) at x_i is defined as,

$$¢ f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$$, where *h* being the interval of differencing.

Let $y_{\frac{1}{2}} = f\left(x_0 + \frac{h}{2}\right)$. Then,

$$¢ y_{\frac{1}{2}} = ¢ f\left(x_0 + \frac{h}{2}\right) = f\left(x_0 + \frac{h}{2} + \frac{h}{2}\right) - f\left(x_0 + \frac{h}{2} - \frac{h}{2}\right)$$

$$= f(x_0 + h) - f(x_0) = f(x_1) - f(x_0) = y_1 - y_0$$

$$\Rightarrow \; ¢ y_{\frac{1}{2}} = \Delta y_0$$

Central differences can be written in a tabular form as follows:

x	y	$¢ y$	$¢^2 y$	$¢^3 y$
x_o	$y_0 = f(x_o)$			
		$¢ y_{\frac{1}{2}} = y_1 - y_0$		
x_1	$y_1 = f(x_1)$		$¢^2 y_1 = ¢ y_{\frac{3}{2}} - ¢ y_{\frac{1}{2}}$	
		$¢ y_{\frac{3}{2}} = y_2 - y_1$		$¢^3 y_{\frac{3}{2}} = ¢^2 y_2 - ¢^2 y_1$
x_2	$y_2 = f(x_2)$		$¢^2 y_2 = ¢ y_{\frac{5}{2}} - ¢ y_{\frac{3}{2}}$	
		$¢ y_{\frac{5}{2}} = y_3 - y_2$		
x_3	$y_3 = f(x_3)$			

Central Difference Table

U	C	δC	δ*C	δ+C	δ,C
U(	C(				
U)	C)	δC) ' *	δ*C)		
				δ+C+ ' *	
U*	C*	δC+ ' *	δ*C*		δ,C*
				δ+C- ' *	
U+	C+	δC- ' *	δ*C+		
		δC/ ' *			
U,	C,				

Example IY`h eYRe

R! $\delta^2 f_m = f_{m+1} - 2f_m + f_{m-1}$

S! $\delta^3 f_{m+\frac{1}{2}} = f_{m+2} - 3f_{m+1} + 3f_m - f_{m-1}$

R! $\delta^2 f_m = \delta f_{m+1/2} - \delta f_{m-1/2} = (f_{m+1} - f_m) - (f_m - f_{m-1})$

$$= f_{m+1} - 2f_m + f_{m-1}$$

S! $\delta^3 f_{m+1/2} = \delta^2 f_{m+1} - \delta^2 f_m = \left(f_{m+2} - 2f_{m+1} + f_m\right) -$

$$\left(f_{m+1} - 2f_m + f_{m-1}\right) = f_{m+2} - 3f_{m+1} + 3f_m - f_{m-1}$$

Shift operator, O

CVe V - C U! SV R Wf_TeZ`_ `W U$ R_U]Ve U eR\Vd eYV T`_dVTfeZgV gR]fVd U$ U ! E$ U ! *E$ VeT& MV eYV_ UVWZ_V R_ `aVcRe`c O" TR]]VU **the shift operator** YRgZ_X eYV ac`aVcej

O C U! 5 C U ! E! t)!

JYfd$ hYV_ O `aVcReVd `_ C U!$ eYV cVdf]e Zd eYV _Vie gR]fV `W eYV Wf_TeZ`_& @W hV Raa]j eYV `aVcRe`c e hZTV `_ C U!$ hV XVe

O* C U! 5 O PO C U!Q 5 C U! *E!&

JYfd$ Z_ XV_VcR]$ ZW hV Raa]j eYV dYZWe `aVcRe`c K eZ^Vd `_ C U!$ hV RccZgV Re

O K C U! 5 C U! KE! t *!

W`c R]] cVR] gR]fVd `W K&

@W C(5 V(!$ C) 5 V)! t RcV eYV T`_dVTfeZgV gR]fVd `W eYV Wf_TeZ`_

V 5 C U!$ eYV_ hV TR_ R]d` hcZeV

O C(- C) `c O V( - V)!$ O C) - C* `c O V) - V*! t

O*C(- C* `c O *V( - V*!$ O * C) - C+ `c O V) - V+! t

O+C(- C+ `c O *V( - V+!$ O + C) - C, `c O V) - V,! t

R_U d` `_& JYV **inverse operator** O$^{J)}$ Zd UVWZ_VU Rd2

0J) C U! 5 C U J E! t +!

R_U dZ ^ Z]Rc] j

0JK C U! 5 C U J KE! t ,!

Average Operator «

JYV **average operator «** Zd UVWZ_VU Rd

$$« f(x) = \frac{1}{2}\left[f(x+\tfrac{h}{2}) + f(x-\tfrac{h}{2}) \right.$$

Differential operator /

JYV **differential operator /** YRd eYV ac`aVce j

$$Df(x) = \frac{d}{dx} f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) = f''(x)$$

Relations between the operators:

Operators∆$∇$δ$« and / in terms of 0

=c` ^ eYV UVWZ_ZeZ`_ `W `aVcRe`cd ∆ R_U 0$ hV YRgV

∆ C U! 5 C U ! E! – C U! 5 0 C U! – C U! 5 0 J)! C U!&

JYVcVW`cV$

∆ 5 0 J)

=c` ^ eYV UVWZ_ZeZ`_ `W `aVcRe`cd ∇ R_U 0 J)$ hV YRgV

∇ C U! 5 C U! – C U J E! 5 C U! – 0 J)C U! 5) J 0 J)! C U!&

JYVcVW`cV$

$$\nabla = 1 - E^{-1} = \frac{E-1}{E}.$$

JYV UVWZ_ZeZ`_ `W eYV `aVcRe`cd δ R_U 0 XZgVd

δC U! 5 C U ! E%*! – C U J E%*! 5 0)'*C U! – 0 J)'*C U!

5 0)'* – 0 J)'*! C U!&

JYVcVW`cV$

δ 5 0)'* – 0 J)'*

JYV UVWZ_ZeZ`_ `W eYV `aVcRe`cd « R_U 0 jZV]Ud

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right] = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x).$$

JYVcVW`cV$

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right).$$

@e Zd _` h_ eYRe

0 C U! 5 C U ! E!&

KdZ_X eYV JRj]`c dVcZVd V i aR_dZ`_$ hV YRgV

$$Ef(x) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \ldots$$

$$= f(x) + h\, Df(x) + \frac{h^2}{2!} D^2(x) + \ldots$$

$$= \left(1 + \frac{h\,D}{1!} + \frac{h^2 D^2}{2!} + \ldots\right) f(x) = e^{hD} f(x) \text{ \&}$$

JYfd $E = e^{hD}$ & Fc$

E/ -]`X 0&

Example @W Δ$ ∇$ δ UV_`eV W`c hRcU$ SRT\ hRcU R_U TV_ecR] UZWWVcV_TV `aVcRe`cd$ 0 R_U « cVdaVTeZgV]j eYV dYZWe `aVcRe`c R_U RgVcRXV `aVcRe`cd$ Z_ eYV R_R]jdZd `W UReR hZeY Vbf R] daRTZ_X E$ ac`gV eYV W`]]`hZ_X2

$(i)\ 1 + \text{¢}^2 \text{«}^2 = \left(1 + \frac{\text{¢}^2}{2}\right)^2$ $\qquad (ii)\ E^{1/2} = \text{«} + \frac{\text{¢}}{2}$

$(iii)\ \Delta = \frac{\text{¢}^2}{2} + \text{¢}\sqrt{1 + \left(\text{¢}^2/4\right)}$

$(iv)\ \mu\delta = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}$ $\qquad (v)\ \mu\delta = \frac{\Delta + \nabla}{2}.$

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Z! =c`^ eYV UVWZ_ZeZ`_ `W `aVcRe`cd$ hV YRgV

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left(E - E^{-1}\right)\&$$

JYVcVW`cV

$$1+\mu^2\delta^2 = 1+\frac{1}{4}\left(E^2 - 2 + E^{-2}\right) = \frac{1}{4}\left(E + E^{-1}\right)^2$$

8]d`$

$$1+\frac{\delta^2}{2} = 1+\frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2 = \frac{1}{2}\left(E + E^{-1}\right)$$

=c` ^ Vb f ReZ`_d)! R_U *!$ hV XVe

$$1+\delta^2\mu^2 = \left(1+\frac{\delta^2}{2}\right)^2.$$

FF! $\mu + \frac{\delta}{2} = \frac{1}{2}\left(E^{1/2} + E^{-1/2} + E^{1/2} - E^{-1/2}\right) = E^{1/2}.$

FFF! MV TR_ hcZeV

$$\frac{\delta^2}{2} + \delta\sqrt{1+\left(\delta^2/4\right)} = \frac{\left(E^{1/2} - E^{-1/2}\right)^2}{2} + \left(E^{1/2} - E^{-1/2}\right)\sqrt{1+\frac{1}{4}\left(E^{1/2} - E^{-1/2}\right)^2}$$

$$= \frac{E - 2 + E^{-1}}{2} + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)\left(E^{1/2} + E^{-1/2}\right)$$

$$= \frac{E - 2 + E^{-1}}{2} + \frac{E - E^{-1}}{2}$$

5 0 –)

5 Δ

FS! MV hcZeV

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left(E - E^{-1}\right)$$

$$= \frac{1}{2}\left(1+\Delta - E^{-1}\right) = \frac{\Delta}{2} + \frac{1}{2}\left(1 - E^{-1}\right) = \frac{\Delta}{2} + \frac{1}{2}\left(\frac{E-1}{E}\right) = \frac{\Delta}{2} + \frac{\Delta}{2E}.$$

S! MV TR_ hcZeV

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left(E - E^{-1}\right)$$

$$= \frac{1}{2}\left(1+\Delta - \left(1-\nabla\right)\right) = \frac{1}{2}\left(\Delta + \nabla\right).$$

Example Gc`gV eYRe

$$hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta).$$

KdZ_X eYV deR_URcU cV]ReZ`_d XZgV_ Z_ S` i Vd Z_ eYV]Rde dVTeZ`_$ hV YRgV

$$hD = \log E = \log(1+\Delta) = \log E = -\log E^{-1} = -\log(1+\nabla)$$

8]d`$

$$\mu\delta = \frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)\left(E^{1/2}-E^{-1/2}\right) = \frac{1}{2}\left(E+E^{-1}\right)$$

$$= \frac{1}{2}\left(e^{hD}-e^{-hD}\right) = \sin(hD)$$

JYVcVW`cV

$$hD = \sinh^{-1}(\mu\delta).$$

Example IY`h eYRe eYV `aVcReZ`_d « R_U O T`^^ feV&

9LIRQFLK

=c`^ eYV UVWZ_ZeZ`_ `W `aVcRe`cd « R_U O$ hV YRgV

$$\mu E f_0 = \mu f_1 = \frac{1}{2}\left(f_{3/2}+f_{1/2}\right)$$

R_U R]d`

$$E\mu f_0 = \frac{1}{2}E\left(f_{1/2}+f_{-1/2}\right) = \frac{1}{2}\left(f_{3/2}+f_{1/2}\right)$$

?V_TV

$$\mu E = E\mu.$$

JYVcVW`cV$ eYV `aVcRe`cd « R_U O T`^^ feV&

Example IY`h eYRe

$$e^x\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \dots\right) = u_0 + u_1 x + u_2\frac{x^2}{2!} + \dots$$

$$e^x\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \dots\right) = e^x\left(1 + x\Delta + \frac{x^2\Delta^2}{2!} + \dots\right)u_0$$

$$= e^x e^{x\Delta} u_0 = e^{x(1+\Delta)} u_0$$

$$= e^{xE} u_0$$

$$= \left(1 + xE + \frac{x^2E^2}{2!} + \ldots\right)u_0$$

$$= u_0 + xu_1 + \frac{x^2}{2!}u_2 + \ldots,$$

Rd UVdZcVU&

Example KdZ_X eYV ^VeY`U `W dVaRcReZ`_ `W dj^S`]d$ dY`h eYRe

$$\Delta^n u_{x-n} = u_x - nu_{x-1} + \frac{n(n-1)}{2}u_{x-2} + \cdots + (-1)^n u_{x-n}.$$

J` ac`gV eYZd cVdf]e$ hV deRce hZeY eYV cZXYe%YR_U dZUV& JYfd$

H&?&I $= u_x - nu_{x-1} + \frac{n(n-1)}{2}u_{x-2} + \cdots + (-1)^n u_{x-n}.$

$$= u_x - nE^{-1}u_x + \frac{n(n-1)}{2}E^{-2}u_x + \cdots + (-1)^n E^{-n}u_x$$

$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2}E^{-2} + \cdots + (-1)^n E^{-n}\right]u_x$$

$$= \left(1 - E^{-1}\right)^n u_x$$

$$= \left(1 - \frac{1}{E}\right)^n u_x$$

$$= \left(\frac{E-1}{E}\right)^n u_x$$

$$= \frac{\Delta^n}{E^n}u_x$$

$$= \Delta^n E^{-n} u_x$$

$$= \Delta^n u_{x-n},$$

5 C&?&I

Differences of a Polynomial

CVe fd T`_dZUVc eYV a`]j_`^ZR] `W UVXcVV K Z_ eYV W`c^

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,$$

hYVcV $a_0 \neq 0$ R_U $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ RcV T`_deR_ed& CVe E SV eYV Z_eVcgR] `W UZWWVcV_TZ_X& JYV_

$$f(x+h) = a_0(x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + \ldots \quad + a_{n-1}(x+h) + a_n$$

E`h eYV UZWWVcV_TV `W eYV a`]j_`^ZR]d Zd2

$$\Delta f(x) = f(x+h) - f(x) = a_0\left[(x+h)^n - x^n\right] + a_1\left[(x+h)^{n-1} - x^{n-1}\right] + \ldots \qquad + a_{n-1}(x+h-x)$$

9Z_`^ZR] ViaR_dZ`_ jZV]Ud

$$\Delta f(x) = a_0\left[x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \ldots + h^n - x^n\right]$$

$$+ a_1[x^{n-1} + {}^{(n-1)}C_1 x^{n-2}h + {}^{(n-1)}C_2 x^{n-3}h^2$$

$$+ \ldots + h^{n-1} - x^{n-1}] + \ldots + a_{n-1}h$$

$$= a_0 nhx^{n-1} + \left[a_0\, {}^nC_2 h^2 + a_1\, {}^{(n-1)}C_1 h\right] x^{n-2} + \ldots + a_{n-1}h.$$

JYVcVW`cV$

$$\Delta f(x) = a_0 nhx^{n-1} + b' x^{n-2} + c' x^{n-3} + \ldots + k'x + l',$$

hYVcV ?à$ @à$ &&&$ Hà$ Ià RcV T`_deR_ed Z_g`]gZ_X E Sfe _`e U& JYfd$ eYV WZcde UZWWVcV_TV `W R a`]j_`^ZR] `W UVXcVV K Zd R_`eYVc a`]j_`^ZR] `W UVXcVV K J)!& IZ^Z]Rc]j$

$$\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta f(x+h) - \Delta f(x)$$

$$= a_0 nh\left[(x+h)^{n-1} - x^{n-1}\right] + b'\left[(x+h)^{n-2} - x^{n-2}\right]$$

$$+ \ldots + k'(x+h-x)$$

F_ dZ^a]ZWZTReZ`_$ Ze cVUfTVd e` eYV W`c^

$$\Delta^2 f(x) = a_0 n(n-1)h^2 x^{n-2} + b'' x^{n-3} + c'' x^{n-4} + \ldots + q''$$ &

JYVcVW`cV$ $\Delta^2 f(x)$ Zd R a`]j_`^ZR] `W UVXcVV K J *! Z_ U& IZ^Z]Rc]j$ hV TR_ W`c^ eYV YZXYVc `cUVc UZWWVcV_TVd$ R_U VgVcj eZ^V hV `SdVcgV eYRe eYV UVXcVV `W eYV a`]j_`^ZR] Zd cVUfTVU Sj)& 8WeVc UZWWVcV_TZ_X K eZ^Vd$ hV RcV]VWe hZeY `_]j eYV WZcde eVc^ Z_ W`c^

$$\Delta^n f(x) = a_0 n(n-1)(n-2)(n-3) \ldots (2)(1)h^n$$

$$= a_0 (n!) h^n = \text{constant}.$$

JYZd T`_deR_e Zd Z_UVaV_UV_e `W U& IZ_TV $\Delta^n f(x)$ Zd R T`_deR_e $\Delta^{n+1} f(x) = 0.$ ?V_TV eYV K ! )!QE R_U YZXYVc `cUVc UZWWVcV_TVd `W R a`]j_`^ZR] `W UVXcVV K RcV (&

: `_gVcdV]j$ ZW eYV $n$ eY UZWWVcV_TVd `W R eRSf]ReVU Wf_TeZ`_ RcV T`_deR_e R_U eYV $(n+1)$th$ $(n+2)$th,..., UZWWVcV_TVd R]] gR_ZdY$ eYV_ eYV eRSf]ReVU Wf_TeZ`_ cVacVdV_ed R a`]j_`^ZR] `W UVXcVV K& @e dY`f]U SV _`eVU eYRe eYVdV cVdf]ed Y`]U X``U `_]j ZW eYV gR]fVd `W U RcV Vbf R]]j daRTVU& JYV T`_gVcdV Zd Z^a`ceR_e Z_ _f^VcZTR] R_R]jdZd dZ_TV Ze V_RS]Vd fd e` Raac`iZ^ReV R Wf_TeZ`_ Sj R a`]j_`^ZR] ZW Zed UZWWVcV_TVd `W d`^V `cUVc SVT`^V _VRc]j T`_deR_e&

Theorem /FCCBOBK@BP LC > MLIVKLJ F>I! JYV KeY UZWWVcV_TVd `W R a`]j_`^ZR] `W UVXcVV K Zd R T`_deR_e$ hYV_ eYV gR]fVd `W eYV Z_UVaV_UV_e gRcZRS]V RcV XZgV_ Re Vbf R] Z_eVcgR]d&

Exercises

1. :R]Tf]ReV $f(x)=\frac{1}{x+1}$, $x=0(0.2)1$ e` >! * UVTZ^R] a]RTVd$ ?! + UVTZ^R] a]RTVd R_U @!, UVTZ^R] a]RTVd& JYV_ T`^aRcV eYV VWWVTe `W c`f_UZ_X Vcc`cd Z_ eYV T`ccVda`_UZ_X UZWWVcV_TV eRS]Vd&

2. <iacVdd Δ*V) Z&V& Δ*C) ! R_U Δ,V(Z&V& Δ,C(! Z_ eVc^d `W eYV gR]fVd `W eYV Wf_TeZ`_ V - C U!&

3. IVe fa R UZWWVcV_TV eRS]V `W $f(x)=x^2$ W`c $x=0(1)10$& ;` eYV dR^V hZeY eYV TR]Tf]ReVU gR]fV *- `W $f(5)$ cVa]RTVU Sj *.& FSdVcgV eYV dacVRU `W eYV Vcc`c&

4. :R]Tf]ReV $f(x)=\frac{1}{x+1}$, $x=0(0.2)1$ e` >!* UVTZ^R] a]RTVd$ ?!+ UVTZ^R] a]RTVd R_U @!, UVTZ^R] a]RTVd& JYV_ T`^aRcV eYV VWWVTe `W c`f_UZ_X Vcc`cd Z_ eYV T`ccVda`_UZ_X UZWWVcV_TV eRS]Vd&

5. IVe fa R W`chRcU UZWWVcV_TV eRS]V `W C U - U* W`c U 5 ( )!)(& ;` eYV dR^V hZeY eYV TR]Tf]ReVU gR]fV *- `W C -! cVa]RTVU Sj *.& FSdVcgV eYV dacVRU `W eYV Vcc`c&

6. : `_dec f Te eYV UZWWVcV_TV eRS]V SRdVU `_ eYV W`]]`hZ_X eRS]V&

U	(&(	(&)	(&*	(&+	(&,	(&-
T`dU	)&(((((	(&11- ((	(&10((/	(&1-- +,	(&1*) (.	(&0// -0

7. : `_dec f Te eYV UZWWVcV_TV eRS]V SRdVU `_ eYV W`]]`hZ_X eRS]V&

U	(&(	(&)	(&*	(&+	(&,	(&-
dZ_U	(&(((((	(&(11 0+	(&)10 ./	(&*1- -*	(&+01 ,*	(&,/1

8. : `_dec f Te eYV SRT\ hRcU UZWWVcV_TV eRS]V$ hYVcV

$f(x)=\sin x$ " U 5)&((&)!)&-$, ;&

9. IY`h eYRe $E\nabla = \Delta = \delta E^{1/2}$.

10. Gc`gV eYRe

11. $(i)\ \delta = 2\sinh(hD/2)$ and $(ii)\ \mu = 2\cosh(hD/2)$.

12. IY`h eYRe eYV `aVcRe`cd $\delta$$ « $ O$ $\Delta$ R_U $\nabla$ T`^^ feV hZeY VRTY `eYVc&

13. : `_dec f Te eYV SRT\ hRcU UZWWVcV_TV eRS]V SRdVU `_ eYV W`]]`hZ_X eRS]V&

U	(&(	(&)	(&*	(&+	(&,	(&-
T`d U	)&(((((	(&11- ((	(&10((/	(&1-- +,	(&1*) (.	(&0// -0

: `_dec f Te eYV UZWWVcV_TV eRS]V SRdVU `_ eYV W`]]`hZ_X eRS]V&

U	(&(	(&)	(&*	(&+	(&,	(&-
dZ_ U	(&(((((	(&(11 0+	(&)10 ./	(&*1- -*	(&+01 ,*	(&,/1 ,+

6. : `_dec f Te eYV SRT\ hRcU UZWWVcV_TV eRS]V$ hYVcV

C U! 5 dZ_ U$ U 5)&((&)!)&-$, ; &

7. <gR]fReV *a # +! 0 # *! +U* # *!$ Z_eVcgR] `W UZWWVcV_TZ_X SVZ_X f_Zej&

8. : `^afeV eYV ^ZddZ_X gR]fVd `W y_n R_U Δy_n Z_ eYV W`]]`hZ_X eRS]V2

y_n	Δy_n	$\Delta^2 y_n$
%		
%	%	)
%	%	,
.	-	)+
%	%	)0
%	%	*,
%	%	

5

NUMERICAL INTERPOLATION

: `_dZUVc R dZ_X]V gR]fVU T`_eZ_f`fd Wf_TeZ`_ $y = f(x)$ UVWZ_VU `gVc PR$SQ hYVcV $f(x)$ Zd \_`h_ Via]ZTZe]j& @e Zd VRdj e` WZ_U eYV gR]fVd `W pVq W`c R XZgV_ dVe `W gR]fVd `W pUZ Z_ PR$SQ& Z&V&$ Ze Zd a`ddZS]V e` XVe Z_W`c^ReZ`_ `W R]] eYV a`Z_ed (x, y) hYVcV $a \le x \le b.$

9fe eYV T`_gVcdV Zd _`e d` VRdj& JYRe Zd$ fdZ_X `_]j eYV a`Z_ed $(x_0, y_0)$ $ $(x_1, y_1)$ $ t $ $(x_n, y_n)$ hYVcV $a \le x_i \le b, i = 0,1,2,...,n$ $ Ze Zd _`e d` VRdj e` WZ_U eYV cV]ReZ`_ SVehVV_ i R_U j Z_ eYV W`c^ $y = f(x)$ Via]ZTZe]j& JYRe Zd `_V `W eYV ac`S]V^ hV WRTV Z_ _f^VcZTR] UZWWVcV_eZReZ`_ `c Z_eVXcReZ`_&

E`h hV YRgV WZcde e` WZ_U R dZ^a]Vc Wf_TeZ`_$ dRj $g(x)$ $ dfTY eYRe $f(x)$ R_U $g(x)$ RXcVV Re eYV XZgV_ dVe `W a`Z_ed R_U RTTVae eYV gR]fV `W $g(x)$ Rd eYV cVbfZcVU gR]fV `W $f(x)$ Re d`^V a`Z_e U Z_ SVehVV_ > R_U ?$ IfTY R ac`TVdd Zd TR]]VU **interpolation.** @W $g(x)$ Zd R a`]j_`^ZR]$ eYV_ eYV ac`TVdd Zd TR]]VU a`]j_`^ZR] Z_eVca`]ReZ`_&

MYV_ R Wf_TeZ`_ C U! Zd _`e XZgV_ Via]ZTZe]j R_U `_]j gR]fVd `W $f(x)$ RcV XZgV_ Re R dVe `W UZdeZ_Te a`Z_ed TR]]VU KLABP `c Q>?RI>O MLFKQP$ fdZ_X eYV Z_eVca`]ReVU Wf_TeZ`_ $g(x)$ e` eYV Wf_TeZ`_ C U!$ eYV cVbfZcVU `aVcReZ`_d Z_eV_UVU W`c $f(x)$ $]Z\V UVeVc^Z_ReZ`_ `W c``ed$ UZWWVcV_eZReZ`_ R_U Z_eVXcReZ`_ VeT& TR_ SV TRccZVU `fe& JYV Raac`iZ^ReZ_X a`]j_`^ZR] $g(x)$ TR_ SV fdVU e` acVUZTe eYV gR]fV `W $f(x)$ Re R _`_%eRSf]Rc a`Z_e& JYV UVgZReZ`_ `W $g(x)$ Wc`^ $f(x)$ $ eYRe Zd $|f(x) - g(x)|$ Zd TR]]VU eYV BOOLO LC >MMOLUFJ>QFLK&

: `_dZUVc R T`_eZ_f`fd dZ_X]V gR]fVU Wf_TeZ`_ $f(x)$ UVWZ_VU `_ R_ Z_eVcgR] P>" ?Q& >ZgV_ eYV gR]fVd `W eYV Wf_TeZ`_ W`c K #) UZdeZ_Te eRSf]Rc a`Z_ed $x_0, x_1, ..., x_n$ dfTY eYRe $a \le x_0 \le x_1 \le ... \le x_n \le b$& JYV ac`S]V^ `W a`]j_`^ZR] Z_eVca`]ReZ`_ Zd e` WZ_U R a`]j_`^ZR] D U LO $p_n(x)$ $ `W UVXcVV K$ hYZTY WZed eYV XZgV_ UReR& JYV Z_eVca`]ReZ`_ a`]j_`^ZR] WZeeVU e` R XZgV_ UReR Zd f_Zbf V&

@W hV RcV XZgV_ eh` a`Z_ed dReZdWjZ_X eYV Wf_TeZ`_ dfTY Rd $(x_0, y_0);(x_1, y_1)$ $ hYVcV $y_0 = f(x_0)$ R_U $y_1 = f(x_1)$ Ze Zd a`ddZS]V e` WZe R f_Zbf V a`]j_`^ZR] `W UVXcVV)& @W eYcVV UZdeZ_Te a`Z_ed RcV XZgV_$ R a`]j_`^ZR] `W UVXcVV _`e XcVReVc eYR_ eh` TR_ SV WZeeVU f_Zbf V]j& @_ XV_VcR]$ ZW _! ' AFPQFK@Q MLFKQP RcV XZgV_$ R a`]j_`^ZR] `W UVXcVV _`e XcVReVc eYR_ K TR_ SV WZeeVU f_Zbf V]j&

@_eVca`]ReZ`_ ved R cVR] Wf_TeZ`_ e` UZdTcVeV UReR& >ZgV_ eYV dVe `W eRSf]Rc gR]fVd $(x_0, y_0), (x_1, y_1)......(x_n, y_n)$ dReZdWjZ_X eYV cV]ReZ`_ $y = f(x)$ $ hYVcV eYV Via]ZTZe _RefcV `W

$f(x)$ is not known, and it is required to find the values of $f(x)$ corresponding to certain given values of x in between x_0 and x_n. To do this we have first to find a simpler function, say $g(x)$, such that $f(x)$ and $g(x)$ agree at the set of tabulated points and accept the value of $g(x)$ as the required value of $f(x)$ at some point x in between x_0 and x_n. Such a process is called **interpolation**. If $g(x)$ is a polynomial, then the process is called **polynomial interpolation**.

In interpolation, we have to determine the function $g(x)$, in the case that $f(x)$ is difficult to be obtained, using the **pivotal values** $f_0 = f(x_0),\ f_1 = f(x_1), \ldots, f_n = f(x_n)$.

Linear interpolation

In linear interpolation, we are given with two pivotal values $f_0 = f(x_0)$ and $f_1 = f(x_1)$, and we approximate the curve of f by a chord (straight line) P_1 passing through the points (x_0, f_0) and (x_1, f_1). Hence the approximate value of f at the intermediate point $x = x_0 + rh$ is given by the **linear interpolation formula**

$$f(x) \approx P_1(x) = f_0 + r(f_1 - f_0) = f_0 + r\Delta f_0$$

where $r = \dfrac{x - x_0}{h}$ and $0 \le r \le 1$.

Example Evaluate $\ln 9.2$, given that $\ln 9.0 = 2.197$ and $\ln 9.5 = 2.251.$

Here $x_0 = 9.0$, $x_1 = 9.5$, $h = x_1 - x_0 = 9.5 - 9.0 = 0.5$, $f_0 = f(x_0) = \ln 9.0 = 2.197$ and $f_1 = f(x_1) = \ln 9.5 = 2.251.$ Now to calculate $\ln 9.2 = f(9.2)$, take $x = 9.2$, so that

$$r = \frac{x - x_0}{h} = \frac{9.2 - 9.0}{0.5} = \frac{0.2}{0.5} = 0.4 \text{ and hence}$$

$$\ln 9.2 = f(9.2) \approx P_1(9.2) = f_0 + r(f_1 - f_0) = 2.197 + 0.4\,(2.251 - 2.197) = 2.219$$

Example Evaluate $f(15)$, given that $f(10) = 46$, $f(20) = 66$.

Here $x_0 = 10$, $x_1 = 20$, $h = x_1 - x_0 = 20 - 10 = 10$,

$f_0 = f(x_0) = 46$ and $f_1 = f(x_1) = 66$.

Now to calculate $f(15)$, take $x = 15$, so that

$$r = \frac{x - x_0}{h} = \frac{15 - 10}{10} = \frac{5}{10} = 0.5$$

and hence $f(15) \approx P_1(15) = f_0 + r(f_1 - f_0) = 46 + 0.5\,(66 - 46) = 56$

Example Evaluate $e^{1.24}$, given that $e^{1.1} = 3.0042$ and $e^{1.4} = 4.0552$.

Here $x_0 = 1.1$, $x_1 = 1.4$, $h = x_1 - x_0 = 1.4 - 1.1 = 0.3$, $f_0 = f(x_0) = 1.1$ and $f_1 = f(x_1) = 1.24$. Now to calculate $e^{1.24} = f(1.24)$, take $x = 1.24$, so that $r = \frac{x - x_0}{h} = \frac{1.24 - 1.1}{0.3} = \frac{0.14}{0.3} = 0.4667$ and hence

$e^{1.24} \approx P_1(1.24) = f_0 + r(f_1 - f_0) = 3.0042 + 0.4667(4.0552 - 3.0042) = 3.4933$, while the exact value of $e^{1.24}$ is 3.4947.

Quadratic Interpolation

In quadratic interpolation we are given with three pivotal values $f_0 = f(x_0)$, $f_1 = f(x_1)$ and $f_2 = f(x_2)$ and we approximate the curve of the function f between x_0 and $x_2 = x_0 + 2h$ by the quadratic parabola P_2, which passes through the points $(x_0, f_0), (x_1, f_1), (x_2, f_2)$ and obtain the quadratic interpolation formula

$$f(x) \approx P_2(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2}\Delta^2 f_0$$

where $r = \frac{x - x_0}{h}$ and $0 \le r \le 2$.

Example: Evaluate ln 9.2, using quadratic interpolation, given that

ln 9.0 = 2.197, ln 9.5 = 2.251 and ln 10.0 = 2.3026.

Here $x_0 = 9.0$, $x_1 = 9.5$, $x_2 = 10.0$, $h = x_1 - x_0 = 9.5 - 9.0 = 0.5$, $f_0 = f(x_0) = \ln 9.0 = 2.197$, $f_1 = f(x_1) = \ln 9.5 = 2.251$ and $f_2 = f(x_2) = \ln 10.0 = 2.3026$. Now to calculate $\ln 9.2 = f(9.2)$, take $x = 9.2$, so that $r = \frac{x - x_0}{h} = \frac{9.2 - 9.0}{0.5} = \frac{0.2}{0.5} = 0.4$ and

$$\ln 9.2 = f(9.2) \approx P_2(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2}\Delta^2 f_0$$

So proceed further, we have to construct the following forward difference table.

x	f	Δf	$\Delta^2 f$
9.0	2.1972		
9.5	2.2513	0.0541 0.0513	−0.0028
10.0	2.3026		

Hence,

$\ln 9.2 = f(9.2) \approx P_2(9.2) = 2.1972 + 0.4(0.0541) + \dfrac{0.4(0.4-1)}{2}(-0.0028)$ 5 *&*)1*$ hYZTY ViRTe e` , ; e` eYV ViRTe gR]fV `W $\ln 9.2 = 2.2192.$

Example KdZ_X eYV gR]fVd XZgV_ Z_ eYV W`]]` hZ_X eRS]V$ WZ_U T`d(&*0 Sj ]Z_VRc Z_eVca`]ReZ`_ R_U Sj bfRUcReZT Z_eVca`]ReZ`_ R_U T`^aRcV eYV cVdf]ed hZeY eYV gR]fV (&1.)(. ViRTe e` -;!

U	$f(x) = \cos x$	=Zcde UZWWVcV_TV	IVT`_U UZWWVcV_TV
(&(	)&(((((	%(&()11+	%(&(+1(0
(&*	(&10((/	%(&(-1()	
(&,	(&1*)(.		

?VcV $f(x)$$ hYVcV $x_0 = 0.28$ Zd e` UVeVc^Z_VU& @_ ]Z_VRc Z_eVca`]ReZ`_$ hV _VVU eh` T`_dVTfeZgV U gR]fVd R_U eYVZc T`ccVda`_UZ_X C gR]fVd R_U WZcde UZWWVcV_TV& ?VcV$ dZ_TV U-(&*0]ZVd Z_ SVehVV_ (&* R_U (&,$ hV eR\V U(5 (&*$ U) 5 (&,& **Attention!** :Y``dZ_X $x_0 = 0.2$, $x_1 = 0.4$ Zd gVcj Z^a`ceR_e3 eR\Z_X $x_0 = 0.0$ h`f]U XZgV hc`_X R_dhVc!& JYV_ E - U) – U(5(&,–(&* 5 (&*$ C(5 C U(! 5(&10((/ R_U C) 5 C U)! 5(&1*)(.&

8]d` $r = \dfrac{x - x_0}{h} = \dfrac{0.28 - 0.2}{0.2} = \dfrac{0.08}{0.2} = 0.4$ R_U

$$\cos 0.28 = f(0.28) \approx P_1(0.28) = f_0 + r(f_1 - f_0)$$

$$= 0.98007 + 0.4(0.92106 - 0.98007)$$

5 (&1-.,/$ T`ccVTe e` - ;&

@_ bfRUcReZT Z_eVca`]ReZ`_$ hV _VVU eYcVV T`_dVTfeZgV VbfR]]j daRTVU! U gR]fVd R_U eYVZc T`ccVda`_UZ_X C gR]fVd$ WZcde UZWWVcV_TVd R_U dVT`_U UZWWVcV_TV& ?VcV U(5 (&($ U) 5 (&*$ U) 5 (&,$ E - U) – U(5(&* – (&(5 (*$ C(5)&(((((($ C) 5 (&10((/ R_U C* 5 (&1*)(.$ ΔC(5%(&()11+$ Δ*C(5%(&(+1(0 $r = \dfrac{x - x_0}{h} = \dfrac{0.28 - 0.00}{0.2} = 1.4$ R_U

$$\cos 0.28 \approx P_2(0.28) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2}\Delta^2 f_0$$

$$= 1.00 + 1.4(-0.-1993) + \frac{1.4(1.4-1)}{2}\left(-0.03908\right) = 0.96116$$ e` - ;&

=c`^ eYV RS`gV$ Ze TR_ SV dVV_ eYRe bfRUcReZT Z_eVca`]ReZ`_ XZgVd ^`cV RTTfcReV gR]fV&

Newton's Forward Difference Interpolation Formula

KdZ_X EVhe`_qd W`chRcU UZWWVcV_TV Z_eVca`]ReZ`_ W`c^f]R hV WZ_U eYV K UVXcVV a`]j_`^ZR] 7K hYZTY Raac`iZ^ReVd eYV Wf_TeZ`_ C U Z_ dfTY R hRj eYRe 7K R_U C RXcVVd Re K!) Vbf R]]j daRTVU U gR]fVd$ d` eYRe $P_n(x_0)=f_0,\ P_n(x_1)=f_1,\ldots,P_n(x_n)=f_n$, hYVcV $f_0=f(x_0)$, $f_1=f(x_1),\ \ldots,\ f_n=f(x_n)$ RcV eYV gR]fVd `W f Z_ eYV eRS]V&

Newton's forward difference interpolation formula Zd

$$f(x)\approx P_n(x)=$$

$$=f_0+r\Delta f_0+\frac{r(r-1)}{2!}\Delta^2 f_0+\ldots+\frac{r(r-1)\ldots(r-n+1)}{n!}\Delta^n f_0$$

hYVcV $x=x_0+rh,\ r=\frac{x-x_0}{h},\ 0\le r\le n$&

Derivation of Newton's forward Formulae for Interpolation

>ZgV_ eYV dVe `W $(n+1)$ gR]fVd$ gZk&$ $(x_0,f_0),(x_1,f_1),(x_2,f_2),...,(x_n,f_n)$

`W U R_U C" Ze Zd cVbfZcVU e` WZ_U $p_n(x)$ $ R a`]j_`^ZR] `W eYV KeY UVXcVV dfTY eYRe $f(x)$ R_U $p_n(x)$ RXcVV Re eYV eRSf]ReVU a`Z_ed& CVe eYV gR]fVd `W U SV VbfZUZdeR_e$ Z&V&$]Ve

$$x_i=x_0+rh,\qquad r=0,1,2,...,n$$

IZ_TV $p_n(x)$ Zd R a`]j_`^ZR] `W eYV _eY UVXcVV$ Ze ^Rj SV hcZeeV_ Rd

$$\left.\begin{aligned}p_n(x)&=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)\\&+a_3(x-x_0)(x-x_1)(x-x_2)+...\\&+a_n(x-x_0)(x-x_1)(x-x_2)...(x-x_{n-1})\end{aligned}\right\}$$

@^a`dZ_X _`h eYV T`_UZeZ`_ eYRe $f(x)$ R_U $p_n(x)$ dY`f]U RXcVV Re eYV dVe `W eRSf]ReVU a`Z_ed$ hV `SeRZ_

$$a_0=f_0;\ a_1=\frac{f_1-f_0}{x_1-x_0}=\frac{\Delta f_0}{h};\ a_2=\frac{\Delta^2 f_0}{h^2 2!};\ a_3=\frac{\Delta^3 f_0}{h^3 3!};...;\ a_n=\frac{\Delta^n f_0}{h^n n!};$$

IVeeZ_X $x=x_0+rh$ R_U dfSdeZefeZ_X W`c $a_0,a_1,...,a_n$, hV `SeRZ_ eYV Vi acVddZ`_&

Remark 1:

EVhe`_qd W`chRcU UZWWVcV_TV W`c^f]R YRd eYV aVc^R_V_TV ac`aVcej& @W hV RUU R _Vh dVe `W gR]fV $(x_{n+1},y_{n+1})$$ e` eYV XZgV_ dVe `W gR]fVd$ eYV_ eYV W`chRcU UZWWVcV_TV eRS]V XVed R _Vh T`]f^_ `W _#)!eY W`chRcU UZWWVcV_TV& JYV_ eYV EVhe`_qd =`chRcU UZWWVcV_TV

Interpolation Formula with the already given values will be added with a new term at the end, $(x-x_0)(x-x_1).....(x-x_n)\frac{1}{(n+1)!h^{n+1}}\left[\Delta^{n+1}y_0\right.$ to get the new interpolation formula with the newly added value.

Remark 2:

Newton's forward difference interpolation formula is useful for interpolation near the beginning of a set of tabular values and for extrapolating values of y a short distance backward, that is left from y_0. The process of finding the value of y for some value of x outside the given range is called EXTRAPOLATION,

Example Using Newton's forward difference interpolation formula and the following table evaluate f(15).

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
10	46				
		20			
20	66		-5		
		15		2	
30	81		-3		-3
		12		-1	
40	93		-4		
		8			
50	101				

Here $x = 15$, $x_0 = 10$, $h = x_1 - x_0 = 20 - 10 = 10$, $r = \frac{x-x_0}{h} = \frac{15-10}{10} = 0.5$, $f_0 = 46$, $\Delta f_0 = 20$, $\Delta^2 f_0 = -5$, $\Delta^3 f_0 = 2$, $\Delta^4 f_0 = -3$.

Substituting these values in the Newton's forward difference interpolation formula for $N = 4$, we obtain

$$f(x) \approx P_4(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \ldots + \frac{r(r-1)\ldots(r-4+1)}{4!}\Delta^4 f_0,$$

so that

$$f(15) \approx 46 + (0.5)(20) + \frac{(0.5)(0.5-1)}{2!}(-5) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(2)$$

$$+\frac{(0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(-3)$$

$= 56.8672$, correct to 4 decimal places.

Example =Z_U R TfSZT a`]j_`^ZR] Z_ U hYZTY eR\Vd `_ eYV gR]fVd %+$ +$))$ */$ -/ R_U)(/$ hYV_ U5($)$ *$ +$, R_U - cVdaVTeZgV]j&

U	C U	Δ	Δ*	Δ+
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E`h eYV cVbfZcVU TfSZT a`]j_`^ZR] a`]j_`^ZR] `W UVXcVV +! Zd `SeRZ_VU Wc`^ EVhe`_qd W`chRcU UZWWVcV_TV Z_eVca`]ReZ`_ W`c^f]R

$$f(x) \approx P_3(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-3+1)}{3!}\Delta^3 f_0 \$$$

hYVcV 0- U Y U(! ' E 5 U Y (! ') 5 U$ d` eYRe

$$f(x) \approx P_3(x) = -3 + x(6) + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)(x-3+1)}{3!}(6)$$

`c $f(x) = x^3 - 2x^2 + 7x - 3$

Example KdZ_X eYV EVhe`_qd W`chRcU UZWWVcV_TV Z_eVca`]ReZ`_ W`c^f]R VgR]fReV C *&(-! hYVcV $f(x) = \sqrt{x}$$ fdZ_X eYV gR]fVd2

U	*&(	*&)	*&*	*&+	*&,
$\sqrt{x}$	)&,), *),	)&,,1)+0	)&,0+ *,(	)&-). -/-	)&-,1)1+

JYV W`c h RcU UZWWVcV_TV eRS]V Zd

U	$\sqrt{x}$	Δ	Δ^*	Δ^+	$\Delta^,$
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*&)	)&,,1)+0		%(&(((0**		
		(&(+,)(*		(&((((--	
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*&+	)&-). -/-		%(&(((/)/		
		(&(+* .)0			
*&,	)&-,1)1+				

?VcV $r=\dfrac{x-x_0}{h}$ 5 *&(-o*&((!'(&)5(&-$ d` Sj dfSdeZefeZ_X eYV gR]fVd Z_ EVhe`_qd W`c^f]R W`c , UVXcVV a`]j_`^ZR]!$ hV XVe

$$f(2.05)\approx P_4(2.05)=1.414214+(0.5)(0.034924)+\frac{(0.5)(0.5-1)}{2!}(-0.000822)$$

$$+\frac{(0.5)(0.5-1)(0.5-2)}{3!}(0.000055)$$

$$+\frac{(0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(0.000005)$$ 5)&,+)/0+&

Example =Z_U eYV TfSZT a`]j_`^ZR] hYZTY eR\Vd eYV W`]]`hZ_X gR]fVd3 $f(1)=24,\ f(3)=120,\ f(5)=336,$ and $f(7)=720$ & ?V_TV$ `c `eYVchZdV$ `SeRZ_ eYV gR]fV `W $f(8)$ &

MV W`c^ eYV UZWWVcV_TV eRS]V2

x	y	Δ	Δ^2	Δ^3
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

?VcV $h=2$ hZeY $x_0=1,$ hV YRgV $x=1+2p$ `c $r=(x-1)/2$ & I fSdeZefeZ_X eYZd gR]fV `W O$ hV `SeRZ_

$$f(x)=24+\frac{x-1}{2}(96)+\frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2}(120)$$

$$+\frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{6}(48) \quad = x^3+6x^2+11x+6.$$

J` UVeVc^Z_V $f(9)$$ hV afe $x=9$ Z_ eYV RS`gV R_U `SeRZ_ $f(9)=1320.$

MZeY $x_0=1,\ \ x_r=9,$ R_U $h=2,$ hV YRgV $r=\frac{x_r-x_0}{h}=\frac{9-1}{2}=4$& ?V_TV

$$f(9)\approx p(9)= f_0+r\,\Delta f_0+\frac{r(r-1)}{2!}\Delta^2 f_0+\frac{r(r-1)\,(r-2)}{3!}\Delta^3 f_0$$

$$=24+4\times 96+\frac{4\times 3}{2}\times 120+\frac{4\times 3\times 2}{3\times 2}\times 48=1320$$

Example KdZ_X EVhe`_qd W`chRcU UZWWVcV_TV W`c^f]R$ WZ_U eYV df^

$$S_n=1^3+2^3+3^3+\ldots+n^3.$$

9LIRQFLK

$$S_{n+1}=1^3+2^3+3^3+\ldots+n^3+(n+1)^3$$

R_U YV_TV

$$S_{n+1}-S_n=(n+1)^3,$$

`c

$$\Delta S_n=(n+1)^3\text{\&}$$

ZeW`]]`hd eYRe

$$\Delta^2 S_n=\Delta S_{n+1}-\Delta S_n=(n+2)^3-(n+1)^3=3n^2+9n+7$$

$$\Delta^3 S_n=3(n+1)+9n+7-(3n^2+9n+7)=6n+12$$

$$\Delta^4 S_n=6(n+1)+12-(6n+12)=6$$

IZ_TV $\Delta^5 S_n=\Delta^6 S_n=\ldots=0, S_n$ Zd R W`fceY%UVXcVV a`]j_`^ZR] Z_ eYV gRcZRS]V K&

8]d`$

$$S_1=1,\qquad \Delta S_1=(1+1)^3=8,\qquad \Delta^2 S_1=3+9+7=19,$$

$$\Delta^3 S_1=6+12=18,\qquad \Delta^4 S_1=8.$$

W`c^f]R +! XZgVd hZeY $f_0=S_1$ R_U $r-n-1)$

$$S_n=1+(n-1)(8)+\frac{(n-1)(n-2)}{2}(19)+\frac{(n-1)(n-2)(n-3)}{6}(18)$$

$$+\frac{(n-1)(n-2)(n-3)(n-4)}{24}(6)$$

$$=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2$$

$$=\left[\frac{n(n+1)}{2}\right]^2$$

Problem: The population of a country for various years in millions is provided. Estimate the population for the year 1898.

Year x:	1891	1901	1911	1921	1931
Population y:	46	66	81	93	101

Solution: Here the interval of difference among the arguments *h=10*. Since 1898 is at the beginning of the table values, we use Newton's forward difference interpolation formula for finding the population of the year 1898.

The forward differences for the given values are as shown here.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		$\Delta y_0 = 20$			
1901	66		$\Delta^2 y_0 = -5$		
		$\Delta y_1 = 15$		$\Delta^3 y_0 = 2$	
1911	81		$\Delta^2 y_1 = -3$		$\Delta^4 y_0 = -3$
		$\Delta y_2 = 12$		$\Delta^3 y_1 = -1$	
1921	93		$\Delta^2 y_2 = -4$		
		$\Delta y_3 = 8$			
1931	101				

Let x=1898. Newton's forward difference interpolation formula is,

$$f(x)=y_0+(x-x_0)\frac{1}{h}[\Delta y_0]+(x-x_0)(x-x_1)\frac{1}{2!h^2}[\Delta^2 y_0]$$

$$+(x-x_0)(x-x_1)(x-x_2)\frac{1}{3!h^3}[\Delta^3 y_0]+\ldots+$$

$$(x-x_0)(x-x_1)\ldots(x-x_{n-1})\frac{1}{n!h^n}[\Delta^n y_0]$$

Now, substituting the values, we get,

$$f(1898) = 46 + (1898-1891)\frac{1}{10}[20] + (1898-1891)(1898-1901)\frac{1}{2!10^2}[-5]$$
$$+(1898-1891)(1898-1901)(1898-1911)\frac{1}{3!10^3}[2]+$$
$$(1898-1891)(1898-1901)(1898-1911)(1898-1921)\frac{1}{4!10^4}[-3]$$

$$\Rightarrow f(1898) = 46 + 14 + \frac{21}{40} + \frac{91}{500} + \frac{18837}{40000} = 61.178$$

Example LR]fVd `W U Z_ UVXcVVd! R_U $\sin x$ RcV XZgV_ Z_ eYV W`]]`hZ_X eRS]V2

x (in degrees)	$\sin x$
15	0.2588190
20	0.3420201
25	0.4226183
30	0.5
35	0.5735764
40	0.6427876

; VeVc^Z_V eYV gR]fV `W $\sin 38^0$ &

9LIRQFLK

JYV UZWWVcV_TV eRS]V Zd

x	$\sin x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
15	0.2588190					
		0.0832011				
20	0.3420201		–0.0026029			
		0.0805982		–0.0006136		
25	0.4226183		–0.0032165		0.0000248	
		0.0773817		–0.0005888		0.0000041
30	0.5		–0.0038053		0.0000289	
		0.0735764		–0.0005599		
35	0.5735764		–0.0043652			
		0.0692112				
40	0.6427876					

8d +0 Zd T]`dVc e` $x_n = 40$ eYR_ $x_0 = 15,$ hV fdV EVhe`_qd SRT\hRcU UZWWVcV_TV W`c^f]R hZeY $x_n = 40$ R_U $x = 38$ & JYZd XZgVd

$$r = \frac{x - x_n}{h} = \frac{38-40}{5} = -\frac{2}{5} = -0.4$$

?V_TV$ fdZ_X W`c^ f]R$ hV `SeRZ_

$$f(38) = 0.6427876 - 0.4(0.0692112) + \frac{-0.4(-0.4-1)}{2}(-0.0043652)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)}{6}(-0.0005599)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{24}(0.0000289)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)(-0.4+4)}{120}(0.0000041)$$

$$= 0.6427876 - 0.02768448 + 0.00052382 + 0.00003583$$

$$-0.00000120$$

$$= 0.6156614$$

Example =Z_U eYV ^ZddZ_X eVc^ Z_ eYV W`]]` hZ_X eRS]V2

x	$y = f(x)$
0	1
1	3
2	9
3	$-$
4	81

<i a]RZ_ hYj eYV cVdf]e UZWWVcd Wc`^ $3^3 = 27$?

IZ_TV W`fc a`Z_ed RcV XZgV_$ eYV XZgV_ UReR TR_ SV Raac`iZ^ReVU Sj R eYZcU UVXcVV a`]j_`^ZR] Z_ $x$& ?V_TV $\Delta^4 f_0 = 0$& IfSdeZefeZ_X $\Delta = E - 1$ hV XVe$ $(E-1)^4 f_0 = 0,$ hYZTY `_ dZ^a]ZWZTReZ`_ jZV]Ud

$$E^4 f_0 - 4E^3 f_0 + 6E^2 f_0 - 4Ef_0 + f_0 = 0$$&

IZ_TV $E^r f_0 = f_r$ eYV RS`gV Vbf ReZ`_ SVT`^Vd

$$f_4 - 4f_3 + 6f_2 - 4f_1 + f_0 = 0$$

IfSdeZefeZ_X W`c $f_0,\ f_1,\ f_2$ R_U $f_4$ Z_ eYV RS`gV$ hV `SeRZ_

$$f_3 = 31$$

9j Z_daVTeZ`_ Ze TR_ SV dVV_ eYRe eYV eRSf]ReVU Wf_TeZ`_ Zd 3^x R_U eYV VieRTe gR]fV `W $f(3)$ Zd */& JYV Vcc`c Zd UfV e` eYV WRTe eYRe eYV Via`_V_eZR] Wf_TeZ`_ $3^x$ Zd Raac`iZ^ReVU Sj ^VR_d `W R a`]j_`^ZR] Z_ U `W UVXcVV +&

Example JYV eRS]V SV]`h XZgVd eYV gR]fVd `W $\tan x$ W`c $0.10 \le x \le 0.30$

x	$y = \tan x$
0.10	0.1003
0.15	0.1511
0.20	0.2027
0.25	0.2553
0.30	0.3093

=Z_U2 R! $\tan 0.12$ S! $\tan 0.26$& T! $\tan 0.40$ U! $\tan 0.50$

JYV eRS]V UZWWVcV_TV Zd

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

R! J` WZ_U $\tan(0.12)$, hV YRgV $r = 0.4$ ?V_TV EVhe`_qd W`chRcU UZWWVcV_TV Z_eVca`]ReZ`_ W`c^f]R XZgVd

$$\tan(0.12) = 0.1003 + 0.4(0.0508) + \frac{0.4(0.4-1)}{2}(0.0008)$$

$$+\frac{0.4(0.4-1)(0.4-2)}{6}(0.0002)$$

$$+\frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(0.0002)$$

$$= 0.1205$$

S! J` WZ_U $\tan(0.26)$, hV fdV EVhe`_qd SRT\ hRcU UZWWVcV_TV Z_eVca`]ReZ`_ W`c^f]R hZeY

$$r = \frac{x - x_n}{n}$$

$$= \frac{0.26 - 03}{0.05}$$

$$= -0.8$$

hYZTY XZgVd

$$\tan(0.26) = 0.3093 - 0.8(0.0540) + \frac{-0.8(-0.8+1)}{2}(0.0014)$$

$$+ \frac{-0.8(-0.8+1)(-0.8+2)}{6}(0.0004)$$

$$+ \frac{-0.8(-0.8+1)(-0.8+2)(-0.8+3}{24}(0.0002) \quad = 0.2662$$

Gc`TVVUZ_X Rd Z_ eYV TRdV Z! RS`gV$ hV `SeRZ_

T! $\tan 0.40 = 0.4241,$ R_U

U! $\tan 0.50 = 0.5543$

JYV RTefR] gR]fVd$ T`ccVTe e` W`fc UVTZ^R] a]RTVd$ `W eR_ (&)*!$ tan(0.26) RcV cVdaVTeZgV]j (&)*(. R_U (&*..(& : `^aRcZd`_ `W eYV T`^afeVU R_U RTefR] gR]fVd dY`hd eYRe Z_ eYV WZcde eh` TRdVd Z&V&$ `W Z_eVca`]ReZ`_! eYV cVdf]ed `SeRZ_VU RcV WRZc]j RTTfcReV hYVcVRd Z_ eYV]Rde% eh` TRdVd Z&V&$ `W ViecRa`]ReZ`_! eYV Vcc`cd RcV bfZeV T`_dZUVcRS]V& JYV ViR^a]V eYVcVW`cV UV^`_decReVd eYV Z^a`ceR_e cVdf]ed eYRe ZW R eRSf]ReVU Wf_TeZ`_ Zd `eYVc eYR_ R a`]j_`^ZR]$ eYV_ ViecRa`]ReZ`_ gVcj WRc Wc`^ eYV eRS]V]Z^Zed h`f]U SV UR_XVc`fd%R]eY`fXY Z_eVca`]ReZ`_ TR_ SV TRccZVU `fe gVcj RTTfcReV]j&

Exercises

1. Using the difference table in exercise 1, compute cos0.75 by Newton's forward difference interpolating formula with $n = 1, 2, 3, 4$ and compare with the 5D-value 0.731 69.

2. Using the difference table in exercise 1, compute cos0.28 by Newton's forward difference interpolating formula with $n = 1, 2, 3, 4$ and compare with the 5D-value

3. Using the values given in the table, find cos0.28 (in radian measure) by linear interpolation and by quadratic interpolation and compare the results with the value 0.961 06 (exact to 5D).

x	$f(x)=\cos x$	First difference	Second difference
0.0	1.000 00		
		-0.019 93	
0.2	0.980 07		-0.03908
		-0.059 01	
0.4	0.921 06		-0.03671
		-0.095 72	
0.6	0.825 34		-0.03291
		-0.128 63	
0.8	0.696 71		-0.02778
		-0.156 41	
1.0	0.540 30		

4. Find Lagrangian interpolation polynomial for the function f having $f(4)=1,\ f(6)=3,\ f(8)=8,\ f(10)=16$. Also calculate $f(7)$.

5. The sales in a particular shop for the last ten years is given in the table:

Year	1996	1998	2000	2002	2004
Sales (in lakhs)	40	43	48	52	57

Estimate the sales for the year 2001 using Newton's backward difference interpolating formula.

6. Find $f(3)$, using Lagrangian interpolation formula for the function f having $f(1)=2,\ f(2)=11,\ f(4)=77$.

7. Find the cubic polynomial which takes the following values:

x	0	1	2	3
$f(x)$	1	2	1	10

8. Compute sin0.3 and sin0.5 by Everett formula and the following table.

	sin$x$	δ^2
0.2	0.198 67	-0.007 92
0.4	0.389 42	-0.015 53
.6	0.564 64	-0.022 50

9. The following table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

x =height :	100	150	200	250	300	350	400
y = distance :	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the value of y when $x = 218$ ft (*Ans*: 15.699)

10. Using the same data as in exercise 9, find the value of y when $x = 410$ft.

6

NEWTON' S AND LAGRANGIAN FORMULAE - PART I

Newton's Backward Difference Interpolation Formula

EVhe`_qd SRT\ hRcU UZWWVcV_TV Z_eVca`]ReZ`_ W`c^ f]R Zd

$$f(x) \approx P_n(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \ldots + \frac{r(r+1)\ldots(r+n-1)}{n!}\nabla^n f_n$$

hYVcV $x = x_n + rh,\ r = \frac{x - x_n}{h},\ -n \le r \le 0$&

Derivation of Newton's Backward Formulae for Interpolation

>ZgV_ eYV dVe `W $(n+1)$ gR]fVd$ gZk&$ $(x_0, f_0), (x_1, f_1), (x_2, f_2), \ldots, (x_n, f_n)$

`W U R_U C" Ze Zd cVbfZcVU e` WZ_U $p_n(x)$$ R a`]j_`^ZR] `W eYV KeY UVXcVV dfTY eYRe $f(x)$ R_U $p_n(x)$ RXcVV Re eYV eRSf]ReVU a`Z_ed& CVe eYV gR]fVd `W U SV Vbf ZUZdeR_e$ Z&V&$]Ve

$$x_i = x_0 + rh, \qquad r = 0, 1, 2, \ldots, n$$

IZ_TV $p_n(x)$ Zd R a`]j_`^ZR] `W eYV KeY UVXcVV$ Ze ^Rj SV hcZeeV_ Rd

$$\begin{aligned} p_n(x) = {} & a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) \\ & + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \ldots \\ & + a_n(x - x_n)(x - x_{n-1})\ldots(x - x_1) \end{aligned}$$

@^a`dZ_X eYV T`_UZeZ`_ eYRQ $f(x)$ R_U $p_n(x)$ dY`f]U RXcVV Re eYV dVe `W eRSf]ReVU a`Z_ed hV `SeRZ_ RWeVc d`^V dZ^a]ZWZTReZ`_! eYV RS`gV W`c^ f]R&

Remark 1:

If the values of the k^th^ forward/backward differences are same, then (k+1)^th^ or higher differences are zero. Hence the given data represents a k^th^ degree polynomial.

Remark 2:

The Backward difference Interpolation Formula is commonly used for interpolation near the end of a set of tabular values and for extrapolating values of y a short distance forward that is right from y_n.

Problem: For the following table of values, estimate *f*(7.5), using Newton's backward difference interpolation formula.

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6	
7	343		42		
		169			
8	512				

Solution:

Since the fourth and higher order differences are 0, the Newton's backward interpolation formula is

$$f(x_n+uh)=y_n+u\left[\nabla y_n\right]+\frac{u(u+1)}{2!}\left[\nabla^2 y_n\right]$$
$$+\frac{u(u+1)(u+2)}{3!}\left[\nabla^3 y_n\right]+\ldots+\frac{u(u+1)(u+2)\ldots(u+n-1)}{n!}\nabla^n\left[y_n\right],$$

Where, $u=\frac{x-x_n}{h}=\frac{7.5-8.0}{1}=-0.5$ and

$\nabla y_n=169$, $\nabla^2 y_n=42$, $\nabla^3 y_n=6$ and $\nabla^4 y_n=0$.

Hence,

$$f(7.5)=512+(-0.5)(169)+\frac{(-0.5)(-0.5+1)}{2!}(42)+\frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}6$$

$= 421.875$.

Example =`c eYV W`]]`hZ_X eRS]V `W gR] fVd$ VdeZ^ReV C /&-!$ fdZ_X EVhe`_gd SRT\hRcU UZWWVcV_TV Z_eVca`]ReZ`_ W`c^f]R&

U	C	∇C	∇*C	∇+C	∇,C
)	)				
		/			
*	0		)*		
		)1		.	
+	*/		)0		(
		+/		.	
,	.,		*,		(
		.)		.	
-	)*-		+(		(
		1)		.	
.	*).		+.		(
		)*/		.	
/	+,+		,*		
		).1			
0	-)*				

IZ_TV eYV W`fceY R_U YZXYVc `cUVc UZWWVcV_TVd RcV ($ eYV EVhe`_qd SRT\hRcU Z_eVca`]ReZ`_ W`c^f]R Zd

$f(x) \approx P_n(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 f_n$$ hYVcV

$r = \frac{x - x_n}{h} = \frac{7.5 - 8.0}{1} = -0.5$ R_U ∇CK 5).1$ ∇*CK 5 ,*$ ∇+CK 5 .& ?V_TV

$$f(7.5) \approx 512 + (-0.5)(169) + \frac{(-0.5)(-0.5+1)}{2!}(42) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}6$$

5 ,*)&0/-

Gauss' Central Difference Formulae

MV T`_dZUVc eh` TV_ecR] UZWWVcV_TV W`c^f]RV&

(i) Gauss's forward formula

MV T`_dZUVc eYV W`]]`hZ_X eRS]V Z_ hYZTY eYV TV_ecR] T``cUZ_ReV Zd eR\V_ W`c T`_gV_ZV_TV Rd y_0 T`ccVda`_UZ_X e` $x = x_0$

>R fddqd =`chRcU W`c^f]R Zd

$$f_p = f_0 + G_1\Delta f_0 + G_2\Delta^2 f_{-1} + G_3\Delta^3 f_{-1} + G_4\Delta^4 f_{-2} + \ldots,$$

hYVcV $G_1, G_2, \ldots$ RcV XZgV_ Sj

$$G_1 = p$$

$$G_2 = \frac{p(p-1)}{2!}$$

$$G_3 = \frac{(p+1)p(p-1)}{3!},$$

$$G_4 = \frac{(p+1)p(p-1)(p-2)}{4!},$$

JRS]V2 >R f ddq =`c h RcU =`c^ f]R

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy^2					
x_3	y_3						

Derivation of Gauss's forward interpolation formula:

MV YRgV EVhe`_qd W`c h RcU Z_eVca`]ReZ`_ W`c^ f]R Rd$

$$f(x_0 + uh) = y_0 + u\left[\Delta y_0\right] + \frac{u(u-1)}{2!}\left[\Delta^2 y_0 \quad \right]$$
$$+ \frac{u(u-1)(u-2)}{3!}\left[\Delta^3 y_0 \quad \right] + ... + \frac{u(u-1)(u-2)....(u-n+1)}{n!} \quad \Delta^n\left[y_0\right.$$

h YVcV$ $u = \frac{(x - x_0)}{h}$

h V YRgV$

$$\Delta^2 y_0 = \Delta^2 E y_{-1} = \Delta^2 (1+\Delta) y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 E y_{-1} = \Delta^3 (1+\Delta) y_{-1} = \Delta^3 y_{-1} + \Delta^4 y_{-1}\$$$

@_ dZ^Z]Rc hRj$ $\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1};\quad \Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ R_U d` `_&

I fSdeZefeZ_X eYVdV gR]fVd Z_ EVhe`_qd W`chRcU Z_eVca`]ReZ`_ W`c^f]R$ hV XVe$

$$f(x_0+uh) = y_0 + u\left[\Delta y_0\right] + \frac{u(u-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^3 y_{-1}\right]$$

$$+\frac{u(u-1)(u-2)}{3!}\left[\Delta^3 y_{-1} + \Delta^4 y_{-1}\right] + \frac{u(u-1)(u-2)(u-3)}{4!}\left[\Delta^4 y_{-1} + \Delta^5 y_{-1}\right] + \ldots$$

I`]gZ_X eYV RS`gV Vi acVddZ`_$ hV XVe$

$$f(x_0+uh) = y_0 + u\left[\Delta y_0\right] + {}^{u}C_2\left[\Delta^2 y_{-1}\right] + {}^{u+1}C_3\left[\Delta^3 y_{-1}\right] + {}^{u+1}C_4\left[\Delta^4 y_{-2}\right] + {}^{u+2}C_5\left[\Delta^5 y_{-2}\right] + \ldots$$

JYZd W`c^f]R Zd \_`h_ Rd >Rfdd W`chRcU Z_eVca`]ReZ`_ W`c^f]R&

(ii) Gauss Backward Formula

>Rfdd SRT\hRcU W`c^f]R Zd

$$f_p = f_0 + G_1'\Delta f_{-1} + G_2'\Delta f_{-1} + G_3'\Delta f_{-2} + G_4'\Delta^4 f_{-2} + \ldots$$

hYVcV $G_1', G_2', \ldots$ RcV XZgV_ Sj

$$G_1' = p,$$

$$G_2' = \frac{p(p+1)}{2!},$$

$$G_3' = \frac{(p+1)p(p-1)}{3!},$$

$$G_4' = \frac{(p+2)(p+1)p(p-1)}{4!},$$

Example =c`^ eYV W`]]`hZ_X eRS]V$ WZ_U eYV gR]fV `W $e^{1.17}$ fdZ_X >Rfddq W`chRcU W`c^f]R&

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

9LIRQFLK

?VcV hV eR\V $x_0 = 1.15,\ h = 0.05$&

8]d`$ $x_p = x_0 + ph$

$$1.17 = 1.15 + p(0.05),$$

hYZTY XZgVd

$$p = \frac{0.02}{0.05} = \frac{1}{4}$$

JYV UZWWVcV_TV eRS]V Zd XZgV_ SV]`h2

x	e^x	Δ	Δ^2	Δ^3	Δ^4
1.00	2.7183				
		0.1394			
1.05	2.8577		0.0071		
		0.1465		0.0004	
1.10	3.0042		0.0075		0
		0.1540		0.0004	
1.15	3.1582		0.0079		0
		0.1619		0.0004	
1.20	3.3201		0.0083		0.0001
		0.1702		0.0005	
1.25	3.4903		0.0088		
		0.1790			
1.30	3.6693				

KdZ_X >Rfddqd W`chRcU UZWWVcV_TV W`c^f]R hV `SeRZ_

$$e^{1.17} = 3.1582 + \frac{2}{5}(0.1619) + \frac{(2/5)(2/5-1)}{2}(0.0079)$$

$$+\frac{(2/5+1)(2/5)(2/5-1)}{6}(0.0004)$$

$$= 3.1582 + 0.0648 - 0.0009 \quad = 3.2221\&$$

Derivation of Gauss's backward interpolation formula:

IeRceZ_X eYV dfSdeZefeZ`_ Z_ EVhe`_qd W`chRcU Z_eVca`]ReZ`_ W`c^f]R hZeY $\Delta y_0 = \Delta E y_{-1} = \Delta(1+\Delta) y_{-1} = \Delta y_{-1} + \Delta^2 y_{-1}$ R_U eYV dfSdeZefeZ`_d U`_V Z_ eYV TRdV `W >Rfddqd W`chRcU Z_eVca`]ReZ`_ W`c^f]R $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$3 $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ VeT&$ hV `SeRZ_

$$f(x_0+uh) = y_0 + u\left[\Delta y_{-1} + \Delta^2 y_{-1}\right] + \frac{u(u-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^3 y_{-1}\right]$$

$$+\frac{u(u-1)(u-2)}{3!}\left[\Delta^3 y_{-1} + \Delta^4 y_{-1}\right] + \frac{u(u-1)(u-2)(u-3)}{4!}\left[\Delta^4 y_{-1} + \Delta^5 y_{-1}\right] + \ldots$$

I`]gZ_X eYV Vi acVddZ`_$ hV XVe$

$$f(x_0+uh) = y_0 + u\left[\Delta y_{-1}\right] + {}^{u+1}C_2\left[\Delta^2 y_{-1}\right] + {}^{u+1}C_3\left[\Delta^3 y_{-2}\right] + {}^{u+2}C_4\left[\Delta^4 y_{-2}\right] + {}^{u+2}C_5\left[\Delta^5 y_{-3}\right] + \ldots\&$$

JYZd Zd _`h_ Rd >Rfddqd SRT\hRcU Z_eVca`]ReZ`_ W`c^f]R&

Central difference interpolation formulas:

EVhe`_qd W`chRcU R_U SRT\hRcU Z_eVca`]ReZ`_ W`c^f]R RcV Raa]ZTRS]V W`c Z_eVca`]ReZ`_ _VRc eYV SVXZ__Z_X R_U _VRc eYV V_U `W eYV eRSf]ReVU RcXf^V_ed$ cVdaVTeZgV]j& E`h Z_ eYZd dVddZ`_ hV UZdTfdd Z_eVca`]ReZ`_ _VRc eYV TV_ecV `W eYV eRSf]ReVU RcXf^V_ed& =`c eYZd afca`dV hV fdV TV_ecR] UZWWVcV_TV Z_eVca`]ReZ`_ W`c^f]R& >Rfddqd W`chRcU Z_eVca`]ReZ`_ W`c^f]R$ >Rfddqd SRT\hRcU Z_eVca`]ReZ`_ W`c^f]R$ leVc]Z_Xqd W`c^f]R$ 9VddV]qd W`c^f]R$ CRa]RTV%<gVcVeeqd W`c^f]R RcV d`^V `W eYV gRcZ`fd TV_ecR] UZWWVcV_TV Z_eVca`]ReZ`_ W`c^f]Rd&

CVe fd T`_dZUVc d`^V Vbf ZUZdeR_e RcXf^V_ed hZeY Z_eVcgR] `W UZWWVcV_TV$ dRj 3 E R_U T`ccVda`_UZ_X Wf_TeZ`_ gR]fVd RcV XZgV_& CVe x_0 " SV eYV TV_ecR] a`Z_e R^`_X eYV RcXf^V_ed&

=`c Z_eVca`]ReZ`_ Re eYV a`Z_e U _VRc eYV TV_ecR] gR]fV$]Ve $f(x_0)=y_0$ $ $f(x_0-h)=y_{-1}$ $ $f(x_0+h)=y_1$ $ $f(x_0-2h)=y_{-2}$ $ $f(x_0+2h)=y_2$ $ $f(x_0-3h)=y_{-3}$ $ $f(x_0+3h)=y_3$ R_U d` `_&

=`c eYV gR]fVd $y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3$ eYV W`chRcU UZWWVcV_TV eRS]V Zd Rd W`]]`hd2

x	**y**	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0-3h	y_{-3}						
		Δy_{-3}					
x_0-2h	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_0-h	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_0+h	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_0+2h	y_2		$\Delta^2 y_1$				
		Δy_2					
x_0+3h	y_3						

The above table can also be written in terms of central differences using the operator δ as follows:

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
$x_0 - 3h$	y_{-3}						
		$\delta y_{-\frac{5}{2}}$					
	y_{-2}		$\delta^2 y_{-2}$				
$x_0 - 2h$		$\delta y_{-\frac{3}{2}}$					
				$\delta^3 y_{-\frac{3}{2}}$			
	y_{-1}		$\delta^2 y_{-1}$		$\delta^4 y_{-1}$		
		$\delta y_{-\frac{1}{2}}$				$\delta^5 y_{-\frac{1}{2}}$	
$x_0 - h$				$\delta^3 y_{-\frac{1}{2}}$			$\delta^6 y_0$
	y_0		$\delta^2 y_0$		$\delta^4 y_0$		
x_0		$\delta y_{\frac{1}{2}}$				$\delta^5 y_{\frac{1}{2}}$	
	y_1		$\delta^2 y_1$	$\delta^3 y_{\frac{1}{2}}$	$\delta^4 y_1$		
$x_0 + h$							
	y_2	$\delta y_{\frac{3}{2}}$	$\delta^2 y_2$				
				$\delta^3 y_{\frac{3}{2}}$			
$x_0 + 2h$		$\delta y_{\frac{5}{2}}$					
	y_3						
$x_0 + 3h$							

The difference given in both the tables are same can be established as follows:

We have $\delta = \Delta E^{-\frac{1}{2}}$. Then, $\delta y_{-\frac{5}{2}} = \Delta E^{-\frac{1}{2}}\left(y_{-\frac{5}{2}}\right) = \Delta\left(y_{-\frac{5}{2}-\frac{1}{2}}\right) = \Delta y_{-3}$;

$$\delta^2 y_{-2} = \left(\Delta E^{-\frac{1}{2}}\right)^2 \left(y_{-2}\right) = \Delta^2\left(y_{-2-1}\right) = \Delta^2 y_{-3};$$

$$\delta^3 y_{-\frac{3}{2}} = \left(\Delta E^{-\frac{1}{2}}\right)^3 \left(y_{-\frac{3}{2}-\frac{3}{2}}\right) = \Delta^3 y_{-3} \text{ and so on.}$$

We use the central differences as found in the first table for interpolation near the central value. Among the various formulae for Central Difference Interpolation, first we consider Gauss's forward interpolation formula.

INTERPOLATION - Arbitrarily Spaced *x* values

@_ eYV acVgZ`fd dVTeZ`_d hV YRgV UZdTfddVU Z_eVca`]ReZ`_d hYV_ eYV U%gR]fVd RcV Vbf R]]j daRTVU& JYVdV Z_eVca`]ReZ`_ W`c^f]RV TR__`e SV fdVU hYV_ eYV U%gR]fVd RcV _`e Vbf R]]j daRTVU& @_ eYV W`]]`hZ_X dVTeZ`_d$ hV T`_dZUVc W`c^f]RV eYRe TR_ SV fdVU VgV_ ZW eYV i%gR]fVd RcV _`e Vbf R]]j daRTVU&

Newton's Divided Difference Interpolation Formula

@W U($ U)$ & & & $ UK RcV >O?FQO>OFIV PM>@BA Z&V& ZW eYV UZWWVcV_TV SVehVV_ U(R_U U)$ U) R_U U* VeT& ^Rj _`e SV Vbf R]!$ eYV_ eYV a`]j_`^ZR] `W UVXcVV K eYc`fXY $(x_0, f_0), (x_1, f_1), \cdots, (x_n, f_n),$ hYVcV $f_j = f(x_j),$ Zd XZgV_ Sj eYV **Newton's divided difference interpolation formula** R]d` _`h_ Rd EVhe`_qd XV_VcR] Z_eVca`]ReZ`_ W`c^ f]R! XZgV_ Sj

$$f(x) \approx f_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \ . \ . \ .$$

$$+ (x - x_0) \ . \ . \ . \ (x - x_{n-1}) f[x_0, \ . \ . \ . \ , x_n]$$

hZeY eYV cV^RZ_UVc eVc^ RWeVc $(n+1)$ eVc^d Zd XZgV_ Sj

$$(x - x_0)\ (x - x_1) \cdots (x - x_n)\ f[x,\ x_0,\ x_1 \cdots, x_n]$$

hYVcV $f[x_0, x_1]$$ $f[x_0, x_1, x_2]$$ ... RcV eYV **divided differences** XZgV_ Sj

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}, \ldots$$

$$f[x_0, \ . \ . \ . \ , x_k] = \frac{f[x_1, \ . \ . \ . \ , x_k] - f[x_0, \ . \ . \ . \ , x_{k-1}]}{x_k - x_0}$$

8]d`$ $f[x,\ x_0,\ x_1, \cdots,\ x_n] = \dfrac{f[x_p\ x_1, \cdots,\ x_n] - f(x,\ x_0,]x_n}{x_0 - x}$

Note @W U($ U)$ & & & $ UK RcV Vbf R]]j daRTVU$ Z&V& hYV_ $x_k = x_0 + kh,$ eYV_ $f[x_0, \ . \ . \ . \ , x_k] = \dfrac{\Delta^k f_0}{k!\, h^k}$ R_U EVhe`_qd UZgZUVU UZWWVcV_TV Z_eVca`]ReZ`_ W`c^ f]R eR\Vd eYV W`c^ `W EVhe`_qd W`chRcU UZWWVcV_TV Z_eVca`]ReZ`_ W`c^ f]R&

Derivation of the formula:

=`c R Wf_TeZ`_ $y = f(x)$$]Ve fd XZgV_ eYV dVe `W $(n+1)$ a`Z_ed$ $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)), ..., (x_n, f(x_n))$. JYV gR]fVd $x_1, x_2, ..., x_n$ `W eYV Z_UVaV_UV_e gRcZRS]V U RcV TR]]VU eYV RcXf^V_ed R_U eYV T`ccVda`_UZ_X gR]fVd $y_1 = f(x_1), y_2 = f(x_2), ..., y_n = f(x_n)$ `W eYV UVaV_UZ_X gRcZRS]V j RcV TR]]VU V_ecZVd& MV UVWZ_V eYV WZcde UZgZUVU UZWWVcV_TV `W $f(x)$ SVehVV_ eh` T`_dVTfeZgV RcXf^V_ed x_i *and* x_{i+1} Rd$

$$f(x_i, x_{i+1}) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \quad for\ i = 0, 1, ..., n-1$$

JYV dVT`_U UZgZUVU UZWWVcV_TV SVehVV_ eYcVV T`_dVTfeZgV RcXf^V_ed x_i, x_{i+1} *and* x_{i+2} Zd XZgV_ Sj\$

$$f\left(x_i, x_{i+1}, x_{i+2}\right) = \frac{f\left(x_{i+1}, x_{i+2}\right) - f\left(x_i, x_{i+1}\right)}{x_{i+2} - x_i} \quad for\ i = 0,1,...,n-2$$

@_ XV_VcR] eYV _eY UZgZUVU UZWWVcV_TV `c UZgZUVU UZWWVcV_TV `W `cUVc _! SVehVV_ $x_1, x_2, ..., x_n$ Zd\$

$$f\left(x_0, x_1, ..., x_n\right) = \frac{f\left(x_1, x_2, ..., x_n\right) - f\left(x_0, x_1, ..., x_{n-1}\right)}{x_n - x_0}$$

?V_TV\$ Z_ aRceZTf]Rc\$ eYV WZcde UZgZUVU UZWWVcV_TV SVehVV_ x_0 *and* x_1 Zd\$

$$f\left(x_0, x_1\right) = \frac{f\left(x_1\right) - f\left(x_0\right)}{x_1 - x_0}$$

JYV dVT`_U UZgZUVU UZWWVcV_TV SVehVV_ eYcVV T`_dVTfeZgV RcXf^V_ed x_0, x_1 *and* x_2 Zd

$$f\left(x_0, x_1, x_2\right) = \frac{f\left(x_1, x_2\right) - f\left(x_0, x_1\right)}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0}\left[\frac{f\left(x_2\right) - f\left(x_1\right)}{x_2 - x_1} - \frac{f\left(x_1\right) - f\left(x_0\right)}{x_1 - x_0}\right]$$

$$= \frac{f\left(x_2\right)}{\left(x_2 - x_0\right)\left(x_2 - x_1\right)} - \frac{f\left(x_1\right)}{\left(x_2 - x_0\right)}\left[\frac{1}{\left(x_2 - x_1\right)} + \frac{1}{\left(x_1 - x_0\right)}\right] + \frac{f\left(x_0\right)}{\left(x_2 - x_0\right)\left(x_1 - x_0\right)}$$

$$= \frac{f\left(x_2\right)}{\left(x_2 - x_0\right)\left(x_2 - x_1\right)} - \frac{f\left(x_1\right)}{\left(x_2 - x_1\right)\left(x_1 - x_0\right)} + \frac{f\left(x_0\right)}{\left(x_2 - x_0\right)\left(x_1 - x_0\right)}$$

$$\Rightarrow \quad f\left(x_0, x_1, x_2\right) = \frac{f\left(x_0\right)}{\left(x_0 - x_2\right)\left(x_0 - x_1\right)} + \frac{f\left(x_1\right)}{\left(x_1 - x_0\right)\left(x_1 - x_2\right)} + \frac{f\left(x_2\right)}{\left(x_2 - x_0\right)\left(x_2 - x_1\right)}$$

8d RS`gV\$ eYV _eY UZgZUVU UZWWVcV_TV SVehVV_ $x_1, x_2, ..., x_n$ \$ $f\left(x_0, x_1, ..., x_n\right)$ Zd ViacVddVU Rd

$$f\left(x_0, x_1, ..., x_n\right) = \frac{f\left(x_0\right)}{\left(x_0 - x_1\right)\left(x_0 - x_2\right)...\left(x_0 - x_n\right)} + \frac{f\left(x_1\right)}{\left(x_1 - x_0\right)\left(x_1 - x_2\right)...\left(x_1 - x_n\right)} + ...$$
$$+ \frac{f\left(x_n\right)}{\left(x_n - x_0\right)\left(x_n - x_1\right)...\left(x_n - x_{n-1}\right)}$$

Properties of divided difference:

1. *The divided differences are symmetrical about their arguments.*

We have, $f\left(x_0, x_1\right) = \dfrac{f\left(x_1\right) - f\left(x_0\right)}{x_1 - x_0}$

$$= \frac{f\left(x_0\right) - f\left(x_1\right)}{x_0 - x_1} = f\left(x_1, x_0\right)$$

$\Rightarrow\ f\left(x_0, x_1\right) = f\left(x_1, x_0\right)$. Hence, the order of the arguments has no importance.

When we are considering the nth divided difference also, we can write, $f\left(x_0, x_1, \ldots, x_n\right)$ as

$$f\left(x_0, x_1, \ldots, x_n\right) = \frac{f\left(x_0\right)}{\left(x_0 - x_1\right)\left(x_0 - x_2\right)\ldots\left(x_0 - x_n\right)} + \frac{f\left(x_1\right)}{\left(x_1 - x_0\right)\left(x_1 - x_2\right)\ldots\left(x_1 - x_n\right)} + \ldots + \frac{f\left(x_n\right)}{\left(x_n - x_0\right)\left(x_n - x_1\right)\ldots\left(x_n - x_{n-1}\right)}$$

From this expression it is clear that, whatever be the order of the arguments, the expression is same.

Hence the divided differences are symmetrical about their arguments.

2. *Divided difference operator is linear.*

For example, consider two polynomials $f\left(x\right)$ *and* $g(x)$. Let

$$h(x) = af\left(x\right) + b\,g(x),$$

where '*a*' and '*b*' are any two real constants. The first divided difference of $h(x)$ corresponding to the arguments x_0 *and* x_1 is,

$$h\left(x_0, x_1\right) = \frac{h\left(x_1\right) - h\left(x_0\right)}{x_1 - x_0} = \frac{af\left(x_1\right) + b\,g(x_1) - af\left(x_0\right) + b\,g(x_0)}{x_1 - x_0}$$

$$= \frac{a\left[f\left(x_1\right) - f\left(x_0\right)\right] + b\left[g(x_1) - g(x_0)\right]}{x_1 - x_0}$$

$$= a\,\frac{f\left(x_1\right) - f\left(x_0\right)}{x_1 - x_0} + b\,\frac{g(x_1) - g(x_0)}{x_1 - x_0}$$

$$= a\ f\left(x_0, x_1\right) + bg\left(x_0, x_1\right)$$

3. *The nth divided difference of a polynomial of degree n is its leading coefficient.*

Consider $f\left(x\right) = x^n$, where *n* is a positive number

Now, $f\left(x_0, x_1\right) = \dfrac{f\left(x_1\right) - f\left(x_0\right)}{x_1 - x_0} = \dfrac{x_1{}^n - x_0{}^n}{x_1 - x_0}$

$$= x_1^{n-1} + x_1^{n-2}x_0 + x_1^{n-3}x_0^{\ 2} + ... + x_0^{n-1}$$

This is a polynomial of degree (n-1) and symmetric in arguments x_o *and* x_1 with leading coefficient 1.

The second divided difference,

$$f\left(x_0, x_1, x_2\right) = \frac{f\left(x_1, x_2\right) - f\left(x_0, x_1\right)}{x_2 - x_0}$$

$$= \frac{\left(x_2^{n-1} + x_2^{n-2}x_1 + ... + x_1^{n-1}\right) - \left(x_0^{n-1} + x_0^{n-2}x_1 + ... + x_1^{n-1}\right)}{x_2 - x_0},$$ which can be expressed as a polynomial of degree n-2, is symmetric about x_0, x_1 *and* x_2 with leading coefficient 1.

Proceeding like this, we get the n[th] divided difference of $f\left(x\right) = x^n$ is 1.

Now we consider a general polynomial of degree n as,

$$g\left(x\right) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$$

Since the divided difference operator is linear, we get n[th] divided difference of $g\left(x\right)$ as a_0, which is the leading coefficient of $g\left(x\right)$.

Example KdZ_X eYV W`]]` hZ_X eRS]V WZ_U f(x) Rd R a`]j_` ^ZR] Z_ U

x	f(x)
–1	3
0	– 6
3	39
6	822
7	1611

JYV UZgZUVU UZWWVcV_TV eRS]V Zd

x	f(x)	f[x$_k$, x$_{k+1}$]			
–1	+				
		–9			
(	–6		.		
		)-		-	
+	+1		,)		)
		*.)		)+	
.	0**		)+*		
		/01			
/	).))				

?V_TV

$$f(x) = 3 + (x+1)(-9) + x(x+1)(6) + x(x+1)(x-3)(5)$$

$$+x(x+1)(x-3)(x-6)$$

$$= x^4 - 3x^3 + 5x^2 - 6.$$

Example =Z_U eYV Z_eVca`]ReZ_X a`]j_`^ZR] Sj EVhe`_qd UZgZUVU UZWWVcV_TV W`c^f]R W`c eYV W`]]`hZ_X eRS]V R_U eYV_ TR]Tf]ReV C *&)!&

U	(	)	*	,
C U	)	)	*	-

$

U	C U	=Zcde UZgZUVU UZWWVcV_TV CPUH%)$ UHQ	IVT`_U UZgZUVU UZWWVcV_TV CPUH%)$ UH$ UH!)Q	JYZcU UZgZUVU UZWWVcV_TV CPUH%)$ UH$ UH!)$ UH!*Q
(	)	$f(x_0, x_1) = 0$		
)	)	$f(x_1, x_2) = 1$	$-1/2$	$-\frac{1}{2}$
*	*	$f(x_2, x_3) = 3/2$	$-1/6$	
,	-			

E`h dfSdeZefeZ_X eYV gR]fVd Z_ eYV W`c^f]R$ hV XVe

$$f(x) \approx 1 + (x-0)(0) + (x-0)(x-1)\left(\frac{1}{2}\right) + (x-0)(x-1)(x-2)\left(-\frac{1}{12}\right)$$

$$= -\frac{1}{12}x^3 + \frac{3}{4}x^2 - \frac{2}{3}x + 1$$

I fSdeZefeZ_X U 5 *&) Z_ eYV RS`gV a`]j_`^ZR]$ hV XVe C *&)!5*&)+-$

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NEWTON' S AND LAGRANGIAN FORMULAE - PART II

Problem: FSeRZ_ EVhe`_qd UZgZUVU UZWWVcV_TV Z_eVca`]ReZ_X a`]j_`^ZR] dReZdWZVU Sj $(-4,1245),\ (-1,33), (0,5), (2,9)\ and\ (5,1335)$&

Solution: EVhe`_qd UZgZUVU UZWWVcV_TV Z_eVca`]ReZ_X a`]j_`^ZR] Zd XZgV_ Sj$

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+....+$$
$$(x-x_0)(x-x_1).....(x-x_{n-1})f(x_0,x_1,...,x_n)$$

?VcV i gR]fVd RcV XZgVd Rd$ %,$ %)$ ($ * R_U 1& :`ccVda`_UZ_X W i! gR]fVd RcV)*,-$++$-$1 R_U)++-&

?V_TV eYV UZgZUVU UZWWVcV_TV Rd dY`h_ Z_ eYV W`]]`hZ_X eRS]V2

N	=Zcde UZgZUVU UZWWVcV_TVd	IVT`_U UZgZUVU UZWWVcV_TVd	JYZcU UZgZUVU UZWWVcV_TVd	=`fceY UZgZUVU UZWWVcV_TVd
-4				
	-404			
%)		**94**		
	%*0		**-14**	
(		)(		**3**
	*		)+	
*		00		
	,,*			
-				

>ZgV_ $f(x_0)=1245$ & =c`^ eYV eRS]V$ hV TR_ `SdVcgV eYRe

$$f(x_0,x_1)=-404;\quad f(x_0,x_1,x_2)=94;$$
$$f(x_0,x_1,x_2,x_3)=-14\ and\ f(x_0,x_1,x_2,x_3,x_4)=3$$

?V_TV eYV Z_eVca`]ReZ_X a`]j_`^ZR] Zd$

$$f(x) = 1245 + \left(x-(-4)\right)\times(-404) + \left(x-(-4)\right)\left(x-(-1)\right)\times 94$$
$$+\left(x-(-4)\right)\left(x-(-1)\right)\left(x-0\right)\times 14 + \left(x-(-4)\right)\left(x-(-1)\right)\left(x-0\right)\left(x-2\right)\times 3$$

$$\Rightarrow\ f(x) = 1245 - 404(x+4) + 94(x+4)(x+1)$$
$$+14(x+4)(x+1)(x-0) + 3(x+4)(x+1)(x-0)(x-2)$$

F_ dZ^a]ZWZTReZ`_$ hV XVe$

$$f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5\text{&}$$

Newton's Interpolation formula with divided differences

:`_dZUVc eh` RcXf^V_ed x and x_0& JYV WZcde UZgZUVU UZWWVcV_TV SVehVV_ x and x_0 Zd$

$$f(x, x_0) = \frac{f(x_0) - f(x)}{x_0 - x} = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow\ f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad \text{%%%%} \quad)!$$

:`_dZUVc x , x_0 and x_1& JYV_$

$$f(x, x_0, x_1) = \frac{f(x_0, x_1) - f(x, x_0)}{x_1 - x} = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\Rightarrow f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1)$$

Gfe Ze Z_)!$ hV XVe$

$$f(x) = f(x_0) + (x - x_0)\left[f(x_0, x_1) \quad (+x \quad x_1) f(x, x_0, x_1) \right.$$

JYRe Zd$

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x, x_0, x_1) \quad \text{%%%} \quad *!$$

8XRZ_$ W`c x , x_0 , x_1 and x_2

$$\Rightarrow\ f(x, x_0, x_1, x_2) = \frac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x_2 - x} = \frac{f(x_0, x_1, x_2) - f(x, x_0, x_1)}{x - x_2}$$

$$\Rightarrow f(x, x_0, x_1) = (x_2 - x) f(x, x_0, x_1, x_2) + f(x_0, x_1, x_2)$$

?V_TV *! Z^a]ZVd$

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)\left[(x-x_2)f(x,x_0,x_1,x_2)+f(x_0,x_1,x_2)\right]$$

$$=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x,x_0,x_1,x_2)$$

Gc`TVVUZ_X ]Z\V eYZd$ hV `SeRZ_ W`c $f(x)$ Rd$

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+....+$$
$$(x-x_0)(x-x_1).....(x-x_n)f(x,x_0,x_1,...,x_n)$$

@W W i! Zd R a`]j_`^ZR] `W UVXcVV _$ eYV_ $f(x,x_0,x_1,...,x_n)=0$$ SVTRfdV Ze Zd eYV _#)!eY UZWWVcV_TV&

?V_TV hV XVe$

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+....+$$
$$(x-x_0)(x-x_1).....(x-x_{n-1})f(x_0,x_1,...,x_n)$$

JYZd Zd _`h_ Rd **Newton's interpolation formula with divided difference**&

Note:

)& =`c eYV XZgV_ RcXf^V_ed $x_1,x_2,...,x_n$$ ZW R]] eYV \eY $ \4_! UZgZUVU UZWWVcV_TVd RcV VbfR]$ eYV \#)eY UZgZUVU UZWWVcV_TVd RcV kVc`Vd& JYV_ EVhe`_qd Z_eVca`]ReZ`_ W`c^f]R XZgVd R a`]j_`^ZR] `W UVXcVV \ W`c eYV XZgV_ URER&

*& EVhe`_qd UZgZUVU UZWWVcV_TV Z_eVca`]ReZ`_ W`c^f]R a`ddVddVd eYV aVc^R_V_TV ac`aVcej& 8aRce Wc`^ eYV XZgV_ RcXf^V_ed $x_1,x_2,...,x_n$ R]`_X hZeY eYV T`ccVda`_UZ_X Wf_TeZ`_ gR]fVd$ dfaa`dV eYRe `_ R ]ReVc eZ^V R _Vh RcXf^V_e $x_{n+1}$ hZeY T`ccVda`_UZ_X V_ecj $f(x_{n+1})$ RcV XZgV_& JYV _Vh dVe `W URER gR]fVd TR_ SV cVacVdV_eVU Sj R a`]j_`^ZR] `W UVXcVV _#)!& J` `SeRZ_ eYV cVbfZcVU a`]j_`^ZR] hV RUU eYV eVc^ $(x-x_0)(x-x_1).....(x-x_n)f(x_0,x_1,...,x_n,x_{n+1})$ e` eYV acVgZ`fd]j `SeRZ_VU _eY UVXcVV a`]j_`^ZR]&

Problem 2: JYV W`]]`hZ_X eRS]V XZgVd eYV cV]ReZ`_ SVehVV_ deVR^ acVddfcV R_U eV^aVcRefcV& =Z_U eYV acVddfcV Re eV^aVcRefcV +/-(&

JV^a&2 +.)(+./(+/0(+0/(+11(

GcVddfcV2)-,&1)./&1)1) *)*&- *,,&*

Solution:

J` WZ_U eYV acVddfcV Re eV^aVcRefcV +/-($ Ze Zd e` VdeRS]ZdY eYV cV]ReZ`_ XZgZ_X acVddfcV Z_ eVc^d `W eV^aVcRefcV& CVe fd T`_dZUVc eV^aVcRefcV Rd i gR]fVd R_U acVddfcV Rd T`ccVda`_UZ_X W i! gR]fVd&

JYV XZgV_ i gR]fVd RcV +.)($ +./($ +/0($ +0/(R_U +11(& :`ccVda`_UZ_X W i! gR]fVd RcV)-,&1$)./&1$)1)$*)*&- R_U *,,&*&

W i! Zd `SeRZ_VU Sj EVhe`_qd UZgZUVU UZWWVcV_TV Z_eVca`]ReZ_X a`]j_`^ZR] Rd$

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+....+$$
$$(x-x_0)(x-x_1).....(x-x_n)f(x,x_0,x_1,...,x_n)$$

>ZgV_ $f(x_0)=f(361^0)=154.9$& JYV UZgZUVU UZWWVcV_TVd W`c eYV XZgV_ a`Z_ed RcV Rd dY`h_ Z_ eYV eRS]V&

N	=Zcde UZgZUVU UZWWVcV_TVd	IVT`_U UZgZUVU UZWWVcV_TVd	JYZcU UZgZUVU UZWWVcV_TVd	=`fceY UZgZUVU UZWWVcV_TVd
361				
	2.01666			
+./		**0.00971**		
	*&)0)0)		**0.0000246**	
+/0		(&()(+-		**(.00000074**
	*&+0000		(&((((-*0	
+0/		(&()*(,		
	*&.,)..			
+11				

=c`^ eYV eRS]V$ hV TR_ `SdVcgV eYRe

$$f(x_0,x_1)=2.01666;\quad f(x_0,x_1,x_2)=0.00971;$$
$$f(x_0,x_1,x_2,x_3)=0.0000246\ \ and\ \ f(x_0,x_1,x_2,x_3,x_4)=0.00000074$$

?V_TV$

$$f(x)=154.9+(x-361)\times 2.01666+(x-361)(x-367)\times 0.00971+$$
$$(x-361)(x-367)(x-378)\times 0.0000246+(x-361)(x-367)(x-378)(x-387)\times 0.0000074$$

IfSdeZefeZ_X i5+/- Z_ eYV RS`gV Vi acVddZ`_ XZgVd$ W +/-!5)0,&*)-,0&

Problem 3: FSeRZ_ EVhe`_qd UZgZUVU UZWWVcV_TV Z_eVca`]ReZ_X a`]j_`^ZR] dReZdWjZ_X eYV W`]]`hZ_X gR]fVd2 i2) + , - /)(

W i!2 + +) .1)+) +-))())

8]d` WZ_U W ,&-!$ W O! R_U eYV dVT`_U UVcZgReZgV `W W i! Re i5+&*&

Solution:

J` `SeRZ_ eYV EVhe`_qd UZgZUVU UZWWVcV_TV Z_eVca`]ReZ_X a`]j_`^ZR] W i!$ hV _VVU eYV UZgZUVU UZWWVcV_TV fdZ_X eYV XZgV_ gR]fVd&

@e Zd TR]Tf]ReVU R_U]ZdeVU Z_ eYV W`]]`hZ_X eRS]V

N	=Zcde UZgZUVU UZWWVcV_TVd	IVT`_U UZgZUVU UZWWVcV_TVd	JYZcU UZgZUVU UZWWVcV_TVd	=`fceY UZgZUVU UZWWVcV_TVd
1	**14**	**8**	**1**	(
+	+0	)*	)	(
,	.*	).	)	
-	))(	**		
/	**(			
1				

IZ_TV eYV W`fceY UZgZUVU UZWWVcV_TVd RcV kVc`Vd$ W i! Zd `W UVXcVV + R_U Ze Zd `SeRZ_VU Rd$

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$

$$f(x_0)=f(1)=3;\ f(x_0,x_1)=14;\ f(x_0,x_1,x_2)=8 \quad and \quad f(x_0,x_1,x_2,x_3)=1$$

$$\Rightarrow\ f(x)=3+(x-1)\times 14+(x-1)(x-3)\times 8+(x-1)(x-3)(x-4)\times 1$$

JYRe Zd$

$$f(x)=x^3+x+1$$

?V_TV$ $f(4.5)=(4.5)^3+4.5+1=96.625$ R_U $f(8)=(8)^3+8+1=521$

IVT`_U UVcZgReZgV `W $f(x)$ *is* $6x$& E`h dVT`_U UVcZgReZgV `W W i! Re i5+&* Zd $6\times 3.2=19.2$

Lagrangian Interpolation

8_`eYVc ^VeY`U `W Z_eVca`]ReZ`_ Z_ eYV TRdV `W >0?FQ0>0FIV PM>@BA MFSLQ>I S>IRBP U($ U)$ & & & $ UK Zd CRXcR_XZR_ Z_eVca`]ReZ`_& JYZd ^VeY`U Zd SRdVU `_ CRXcR_XVqd K#) a`Z_e Z_eVca`]ReZ`_ W`c^f]R XZgV_ Sj

$$f(x) \approx L_n(x) = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} f_k \$$$

hYVcV

$l_0(x) = (x - x_1)(x - x_2) \ldots . (x - x_n)$ $

$l_k(x) = (x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)$ $ (4 H 4 K&

$l_n(x) = (x - x_0)(x - x_1) \ldots .(x - x_{n-1})$

HV^Rc\2 $L_k(x_k) = f_k$. =`c$ $l_k(x_j) = 0,\ when\ j \neq k$, d` eYRe W`c $x = x_k$ $ eYV df^ `_ eYV H?I `W eYV W`c^f]R cVUfTVd e` eYV dZ_X]V eVc^ $f_k$ $ hYZTY Z_UZTReVd eYRe C R_U $L_k$ RXcVVd Re K#) eRSf]ReVU a`Z_ed&

Derivation of the formula:

Given the set of $(n+1)$ points, $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)), \ldots, (x_n, f(x_n))$ of x and *f(x),* it is required to fit the unique polynomial $p_n(x)$ of maximum degree *n*, such that $f(x)$ and $p_n(x)$ agree at the given set of points. The values $x_0, x_1, \ldots, x_n$ may not be equidistant.

Since the interpolating polynomial must use all the ordinates $f(x_0), f(x_1), \ldots, f(x_n)$, it can be written as a linear combination of these ordinates. That is, we can write the polynomial as

$$p_n(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + \cdots + l_n(x) f(x_n).$$

where $f(x_i)$ *and* $l_i(x)$, *for* $i = 0,1,2,\ldots,n$ are polynomials of degree *n*.

This polynomial fits the given data exactly.

At $x = x_0$, as $p_n(x)$ and $f(x)$ coincide, we get,

$$f(x_0) = p_n(x_0) = l_0(x_0) f(x_0) + l_1(x_0) f(x_1) + \ldots + l_n(x_0) f(x_n)$$

This equation is satisfied only when $l_0(x_0) = 1$ and $l_i(x_0) = 0, i \neq 0$

At a general point $x = x_i$, we get,

$$f(x_i) = p_n(x_i) = l_0(x_i) f(x_0) + l_1(x_i) f(x_1) + ... + l_n(x_i) f(x_n)$$

This equation is satisfied only when $l_i(x_i) = 1$ and $l_j(x_i) = 0, i \neq j$

Therefore, $l_i(x)$, which are polynomials of degree *n*, satisfy the conditions

$$l_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Since, $l_i(x) = 0$ at $x = x_0, x_1, ..., x_{i-1}, x_{i+1}, ..., x_n$, we know that

$(x - x_0), (x - x_1), ..., (x - x_{i-1}), (x - x_{i+1}), ..., (x - x_n)$ are factors of $l_i(x)$. The product of these factors is a polynomial of degree *n*. Therefore, we can write

$$l_i(x) = C(x - x_0)(x - x_1)...(x - x_{i-1})(x - x_{i+1})...(x - x_n)$$, where *C* is a constant.

Now, since $l_i(x_i) = 1$, we get

$$l_i(x_i) = 1 = C(x_i - x_0)(x_i - x_1)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n)$$

Hence, $C = \dfrac{1}{(x_i - x_0)(x_i - x_1)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n)}$

Therefore,

$$l_i(x) = \frac{(x - x_0)(x - x_1)...(x - x_{i-1})(x - x_{i+1})...(x - x_n)}{(x_i - x_0)(x_i - x_1)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n)}$$

Now the polynomial

$$p_n(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + ... + l_n(x) f(x_n),$$

with $l_i(x) = \dfrac{(x - x_0)(x - x_1)...(x - x_{i-1})(x - x_{i+1})...(x - x_n)}{(x_i - x_0)(x_i - x_1)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n)}$ is called Lagrange interpolating polynomial and $l_i(x)$ are called Lagrange fundamental polynomials.

To fit a polynomial of degree 1, we require at least two points. Let $(x_0, f(x_0)), (x_1, f(x_1))$ are the points. Then the Lagrange polynomial of degree one or a straight line for the given data is,

$$p_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$, where, $l_0(x) = \dfrac{(x - x_1)}{(x_0 - x_1)}$ and $l_1(x) = \dfrac{(x - x_0)}{(x_1 - x_0)}$.

Let $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))$ are the given three points. Then the Lagrange polynomial of degree two for the data is given by

$$p_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$, where,

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \quad l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \text{ and } l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

For the four points $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3))$, the Lagrange polynomial of degree three is given by,

$$p_3(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2) + l_3(x) f(x_3),$$ where, $$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}, \quad l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$ and

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$ and so on.

Problem : Given f(2) = 9, and f(6) = 17. Find an approximate value for f(5) by the method of Lagrange's interpolation.

Solution:

For the given two points (2,9) and (6,17), the Lagrangian polynomial of degree 1 is $p_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$, where, $l_0(x) = \frac{(x-x_1)}{(x_0-x_1)}$ and $l_1(x) = \frac{(x-x_0)}{(x_1-x_0)}$. That is,

$$p_1(x) = \frac{(x-x_1)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1)$$

$$\Rightarrow \quad p_1(x) = \frac{(x-6)}{(2-6)} \times 9 + \frac{(x-2)}{(6-2)} \times 17$$

Hence,

$$f(5) = \quad P_1(5) = \frac{(5-6)}{(2-6)} \times 9 + \frac{(5-2)}{(6-2)} \times 17$$

$$= \frac{1}{4} \times 9 + \frac{3}{4} \times 17$$

$$=15$$

Problem: *Use Lagrange's formula, to find the quadratic polynomial that takes the values*

x	:	0	1	3
$f(x)$	:	0	1	0

For the given three points (0,0) , (1,1) and (3,0), the quadratic polynomial by Lagrange's interpolation is $p_2(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$

We are considering the given x values 0,1, and 3 as x_0, x_1 *and* x_2 . Given, $f(x_0)$ *and* $f(x_2)$ are zeroes. Hence the polynomial is,

$$p_2(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)$$

Then,

$$p_2(x)=\frac{(x-0)(x-3)}{(1-0)(1-3)}\times 1$$

$$\Rightarrow \quad p_2(x)=\frac{x(x-3)}{-2}\times 1=\frac{1}{2}\left(3x-x^2\right).$$

Example =Z_U CRXcR_XVqd Z_eVca`]ReZ`_ a`]j_`^ZR] WZeeZ_X eYV a`Z_ed C)! 5 –+$ C +! 5 ($ C ,! 5 +($ C .! 5)+*& ?V_TV WZ_U C -!&

?VcV , eRS f]ReVU a`Z_ed RcV XZgV_& ?V_TV hV _VVU CRXcR_XVqd a`]j_`^ZR] W`c K ! 5 + #) 5 , a`Z_ed! R_U Zd XZgV_ Sj

$$f(x)\approx L_3(x)=\sum_{k=0}^{3}\frac{l_k(x)}{l_k(x_k)}f_k$$ &

E`h dfSdeZe feZ_X eYV gR]fVd$ hV `SeRZ_

$$L_3(x)=\frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3)+\frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0)$$

$$+\frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30)+\frac{(x-1)(x-3)(x-4)}{(6-1)(6-4)(6-4)}(132)$$

$=\frac{1}{2}(-x^3+27x^2-92x+60),$ `_ dZ^ a]ZWZTReZ`_&

E`h $f(5)\approx L_3(5)=\frac{1}{2}\left(-(5)^3+27(5)^2-92(5)+60\right)=75.$

Example =Z_U $\ln 9.2$ hZeY $n=3$$ fdZ_X CRXcR_XVqd Z_eVca`]ReZ`_ W`c^ f]R hZeY eYV XZgV_ eRS]V2

U	1&(	1&-	)(&(	))&(
]_U	*&)1/	*&*-)	*&+(*	*&+1/

** *1 -1 1(

$$\ln(9.2) = f(9.2) \approx L_3(9.2) = \sum_{k=0}^{3} \frac{l_k(9.2)}{l_k(x_k)} f_k \&$$

$$= \frac{(9.2-9.5)(9.2-10.0)(9.2-11.0)}{(9.0-9.5)(9.0-10.0)(9.0-11.0)}(2.19722)$$

$$+\frac{(9.2-9.0)(9.2-10.0)(9.2-11.0)}{(9.5-9.0)(9.5-10.0)(9.5-11.0)}(2.25129)$$

$$+\frac{(9.2-9.0)(9.2-9.5)(9.2-11.0)}{(10.0-9.0)(10.0-9.5)(10.0-11.0)}(2.30259)$$

$$+\frac{(9.2-9.0)(9.2-9.5)(9.2-10.0)}{(11.0-9.0)(11.0-9.5)(11.0-10.0)}(2.39790)$$

5 *&*)1 *($ hYZTY Zd ViRTe e` - ; &

Example : VceRZ_ T`ccVda`_UZ_X gR] f Vd `W U R_U $\log_{10} x$ RcV (300, 2.4771), (304, 2.4829), (305, 2.4843) R_U (307, 2.4871). =Z_U $\log_{10} 301$.

$$\log_{10} 301 = \frac{(-3)\,(-4)\,(-6)}{(-4)\,(-5)\,(-7)}(2.4771) + \frac{(1)\,(-4)\,(-6)}{(4)\,(-1)\,(-3)}(2.4829)$$

$$+\frac{(1)\,(-3)\,(-6)}{(5)\,(1)\,(-2)}(2.4843) + \frac{(1)\,(-3)\,(-4)}{(7)\,(3)\,(2)}(2.4871)$$

$$= 1.2739 + 4.9658 - 4.4717 + 0.7106$$

$$= 2.4786.$$

Inverse Lagrangian Interpolation Formula

@_eVcTYR_XZ_X x R_U y Z_ eYV CRXcR_XZR_ @_eVca`]ReZ`_ =`c^f]R$ hV `SeRZ_ eYV **inverse Lagrangian interpolation formula** XZgV_ Sj

$$x \approx L_n(y) = \sum_{k=0}^{n} \frac{l_k(y)}{l_k(y_k)} x_k.$$

Example @W $y_1 = 4,\ y_3 = 12,\ y_4 = 19$ R_U $y_x = 7$, WZ_U U& : `^aRcV hZeY eYV RTe f R] gR] f V&

KdZ_X eYV Z_gVcdV Z_eVca`]ReZ`_ W`c^f]R$

$$x \approx L_n(7) = \sum_{k=0}^{2} \frac{l_k(7)}{l_k(y_k)} x_k$$

hYVcV $x_0 = 1,\ y_0 = k,\quad x_y = 3,\ y_1 = 12\quad x_2 = 4,\ y_2 = 19$ R_U $y = 7$

Z&V&$ $x \approx \dfrac{(7-y_1)\,(7-y_2)}{(y_0-y_1)\,(y_0-y_2)}x_0 + \dfrac{(7-y_0)\,(7-y_2)}{(y_1-y_0)\,(y_1-y_2)}$

$$x_1 + \frac{(7-y_0)\,(7-y_1)}{(y_2-y_0)\,(y_2-y_1)}x_2$$

Z&V&$ $x = \dfrac{(-5)\,(-12)}{(-8)\,(-15)}(1) + \dfrac{(3)\,(-12)}{(8)\,(-7)}(3) + \dfrac{(3)\,(-5)}{(15)\,(7)}(4)$

$$= \frac{1}{2} + \frac{27}{14} - \frac{4}{7}$$

5)&0.

JYV RTefR] gR]fV Zd *&(dZ_TV eYV RS`gV gR]fVd hVcV `SeRZ_VU Wc`^ eYV a`]j_`^ZR] $y(x) = x^2 + 3.$

Example =Z_U eYV CRXcR_XV Z_eVca`]ReZ_X a`]j_`^ZR] `W UVXcVV * Raac`iZ^ReZ_X eYV Wf_TeZ`_ $y = \ln x$ UVWZ_VU Sj eYV W`]]`hZ_X eRS]V `W gR]fVd& ?V_TV UVeVc^Z_V eYV gR]fV `W]_ *&/&

x	$y = \ln x$
2	0.69315
2.5	0.91629
3.0	1.09861

IZ^Z]Rc]j$

$l_1(x) = -(4x^2 - 20x + 24)$ R_U $l_2(x) = 2x^2 - 9x + 10.$

?V_TV

$$L_2(x) = \frac{l_0(x)}{l_0(x_k)}f_0 + \frac{l_1(x)}{l_1(x_k)}f_1 + \frac{l_2(x)}{l_0(x_k)}f_2$$

$$= \frac{(x-2.5)\,(x-3.0)}{(-0.5)\,(-1.0)}\cdot f_0 + \frac{(x-2)\,(x-3)}{(2.5-2)\,(3.0-2.5)}f_1 + \frac{(x-2)\,(x-2.5)}{(3-2)\,(3-2.5)}f_2$$

$$= (2x^2 - 11x + 15)\,(0.69315) - (4x^2 - 20x + 24)\,(0.91629)$$

$$+ (2x^2 - 9x + 10)\,(1.09861)$$

$$= -0.08164x^2 + 0.81366x - 0.60761.$$

hYZTY Zd eYV cVbfZcVU bfRUcReZT a`]j_`^ZR]&

G f eeZ_X U 5 *&/$ Z_ eYV RS`gV a`]j_` ^ZR]$ hV `SeRZ_

$\ln 2.7 \approx L_2(2.7) = -0.08164(2.7)^2 + 0.81366(2.7) - 0.60761 = 0.9941164.$ 8Te f R] gR] f V `W
$\ln 2.7 = 0.9932518,$ d` eYRe

$$|\,\text{Error}\,| = 0.0008646.$$

Example JYV W f _TeZ`_ $y = \sin x$ Zd eRS f]ReVU SV]`h

x	$y = \sin x$
0	0
® / 4	0.70711
® / 2	1.0

KdZ_X CRXcR_XVqd Z_eVca`]ReZ`_ W`c^ f]R$ WZ_U eYV gR] f V `W sin(® / 6).

9LIR QFLK MV YRgV

$$\sin\frac{®}{6} \approx \frac{(®/6-0)\,(®/6-®/2)}{(®/4-0)\,(®/4-®/2)}(0.70711) + \frac{(®/6-0)\,(®/6-®/4)}{(®/2-0)\,(®/2-®/4)}(1) \quad = \frac{8}{9}(0.70711) - \frac{1}{9}$$

$$= \frac{4.65688}{9} = 0.51743.$$

Example KdZ_X CRXcR_XVdq Z_eVca`]ReZ`_ W`c^ f]R$ WZ_U eYV W`c^ `W eYV W f _TeZ`_ $y(x)$ Wc`^ eYV W`]]`hZ_X eRS]V&

x	y
0	−12
1	0
3	12
4	24

IZ_TV $y = 0$ hYV_ $x = 1,$ Ze W`]]`hd eYRe $x - 1$ Zd R WRTe`c& CVe $y(x) = (x-1)R(x).$ JYV_ $R(x) = y/(x-1).$ MV _`h eRS f]ReV eYV gR] f Vd `W U R_U $R(x)$ Z =`c $x = 0,$ $R(0) = \frac{-12}{0-1} = 12,$ R_U d` `_&

x	R(x)
0	12
3	6
4	8

8aa]jZ_X CRXcR_XVqd W`c^ f]R e` eYV RS`gV eRS]V$ hV WZ_U

$$R(x) = \frac{(x-3)(x-4)}{(-3)\,(-4)}(12) + \frac{(x-0)(x-4)}{(3-0)\,(3-4)}(6) + \frac{(x-0)(x-3)}{(4-0)\,(4-3)}(8)$$

$= (x-3)(x-4) - 2x(x-4) + 2x(x-3)$

5 $x^2 - 5x + 12.$

?V_TV eYV cVbfZcVU a`]j_`^ZR] Raac`iZ^ReZ`_ e` $y(x)$ Zd XZgV_ Sj

$$y(x) = (x-1)(x^2 - 5x + 12).$$

Example MZeY eYV fdV `W EVhe`_qd UZgZUVU UZWWVcV_TV W`c^f]R$ WZ_U $\log 10^{301}$. >ZgV_ eYV W`]]`hZ_X UZgZUVU UZWWVcV_TV eRS]V

x	$f(x) = \log_{10} x$	$f[x_{k-1}, x_k]$	$f[x_{k-2}, x_k, x_{k+1}]$
+((	*&,//),		
		(&(),-	
+(,	*&,0*1		(&(((()
		(&(),(	
+(-	*&,0,+		(
		(&(),(	
+(/	*&,0/)		

$\log_{10} 301 = 2.4771 + 0.00145 + (-3)(-0.00001) = 2.4786,$ Rd SVW`cV&

@e Zd T]VRc eYRe eYV RcZeY^VeZT Z_ eYZd ^VeY`U Zd ^fTY dZ^a]Vc hYV_ T`^aRcV e` eYRe Z_ CRXcR_XVqd ^VeY`U&

Exercises

9. KdZ_X eYV UZWWVcV_TV eRS]V Z_ ViVcTZdV)$ T`^afeV T`d(&/- Sj EVhe`_qd W`chRcU UZWWVcV_TV Z_eVca`]ReZ_X W`c^f]R hZeY $n = 1, 2, 3, 4$ R_U T`^aRcV hZeY eYV - ; %gR]fV (&/+) .1&

10. KdZ_X eYV UZWWVcV_TV eRS]V Z_ ViVcTZdV)$ T`^afeV T`d(&*0 Sj EVhe`_qd W`chRcU UZWWVcV_TV Z_eVca`]ReZ_X W`c^f]R hZeY $n = 1, 2, 3, 4$ R_U T`^aRcV hZeY eYV - ; %gR]fV

11. KdZ_X eYV gR]fVd XZgV_ Z_ eYV eRS]V$ WZ_U T`d(&*0 Z_ cRUZR_ ^VRdfcV! Sj ]Z_VRc Z_eVca`]ReZ`_ R_U Sj bfRUcReZT Z_eVca`]ReZ`_ R_U T`^aRcV eYV cVdf]ed hZeY eYV gR]fV (&1.) (. ViRTe e` - ; !&

U	C U -T`dU	=Zcde UZWWVcV_TV	IVT`_U UZWWVcV_TV
(&(	)&(((((	%(&()1 1+	%(&(+1(0
(&*	(&10((/	%(&(-1 ()	%(&(+./)
(&,	(&1*) (.	%(&(1- /*	%(&(+*1)

(&.	(&0*- +,	%(&)*0 .+	%(&(*//0
(&0	(&.1. /)	%(&)-. ,)	
)&(	(&-,(+(		

12. =Z_U CRXcR_XZR_ Z_eVca`]ReZ`_ a`]j_`^ZR] W`c eYV Wf_TeZ`_ C YRgZ_X $f(4)=1,\ f(6)=3,\ f(8)=8,\ f(10)=16$& 8]d` TR]Tf]ReV $f(7)$&

13. JYV dR]Vd Z_ R aRceZTf]Rc dY`a W`c eYV]Rde eV_ jVRcd Zd XZgV_ Z_ eYV eRS]V2

OVRc	)11.	)110	*(((	*((*	*((,
IR]Vd Z_]R\Yd!	,(	,+	,0	-*	-/

<deZ^ReV eYV dR]Vd W`c eYV jVRc *(() fdZ_X EVhe`_qd SRT\hRcU UZWWVcV_TV Z_eVca`]ReZ_X W`c^f]R&

14. =Z_U $f(3)$$ fdZ_X CRXcR_XZR_ Z_eVca`]ReZ`_ W`c^f]R W`c eYV Wf_TeZ`_ C YRgZ_X $f(1)=2,\ f(2)=11,\ f(4)=77$&

15. =Z_U eYV TfSZT a`]j_`^ZR] hYZTY eR\Vd eYV W`]]`hZ_X gR]fVd2

U	(	)	*	+
$f(x)$		)	*	)	)(

16. :`^afeV dZ_(&+ R_U dZ_(&- Sj <gVcVee W`c^f]R R_U eYV W`]]`hZ_X eRS]V&

	dZ_U	δ^*
(&*	(&)10./	%(&((/1*
(&,	(&+01,*	%(&()--+
&.	(&-.,.,	%(&(**-(

9& JYV W`]]`hZ_X eRS]V XZgVd eYV UZdeR_TVd Z_ _RfeZTR] ^Z]Vd `W eYV gZdZS]V Y`cZk`_ W`c eYV XZgV_ YVZXYed Z_ WVVe RS`gV eYV VRceYqd dfcWRTV2

U -YVZXYe 2	)((	)-(	*((	*-(	+((	+-(	,((
V 5 UZdeR_TV 2	)(&.+	)+&(+	)-&(,	).&0)	)0&,*	)1&1(	*)&*/

=Z_U eYV gR]fV `W V hYV_ U 5 *)0 We . KP2)-&.11!

10. KdZ_X eYV dR^V UReR Rd Z_ Vi VcTZdV 1$ WZ_U eYV gR]fV `W V hYV_ U 5 ,)(We&

8

INTERPOLATION BY ITERATION

Interpolation by Iteration

>ZgV_ eYV $(n+1)$ a`Z_ed $(x_0, f_0), (x_1, f_1), \cdots, (x_n, f_n)$, hYVcV eYV gR]fVd `W U _VVU _`e _VTVddRcZ]j SV Vbf R]]j daRTVU$ eYV_ e` WZ_U eYV gR]fV `W C T`ccVda`_UZ_X e` R_j XZgV_ gR]fV `W U hV ac`TVVU ZeVcReZgV]j Rd W`]]`hd2 `SeRZ_ R WZcde Raac`iZ^ReZ`_ e` C Sj T`_dZUVcZ_X eYV WZcde%e h` a`Z_ed `_]j3 eYV_ `SeRZ_ Zed dVT`_U Raac`iZ^ReZ`_ Sj T`_dZUVcZ_X eYV WZcde%eYcVV a`Z_ed$ R_U d` `_& MV UV_`eV eYV UZWWVcV_e Z_eVca`]ReZ`_ a`]j_`^ZR]d Sj $\Delta(x)$, hZeY dfZeRS]V dfSdTcZaed$ d` eYRe Re eYV WZcde deRXV `W Raac`iZ^ReZ`_$ hV YRgV

$$\Delta_{01}(x) = f_0 + (x - x_0)\, f[x_0, x_1] = \frac{1}{x_1 - x_0}\begin{vmatrix} f_0 & x_0 & -x \\ f_1 & x_1 & -x \end{vmatrix}\&$$

IZ^Z]Rc]j$ hV TR_ W`c^ $\Delta_{02}(x), \Delta_{03}(x), \cdots$

EVie hV W`c^ $\Delta_{012}$ Sj T`_dZUVcZ_X eYV WZcde%eYcVV a`Z_ed2

$$\Delta_{012}(x) = \frac{1}{x_2 - x_1}\begin{vmatrix} \Delta_{01}(x) & x_1 & -x \\ \Delta_{02}(x) & x_2 & -x \end{vmatrix}\&$$

IZ^Z]Rc]j hV `SeRZ_ $\Delta_{013}(x), \Delta_{014}(x)$, VeT& 8e eYV KeY deRXV `W Raac`iZ^ReZ`_$ hV `SeRZ_

$$\Delta_{012\cdots n}(x) = \frac{1}{x_n - x_{n-1}}\begin{vmatrix} \Delta_{012\cdots \overline{n-1}}(x) & x_{n-1} & -x \\ \Delta_{012\cdots \overline{n-2}n}(x) & x_n & -x \end{vmatrix}\&$$

JYV T`^afeReZ`_d Zd RccR_XVU Rd Z_ eYV W`]]`hZ_X JRS]V

Table 1 8Ze\V_qd ITYV^V

x	f				
x_0	f_0				
		$\Delta_{01}(x)$			
x_1	f_1		$\Delta_{012}(x)$		
		$\Delta_{02}(x)$		$\Delta_{0123}(x)$	
x_2	f_2		$\Delta_{013}(x)$		$\Delta_{01234}(x)$
		$\Delta_{03}(x)$		$\Delta_{0124}(x)$	
x_3	f_3		$\Delta_{014}(x)$		
		$\Delta_{04}(x)$			
x_4	f_4				

8 ^`UZWZTReZ`_ `W eYZd dTYV^V$ UfV e` EVgZ]]V$ Zd XZgV_ Z_ eYV W`]]`hZ_X JRS]V& EVgZ]]Vqd dTYV^V Zd aRceZTf]Rc]j dfZeVU W`c ZeVcReVU Z_gVcdV Z_eVca`]ReZ`_&

Table 2 EVgZ]]Vqd ITYV^V

x	f				
x_0	f_0				
		$\Delta_{01}(x)$			
x_1	f_1		$\Delta_{012}(x)$		
		$\Delta_{12}(x)$		$\Delta_{0123}(x)$	
x_2	f_2		$\Delta_{123}(x)$		$\Delta_{01234}(x)$
		$\Delta_{23}(x)$		$\Delta_{1234}(x)$	
x_3	f_3		$\Delta_{234}(x)$		
		$\Delta_{34}(x)$			
x_4	f_4				

***Example* 26** KdZ_X 8Ze\V_qd dTYV^V R_U eYV W`]]`hZ_X gR]fVd VgR]fReV $\log_{10} 301$.

x	$\log_{10} x$			
+((	*&,//)			
		*&,/0--		
+(,	*&,0*1		*&,/0-0	
		*&,/0-,		*&,/0.(
+(-	*&,0,+		*&,/0-/	
		*&,/0-+		
+(/	*&,0/)			

9LIRQFLK

$\log_{10} 301 = 2.4786.$

Inverse Interpolation

>ZgV_ R dVe `W gR]fVd `W U R_U V$ eYV ac`TVdd `W WZ_UZ_X eYV gR]fV `W U W`c R TVceRZ_ gR]fV `W V Zd TR]]VU FKSBOPB FKQBOMLI>QFLK$ MYV_ eYV gR]fVd `W U RcV Re f_VbfR] Z_eVcgR]d$ eYV ^`de `SgZ`fd hRj `W aVcW`c^Z_X eYZd ac`TVdd Zd Sj Z_eVcTYR_XZ_X U R_U V Z_ CRXcR_XVqd `c 8Ze\V_qd ^VeY`Ud&

Example @W $y_1 = 4,\ y_3 = 12,\ y_4 = 19$ R_U $y_x = 7$, WZ_U U& : `^aRcV hZeY eYV RTefR] gR]fV&

9LIRQFLK

8Ze\V_qd dTYV^V dVV JRS]V)!Zd

y	x		
,	)	)&/-(	)&0-/
)*	+	)&.((	
)1	,		

hYVcVRd EVgZ]]Vqd dTYV^V dVV JRS]V *! XZgVd

y	x		
,	)	)&/-(	)&0-/
)*	+	*&*0.	
)1	,		

@_ eYZd ViR^a]Vd S`eY eYV dTYV^Vd XZgV eYV dR^V cVdf]e&

Method of Successive Approximations

MV deRce hZeY EVhe`_qd W`chRcU UZWWVcV_TV W`c^f]R hYZTY Zd hcZeeV_ Rd

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \cdots$$

=c`^ eYZd hV `SeRZ_

$$u = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u(u-1)}{2}\Delta^2 y_0 - \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 - \cdots\right].$$

EVX]VTeZ_X eYV dVT`_U R_U YZXYVc UZWWVcV_TVd$ hV `SeRZ_ eYV WZcde Raac`iZ^ReZ`_ e` R Rd W`]]`hd

$$u_1 = \frac{1}{\Delta y_0}(y_u - y_0).$$

EVie$ hV `SeRZ_ eYV dVT`_U Raac`iZ^ReZ`_ e` R Sj Z_T]fUZ_X eYV eVc^ T`_eRZ_Z_X eYV dVT`_U UZWWVcV_TVd& JYfd$

$$u_2 = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_1(u_1-1)}{2}\Delta^2 y_0\right]$$

hYVcV hV YRgV fdVU eYV gR]fV `W $u_1$ W`c R Z_ eYV T`VWWZTZV_e `W $\Delta^2 y_0$& IZ^Z]Rc]j$ hV `SeRZ_

$$u_3 = \frac{1}{\Delta y_0}\left[y_u - y_0 - \frac{u_2(u_2-1)}{2}\Delta^2 y_0 - \frac{u_2(u_2-1)(u_2-2)}{6}\Delta^3 y_0\right]$$

R_U d` `_& JYZd ac`TVdd dY`f]U SV T`_eZ_fVU eZ]] e h` dfTTVddZgV Raac`iZ^ReZ`_d e` R RXcVV hZeY VRTY `eYVc e` eYV cVbfZcVU RTTfcRTj& JYV ^VeY`U Zd Z]]fdecReVU Z_ eYV W`]]`hZ_X ViR^a]V&

Example JRSf]ReV $y = x^3$ W`c $x = 2, 3, 4$ R_U -$ R_U TR]Tf]ReV eYV TfSV c``e `W )( T`ccVTe e` QEOBB UVTZ^R] a]RTVd&

9LIRQFLK

x	$y = x^3$	Δ	Δ^2	Δ^3
*	0			
		)1		
+	*/		)0	
		+/		.
,	.,		*,	
		.)		
-	)*-			

?VcV $y_u = 10,\ y_0 = 8,\ \Delta y_0 = 19,\ \Delta^2 y_0 = 18$ R_U $\Delta^3 y_0 = 6.$ JYV dfTTVddZgV Raac`iZ^ReZ`_d e` R RcV eYVcVW`cV

$$u_1 = \frac{1}{19}(2) - 0.1$$

$$u_2 = \frac{1}{19}\left[2 - \frac{0.1(0.1-1)}{2}(18)\right] = 0.15$$

$$u_3 = \frac{1}{19}\left[2 - \frac{0.15(0.15-1)}{2}(18) - \frac{0.15(0.15-1)\ (0.15-2)}{6}(6)\right] = 0.1532$$

$$u_4 = \frac{1}{19}\left[2 - \frac{0.1541(0.1541-1)}{2}(18) - \frac{0.1541(0.1541-1)\ (0.1541-2)}{6}(6)\right.$$

$$= 0.1542.$$

MV eR\V $u = 0.154$ T`ccVTe e` eYcVV UVTZ^R] a]RTVd& ?V_TV eYV gR]fV `W U hYZTY T`ccVda`_Ud e` $y = 10$), Z&V&$ eYV TfSV c``e `W)(Zd XZgV_ Sj $x_0 + uh = 2 + (0.154)1 = 2.154.$

Exercises

)& JYV gR]fVd `W x R_U $u_x$ RcV XZgV_ Z_ eYV W`]]`hZ_X JRS]V&

x	*	+	-
u_x	))+	*0.	.)+

=Z_U eYV gR]fV `W x W`c hYZTY $u_x = 1001.$

*& KdZ_X CRXcR_XV Z_gVcdV W`c^ f]R$ WZ_U eYV gR]fV `W x T`ccVda`_UZ_X e` $y = 100$ Wc`^ eYV W`]]`hZ_X JRS]V&

x	+	-	/	1	))
y	.	*,	-0	)(0	)/,

+& JYV gR]fVd `W $x$ R_U $f(x)$ RcV XZgV_ Z_ eYV W`]]`hZ_X JRS]V&

x	-	.	1	))
f(x)	)*	)+	),	).

=Z_U eYV gR]fV `W $x$ W`c hYZTY $f(x) = 15$.

,& JYV gR]fVd `W $x$ R_U $u_x$ RcV XZgV_ Z_ eYV W`]]`hZ_X JRS]V&

x	(	-	)(	)-
u_x	).&+-	),&00	)+&-1	)*&,.

=Z_U T`ccVTe e` `_V UVTZ^R] a]RTV eYV gR]fV `W x W`c hYZTY $u_x = 14$.

9

NUMERICAL DIFFERENTIATION AND INTEGRATION

Numerical differentiation

JYV ac`S]V^ `W **numerical differentiation** Zd eYV UVeVc^Z_ReZ`_ `W Raac`iZ^ReV gR]fV `W eYV UVcZgReZgV `W R Wf_TeZ`_ f Re R XZgV_ a`Z_e&

Differentiation using Difference Operators

MV Rddf^V eYRe eYV Wf_TeZ`_ V 5 C U! Zd XZgV_ W`c eYV Vbf R]]j daRTVU U gR]fVd UK 5 U(# KE$ W`c K 5 ($ )$ *$ … J` WZ_U eYV UVcZgReZgVd `W dfTY R eRSf]Rc Wf_TeZ`_$ hV ac`TVVU Rd W`]]`hd2

- **Using Forward Difference Operator**

IZ_TV $\Delta = E - 1$ and $hD = \log E$$ hYVcV / Zd R UZWWVcV_eZR] `aVcRe`c$ O R dYZWe `aVcRe`c$ hV YRgV dVV_ VRc]ZVc eYRe

$$hD = \log E = \log(1+\Delta)$$

?V_TV

$$D = \frac{1}{h}\log(1+\Delta) = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \ . \ . \ . \ \right)$$

8]d`$

$$D^2 = \frac{1}{h^2}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \ . \ . \ . \ \right)^2$$

$$= \frac{1}{h^2}\left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \ . \ . \ . \ \right)$$

JYVcVW`cV$

$$f'(x) = \frac{d}{dx}f(x) = Df(x) = \frac{1}{h}\left(\Delta f(x) - \frac{\Delta^2 f(x)}{2} + \frac{\Delta^3 f(x)}{3} - \frac{\Delta^4 f(x)}{4} + \frac{\Delta^5 f(x)}{5} - \ . \ . \ . \ \right)$$

$$f''(x) = D^2 f(x) = \frac{1}{h^2}\left(\Delta^2 f(x) - \Delta^3 f(x) + \frac{11}{12}\Delta^4 f(x) - \frac{5}{6}\Delta^5 f(x) + \ . \ . \ . \ \right)$$

- **Using Backward Difference Operator** d′&

HVTR]] eYRe

$$hD = -\log(1-\nabla)\&$$

On expansion, we have

$$D = \frac{1}{h}\left(\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \ldots \right)$$

Also,

$$D^2 = \frac{1}{h^2}\left(\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \ldots \right)^2$$

$$= \frac{1}{h^2}\left(\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \ldots \right)$$

Hence,

$$f'(x) = \frac{d}{dx} f(x) = Df(x)$$

$$= \frac{1}{h}\left(\nabla f(x) + \frac{\nabla^2 f(x)}{2} + \frac{\nabla^3 f(x)}{3} + \frac{\nabla^4 f(x)}{4} + \ldots \right)$$

$$f''(x) = D^2 f(x) = \frac{1}{h^2}\left(\nabla^2 f(x) + \nabla^3 f(x) + \frac{11}{12}\nabla^4 f(x) + \frac{5}{6}\nabla^5 f(x) + \ldots \right)$$

Example : Compute $f'(0.2)$ and $f''(0.2)$ from the following tabular data.

x	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.00	1.16	3.56	13.96	41.96	101.00

Since $x = 0$ and 0.2 appear at and near beginning of the table, it is appropriate to use formulae based on forward differences to find the derivatives. The forward difference table for the given data is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0.0	1.00	0.16	2.24	5.76	3.84	0.00
0.2	1.16	2.40	8.00	9.60	3.84	
0.4	3.56	10.40	17.60	13.44		
0.6	13.96	28.00	31.04			
0.8	41.96	59.04				
1.0	101.00					

KdZ_X $$f'(x) = Df(x) = \frac{1}{h}\left(\Delta f(x) - \frac{\Delta^2 f(x)}{2} + \frac{\Delta^3 f(x)}{3} - \frac{\Delta^4 f(x)}{4} + \ . \ . \ . \ \right)$$

hV `SeRZ_

$$f'(0.2) = \frac{1}{0.2}\left[2.40 - \frac{8.00}{2} + \frac{9.60}{3} - \frac{3.84}{4}\right] = 3.2$$

KdZ_X

$$f''(x) = D^2 f(x) = \frac{1}{h^2}\left(\Delta^2 f(x) - \Delta^3 f(x) + \frac{11}{12}\Delta^4 f(x) - \ . \ . \ . \ \right)$$

hV `SeRZ_

$$f''(0) = \frac{1}{(0.2)^2}\left[2.24 - 5.76 + \frac{11}{12}(3.84) - \frac{5}{6}(0)\right] = 0.0$$

Example : ` ^ a f eV C ′ \*&\*! R_U Càà \*&\*! Wc` ^ eYV W`]]` hZ_X eRS f]Rc UReR&

U	)&,	)&.	)&0	*&(	*&*
C U!	,&(--*	,&1-+(	.&(,1.	/&+10)	1&(*-(

IZ_TV U 5 *&* RaaVRcd Re eYV V_U `W eYV eRS]V$ Ze Zd Raac`acZReV e` fdV W`c^ f]RV SRdVU `_ SRT\ hRcU UZWWVcV_TVd e` WZ_U eYV UVcZgReZgVd& JYV SRT\ hRcU UZWWVcV_TV eRS]V W`c eYV XZgV_ UReR Zd2

U	C U!	∇ C U!	∇* C U!	∇+ C U!	∇, C U!
)&,)&.)&0 *&(*&*	,&(--* ,&1-+(.&(,1. /&+01) 1&(*-(	(&01/0)&(1..)&++1-)&.+-1	(&)100 (&*,*1 (&*1.,	(&(,,) (&(-+-	(&((1,

KdZ_X eYV SRT\ hRcU UZWWVcV_TV W`c^ f]R

$$f'(x) = Df(x) = \frac{1}{h}\left(\nabla f(x) + \frac{\nabla^2 f(x)}{2} + \frac{\nabla^3 f(x)}{3} + \frac{\nabla^4 f(x)}{4} + \ . \ . \ . \ \right)$$

hV `SeRZ_

$$f'(2.2) = \frac{1}{0.2}\left[1.6359 + \frac{0.2964}{2} + \frac{0.0535}{3} + \frac{0.0094}{4}\right] = 9.0215$$

8]d`$ fdZ_X SRT\ hRcU UZWWVcV_TV W`c^ f]R W`c /*C U!$ Z&V&

$$f''(x) = D^2 f(x) = \frac{1}{h^2}\left(\nabla^2 f(x) + \nabla^3 f(x) + \frac{11}{12}\nabla^4 f(x) + \ldots\right)$$

hV `SeRZ_

$$f''(2.2) = \frac{1}{(0.2)^2}\left[0.2964 + 0.0535 + \frac{11}{12}(0.0094)\right] = 8.9629$$

Example =c`^ eYV W`]]`hZ_X eRS]V `W gR]fVd `W U R_U V" `SeRZ_ $\frac{dy}{dx}$ R_U $\frac{d^2y}{dx^2}$ W`c $x = 1.2$:

x	)&(	)&*	)&,	)&.	)&0	*&(	*&*
y	*&/)0+	+&+*()	,&(--*	,&1-+(	.&(,1.	/&+01)	1&(*-(

JYV UZWWVcV_TV eRS]V Zd

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
)&(	*&/)0+						
		(&.()0					
)&*	+&+*()		(&)++				
		(&/+-)		(&(*1,			
)&,	,&(--*		(&).*/		(&((./		
		(&01/0		(&(+.)		(&(()+	
)&.	,&1-+(		(&)100		(&((0(		(&(()
		)&(1..		(&(,,)		(&((),	
)&0	.&(,1.		(&*,*1		(&((1,		
		)&++1-		(&(-+-			
*&(	/&+01)		(&*1.,				
		)&.+-1					
&	1&(*-(						

?VcV $x = 1.2,\ \ f(x) = 3.3201$ R_U $h = 0.2.$?V_TV

$$\left[\frac{dy}{dx}\right]_{x=1.2} = f'(1,\ 2)$$

$$= \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014)\right.$$

$= 3.3205$ &

IZ^Z]Rc]j$ $\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{0.04}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right] = 3.318.$

Example : R]Tf]ReV eYV WZcde R_U dVT`_U UVcZgReZgVd `W eYV Wf_TeZ`_ eRSf]ReVU Z_ eYV acVTVUZ_X VlR^a]V Re eYV a`Z_e $x = 2.2$ R_U R]d` $\frac{dy}{dx}$ Re $x = 2.0$ &

MV fdV eYV eRS]V `W UZWWVcV_TVd `W <iR^a]V)& ?VcV $x_n = 2.2, \quad y_n = 9.0250$ R_U $h = 0.2.$?V_TV SRT\ hRcU UZWWVcV_TV W`c UVcZgReZgV XZgVd

$$\left[\frac{dy}{dx}\right]_{x=2.2} = f'(2.2) = \frac{1}{0.2}$$

$$\left[1.6359 + \frac{1}{2}(0.2964) + \frac{1}{3}(0.0535) + \frac{1}{4}(0.0094) + \frac{1}{5}(0.0014)\right.$$

5 1&(**0&

$$\left[\frac{d^2y}{dx^2}\right]_{x=2.2} = f''(2.2) = \frac{1}{0.04}$$

$$\left[0.2964 + 0.0535 + \frac{11}{12}(0.0094) + \frac{5}{6}(0.0014)\right] = 8.992.$$

8]d`$

$$\left[\frac{dy}{dx}\right]_{x=2.0} = f'(2.2) = \frac{1}{0.2}$$

$$\left[1.3395 + \frac{1}{2}(0.2429) + \frac{1}{3}(0.0441) + \frac{1}{4}(0.0080) + \frac{1}{5}(0.0013)\frac{1}{6}(0.0001)\right.$$

5 /&+01.&

- **Derivative using Newton's Forward difference Formula**

=`c WZ_UZ_X eYV UVcZgReZgV Re R a`Z_e _VRc e` eYV SVXZ__Z_X `W eYV eRSf]Rc gR]fVd$ EVhe`_qd =`chRcU UZWWVcV_TV =`c^f]R Zd fdVU& =`c eYV gR]fVd $y_0, y_1, ..., y_n$ `W R Wf_TeZ`_ j5W i!$ T`ccVda`_UZ_X e` eYV VbfZUZdeR_e gR]fVd $x_0, x_1, x_2, ..., x_n$ $ hYVcV $x_1 = x_0 + h, x_2 = x_0 + 2h, \; x_3 = x_0 + 3h, ..., x_n = x_0 + nh$ $ EVhe`_qd =`chRcU UZWWVcV_TV =`c^f]R Zd$

$$f(x) = f(x_0 + uh) = y_0 + u\left[\Delta y_0\right] + \frac{u(u-1)}{2!}\left[\Delta^2 y_0\right]$$
$$+ \frac{u(u-1)(u-2)}{3!}\left[\Delta^3 y_0\right] + ... + \frac{u(u-1)(u-2)....(u-n+1)}{n!}\left[\Delta^n y_0\right]$$

hYVcV$ $u = \frac{x - x_0}{h}$&

JYV UVcZgReZgV `W $f(x)$ hZeY cVdaVTe e` i$ hYVcV i Zd R_j a`Z_e Z_ eYV Z_eVcgR] $[x_0, x_n]$ Zd `SeRZ_VU Rd W`]]`hd2

$$\frac{d}{dx} f(x) = \frac{d}{du} f(x) \times \frac{du}{dx}, \text{ by chain rule}$$
$$= \frac{d}{du} f(x) \times \frac{d}{dx}\left(\frac{(x - x_0)}{h}\right) = \frac{d}{du} f(x) \times \frac{1}{h}$$

$$\Rightarrow \frac{d}{dx} f(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + ...\right]$$

MYV_ $x = x_0$ $ hV XVe f5(& JYfd$

$$\frac{d}{dx} f(x) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{2}{6}\Delta^3 y_0 - \frac{6}{24}\Delta^4 y_0 + ...\right]$$

JYV dVT`_U UVcZgReZgV `W $f(x)$ Zd

$$\frac{d^2}{dx^2} f(x) = \frac{d}{du}\left(\frac{d}{dx} f(x)\right)\times\frac{du}{dx}$$

$$= \frac{d}{du}\left(\frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + ...\right]\right)\times\frac{1}{h}$$

$$= \frac{1}{h^2}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6u-6}{3!}\Delta^3 y_0 + \frac{12u^2-36u+22}{24}\Delta^4 y_0 + ...\right]$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + ...\right]$$

@_ dZ^Z]Rc hRj$

$$\frac{d^3}{dx^3} f(x) = \frac{d}{du}\left[\frac{d^2}{dx^2} f(x)\right]\times\frac{du}{dx} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{12u-18}{12}\Delta^4 y_0 + ...\right]$$

MYV_ $x = x_0$, and $u = 0$, hV YRgV

$$\frac{d^2}{dx^2} f(x) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - .....\right] \quad \text{R\_U}$$

$$\frac{d^3}{dx^3} f(x) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + ...\right]$$

- **Derivative using Newton's Backward difference Formula**

J` WZ_U eYV UVcZgReZgV Re R a`Z_e _VRc e` eYV V_U `W eYV eRSf]Rc gR]fVd$ EVhe`_qd SRT\ hRcU UZWWVcV_TV =`c^f]R Zd fdVU& =`c eYV VbfZUZdeR_e RcXf^V_ed$ EVhe`_qd SRT\ hRcU UZWWVcV_TV =`c^f]R Zd$

$$f(x) = f(x_n + uh) = y_n + u\left[\nabla y_n\right] + \frac{u(u+1)}{2!}\left[\nabla^2 y_n\right] + \frac{u(u+1)(u+2)}{3!}\left[\nabla^3 y_n\right] + ... + \frac{u(u+1)(u+2)....(u+n-1)}{n!}\nabla^n y_n$$

hYVcV $u = \frac{x - x_n}{h}$

$$\frac{d}{dx} f(x) = \frac{d}{du} f(x) \times \frac{du}{dx}$$

$$= \frac{d}{du} f(x) \times \frac{d}{dx} \left(\frac{(x - x_n)}{h} \right) = \frac{d}{du} f(x) \times \frac{1}{h}$$

$$\Rightarrow \frac{d}{dx} f(x) = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n + \frac{4u^3+18u^2+22u+6}{24}\nabla^4 y_n + \ldots \right]$$

$$\frac{d^2}{dx^2} f(x) = \frac{d}{du}\left[\frac{d}{dx} f(x) \right] \times \frac{du}{dx} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \frac{6u^2+18u+11}{12}\nabla^4 y_n + \ldots \right]$$ $ R_U

$$\frac{d^3}{dx^3} f(x) = \frac{d}{du}\left[\frac{d^2}{dx^2} f(x) \right] \times \frac{du}{dx} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{12u+18}{12}\nabla^4 y_n + \ldots \right]$$

8e $x = x_n,\ u = 0.$ JYV RS`gV XZgVd$

$$\frac{d}{dx} f(x) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots \right]$$

$$\frac{d^2}{dx^2} f(x) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots \right]$$ R_U

$$\frac{d^3}{dx^3} f(x) = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots \right]$$ &

***Problem*:** : `^afeV Càà (! R_U C′ (&*! Wc`^ eYV W`]]`hZ_X eRSf]Rc UReR&

U	(&(	(&*	(&,	(&.	(&0	)&(
C U!	)&((	)&).	+&-.	)+&1.	,)&1.	)()&((

Solution:

IZ_TV U 5 (&(R_U (&* RaaVRc Re R_U _VRc SVXZ__Z_X `W eYV eRS]V$ Ze Zd Raac`acZReV e` fdV W`c^f]RV SRdVU `_ W`chRcU UZWWVcV_TVd e` WZ_U eYV UVcZgReZgVd& JYV W`chRcU UZWWVcV_TV eRS]V W`c eYV XZgV_ UReR Zd2

U	V-C U!	∆V	∆* V	∆+ V	∆, V	∆- V
(&((&* (&, (&. (&0)&(	)&(( )&). +&-.)+&1. ,)&1.)()&((	(&). *&,( )(&,(*0&((-1&(,	*&*, 0&(( )/&.(+)&(,	-&/. 1&.( )+&,,	+&0, +&0,	(&((

?VcV $x_0 = 0$ \$ R_U Y5(&*& 8e $x=0,\ u=\frac{(x-x_0)}{h}=0$ \$

JYV dVT`_U UVcZgReZgV Re $x=0$ Zd XZgV_ Sj EVhe`_qd W`chRcU W`c^ f]R2

$$\frac{d^2}{dx^2}f(x)=\frac{1}{h^2}\left[\Delta^2 y_0-\Delta^3 y_0+\frac{11}{12}\Delta^4 y_0-.....\right.$$

$$f''(0)=\frac{1}{(0.2)^2}\left[2.24-5.76+\frac{11}{24}(3.84)-\frac{5}{6}(0)\right]=0$$ &

=`c i5(&*\$ $u=\frac{(0.2-0.0)}{0.2}=1$ &

9j EVhe`_qd W`chRcU W`c^ f]R\$ hV YRgV eYV UVcZgReZgV `W W i! Re R a`Z_e i Zd\$

$$\frac{d}{dx}f(x)=\frac{1}{h}\left[\Delta y_0+\frac{2u-1}{2!}\left[\Delta^2 y_0\right]+\frac{3u^2-6u+2}{3!}\Delta^3\left[y_0\right]+\frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0\left[+.....\right.\right.$$

?V_TV\$

$$\left.\frac{d}{dx}f(x)\right|_{x=0.2}=\frac{1}{0.2}\left[0.16+\frac{2\times1-1}{2!}[2.24]+\frac{3\times1^2-6\times1+2}{3!}[5.76]+\frac{4\times1^3-18\times1^2+22\times1-6}{24}[3.84]\right.$$

5 +&*\$ `_ dZ^a]ZWZTReZ`_&

@W eYV RcXf^V_ed RcV _`e Vbf ZUZdeR_e\$ eYV Raac`iZ^ReZ_X a`]j_`^ZR] W`c eYV XZgV_ eRSf]Rc a`Z_ed Zd W`f_U Sj EVhe`_qd UZgZUVU UZWWVcV_TV W`c^f]R `c CRXcR_XVqd Z_eVca`]ReZ`_ W`c^f]R& JYV_ eYV UVcZgReZgV `W eYV Wf_TeZ`_ TR_ XVe Re R_j i Z_ eYV cR_XV&

For example: MV WZ_U eYV WZcde UVcZgReZgV `W R Wf_TeZ`_ Re (\$ fdZ_X eYV a`Z_ed $(-4,1245),(-1,33),(0,5),(2,9)$ *and* $(5,1335)$ hYVcV i gR]fVd RcV _`e Vbf ZUZdeR_e& MV TR_ XVe eYV Raac`iZ^ReZ_X a`]j_`^ZR] Sj EVhe`_qd UZgZUVU UZWWVcV_TV W`c^f]R&

JYV eRS]V `W UZgZUVU UZWWVcV_TVd Zd\$

i	j	=Zcde UZgZUVU UZWWVcV_TVd	IVT`_U UZgZUVU UZWWVcV_TVd	JYZcU UZgZUVU UZWWVcV_TVd	=`fceY UZgZUVU UZWWVcV_TVd
%,	)*,-	%,(,			
%)	++	%*0	1,		
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*	1	,,*	)(		+
-	)++-		00	)+	

>ZgV_ $f\left(x_0\right)=1245$ & =c` ^ eYV eRS]V$ hV TR_ `SdVcgV eYRe

$$f\left(x_0,x_1\right)=-404;\quad f\left(x_0,x_1,x_2\right)=94;$$
$$f\left(x_0,x_1,x_2,x_3\right)=-14\ \ and\ \ f\left(x_0,x_1,x_2,x_3,x_4\right)=3$$

?V_TV eYV Z_eVca`]ReZ_X a`]j_` ^ZR] Zd

$$f(x)=1245+\left(x-(-4)\right)\times(-404)+\left(x-(-4)\right)\left(x-(-1)\right)\times 94$$
$$+\left(x-(-4)\right)\left(x-(-1)\right)\left(x-0\right)\times\left(-14\right)+\left(x-(-4)\right)\left(x-(-1)\right)\left(x-0\right)\left(x-2\right)\times 3$$

F_ dZ^a]ZWZTReZ`_$ hV XVe

$$f(x)=3x^4-5x^3+6x^2-14x+5$$ &

JYV_$

$$f'(x)=12x^3-15x^2+12x-14$$

?V_TV$ $f'(0)=-14$ &

Exercises

1. =c` ^ eYV W`]]` hZ_X eRS]V `W gR] fVd$ VdeZ^ ReV C à)&) (! R_U $f''(1.10)$:

U	)&((	)&(-	)&)(	)&)-	)&*(	)&*-	)&+(
$f(x)$	)&((((	)&(*,/	)&(,00	)&(/*,	)&(1-,	)&))0(	)&),(*

2. =Z_U eYV WZcde UVcZgReZgV `W C U! Re U - (&, Wc`^ eYV W`]]` hZ_X eRS]V2

U	(&)	(&*	(&+	(&,
C U!	)&)(-)/	)&**),(	)&+,10.	)&,1)0*

3. 8 d]ZUVc Z_ R ^RTYZ_V ^`gVd R]`_X R WZiVU decRZXYe c`U& @ed UZdeR_TV U T^ R]`_X eYV c`U Zd XZgV_ SV]` h W`c gRcZ`fd gR]fVd `W eZ^V Q dVT`_Ud!& =Z_U eYV gV]`TZej `W eYV d]ZUVc R_U Zed RTTV]VcReZ`_ Re eZ^V Q - (&+ dVT&

Q	(&(	(&)	(&*	(&+	(&,	(&-	(&.
U	+&()+	+&).*	+&*0/	+&+.,	+&+1-	+&+0)	+&+*,

KdV S`eY eYV W`chRcU UZWWVcV_TV W`c^ f]R R_U eYV TV_ecR] UZWWVcV_TV W`c^ f]R e` WZ_U eYV gV]`TZej R_U T`^aRcV eYV cVdf]ed&

4. KdZ_X eYV gR]fVd Z_ eYV eRS]V$ VdeZ^ReV V′′)&+! 2

U	)&+	)&-	)&/	)&1	*&)	*&+
V	*&1.,0	*&.-11	*&++++	)&11**	)&.,,*	)&*1.1

10

NUMERICAL INTEGRATION

THE TRAPEZOIDAL RULE

@_ eYZd ^VeY`U e` VgR]fReV $\int_a^b f(x)dx$$ hV aRceZeZ`_ eYV Z_eVcgR] `W Z_eVXcReZ`_ $[a, b]$ R_U cVa]RTV $f$ Sj R decRZXYe ]Z_V dVX^V_e `_ VRTY dfSZ_eVcgR]& JYV gVceZTR]]Z_Vd Wc`^ eYV V_Ud `W eYV dVX^V_ed e` eYV aRceZeZ`_ a`Z_ed TcVReV R T`]]VTeZ`_ `W ecRaVk`ZUd eYRe Raac`iZ^ReV eYV cVXZ`_ SVehVV_ eYV TfcgV R_U eYV U%RiZd& MV RUU eYV RcVRd `W eYV ecRaVk`ZUd T`f_eZ_X RcVR RS`gV eYV U#RiZd Rd a`dZeZgV R_U RcVR SV]`h eYV RiZd Rd _VXReZgV R_U UV_`eV eYV df^ Sj T. JYV_

$$T = \frac{1}{2}(y_0 + y_1)h + \frac{1}{2}(y_1 + y_2)h + \cdots + \frac{1}{2}(y_{n-2} + y_{n-1})h + \frac{1}{2}(y_{n-1} + y_n)h$$

$$= h\left(\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right)$$

$$= \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$$

hYVcV

$y_0 = f(a),\quad y_1 = f(x_1),\ \ldots\ ,\quad y_{n-1} = f(x_{n-1}),\quad y_n = f(b)$&

The Trapezoidal Rule

J` Raac`iZ^ReV $\int_a^b f(x)dx$$

W`c K dfSZ_eVcgR]d `W]V_XeY $h = \dfrac{b-a}{n}$ and $y_j = f(x_j)$).

fdV

$$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$$

`c

$$T = \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})]$$

Example KdV eYV ecRaVk`ZUR] cf]V hZeY $n = 4$ e` VdeZ^ReV

$$\int_1^2 x^2 dx$$&

: ` ^aRcV eYV VdeZ^ReV hZeY eYV ViRTe gR]fV `W eYV Z_eVXcR]&

J` WZ_U eYV ecRaVk`ZUR] Raac`iZ^ReZ`_$ hV UZgZUV eYV Z_eVcgR] `W Z_eVXcReZ`_ Z_e` W`fc dfSZ_eVcgR]d `W Vbf R] ]V_XeY R_U ]Zde eYV gR]fVd `W $y = x^2$ Re eYV V_Ua`Z_ed R_U aRceZeZ`_ a`Z_ed&

G	UG	$y_j = x_j^2$	
(	)&(	)&((((	
)	)&*-		)&-.*-
*	)&-(		*&*-((
+	)&/-		+&(.*-
,	*&((	,&((((	
	I f ^	-&((((	.&0/-(

MZeY $n = 4$ R_U $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$2

$$T = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{8}\left[1.4 + 2(6.875)\right.$$

5 *&+,+/-

JYV ViRTe gR]fV `W eYV Z_eVXcR] Zd

$$\int_1^2 x^2 dx = \left.\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = 2.33334$$

JYV Raac`iZ^ReZ`_ Zd R d]ZXYe `gVcVdeZ^ReV& <RTY ecRaVk`ZU T`_eRZ_d d]ZXYe]j ^`cV eYR_ eYV T`ccVda`_UZ_X decZa f_UVc eYV TfcgV&

Problem2 KdZ_X JcRaVk`ZUR] cf]V d`]gV eYV Z_eVXcR]$ $\int_0^1 \frac{1}{x^2 + 6x + 10} dx$ hZeY W`fc dfSZ_eVcgR]d&

Solution:

=`c K dfSZ_eVcgR]d$ eYV ecRaVk`ZUR] cf]V W`c eYV Z_eVXcR] `W R Wf_TeZ`_ Z_ eYV cR_XV PR$SQ Zd$

$$\int_a^b f(x)dx = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + ... + 2y_{n-1} + y_n]$$

?VcV e` T`_dZUVc K-*$

6LT" $$\int_a^b f(x)dx = \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right]$$

@_ `fc Z_eVXcR]$ $\int_0^1 \frac{1}{x^2+6x+10}dx$$ eYV cR_XV `W Z_eVXcR] P($)Q Zd UZgZUVU Z_e` W`fc VbfR] dfSZ_eVcgR] `W hZUeY E-&$(+" Sj eYV a`Z_ed$ (&($(&*-$(&-($(&/- R_U)&

: `_dZUVcZ_X eYV^ Rd eYV i gR]fVd$ T`ccVda`_UZ_X gR]fVd `W eYV Z_eVXcR_U $\frac{1}{x^2+6x+10}$ UV_`eVU Sj y_0, y_1, y_2, y_3, y_4 RcV (&)($ (&(0.,1$ (&(/-,/$ (&(..+1 R_U (&(-00* cVdaVTeZgV]j&

?V_TV$

$$\int_0^1 \frac{1}{x^2+6x+10}dx = \frac{0.25}{2}\left[0.10 + 2\times 0.08649 + 2\times 0.07547 + 2\times 0.06639 + 0.05882\right]$$

5 (&(/.1,&

Example KdV eYV ecRaVk`ZUR] cf]V hZeY $n = 4$ e` VdeZ^ReV

$$\int_1^2 \frac{1}{x}dx$$&

: `^aRcV eYV VdeZ^ReV hZeY eYV ViRTe gR]fV `W eYV Z_eVXcR]&

J` WZ_U eYV ecRaVk`ZUR] Raac`iZ^ReZ`_$ hV UZgZUV eYV Z_eVcgR] `W Z_eVXcReZ`_ Z_e` W`fc dfSZ_eVcgR]d `W VbfR] ]V_XeY R_U ]Zde eYV gR]fVd T`ccVTe e` WZgV UVTZ^R] a]RTVd! `W $y = \frac{1}{x}$ Re eYV V_Ua`Z_ed R_U aRceZeZ`_ a`Z_ed&

G	UG	$y_j = \frac{1}{x_j}$	
(	)&(	)&(((((	
)	)&*-		(&0((((
*	)&-(		(&..../
+	)&/-		(&-/),+
,	*&((	(&-((((	
	If^	)&-((((	*&(+0)

MZeY $n = 4$ and $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} = 0.25$ 2

$$T = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{8}\left[1.5 + 2(2.0381)\right. \quad 5 \ (\&.1/(*\&$$

JYV ViRTe gR]fV `W eYV Z_eVXcR] Zd

$$\int_1^2 \frac{1}{x} dx = \ln x \Big]_1^2 = \ln 2 - \ln 1 = 0.69315$$

JYV Raac`iZ^ReZ`_ Zd R d]ZXYe `gVcVdeZ^ReV&

Example <gR]fReV $\int_0^1 e^{-x^2} dx$ Sj ^VR_d `W JcRaVk`ZUR] cf]V hZeY K5)(&

?VcV $h = \frac{b-a}{n} = \frac{1-0}{1} = 0.1$ R_U

$$\int_0^1 e^{-x^2} dx \approx T = \frac{0.1}{2}\left[y_0 + y_{10} + 2(y_1 + y_2 + \cdots + y_9)\right]$$

G	UG	UG*	$f(x_j) = e^{-x_j^2}$	
(	(&(	(&((	)&((((((	(&11((-(
)	(&)	(&()		(&1.(/01
*	(&*	(&(,		(&1)+ 1+)
+	(&+	(&(1		(&0-*),,
,	(&,	(&).		(&//0 0()
-	(&-	(&*-		(.1/ ./.
.	(&.	(&+.		(&.)* .*.
/	(&/	(&,1		(&.)* .*.
0	(&0	(&.,		(&-*/ *1*
1	(&1	(&0)		(&,,, 0-0
) (	)&(	)&((	(&+./ 0/1	
I f ^d			)&+./ 0/1	.&//0)./

?V_TV $\int_0^1 e^{-x^2} dx \approx T = \frac{0.1}{2}\left[1.367879 + 2(6.778167)\right] = 0.746211$

SIMPSON'S 1/3 RULE

IZ^ad`_qd cf]V W`c Raac`iZ^ReZ_X $\int_a^b f(x)\,dx$ Zd SRdVU `_ Raac`iZ^ReZ_X f hZeY bfRUcReZT a`]j_`^ZR]d Z_deVRU `W]Z_VRc a`]j_`^ZR]d& MV Raac`iZ^ReV eYV XcRaY hZeY aRcRS`]ZT RcTd Z_deVRU `W]Z_V dVX^V_ed&

JYV Z_eVXcR] `W eYV bfRUcReZT a`]j_`^ZR] $y = Ax^2 + Bx + C$ Z_ =ZX&+ Wc`^ $x = -h$ to $x = h$ Zd

$$\int_{-h}^{h}(Ax^2 + Bx + C)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

IZ^ad`_qd cf]V W`]]`hd Wc`^ aRceZeZ`_Z_X [a, b] Z_e` R_ VgV_ _f^SVc `W dfSZ_eVcgR]d `W VbfR]]V_XeY h, Raa]jZ_X <b& e` dfTTVddZgV Z_eVcgR] aRZcd$ R_U RUUZ_X eYV cVdf]ed&

Algorithm: Simpson's 1/3 Rule

J` Raac`iZ^ReV $\int_a^b f(x)\,dx$ $ fdV

$$S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n).$$

JYV Vqd RcV eYV gR]fVd `W f Re eYZd aRceZeZ`_ a`Z_ed

$$x_0 = a, x_1 = a + h, x_2 = a + 2h, \ldots, x_{n-1} = a + (n-1)h, x_n = b$$

JYV _f^SVc n Zd **even**$ $h = \dfrac{b-a}{n}$ and $y_j = f(x_j))$.

Simpson's 1/3 Rule given by (5) can be simplified as below:

$$S = \frac{h}{3}(s_0 + 4s_1 + 2s_2), \qquad \text{t -8!}$$

hYVcV $s_0 = y_0 + y_n,\ s_1 = y_1 + y_3 + \ldots + y_{n-1},\ s_2 = y_2 + y_4 + \ldots + y_{n-2}.$

Example =Z_U R_ Raac`iZ^ReV gR]fV `W]`X$_B$- Sj TR]Tf]ReZ_X $\int_0^5 \frac{dx}{4x+5}$ $ Sj IZ^ad`_qd)'+ cf]V `W Z_eVXcReZ`_&

MV _`eV eYRe

$$\int_0^5 \frac{dx}{4x+5} = \left[\frac{1}{4}\log(4x+5)\right]_0^5 = \frac{1}{4}[\log 25 - \log 5] = \frac{1}{4}\log\frac{25}{5} = \frac{1}{4}\log 5.$$

E`h e` TR]Tf]ReV eYV gR]fV `W $\int_0^5 \frac{dx}{4x+5}$ $ Sj IZ^ad`_qd cf]V `W Z_eVXcReZ`_$ UZgZUV eYV Z_eVcgR] P($ -Q Z_e` K 5 )( VbfR] dfSZ_eVcgR]d$ VRTY `W]V_XeY $h = \frac{b-a}{n} = \frac{5-0}{10} = 0.5.$

G	UG	, UG#-	$f_j = f(x_j) = \frac{1}{4x_j+5}$		
(	(&(	-	(&*(		
)	(&-	/		(&),*1	
*	)&(	1			(&))))
+	)&-	))		(&(1(1	
,	*&(	)+			(&(/.1
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.	+&(	)/			(&(-00
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0	,&(	*)			(&(,/.
1	,&-	*+		(&(,+,	
)(	-&(	*-	(&(,		
lf^d			P(5(&*,	P) 5(&+1.+	P* 5(&*1,,

?V_TV$

$$\int_0^5 \frac{dx}{4x+5} \approx S = \frac{0.5}{3}\left[0.24+4(0.3963)+2(0.2944)\right] = 0.4023.$$

R_U]`XB - 5 , (&,(*+! 5)&.(1*&

Problem2 =Z_U $\int_0^{10} \frac{1}{1+x^2}\,dx$ fdZ_X lZ^ad`_qd `_V eYZcU cf]V&

l**olution:**

9j lZ^ad`_qd `_V eYZcU cf]V$ $\int_a^b f(x)dx = \frac{h}{3}\left[y_0 + 4\left(y_1+y_3+...\right)+2\left(y_2+y_4+....\right)+y_n\right.$

@_ `fc Z_eVXcR]$ $\int_0^{10} \frac{1}{1+x^2}\,dx$$ ]Ve eYV cR_XV P($)(Q Zd dfSUZgZUVU Z_e`)(VbfR] Z_eVcgR] `W hZUeY E-'" Sj eYV i gR]fVd ($)$*$+$,$-$.$/$O$1 R_U )(& :`ccVda`_UZ_X j gR]fVd `W eYV Wf_TeZ`_ $\frac{1}{1+x^2}$ RcV ]ZdeVU SV]`h2

i	(	)	*	+	,	-	.	/	0	1	)(
j	)	(&-	(&*	(&)	(&(-00	(&(+0-	(&(*/(	(&(*	(&()-,	(&()**	(&((11

JYfd$

$$\int_0^{10} \frac{1}{1+x^2}\,dx = \frac{1}{3}\big[1+4\left(0.5+0.1+0.0385+0.02+0.0122\right)+2\left(0.2+0.0588+0.027+0.0154\right)+0.0099$$

$$= \frac{1}{3}\big[1.0099+4\left(0.6707\right)+2\left(0.3012\right)$$

$$= \frac{1}{3}\left[4.2951\right] = 1.4317 \text{ \&}$$

Problem2 <gR] f ReV $\int_0^6 \frac{1}{3+x^2}\,dx$ f dZ_X lZ^ad`_qd eYcVV VZXYe c f]V&

lolution:

9 j lZ^ad`_qd eYcVV VZXYe c f]V$

$$\int_a^b f(x)dx = \frac{3h}{8}\big[y_0+3\left(y_1+y_2+y_4+....+y_{n-1}\right)+2\left(y_3+y_6+y_9+...\right)+y_n$$

CVe eYV]Z^Ze `W Z_eVXcR] P($.Q SV UZgZUVU Z_e` dZi VbfR] aRced hZeY Z_eVcgR] E-'" f dZ_X eYV i gR] f Vd ($)$*$+$,$- R_U .& :`ccVda`_UZ_X j gR] f Vd `W eYV XZgV_ Z_eVXcR_U Wf_TeZ`_ $\frac{1}{3+x^2}$ RcV$

i	(	)	*	+	,	-	.
j	(&+++	(&*-	(&),*1	(&)	(&(-*.	(&(+-/	(&(*-.

JYfd$

$$\int_0^6 \frac{1}{3+x^2}\,dx = \frac{3\times 1}{8}\big[y_0+3\left(y_1+y_2+y_4+....+y_{n-1}\right)+2\left(y_3+y_6+y_9+...\right)+y_n$$

=`c _5.$

$$\int_0^6 \frac{1}{3+x^2}\,dx = \frac{3\times 1}{8}\big[y_0+3\left(y_1+y_2+y_4+y_5\right)+2y_3+y_6$$

$$\int_0^6 \frac{1}{3+x^2}\,dx = \frac{3\times 1}{8}\big[0.333+3\left(0.25+0.1429+0.0526+0.0357\right)+2\left(0.1\right)+0.0256$$

$$= \frac{3}{8}\left[0.333+1.4436+0.2+0.0256\right] = \frac{3}{8}\left[2.0022\right]$$

$$\Rightarrow \int_0^6 \frac{1}{3+x^2}\,dx = 0.7508 \text{ \&}$$

Example =Z_U R_ Raac`iZ^ReZ`_ gR] f V `W $\int_0^1 x^2 dx$ Sj lZ^ad`_qd)'+ c f]V hZeY K 5)(&

Here $h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$

j	x_j	$y_j = f(x_j) = x_j^2$		
0	0.0	0.00		
1	0.1		0.01	
2	0.2			0.04
3	0.3		0.09	
4	0.4			0.16
5	0.5		0.25	
6	0.6			0.36
7	0.7		0.49	
8	0.8			0.64
9	0.9		0.81	
10	1.0	1.00		
Sums		$S_0 = 1.00$	$S_1 = 1.65$	$S_2 = 1.20$

Hence,

$$\int_0^1 x^2\,dx \approx S = \frac{0.1}{3}\left[1.00 + 4(1.65) + 2(1.20)\right] = 0.3333.$$

Also, the exact value is given by

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1-0}{3} = 0.3333.$$

***Example* 11** A town wants to drain and fill a small-polluted swamp (See the adjacent figure). The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?

Solution To calculate the volume of the swamp, we estimate the surface area and multiply by 5. To estimate the area, we use Simpson's rule with $h = 20$ ft and the y's equal to the distances measured across the swamp, as shown in the adjacent figure.

$$S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$

$$= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100$$

JYV g`]f ^V Zd RS` fe $(8100)(5) = 40,500\,\text{ft}^3$ or $1500\,\text{yd}^3$ &

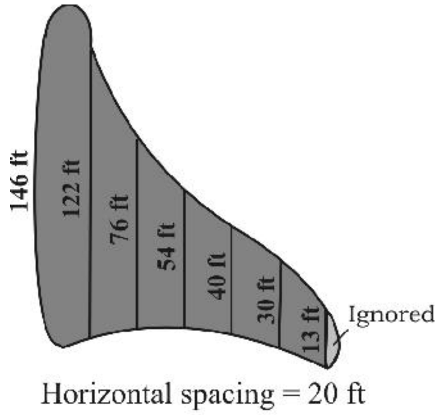

Fig.4

Example : `^afeV eYV Z_eVXcR] $I = \sqrt{\frac{2}{\pi}}\int_0^1 e^{-x^2/2}dx$ fdZ_X

IZ^ad`_qd)'+ cf]V$ eR\Z_X E5 (&)*-&

G	UG	$f_j = f(x_j) = \sqrt{\frac{2}{\pi}}e^{-x_j^2/2}$		
(	(&(((	(&/1/1		
)	(&)*-		(&/1)/	
*	(&*-(			(&//++
+	(&+/-		(&/,+/	
,	(&-((			(&/(,)
-	(&.*-		(&.-.+	
.	(&/-(			(&.(*+
/	(&0/-		(&-,,)	
0	)&(((	(&,0+1		
Ifd		P(5)&*0)0	P)5*&/+-0	P*5*&(/1/

?V_TV $I = \sqrt{\frac{2}{®}}\int_0^1 e^{-x^2/2}dx \approx S = \frac{0.125}{3}\left[1.2818 + 4(2.7358) + 2(2.0797)\right]$

$$= 0.6827$$

Derivation of Trapezoidal and Simpson's 1/3 rules of integration from Lagrangian Interpolation

@_eVXcReZ_X eYV W`c^ f]R Z_ CRXcR_XZR_ Z_eVca`]ReZ`_$ hV `SeRZ_

$$\int_a^b f(x)dx \approx \int_a^b L_n(x)dx = \sum_{k=0}^{n}\frac{f_k}{l_k(x_k)}\int_a^b l_k(x)dx$$

=`cK 5 )$ hV YRgV `_]j `_V Z_eVcgR] PU($U)Q dfTY eYRe > - U( R_U ? - U) R_U eYV_ eYV RS`gV Z_eVXcReZ`_ W`c^f]R XZgVd ecRaVk`ZUR] cf]V&

=`cK 5 *$ hV YRgV eh` dfSZ_eVcgR]d PU($U)Q R_U PU)$U*Q `W Vbf R] hZUeY E dfTY eYRe > - U( R_U ? - U* R_U eYV_ eYV RS`gV Z_eVXcReZ`_ W`c^f]R SVT`^Vd

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}\left(f_0 + 4f_1 + f_2\right)\$$$

R_U Zd eYV IZ^ad`_qd )'+ cf]V `W Z_eVXcReZ`_&

=`cK 5 + eYV RS`gV Z_eVXcReZ`_ W`c^f]R ,! SVT`^Vd

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx \approx \frac{3}{8}h\left(f_0 + 3f_1 + 3f_2 + f_3\right)\$$$

R_U Zd _`h_ Rd IZ^ad`_qd +'0 cf]V `W Z_eVXcReZ`_&

Simpson's three eight (3/8) rule

MYV_ K-)$ R]] eYV UZWWVcV_TVd `W `cUVc W`fc `c YZXYVc SVT`^Vd kVc`&

?V_TV$

$$\int_{x_0}^{x_3=x_0+3h} f(x)dx = h\left[3\times y_0 + \frac{3^2}{2}\left[\Delta y_0\right] + \frac{1}{2}\left[\frac{3^3}{3} - \frac{3^2}{2}\right]\Delta^2 y_0 + \frac{1}{6}\left[\frac{3^4}{4} - 3^3 + 3^2\right]\Delta^3 y_0\right] + 0$$

$$= h\left[3y_0 + \frac{9}{2}\left[y_1 - y_0\right] + \frac{1}{2}\left[\frac{27}{3} - \frac{9}{2}\right]\left[y_2 - 2y_1 + y_0\right] + \frac{1}{6}\left[\frac{81}{4} - 27 + 9\right]\left[y_3 - 3y_2 + 3y_1 - y_0\right]\right]$$

$$= \frac{h}{24}\left[72y_0 + 108\left[y_1 - y_0\right] + 54\left[y_2 - 2y_1 + y_0\right] + 9\left[y_3 - 3y_2 + 3y_1 - y_0\right]\right]$$

$$= \frac{h}{24}\left[9y_0 + 27y_1 + 27y_2 + 9y_3\right]$$

$$\Rightarrow \int_{x_0}^{x_3=x_0+3h} f(x)dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

IZ^Z]Rc]j$

$$\int_{x_3}^{x_6=x_0+6h} f(x)dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

=Z_R]]j$ f_UVc eYV Rddf^aeZ`_ eYRe K Zd R ^f]eZa]V `W eYcVV$

$$\int_{x_{n-3}}^{x_n=x_0+nh} f(x)dx = \frac{3h}{8}\left[y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\right]$$

8UUZ_X eYVdV Z_eVXcR]d$ hV XVe$

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\left[\left(y_0+3y_1+3y_2+y_3\right)+\left(y_3+3y_4+3y_5+y_6\right)+...+\left(y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\right)\right)$$

JYRe Zd$

$$\int_a^b f(x)dx = \frac{3h}{8}\left[\left(y_0+3y_1+3y_2+y_3\right)+\left(y_3+3y_4+3y_5+y_6\right)+...+\left(y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\right)\right)$$

$$\int_a^b f(x)dx = \frac{3h}{8}\left[y_0+3y_1+3y_2+2y_3+3y_4+3y_5+2y_6+3y_7+...+3y_{n-1}+y_n\right]$$

$$\Rightarrow \int_a^b f(x)dx = \frac{3h}{8}\left[y_0+3\left(y_1+y_2+y_4+....+y_{n-1}\right)+2\left(y_3+y_6+y_9+...\right)+y_n\right.$$

Exercises

<deZ^ReV eYV Z_eVXcR] fdZ_X

R! ecRaVk`ZUR] cf]V R_U S! IZ^ad`_qd)'+ cf]V&

)& $\int_1^2 \frac{1}{s^2}ds$ *& $\int_0^{\text{®}} \sin t\,dt$ +& $\int_0^2 x^3dx$

,& $\int_1^2 x\,dx$ -& $\int_{-1}^1 (x^2+1)dx$.& $\int_0^{-2} (t^3+t)dt$

/& $\int_0^1 \frac{\sin x}{x}dx$ 0& $\int_0^1 \frac{1}{1+x}dx$ 1& $\int_0^6 \frac{1}{1+x^2}dx$

)(& $\ln 2 = \int_0^1 \frac{dx}{x}$))& $\int_1^7 \frac{1}{x}dx$)*& $\int_1^3 (2x-1)dx$

)+& $\int_0^1 x\sqrt{1-x^2}\,dx$

x	$x\sqrt{1-x^2}$
(	(&(
(&)*-	(&)*,(*
(&*-	(&*,*(.
(&+/-	(&+,/.+
(&-	(&,++()
(&.*-	(&,0/01
(&/-	(&,1.(0
(&0/-	(&,*+.)
)&(	(

),& $\int_{-\frac{®}{2}}^{\frac{®}{2}} \frac{3\cos¯}{(2+\sin¯)^2}\,d¯$

¯	$\frac{3\cos¯}{(2+\sin¯)^2}$
−)&-/(0(	(&(
−)&)/0)(	(&11)+0
−(&/0-,(	)&*.1(.
−(&+1*/(	)&(-1.)
(	(&/-
(&+1*/(	(&,00*)
(&/0-,(	(&*01,.
)&)/0)(	(&)+,*1
)&-/(0(	(

)-& $\int_{-2}^{0}(x^2-1)\,dx$).& $\int_{-1}^{1}(t^3+1)\,dt$)/& $\int_{2}^{4}\frac{1}{(S-1)^2}\,ds$

)0& $\int_0^1 \sin® t\,dt$

)1& JYV W`]]` hZ_X eRS]V XZgVd gR]fVd `W $x$ R_U $f(x)$& =Z_U eYV RcVR S` f_UVU Sj eYV TfcgV $y=f(x)$, eYV x%RiZd R_U eYV `cUZ_ReVd x 5 /&,/ R_U /&-*&

x	/&,/	/&,0	/&,1	/&-(	/&-)	/&-*
$F(x)$	)&1+	)&1-	)&10	*&()	*&(+	*&(.

*(& =Z_U eYV Raac`iZ^ReV gR]fV `W $\int_{1.2}^{1.6} e^{-x^2}\,dx$ Wc`^ eYV W`]]`hZ_X eRS]V2

x	)&*	)&+	)&,	)&-	)&.
$f(x)=e^{-x^2}$	(&*+/	(&)0-	(&),)	(&)(.	(&(//

*)& <deZ^ReV eYV Vcc`cd Z_ eYV cVdf]ed `SeRZ_VU Sj VgR]fReZ_X eYV Z_eVXcR] $\int_0^1 \frac{dx}{1+x}$ Sj ecRaVk`ZUR] R_U IZ^ad`_qd cf]V&

11

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

Solution of system of linear equations

A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

where the coefficients a_{jk} and the b_j are given numbers. The system is said to be **homogeneous** if all the b_j are zero; otherwise, it is said to be **non-homogeneous**.

The system of linear equations is equivalent to the matrix equation or the single vector equation

$$Ax = b$$

where the **coefficient matrix** $A=[a_{ij}]$ is the $m \times n$ matrix and **x** and **b** are the column matrices (vectors) given by:

$$A=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & \ldots & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad x=\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} \quad \text{and} \quad b=\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$$

A **solution** of the system is a set of numbers $x_1, x_2, \ldots, x_n$ which satisfy all the m equations, and a **solution vector** of (1) is a column matrix whose components constitute a solution of system. The method of solving such a system using methods like Cramer's rule is impracticable for large systems. Hence, we use other methods like Gauss elimination.

Gauss Elimination Method

In the Gauss elimination method, the solution to the system of equations is obtained in two stages. In the first stage, the given system of equations is reduced to an equivalent upper triangular form using elementary transformations. In the second stage, the upper triangular system is solved using back substitution procedure by which we obtain the solution in the order $x_n, x_{n-1}, x_{n-2}, \cdots, x_2, x_1$.

Example l`]gV eYV djdeV^

$$2x_1 + x_2 + 2x_3 + x_4 = 6 \qquad \text{t)!}$$

$$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36 \qquad \text{t *!}$$

$$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1 \qquad \text{t +!}$$

$$2x_1 + 2x_2 - x_3 + x_4 = 10 \qquad \text{t ,!}$$

J` V]Z^Z_ReV U) Wc`^ VbfReZ`_d *!$ +! R_U ,!$ hV dfSecRTe dfZeRS]V ^f]eZa]Vd `W VbfReZ`_ )! R_U hV XVe eYV W`]]`hZ_X djdeV^ `W VbfReZ`_d2

! – + ·)! → –1U # (U+ # 1U, 5)0 t -!

+! – * ·)! → U* – U+ ––U, 5 –)+ t .!

,! –) ·)! → U* – +U+ #(U, 5 , t /!

J` V]Z^Z_ReV U* Wc`^ VbfReZ`_d .! R_U /!$ dfSecRTe dfZeRS]V ^f]eZa]Vd `W VbfReZ`_ -! R_U XVe eYV W`]]`hZ_X djdeV^ `W VbfReZ`_d2

.! – %) ' 1! -! → – U+ – ,U, 5 –)) t 0!

/! – %) ' 1! -! → – +U+ # U, 5 . t 1!

J` V]Z^Z_ReV U* Wc`^ VbfReZ`_ 1!$ dfSecRTe +× 0! R_U XVe eYV W`]]`hZ_X VbfReZ`_2

)+ U, 5 +1 t)(!

=c`^ VbfReZ`_)(!$ U, 5 +1 ')+ 5 +3 fdZ_X eYZd gR]fV `W U,$ 1! XZgVd U+ 5 %)3 fdZ_X eYVdV gR]fVd `W U, R_U U+$ /! XZgVd U* 5)3 fdZ_X R]] eYVdV gR]fVd)! XZgVd U) 5 *& ?V_TV eYV d`]feZ`_ e` eYV djdeV^ Zd U) 5 *$ U* 5)$ U+ 5 –)$ U, 5 +&

E`eV2 JYV RS`gV ^VeY`U TR_ SV dZ^a]ZWZVU fdZ_X eYV ^RecZi _`eReZ`_& JYV XZgV_ djdeV^ `W VbfReZ`_d TR_ SV hcZeeV_ Rd

$$Ax = b$$

R_U eYV RfX^V_eVU ^RecZi Zd

$$\left[\begin{array}{rrrr|r} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right.$$

hYZTY `_ dfTTVddZgV c`h ecR_dW`c^ReZ`_d XZgV

$$\left[\begin{array}{ccccc} 2 & 1 & 2 & 1 & 6 \\ 0 & -9 & 0 & 9 & 18 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & 0 & 13 & 39 \end{array}\right] \&$$

?V_TV

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & -9 & 0 & 9 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ -11 \\ 39 \end{bmatrix}$$

9RT\ dfSdeZefeZ`_ XZgVd

$$x_1 = 2\$ \quad x_2 = 1\$ \quad x_3 = -1\$ \quad x_4 = 3$$

@_ eYV ViR^a]V$ hV YRU >)) $\neq$ (& FeYVc hZdV hV h`f]U _`e YRgV SVV_ RS]V e` V]Z^Z_ReV U) Sj fdZ_X eYV VbfReZ`_d Z_ eYV XZgV_ `cUVc& ?V_TV ZW >)) $\neq$ ( Z_ eYV djdeV^ `W VbfReZ`_d hV YRgV e` cV`cUVc eYV VbfReZ`_d R_U aVcYRad VgV_ eYV f__`h_d Z_ VRTY VbfReZ`_! Z_ R dfZeRS]V WRdYZ`_3 dZ^Z]Rc]j$ Z_ eYV WfceYVc deVad& IfTY R dZefReZ`_ TR_ SV dVV_ Z_ eYV W`]]`hZ_X <iR^a]V&

Example KdZ_X >Rfdd V]Z^Z_ReZ`_ d`]gV2

$$\begin{aligned} y + 3z &= 9 \\ 2x + 2y - z &= 8 \\ -x + 5z &= 8 \end{aligned}$$

?VcV eYV]VRUZ_X T`VWWZTZV_e Z&V&$ T`VWWZTZV_e `W U! Zd (& ?V_TV e` ac`TVVU WfceYVc hV YRgV e` Z_eVcTYR_XV c`hd ) R_U *$ d` eYRe

*U ! *V J W - 0 t)!

V ! +W - 1 t *!

JU ! -W - 0 t +!

<]Z^Z_ReZ`_ `W U Wc`^ ]Rde eh` VbfReZ`_d2

*U ! *V J W - 0

V ! +W - 1

+! # $\frac{1}{2}$)! $\rightarrow$ V ! $\frac{9}{2}$ W -)* t ,!

<]Z^Z_ReZ`_ `W V Wc`^ ]Rde VbfReZ`_2

*U ! *V J W - 0

V ! +W - 1

,! – *! → $\frac{3}{2}$ W - + t -!

?V_TV W - *" V - 1 o . 5 +" U - *&

?V_TV

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}.$$

Partial and Full Pivoting

@_ VRTY deVa Z_ eYV >R f dd V]Z ^ Z_ReZ`_ ^VeY`U$ eYV T`VWWZTZV_e `W eYV WZcde f__`h_ Z_ eYV WZcde Vb f ReZ`_ Zd TR]]VU **pivotal coefficient**& 9j eYV RS`gV <i R ^ a]V$ eYV >R f dd V]Z ^ Z_ReZ`_ ^VeY`U WRZ]d ZW R_j `_V `W eYV aZg`eR] T`VWWZTZV_ed SVT`^Vd kVc`& @_ d f TY R dZe f ReZ`_$ hV cV h cZeV eYV Vb f ReZ`_d Z_ R UZWWVcV_e `cUVc e` Rg`ZU kVc` aZg`eR] T`VWWZTZV_e& :YR_XZ_X eYV `cUVc `W Vb f ReZ`_d Zd TR]]VU **pivoting**&

@_ **partial pivoting**$ ZW eYV aZg`eR] T`VWWZTZV_e a_{ii} YRaaV_d e` SV kVc` `c _VRc e` kVc`$ eYV FeY T`] f ^_ V]V ^ V_ed RcV dVRcTYVU W`c eYV _ f ^ VcZTR]]j ]RcXVde V]V ^ V_e& CVe eYV GeY c` h G6F! T`_eRZ_d eYZd V]V ^ V_e$ eYV_ hV Z_eVcTYR_XV eYV FeY Vb f ReZ`_ hZeY eYV GeY Vb f ReZ`_ R_U ac`TVVU W`c V]Z ^ Z_ReZ`_& JYZd ac`TVdd Zd T`_eZ_ f VU hYV_VgVc aZg`eR] T`VWWZTZV_ed SVT`^V kVc` U f cZ_X V]Z ^ Z_ReZ`_&

@_ **total pivoting**$ hV]``\ W`c R_ RSd`] f eV]j]RcXVde T`VWWZTZV_e Z_ eYV V_eZcV d j deV ^ R_U deRce eYV V]Z ^ Z_ReZ`_ hZeY eYV T`ccVda`_UZ_X gRcZRS]V$ f dZ_X eYZd T`VWWZTZV_e Rd eYV aZg`eR] T`VWWZTZV_e W`c eYZd hV YRgV e` Z_eVcTYR_XV OLTP R_U @LIR J KP$ ZW _VTVddRcj!3 dZ ^ Z]Rc]j Z_ eYV W f ceYVc deVad& J`eR] aZg`eZ_X$ Z_ WRTe$ Zd ^`cV T`^a]ZTReVU eYR_ eYV aRceZR] aZg`eZ_X& GRceZR] aZg`eZ_X Zd acVWVccVU W`c YR_U TR]T f]ReZ`_&

Example I`]gV eYV d j deV ^

$$0.0004x_1 + 1.402x_2 = 1.406 \qquad \text{t)!}$$

$$0.4003x_1 - 1.502x_2 = 2.501 \qquad \text{t *!}$$

Sj >R f dd V]Z ^ Z_ReZ`_ >! hZeY` f e aZg`eZ_X ?! hZeY aRceZR] aZg`eZ_X&

>! hZeY` f e aZg`eZ_X TY``dZ_X eYV WZcde Vb f ReZ`_ Rd eYV aZg`eR] Vb f ReZ`_!

$$0.0004x_1 + 1.402x_2 = 1.406 \qquad \text{t)>!}$$

$$(2) - \frac{0.40031}{0.0001} \times (1a) \rightarrow \quad -1405x_2 = -1404 \qquad \text{t *R!}$$

R_U d` $x_2 = \frac{1404}{1405} = 0.9993$

R_U YV_TV Wc`^)>!$

$$x_1 = \frac{1}{0.0004}\left(1.406 - 1.402 \times 0.9993\right) = \frac{0.005}{0.0004} = 12.5.$$

?! hZeY aRceZR] aZg`eZ_X !

IZ_TV $|a_{11}|$ Zd d^R]] R_U Zd _VRcVc e` kVc` Rd T`^aRcVU hZeY $|a_{21}|$$ hV RTTVae $a_{21}$ Rd eYV aZg`eR] T`VWWZTZV_e Z&V& dVT`_U Vbf ReZ`_ SVT`^Vd eYV aZg`eR] Vbf ReZ`_!& J` deRce hZeY hV cVRccR_XV eYV XZgV_ djdeV^ Rd W`]]`hd2

$$0.4003x_1 - 1.502x_2 = 2.501 \qquad t \ +!$$

$$0.0004x_1 + 1.402x_2 = 1.406 \qquad t \ ,!$$

E`h Sj >RfddV]Z^Z_ReZ`_ eYV djdeV^ SVT`^Vd$

$$0.4003x_1 - 1.502x_2 = 2.501 \qquad t \ +>!$$

$$(4) - \frac{\cdot 0004}{\cdot 4003}(3) \quad 1.404x_2 = 1.404 \qquad t \ ,>!$$

R_U d` $\qquad x_2 = \frac{1.404}{1.404} = 1$

R_U Wc`^ +>! $\qquad x_1 = \frac{1}{0.4003}(2.501 + 1.502 \times 1) = 10.$

Example I`]gV eYV W`]]`hZ_X djdeV^ Z! hZeY`fe aZg`eZ_X ZZ! hZeY aZg`eZ_X

$$0.0002x + 0.3003y = 0.1002 \qquad \&\&\& \ )!$$

$$2.0000x + 3.0000y = 2.0000. \qquad \&\&\& \ *!$$

Z! hZeY`fe aZg`eZ_X

$$0.0002x + 0.3003y = 0.1002$$

$$(2) - \frac{2}{\cdot 0002}(1) \rightarrow \left(3.000 - \frac{0.3003 \times 2}{0.0002}\right)y = 2.0000 - \frac{0.1002 \times 2}{0.0002}$$

Z&V&$ $\qquad 1498.5y = 499.$

E`h Sj SRT\dfSdeZefeZ`_$ eYV d`]feZ`_ e` eYV djdeV^ Zd XZgV_ Sj $y = 0.3330$ R_U $x = 0.5005$3

ZZ! MZeY aZg`eZ_X2

IZ_TV $|a_{11}|$ Zd d^R]] R_U Zd _VRcVc e` kVc` Rd T`^aRcVU hZeY $|a_{21}|$$ hV RTTVae $a_{21}$ Rd eYV aZg`eR] T`VWWZTZV_e Z&V& dVT`_U Vbf ReZ`_ SVT`^Vd eYV aZg`eR] Vbf ReZ`_!& J` deRce hZeY hV cVRccR_XV eYV XZgV_ djdeV^ Rd W`]]`hd2

$$2.0000x + 3.0000y = 2.0000 \qquad \&\&\& \ +!$$

$$0.0002x + 0.3003y = 0.1002 \qquad \&\&\& \ ,!$$

$$(4)-\frac{\cdot 0002}{2}(3)\rightarrow\left(0.3003-\frac{3.0000\times 0.0002}{2}\right)y=0.1002-\frac{2\times 0.0002}{2}$$

hYZTY dZ^a]ZWZVd e`

$$0.3000y=0.1000.$$

?V_TV Sj SR_\ dfSdeZefeZ`_$ eYV d`]feZ`_ Zd

$$y=\frac{1}{3}\quad \text{R\_U}\quad x=\frac{1}{2}.$$

:Y`]Vd\j DVeY`U D`UZWZTReZ`_ `W eYV >Rfdd ^VeY`U!

:Y`]Vd\j ^VeY`U$ hYZTY Zd R ^`UZWZTReZ`_ `W eYV >Rfdd ^VeY`U$ Zd SRdVU `_ eYV cVdf]e eYRe R_j a`dZeZgV UVWZ_ZeV dbfRcV ^ReZi **A** TR_ SV cVacVdV_eVU Z_ eYV W`c^ **A = LU**$ hYVcV **L** R_U **U** RcV eYV f_ZbfV ]`hVc R_U faaVc ecZR_Xf]Rc ^ReZTVd& JYV ^VeY`U Zd Z]]fdecReVU eYc`fXY eYV W`]]`hZ_X ViR^a]Vd&

Example KdZ_X :Y`]Vd\jqd ^VeY`U$ d`]gV eYV djdeV^2

U) # *U* # +U+ 5),

U) # +U # ,U+ 5 *(

+U) # ,U* # U+ 5),

LU AB@LJMLPFQFLK LC QEB @LBCCF@FBKQ J>QOFU **A**!

$$\mathbf{A}=\begin{bmatrix}1&2&3\\2&3&4\\3&4&1\end{bmatrix}\qquad \begin{matrix}R_2\rightarrow R_2+(-2)R_1 & m_{21}=-2\\ R_3\rightarrow R_3+(-3)R_1 & m_{31}=-3\end{matrix}$$

$$\sim\begin{bmatrix}1&2&3\\0&-1&-2\\0&-2&-8\end{bmatrix}$$

$$\sim\begin{bmatrix}1&2&3\\0&-1&-2\\0&0&-4\end{bmatrix}\qquad R_3\rightarrow R_3+(-2)R_2\quad m_{32}=-3$$

MV eR\V $U=\begin{bmatrix}1&2&3\\0&-1&-2\\0&0&-4\end{bmatrix}$ Rd eYV faaVc ecZR_Xf]Rc ^ReZi&

KdZ_X eYV ^f]eZa]ZVcd $m_{21}=-2,\ m_{31}=-3,\ m_{32}=-2$$ hV XVe eYV]`hVc ecZR_Xf]Rc ^ReZi Rd W`]]`hd2

$$L=\begin{bmatrix}1&0&0\\-m_{21}&1&0\\-m_{31}&-m_{32}&1\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}$$ &

9LIRQFLK LC QEB PVPQBJ!

JYV XZgV_ djdeV^ `WVbfReZ`_d TR_ SV hcZeeV_ Rd

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} 5 \begin{bmatrix} 14 \\ 20 \\ 14 \end{bmatrix} \qquad \text{&&&)!}$$

JYV RS`gV TR_ SV hcZeeV_ Rd

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} 5 \begin{bmatrix} 14 \\ 20 \\ 14 \end{bmatrix} \qquad \text{&&& *!}$$

hYVcV

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} 5 \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \text{&&& +!}$$

I`]gZ_X eYV djdeV^ Z_ *! Sj W`chRcU dfSdeZefeZ`_$ hV XVe

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} 5 \begin{bmatrix} 14 \\ -8 \\ -12 \end{bmatrix}$$

MZeY eYVdV gR]fVd `W $y_1$$ $y_2$$ $y_3$$ <b& +! TR_ _`h SV d`]gVU Sj SRT\ dfSdeZefeZ`_ R_U hV `SeRZ_

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} 5 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Example I`]gV eYV Vbf ReZ`_d

$$2x + 3y + z = 9$$

$$x + 2y + 3z = 6$$

$$3x + y + 2z = 8$$

Sj LU UVT`^a`dZeZ`_&

LU UVT`^a`dZeZ`_ `W eYV T`VWWZTZV_e ^RecZi **A**!

Gc`TVVUZ_X Rd Z_ eYV RS`gV Vi R^a]V$

$$U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \quad \text{R\_U} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}$$

9LIRQFLK LC QEB PVPQBJ!

JYV XZgV_ djdeV^ `W Vbf ReZ`_d TR_ SV hcZeeV_ Rd

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$ t Zg!

`c$ Rd $$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix},$$ t g!

hYVcV $$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$ t gZ!

I`]gZ_X eYV djdeV^ Z_ g! Sj W`chRcU dfSdeZefeZ`_$ hV XVe

$$y_1 = 9, \qquad y_2 = \frac{3}{2}, \qquad y_3 = 5$$ &

MZeY eYVdV gR]fVd `W $y_1, y_2, y_3,$ Vb& gZ! TR_ _`h SV d`]gVU Sj eYV SRT\ dfSdeZefeZ`_ ac`TVdd R_U hV `SeRZ_

$$x = \frac{35}{18}, \qquad y = \frac{29}{18}, \qquad z = \frac{5}{18}.$$

Gauss Jordan Method

JYV ^VeY`U Zd SRdVU `_ eYV ZUVR `W cVUfTZ_X eYV XZgV_ djdeV^ `W Vbf ReZ`_d **Ax = b,** e` R UZRX`_R] djdeV^ `W Vbf ReZ`_d **Ix = d,** hYVcV **I** Zd eYV ZUV_eZej ^ReciZi$ fdZ_X V]V^V_eRcj c`h `aVcReZ`_d& MV _`h eYRe eYV d`]feZ`_d `W S`eY eYV djdeV^d RcV ZUV_eZTR]& JYZd cVUfTVU djdeV^ XZgVd eYV d`]feZ`_ gVTe`c **x**& JYZd cVUfTeZ`_ Zd VbfZgR]V_e e` WZ_UZ_X eYV d`]feZ`_ Rd $x = A^{-1}b$ &

@_ eYZd TRdV$ R djdeV^ `W + Vbf ReZ`_d Z_ + f__`h_d

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_2 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_2 + a_{32}x_2 + a_{33}x_3 = b_3$$

Zd hcZeeV_ Rd

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad --- (*)$$

8WeVc d`^V ]Z_VRc ecR_dW`c^ReZ`_d$ hV `SeRZ_ eYV + m + djdeV^ Rd

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}---(**)$$

J` `SeRZ_ eYV djdeV^ Rd XZgV_ Z_ ""!$ WZcde hV RfX^V_e eYV ^RecZTVd XZgV_ Zd "! Rd$

$$\begin{bmatrix}a_{11} & a_{12} & a_{13} & b_1\\a_{21} & a_{22} & a_{23} & b_2\\a_{31} & a_{32} & a_{33} & b_3\end{bmatrix}$$ R_U RWeVc d` ^V V]V^V_eRcj `aVcReZ`_d$ Ze Zd hcZeeV_ Rd$

$$\begin{bmatrix}1 & 0 & 0 & d_1\\0 & 1 & 0 & d_2\\0 & 0 & 1 & d_3\end{bmatrix}---(***)$$ $ eYZd YV]ad fd e` hcZeV eYV XZgV_ djdeV^ Rd XZgV_ Z_ ""!& JYV_ Ze Zd VRdj e` XVe eYV d`]feZ`_ `W eYV djdeV^ Rd $x_1 = d_1, x_2 = d_2$ *and* $x_3 = d_3$&

Elimination procedure, JYV WZcde deVa Zd dR^V Rd Z_ >Rfdd V]Z^Z_ReZ`_ ^VeY`U$ hYZTY Zd$ hV ^R\V eYV V]V^V_ed SV]`h eYV WZcde aZg`e Z_ eYV RfX^V_eVU ^RecZi Rd kVc`d$ fdZ_X eYV V]V^V_eRcj c`h ecR_dW`c^ReZ`_d& =c`^ eYV dVT`_U deVa `_hRcUd$ hV ^R\V eYV V]V^V_ed SV]`h R_U RS`gV eYV aZg`ed Rd kVc`d fdZ_X eYV V]V^V_eRcj c`h ecR_dW`c^ReZ`_d& CRde]j$ hV UZgZUV VRTY c`h Sj Zed aZg`e d` eYRe eYV WZ_R] ^RecZi Zd `W eYV W`c^ """!& GRceZR] aZg`eZ_X TR_ R]d` SV fdVU Z_ eYV d`]feZ`_& MV ^Rj R]d` ^R\V eYV aZg`ed Rd ) SVW`cV aVcW`c^Z_X eYV V]Z^Z_ReZ`_&

Problem: I`]gV eYV W`]]`hZ_X djdeV^ `W VbfReZ`_d

$$\begin{aligned}x_1 + x_2 + x_3 &= 1\\4x_1 + 3x_2 - x_3 &= 6\\3x_1 + 5x_2 + 3x_3 &= 4\end{aligned}$$

fdZ_X eYV >Rfdd%A`cUR_ ^VeY`U hZeY`fe aRceZR] aZg`eZ_X

Solution:

MV YRgV eYV ^RecZi W`c^ Rd

$$\begin{bmatrix}1 & 1 & 1\\4 & 3 & -1\\3 & 5 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\6\\4\end{bmatrix}$$ & JYV_ eYV RfX^V_eVU ^RecZi Zd$

$$\begin{bmatrix}1 & 1 & 1 & 1\\4 & 3 & -1 & 6\\3 & 5 & 3 & 4\end{bmatrix}$$

F! J` U` eYV V]Z^Z_ReZ`_d W`]]`h eYV `aVcReZ`_d$

8*5 8* o ,8)$R_U H+5 8+ o +8)& JYZd XZgVd$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -5 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

:EBK" 8)58) # 8* R_U 8+ 58+# *8* XZgVd$

$$\begin{bmatrix} 1 & 0 & -4 & 3 \\ 0 & -1 & -5 & 2 \\ 0 & 0 & -10 & 5 \end{bmatrix}$$

8)58) o ,')(! 8)$ 8*58* o -')(! 8+ XZgVd$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -\frac{1}{2} \\ 0 & 0 & -10 & 5 \end{bmatrix}$$

E`h$ ^R\Z_X eYV aZg`ed Rd)$ 8*5 o 8*! R_U 8+5 8+' o)(!!!$ hV XVe

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

?V_TV$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

JYVcVW`cV$ eYV d`]feZ`_ `W eYV djdeV^ Zd$

$$x_1 = 1,\ x_2 = \frac{1}{2},\ x_3 = -\frac{1}{2}\ \&$$

Note: JYV >Rfdd%A`cUR_ ^VeY`U]``\d gVcj V]VXR_e Rd eYV d`]feZ`_ Zd `SeRZ_VU UZcVTe]j& ?`hVgVc$ Ze Zd T`^afeReZ`_R]]j ^`cV ViaV_dZgV eYR_ >Rfdd V]Z^Z_ReZ`_& =`c ]RcXV K" eYV e`eR] _f^SVc `W UZgZdZ`_d R_U ^f]eZa]ZTReZ`_d W`c >Rfdd%A`cUR_ ^VeY`U Zd R]^`de )&- eZ^Vd eYV e`eR] _f^SVc `W UZgZdZ`_d R_U ^f]eZa]ZTReZ`_d cVbfZcVU W`c >Rfdd V]Z^Z_ReZ`_& ?V_TV$ hV U`_`e _`c^R]]j fdV eYZd ^VeY`U W`c eYV d`]feZ`_ `W eYV djdeV^ `W Vbf ReZ`_d&

JYV ^`de Z^a`ceR_e Raa]ZTReZ`_ `W eYZd ^VeY`U Zd e` WZ_U eYV Z_gVcdV `W R _`_% dZ_Xf]Rc ^RecZi& J` `SeRZ_ Z_gVcdV `W R ^RecZi$ hV deRce hZeY eYV RfX^V_eVU ^RecZi `W **A** hZeY eYV ZUV_eZej ^RecZi **I** `W eYV dR^V `cUVc&

MYV_ eYV >Rfdd%A`cUR_ ac`TVUfcV Zd T`^a]VeVU$ hV `SeRZ_$ eYV ^RecZi 8 RfX^V_eVU hZeY 2$ $[A|I]$ Z_ eYV W`c^ $[I|A^{-1}]$ $dZ_TV $AA^{-1} = I$ &

Example KdZ_X >R f dd A`cUR_ ^VeY`U d`]gV eYV djdeV^ `W Vbf ReZ`_d2

U # *V # W 5 0 t)!

*U # +V # ,W 5 *(t *!

,U # +V # *W 5). t +!

POIF J FK>QFLK LC U COLJ ONP$ *! >KA +!$ f dZ_X)!Q

U # *V # W 5 0 t)>!

– V # *W 5 , t *>!

–-V J *W 5 –). t +>!

POIF J FK>QFLK LC V COLJ)>! >KA +>!$ f dZ_X *>!Q

U # -W 5). t)?!

– V # *W 5 , t *?!

–)*W 5 –+. t +?!

POIF J FK>QFLK LC W COLJ)?! >KA *?!$ f dZ_X +?!Q

U 5) t)@!

– V 5 –* t *@!

–)*W 5 –+. t +@!

?V_TV$ U 5)$ V 5 *$ W 5 +&

8 ddZX_^V_ed

1. 8 aa]j >R f dd V]Z^Z_ReZ`_ ^VeY`U e` d`]gV eYV Vbf ReZ`_d2

$$2x + 3y - z = 5$$
$$4x + 4y - 3z = 3$$
$$-2x + 3y - z = 1$$

2. 8 aa]j >R f dd V]Z^Z_ReZ`_ ^VeY`U e` d`]gV eYV Vbf ReZ`_d2

$$3x_1 + 6x_2 + x_3 = 16$$
$$2x_1 + 4x_2 + 3x_3 = 13$$
$$x_1 + 3x_2 + 2x_3 = 9$$

3. 8 aa]j >R f dd V]Z^Z_ReZ`_ ^VeY`U e` d`]gV eYV Vbf ReZ`_d2

$$10x + 2y + z = 9$$
$$2x + 20y - 2z = -44$$
$$-2x + 3y + 10z = 22$$

4. 8aa]j >Rfdd V]Z^Z_ReZ`_ ^VeY`U e` d`]gV eYV Vb fReZ`_d2

$$\begin{aligned} x + y + z &= 10 \\ 2x + y + 2z &= 17 \\ 3x + 2y + z &= 17 \end{aligned}$$

5. I`]gV eYV djdeV^$ fdZ_X >Rfdd V]Z^Z_ReZ`_ ^VeY`U2

$$\begin{aligned} 5x_1 + x_2 + x_3 + x_4 &= 4 \\ x_1 + 7x_2 + x_3 + x_4 &= 12 \\ x_1 + x_2 + 6x_3 + x_4 &= -5 \\ x_1 + x_2 + x_3 + 4x_4 &= -6 \end{aligned}$$

6. 8aa]j >Rfdd V]Z^Z_ReZ`_ ^VeY`U e` d`]gV eYV Vb fReZ`_d2

$$\begin{aligned} x + 4y - z &= -5 \\ x + y - 6z &= -12 \\ 3x + y - z &= 4 \end{aligned}$$

7. I`]gV eYV W`]]`hZ_X djdeV^$ fdZ_X :Y`]Vd\j ^VeY`U

$$\begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ 2x + 2y - 10z &= 14 \end{aligned}$$

8. I`]gV eYV W`]]`hZ_X djdeV^$ fdZ_X :Y`]Vd\j ^VeY`U

$$\begin{aligned} 2x + 3y - z &= 5 \\ 4x + 4y - 3z &= 3 \\ -2x + 3y - z &= 1 \end{aligned}$$

9. I`]gV eYV W`]]`hZ_X djdeV^$ fdZ_X :Y`]Vd\j ^VeY`U

$$\begin{aligned} 2x + 3y + z &= 9 \\ x + 2y + 3z &= 6 \\ 3x + y + 2z &= 8 \end{aligned}$$

10. I`]gV eYV W`]]`hZ_X fdZ_X :Y`]Vd\j ^VeY`U2

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}.$$

11. =Z_U eYV Z_gVcdV `W eYV W`]]` hZ_X ^RecZi fdZ_X :Y`]Vd\j ^VeY`U2

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 2 & 2 \end{bmatrix}.$$

12. I`]gV eYV W`]]` hZ_X djdeV^ fdZ_X >Rfdd A`cUR_ ^VeY`U2

$$2x - 3y + z = -1$$

$$x + 4y + 5z = 25$$

$$3x - 4y + z = 2$$

13. I`]gV eYV W`]]` hZ_X djdeV^ fdZ_X >Rfdd A`cUR_ ^VeY`U2

$$2x - 3y + 4z = 7$$

$$5x - 2y + 2z = 7$$

$$6x - 3y + 10z = 23$$

D8JH@N @EL<HI@FE KI@E> >8KII <C@D@E8J@FE

MV _`h eYRe $X$ hZ]] SV eYV Z_gVcdV `W R_ K#dbfRcV _`_%dZ_Xf]Rc ^RecZi A ZW

$$AX = I, \qquad \textbf{... (1)}$$

hYVcV I Zd eYV $n \times n$ ZUV_eZej ^RecZi$

<gVcj dbfRcV _`_%dZ_Xf]Rc ^RecZi hZ]] YRgV R_ Z_gVcdV& >Rfdd V]Z^Z_ReZ`_ R_U >Rfdd% A`cUR_ ^VeY`Ud RcV a`af]Rc R^`_X ^R_j ^VeY`Ud RgRZ]RS]V W`c WZ_UZ_X eYV Z_gVcdV `W R _`_%dZ_Xf]Rc ^RecZi&

For the third order matrices, (1) may be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

:]VRc]j eYV RS`gV VbfReZ`_ Zd VbfZgR]V_e e` eYV eYcVV VbfReZ`_d

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can therefore solve each of these systems using Gaussian elimination method and the result in each case will be the corresponding column of $X = A^{-1}$. We solve all the three equations simultaneously as illustrated in the following examples.

Example Using Gaussian elimination, find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$.

In this method, we place an identity matrix, whose order is same as that of A, adjacent to A which we call AUGMENTED MATRIX. Then the inverse of A is computed in two stages. In the first stage, A is converted into an upper triangular form, using Gaussian elimination method.

We write the augmented system first and then apply low transformations:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & \frac{7}{2} & \frac{17}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right] \begin{array}{l} \text{by } R_2 \to R_2 - \frac{3}{2}R_1 \\ \text{by } R_3 \to R_3 - \frac{1}{2}R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right] \text{ by } R_3 \to R_3 - 7R_{21}$$

The above is equivalent to the following three systems:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & -2 & 10 \end{array}\right] \qquad (1)$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 0 & -2 & -7 \end{array}\right] \qquad (2)$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 0 & -2 & 1 \end{array}\right] \qquad (3)$$

Now the matrix equation of the system of equations corresponding to (1) is

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 10 \end{bmatrix}$$

hYZTY `_ SRT\ dfSdeZefeZ`_ XZgVd $x_{31}=-5,\ \ x_{21}=12,\ \ x_{11}=-3.$

IZ^Z]Rc]j fdZ_X eYV `eYVc eh` djdeV^d `eYVc U gR]fVd RcV UVeVc^Z_VU R_U YV_TV eYV Z_gVcdV Zd XZgV_ Sj

$$A^{-1}=\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}=\begin{bmatrix} -3 & \frac{5}{2} & -\frac{1}{2} \\ 12 & -\frac{17}{2} & \frac{3}{2} \\ -5 & \frac{7}{2} & -\frac{1}{2} \end{bmatrix}.$$

8]] eYVdV `aVcReZ`_d RcV R]d` aVcW`c^VU `_ eYV RU[RTV_e]j a]RTVU ZUV_eZej ^RecZi&

Example KdV eYV >RfddZR_ V]Z^Z_ReZ`_ ^VeY`U e` WZ_U eYV Z_gVcdV `W eYV ^RecZi

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}&$$

8e WZcde$ hV a]RTV R_ ZUV_eZej ^RecZi `W eYV dR^V `cUVc RU[RTV_e e` eYV XZgV_ ^RecZi& JYfd$ eYV RfX^V_eVU ^RecZi TR_ SV hcZeeV_ Rd

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \qquad \&\&\&\ )!$$

@_ `cUVc e` Z_TcVRdV eYV RTTfcRTj `W eYV cVdf]e$ Ze Zd VddV_eZR] e` V^a]`j aRceZR] aZg`eZ_X& MV]``\ W`c R_ RSd`]feV]j]RcXVde T`VWWZTZV_e FK QEB CFOPQ @LIRJK R_U hV fdV eYZd T`VWWZTZV_e Rd eYV aZg`eR] T`VWWZTZV_e W`c eYZd hV YRgV e` Z_eVcTYR_XV OLTP ZW _VTVddRcj!

@_ WZcde T`]f^_ `W ^RecZi)!$, Zd eYV]RcXVde V]V^V_e$ R_U YV_TV Zd eYV aZg`eR] V]V^V_e& @_ `cUVc e` ScZ_X , Z_ eYV WZcde c`h hV Z_eVcTYR_XV eYV WZcde R_U dVT`_U c`hd R_U `SeRZ_ eYV RfX^V_eVU ^RecZi Z_ eYV W`c^

$$\left[\begin{array}{ccc|ccc} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \qquad \&\&\&\ *!$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \quad \text{by } R_1 \to \frac{1}{4}R_1$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \end{array}\right] \quad \begin{array}{l} \text{by } R_2 \to R_2 - R_1 \\ \text{by } R_3 \to R_3 - 3R_1 \end{array}$$

MV _`h dVRcTY W`c R_ RSd`]feV]j ]RcXVde T`VWWZTZV_e FK QEB PB@LKA @LIRJK R_U _`e Z_ eYV WZcde c`h! R_U hV fdV eYZd T`VWWZTZV_e Rd eYV aZg`eR] T`VWWZTZV_e& JYV aZg`e V]V^V_e Zd eYV ^Ri)',$))',! R_U Zd))',& JYVcVW`cV$ hV Z_eVcTYR_XV dVT`_U R_U eYZcU c`hd `W eYV RS`gV&

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{array}\right]$$

E`h$ UZgZUV 8* Sj eYV aZg`e V]V^V_e >** 5))',$ R_U `SeRZ_

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{array}\right]$$

@_ `cUVc e` ^R\V eYV V_ecZVd SV]`h ) Z_ eYV dVT`_U T`]f^_ hV aVcW`c^

8+ $\to$ 8+ $-$)',!8) Z_ eYV RS`gV ^RecZi R_U `SeRZ_

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & 0 & \frac{10}{11} & 1 & -\frac{2}{11} & -\frac{1}{11} \end{array}\right]$$

JYZd Zd Vbf ZgR]V_e e` eYV W`]]`hZ_X eYcVV ^RecZTVd

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 \\ 0 & 0 & \frac{10}{11} & 1 \end{array}\right] 3 \quad \left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & \frac{15}{10} & -\frac{3}{11} \\ 0 & 0 & \frac{10}{11} & -\frac{2}{11} \end{array}\right] 3 \quad \left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & \frac{4}{11} \\ 0 & 0 & \frac{10}{11} & -\frac{1}{11} \end{array}\right]$$

JYfd hV YRgV

$$A^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

Matrix Inversion using Gauss-Jordan method

JYZd ^VeY`U Zd dZ^Z]Rc e` >RfddZR_ V]Z^Z_ReZ`_ ^VeY`U W`c ^RecZi Z_gVcdZ`_$ deRceZ_X hZeY eYV RfX^V_eVU ^RecZi $[AI]$ R_U cVUfTZ_X . e` eYV ZUV_eZej ^RecZi fdZ_X V]V^V_eRcj c`h ecR_dW`c^ReZ`_d& JYV ^VeY`U Zd Z]]fdecReVU Z_ eYV W`]]`hZ_X ViR^a]V&

Example =Z_U eYV Z_gVcdV `W eYV W`]]`hZ_X ^RecZi . Sj >Rfdd%A`cUR_ ^VeY`U&

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}.$$

JYV RfX^V_eVU ^RecZi Zd XZgV_ Sj

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \text{by } R_2 \to R_2 - 4R_1 \\ \text{by } R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \text{by } R_2 \to -R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \begin{array}{l} \text{by } R_1 \to R_1 - R_2 \\ \text{by } R_3 \to R_3 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] \text{by } R_3 \to -\frac{1}{10}R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] \begin{array}{l} \text{by } R_1 \to R_1 + 4R_3 \\ \text{by } R_2 \to R_2 - 5R_1 \end{array}$$

JYfd hV YRgV

$$A^{-1} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}.$$

- **Triangulation Method (LU Decomposition Method):**

@_]Z_VRc R]XVScR$ **LU decomposition** R]d` TR]]VU **LU factorization**! WRTe`cZkVd R ^RecZi Rd eYV ac`UfTe `W R]`hVc ecZR_Xf]Rc ^RecZi R_U R_ faaVc ecZR_Xf]Rc ^RecZi

CVe . SV R _`_%dZ_Xf]Rc dbfRcV ^RecZi& CK **decomposition** Zd R UVT`^a`dZeZ`_ `W eYV W`c^

85CK

hYVcV 5 Zd R]`hVc ecZR_Xf]Rc ^RecZi R_U ; Zd R_ faaVc ecZR_Xf]Rc ^RecZi& JYZd ^VR_d eYRe 5 YRd `_]j kVc`d RS`gV eYV UZRX`_R] R_U ; YRd `_]j kVc`d SV]`h eYV UZRX`_R]& =`c ViR^a]V$ W`c R +%Sj%+ ^RecZi .$ Zed CK UVT`^a`dZeZ`_]``\d]Z\V eYZd2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

: `_dZUVc R djdeV^ `W]Z_VRc Vbf ReZ`_d$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

JYZd TR_ SV hcZeeV_ Z_ eYV W`c^$

8 i 5S$

hYVcV $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $ $x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ R_U $b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$

J` d`]gV eYV djdeV^ `W Vbf ReZ`_d Sj CK UVT`^a`dZeZ`_$ WZcde hV UVT`^a`dV 8 Rd CK$ hYVcV$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad and \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

JYZd XZgVd$

CKi 5 S&

CVe Ki5j& JYZd Z^a]ZVd$ Cj5S&

JYRe Zd$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

JYfd$

$$y_1 = b_1$$
$$l_{21}y_1 + y_2 = b_2$$
$$l_{31}y_1 + l_{32}y_2 + y_3 = b_3$$

JYZd XZgVd eYV V gR]fVd Sj W`chRcU dfSdeZefeZ`_\$ hYZTY ^VR_d\$ dfSdeZefeV eYV gR]fV `W $y_1$ XZgV_ Sj eYV WZcde Vbf ReZ`_ Z_ eYV dVT`_U R_U d`]gV y_2\$ eYV_ fdV eYVdV gR]fVd `W $y_1$ *and* $y_2$ Z_ eYV eYZcU R_U d`]gV y_3&

JYV_ eYV djdeV^ `W Vbf ReZ`_d

$$Ux = y\ ;\ \text{that is} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

XZgVd eYV cVbfZcVU gR]fVd `W $x_1, x_2$ *and* $x_3$ Rd eYV d`]feZ`_ `W eYV `cZXZ_R] djdeV^ `W]Z_VRc Vbf ReZ`_d Sj SRT\ hRcU dfSdeZefeZ`_&

J` UVT`^a`dV R ^RecZi $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ \$Z_ eYV W`c^

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

\$ hV ac`TVVU Rd W`]]`hd&

F_ ^f]eZa]jZ_X $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ *and* $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ \$ hV XVe\$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

<bfReZ_X Ze hZeY eYV T`ccVda`_UZ_X eVc^d `W 8\$ hV XVe\$

$$u_{11} = a_{11}; \quad u_{12} = a_{12}; \quad u_{13} = a_{13}$$

$$l_{21}u_{11} = a_{21} \ \Rightarrow\ l_{21} = \frac{a_{21}}{u_{11}}\ ; \quad l_{31}u_{11} = a_{31} \ \Rightarrow\ l_{31} = \frac{a_{31}}{u_{11}}$$

$$l_{21}u_{12} + u_{22} = a_{22} \ \Rightarrow u_{22} = a_{22} - l_{21}u_{12};$$

&

$$l_{21}u_{13} + u_{23} = a_{23} \ \Rightarrow u_{23} = a_{23} - l_{21}u_{13};$$

simililarly,

$$l_{31}u_{12} + l_{32}u_{22} = a_{32}, \quad l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33} \ \text{gives}\ \ l_{32}\ \text{and}\ u_{33}$$

Example: Solve the following system of equations by LU decomposition.

$2x+3y+z=9$

$x+2y+3z=6$

$3x+y+2z=8.$

Solution:

The above system of equations is written as:

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

To decompose the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ in the form of LU, we equate the corresponding terms of A and LU as already illustrated, and obtain

$$u_{11}=2; \quad u_{12}=3; \quad u_{13}=1$$

$$l_{21}=\frac{a_{21}}{u_{11}}=\frac{1}{2}\ ; \quad l_{31}=\frac{a_{31}}{u_{11}}=\frac{3}{2}$$

$$u_{22}=a_{22}-l_{21}u_{12}=2-\frac{1}{2}\times 3=\frac{1}{2};$$

$$u_{23}=a_{23}-l_{21}u_{13}=3-\frac{1}{2}\times 1=\frac{5}{2};$$

$$l_{32}=\frac{a_{32}-l_{31}u_{12}}{u_{22}}=\frac{1-\frac{3}{2}\times 3}{\frac{1}{2}}=-7 \quad \textit{and}$$

$$u_{33}=u_{33}=a_{33}-\left(l_{31}u_{13}+l_{32}u_{23}\right)=2-\left(\frac{3}{2}\times 1+(-7)\times\frac{5}{2}\right)=2-\left(\frac{3}{2}-\frac{35}{2}\right)=18$$

Hence,

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

This implies,

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Consider

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \; \$ \; \text{eYV\_} \; \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \; \$$$

I`]gZ_X eYVdV$ hV XVe$ $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$

JYRe Zd$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

E`h$ d`]gZ_X eYV RS`gV ViacVddZ`_ hV `SeRZ_ eYV gR]fVd `W i$ j R_U k Rd R d`]feZ`_ `W eYV XZgV_ djdeV^ `W Vbf ReZ`_d Rd$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{35}{18} \\ \frac{29}{18} \\ \frac{5}{18} \end{bmatrix} \; \&$$

Assignments

1. KdZ_X >Rfdd%A`cUR_ ^VeY`U$ WZ_U eYV Z_gVcdV `W eYV W`]]`hZ_X ^ReeZTVd2

Z! $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ ZZ! $B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 4 & 7 \end{bmatrix}$

2. KdZ_X >RfddZR_ V]Z^Z_ReZ`_ ^VeY`U$ WZ_U eYV Z_gVcdV `W eYV W`]]`hZ_X ^ReeZTVd2

Z! $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ ZZ! $B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix}$

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SOLUTION BY ITERATIONS

SOLUTION BY ITERATION: Jacobi's iteration method and Gauss Seidel iteration method

JYV ^VeY`Ud UZdTfddVU Z_ eYV acVgZ`fd dVTeZ`_ SV]`_X e` eYV **direct methods** W`c d`]gZ_X djdeV^d `W]Z_VRc Vbf ReZ`_d3 eYVdV RcV ^VeY`Ud eYRe jZV]U d`]feZ`_d RWeVc R_ R^`f_e `W T`^afeReZ`_ eYRe TR_ SV daVTZWZVU Z_ RUgR_TV&

@_ eYZd dVTeZ`_$ hV UZdTfdd **indirect** `c **iterative methods** Z_ hYZTY hV deRce Wc`^ R_ Z_ZeZR] gR]fV R_U `SeRZ_ SVeeVc R_U SVeeVc Raac`iZ^ReZ`_d Wc`^ R T`^afeReZ`_R] TjT]V cVaVReVU Rd `WeV_ Rd ^Rj SV _VTVddRcj$ W`c RTYZVgZ_X R cVbfZcVU RTTfcRTj$ d` eYRe eYV R^`f_e `W RcZeY^VeZT UVaV_Ud fa`_ eYV RTTfcRTj cVbfZcVU&

Jacobi's iteration method and Gauss Seidel iteration method

:`_dZUVc R ]Z_VRc djdeV^ `W n]Z_VRc Vbf ReZ`_d Z_ $n$ f_\_`h_d $x_1, x_2, \ldots, x_n$ `W eYV W`c^

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad \&\&\& \ )!$$

Z_ hYZTY eYV UZRX`_R] V]V^V_ed $a_{ii}$ U` _`e gR_ZdY&

E`h eYV djdeV^)! TR_ SV hcZeeV_ Rd

$$\left.\begin{aligned} x_1 &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \cdots - \frac{a_{1n}}{a_{11}}x_n \\ x_2 &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{22}}x_3 - \cdots - \frac{a_{2n}}{a_{22}}x_n \\ x_3 &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 - \cdots - \frac{a_{2n}}{a_{33}}x_n \\ &\vdots \\ x_n &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_2 - \cdots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1} \end{aligned}\right\} \quad t \ *!$$

Ifaa`dV hV deRce hZeY $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$ Rd Z_ZeZR] gR]fVd e` eYV gRcZRS]Vd $x_1, x_2, \ldots, x_n$& JYV_ hV TR_ WZ_U SVeeVc Raac`iZ^ReZ`_d e` $x_1, x_2, \ldots, x_n$ fdZ_X eYV W`]]`hZ_X eh` ZeVcReZgV ^VeY`Ud2

(i) Jacobi's iteration method

ART`SZqd ZeVcReZ`_ ^VeY`U$ R]d` TR]]VU eYV JBQELA LC PFJRIQ>KBLRP AFPMI>@BJBKQP$ Zd Rd W`]]`hd2

IeVa)2 ; VeVc^Z_ReZ`_ `W WZcde Raac`iZ^ReZ`_ $x_1^{(1)},\ x_2^{(1)},\ \ldots,x_n^{(1)}$ fdZ_X $x_1^{(0)},\ x_2^{(0)},\ \ldots,x_n^{(0)}$ &

$$\left.\begin{aligned}
x_1^{(1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(0)} - \frac{a_{13}}{a_{11}}x_3^{(0)} - \;\cdots\; - \frac{a_{1n}}{a_{11}}x_n^{(0)}\\
x_2^{(1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(0)} - \frac{a_{23}}{a_{22}}x_3^{(0)} - \;\cdots\; - \frac{a_{2n}}{a_{22}}x_n^{(0)}\\
x_3^{(1)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x_1^{(0)} - \frac{a_{32}}{a_{33}}x_2^{(0)} - \;\cdots\; - \frac{a_{2n}}{a_{33}}x_n^{(0)}\\
&\;\vdots\\
x_n^{(1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(0)} - \frac{a_{n2}}{a_{nn}}x_2^{(0)} - \;\cdots\; - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}^{(0)}
\end{aligned}\right\} \qquad \text{t +!}$$

IeVa *2 IZ^Z]Rc]j$ $x_1^{(2)},\ x_2^{(2)},\ \ldots,x_n^{(2)}$ RcV VgR]fReVU Sj [fde cVa]RTZ_X $x_r^{(0)}$ Z_ eYV cZXYe YR_U dZUVd Vb fReZ`_d Z_ +! Sj $x_r^{(1)}$ &

IeVa $n+1$: @_ XV_VcR]$ ZW $x_1^{(n)},\ x_2^{(n)},\ \ldots,x_n^{(n)}$ RcV R djdeV^ `W $n$ eY Raac`iZ^ReZ`_d$ eYV_ eYV _Vie Raac`iZ^ReZ`_ Zd XZgV_ Sj eYV W`c^ f]R

$$\left.\begin{aligned}
x_1^{(n+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(n)} - \frac{a_{13}}{a_{11}}x_3^{(n)} - \;\cdots\; - \frac{a_{1n}}{a_{11}}x_n^{(n)}\\
x_2^{(n+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(n)} - \frac{a_{23}}{a_{22}}x_3^{(n)} - \;\cdots\; - \frac{a_{2n}}{a_{22}}x_n^{(n)}\\
x_3^{(n+1)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x_1^{(n)} - \frac{a_{32}}{a_{33}}x_2^{(n)} - \;\cdots\; - \frac{a_{2n}}{a_{22}}x_n^{(n)}\\
&\;\vdots\\
x_n^{(n+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(n)} - \frac{a_{n2}}{a_{nn}}x_2^{(n)} - \;\cdots\; - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}^{(n)}
\end{aligned}\right\} \qquad \text{t ,!}$$

JYV djdeV^ Z_ ,! TR_ R]d` SV ScZVW]j UVdTcZSVU Rd W`]]`hd2

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{a_{ij}}{a_{ii}}x_j^{(r)} \qquad \left(r=0,1,2,\ldots,\quad i=1,\,2,\,\ldots,\,n\right)$$

8 dfWWZTZV_e T`_UZeZ`_ W`c `SeRZ_Z_X R d`]feZ`_ Sj ART`SZqd ZeVcReZ`_ ^VeY`U Zd eYV UZRX`_R] U`^Z_R_TV$

Z&V&$ $$\left|a_{ii}\right| > \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}, \quad i=1,\,2,\,\ldots,\,n.$$

Z&V&$ Z_ VRTY c`h `W $\mathbf{A}$ eYV ^`Uf]fd `W eYV UZRX`_R] V]V^V_e ViTVVUd eYV df^ `W eYV `WW UZRX`_R] V]V^V_ed R_U R]d` eYV UZRX`_R] V]V^V_ed $a_{ii}\neq 0$ & @W R_j UZRX`_R] V]V^V_e Zd ($ eYV Vb fReZ`_d TR_ R]hRjd SV cV%RccR_XVU e` dReZdWj eYZd T`_UZeZ`_ &

(ii) Gauss Seidel iteration method

8 dZ^a]V ^`UZWZTReZ`_ e` ART`SZqd ZeVcReZ`_ ^VeY`U Zd XZgV_ Sj 1>RPP#9BFABI ^VeY`U&

lVeVa) 1>RPP#9BFABI JBQELA!2 ;VeVc^Z_ReZ`_ `W WZcde Raac`iZ^ReZ`_ $x_1^{(1)},\ x_2^{(1)},\ \dots ,x_n^{(1)}$ fdZ_X $x_1^{(0)},\ x_2^{(0)},\ \dots ,x_n^{(0)}$&

$$\left.\begin{aligned}
x_1^{(1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(0)} - \frac{a_{13}}{a_{11}} x_3^{(0)} - \cdots - \frac{a_{1n}}{a_{11}} x_n^{(0)} \\
x_2^{(1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - \frac{a_{23}}{a_{22}} x_3^{(0)} - \cdots - \frac{a_{2n}}{a_{22}} x_n^{(0)} \\
x_3^{(1)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(1)} - \frac{a_{32}}{a_{33}} x_2^{(1)} - \cdots - \frac{a_{2n}}{a_{33}} x_n^{(0)} \\
&\ \vdots \\
x_n^{(1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(1)} - \frac{a_{n2}}{a_{nn}} x_2^{(1)} - \cdots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(1)}
\end{aligned}\right\} \qquad t \quad -!$$

lVeVa $n+1$: @_ XV_VcR]$ ZW $x_1^{(n)},\ x_2^{(n)},\ \dots ,x_n^{(n)}$ RcV R djdeV^ `W $n$eY Raac`iZ^ReZ`\_d$ eYV\_ eYV \_Vie Raac`iZ^ReZ`\_ Zd XZgV\_ Sj eYV W`c^ f]R

$$\left.\begin{aligned}
x_1^{(n+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(n)} - \frac{a_{13}}{a_{11}} x_3^{(n)} - \cdots - \frac{a_{1n}}{a_{11}} x_n^{(n)} \\
x_2^{(n+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(n+1)} - \frac{a_{23}}{a_{22}} x_3^{(n)} - \cdots - \frac{a_{2n}}{a_{22}} x_n^{(n)} \\
x_3^{(n+1)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(n+1)} - \frac{a_{32}}{a_{33}} x_2^{(n+1)} - \cdots - \frac{a_{2n}}{a_{33}} x_n^{(n)} \\
&\ \vdots \\
x_n^{(n+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n+1)} - \frac{a_{n2}}{a_{nn}} x_2^{(n+1)} - \cdots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(n+1)}
\end{aligned}\right\} \qquad t \quad .!$$

.! TR_ SV ScZVW]j UVdTcZSVU Rd W`]]`hd2

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{(r+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(r)} \quad (r = 0,1,2,\dots, \quad i = 1,\ 2,\ \dots,\ n).$$

Remark MV _`eV eYV UZWWVcV\_TV SVehVV\_ ART`SZqd ^VeY`U R\_U 1>RPP#9BFABI ^VeY`U&

Attention! @_ eYV W`]]`hZ_X eYV S`]U WRTV ]VeeVcd ^fde SV TRcVWf]]j \_`eVU!2

3>@L?FZP JBQELA, @_ eYV WZcde Vbf ReZ`\_ `W +!$ hV dfSdeZefeV eYV Z_ZeZR] Raac`iZ^ReZ`_d $x_2^{(0)},\ x_3^{(0)}, \dots ,x_n^{(0)}$ Z_e` eYV cZXYe%YR\_U dZUV R\_U UV\_`eV eYV cVdf]e Rd $x_1^{(1)}$. @_ eYV dVT`\_U Vbf ReZ`_$ hV dfSdeZefeV $\mathbf{x_1^{(0)}},\ \mathbf{x_3^{(0)}}, \dots ,x_n^{(0)}$ R_U UV_`eV eYV cVdf]e Rd $x_2^{(1)}$. @\_ eYZcU$ hV dfSdeZefeV $\mathbf{x_1^{(0)}},\ \mathbf{x_2^{(0)}},\ \dots ,x_n^{(0)}$ R\_U TR]] eYV cVdf]e Rd $x_3^{(1)}$. JYV ac`TVdd Zd cVaVReVU Z_ eYZd ^R__Vc&

1>RPP#9BFABI JBQELA, @_ eYV WZcde Vbf ReZ`_ `W +!$ hV dfSdeZefeV eYV Z_ZeZR] Raac`iZ^ReZ`_ $x_2^{(0)}, \ldots, x_n^{(0)}$ Z_e` eYV cZXYe%YR_U dZUV R_U UV_`eV eYV cVdf]e Rd $x_1^{(1)}$. @_ eYV dVT`_U Vbf ReZ`_$ hV dfSdeZefeV $\mathbf{x_1^{(1)}}, x_3^{(0)}, \ldots, x_n^{(0)}$ R_U UV_`eV eYV cVdf]e Rd $x_2^{(1)}$. @_ eYZcU$ hV dfSdeZefeV $\mathbf{x_1^{(1)}}, \mathbf{x_2^{(1)}}, \ldots, x_n^{(0)}$ R_U TR]] eYV cVdf]e Rd $x_3^{(1)}$. JYV ac`TVdd Zd cVaVReVU Z_ eYZd ^R__Vc R_U Z]] f decReVU SV]`h2

***Example* 11** I`]gV eYV W`]]`hZ_X djdeV^ `W Vbf ReZ`_d fdZ_X R! ART`SZqd ZeVcReZ`_ ^VeY`U R_U S! >Rfdd%IVZUV] ZeVcReZ`_ ^VeY`U&

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$

$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$

$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$

$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$ &

9LIRQFLK

J` d`]gV eYVdV Vbf ReZ`_d Sj eYV ZeVcReZgV ^VeY`Ud$ hV cV%hcZeV eYV^ Rd W`]]`hd2

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3$$

@e TR_ SV gVcZWZVU eYRe eYVdV Vbf ReZ`_d dReZdWj eYV UZRX`_R] U`^Z_R_TV T`_UZeZ`_& JYV ac`TVdd R_U XZgV_ Z_ eYV W`]]`hZ_X JRS]Vd&

Table 1. ART`SZqd DVeY`U

n	x_1	x_2	x_3	x_4
1	0.3	1.56	2.886	−0.1368
2	0.8869	1.9523	2.9566	−0.0248
3	0.9836	1.9899	2.9924	−0.0042
4	0.9968	1.9982	2.9987	−0.0008
5	0.9994	1.9997	2.9998	−0.0001
6	0.9999	1.9999	3.0	0.0
7	1.0	2.0	3.0	0.0

Table 2. >Rfdd%I VZUV] ^VeY`U

n	x_1	x_2	x_3	x_4
1	0.3	1.5	2.7	−0.9
2	0.78	1.74	2.7	−0.18
3	0.9	1.908	2.916	−0.108
4	0.9624	1.9608	2.9592	−0.036
5	0.9845	1.9848	2.9851	−0.0158
6	0.9939	1.9938	2.9938	−0.006
7	0.9975	1.9975	2.9976	−0.0025
8	0.9990	1.9990	2.9990	−0.0010
9	0.9996	1.9996	2.9996	−0.0004
10	0.9998	1.9998	2.9998	−0.0002
11	0.9999	1.9999	2.9999	−0.0001
12	1.0	2.0	3.0	0.0

=c`^ JRS]Vd ) R_U *$ Ze Zd T]VRc eYRe ehV]gV ZeVcReZ`_d RcV cVbfZcVU Sj ART`SZqd ^VeY`U e` RTYZVgV eYV dR^V RTTfcRTj Rd dVgV_ >Rfdd%IVZUV] ZeVcReZ`_d&

Example **12** I`]gV Sj ART`SZqd ZeVcReZ`_ ^VeY`U$ eYV dj deV^ `W Vbf ReZ`_d

$$20x_1 + x_2 - 7x_3 = 17$$

$$3x_1 + 20x_2 - x_3 = -18$$

$$2x_1 - 3x_2 + 20x_3 = 25$$

9LIRQFLK JYV XZgV_ dj deV^ `W Vbf ReZ`_d TR_ SV hcZeeV_ Rd

$$\left.\begin{aligned} x_1 &= \frac{17}{20} - \frac{1}{20}x_2 + \frac{7}{20}x_3 \\ x_2 &= -\frac{18}{20} - \frac{3}{20}x_1 + \frac{1}{20}x_3 \\ x_3 &= \frac{25}{20} - \frac{2}{20}x_1 + \frac{3}{20}x_2 \end{aligned}\right\} \quad +!$$

MV deRce Wc`^ R_ Raac`iZ^ReZ`_ $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$ e` x_1, x_2, x_3 cVdaVTeZgV]j& I fSdeZefeZ_X eYVdV gR]fVd `_ eYV cZXYe dZUVd `W VbfReZ`_d Z_ +!$ hV XVe eYV WZcde Raac`iZ^ReZ`_ gR]fVd $x_1^{(1)} = \frac{17}{20} = 0.85$$ $x_2^{(1)} = -\frac{18}{20} = -0.90$ R_U $x_3^{(1)} = \frac{25}{20} = 1.25$

GfeeZ_X eYVdV gR]fVd `_ eYV cZXYe dZUV `W eYV VbfReZ`_d Z_ *!$ hV `SeRZ_ eYV dVT`_U Raac`iZ^ReZ`_ gR]fVd$ $x_1^{(2)} = 1.02$$ $x_2^{(2)} = -0.965$ R_U $x_3^{(2)} = 1.03$& IZ^Z]Rc]j$ eYZcU Raac`iZ^ReZ`_ gR]fVd RcV $x_1^{(3)} = 1.00125$$ $x_2^{(3)} = -1.0015$ R_U $x_3^{(3)} = 1.004$ R_U W`fceY Raac`iZ^ReZ`_ gR]fVd RcV $x_1^{(4)} = 1.000475$$ $x_2^{(4)} = -0.9999875$ R_U $x_3^{(4)} = 0.99965$& @e TR_ SV dVV_ eYRe eYV gR]fVd Raac`RTY eYV ViRTe d`]feZ`_ $x_1 = 1$$ $x_2 = -1$$ $x_3 = 1$&

Example 13 I`]gV$ fdZ_X >Rfdd%IVZUV] ZeVcReZ`_ ^VeY`U$ eYV djdeV^2

U) % (&*-U* % (&*-U+ 5 -(

%(&*-U) # U* % (&*-U, 5 -(

%(&*-U) # U+ % (&*-U, 5 *-

%(&*-U* %(&*-U+ # U, 5 *-

9LIRQFLK

JYV XZgV_ djdeV^ `W VbfReZ`_d TR_ SV hcZeeV_ Rd

$$\left.\begin{aligned} x_1 &= 50 + 0.25x_2 + 0.25x_3 \\ x_2 &= 50 + 0.25x_1 + 0.25x_4 \\ x_3 &= 25 + 0.25x_1 + 0.25x_4 \\ x_4 &= 25 + 0.25x_2 + 0.25x_3 \end{aligned}\right\} \quad \text{t *!}$$

MV deRce Wc`^ R_ Raac`iZ^ReZ`_ $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 100$ e` x_1, x_2, x_3 cVdaVTeZgV]j& JYV_ hV XVe Raac`iZ^ReZ`_ gR]fVd Rd W`]]`hd2

$$x_1^{(1)} = 50 + 0.25\,x_2^{(0)} + 0.25x_3^{(0)} = 100.00$$

$$x_2^{(1)} = 50 + 0.25x_1^{(1)} + 0.25x_4^{(0)} = 100.00 \qquad x_3^{(1)} = 50 + 0.25x_1^{(1)} + 0.25x_4^{(0)} = 75.00$$

$$x_4^{(1)} = 25 + 0.25x_2^{(1)} + 0.25x_3^{(1)} = 68.75$$

E`h dVT`_U Raac`iZ^ReZ`_ gR]fVd RcV XZgV_ Sj2

$$x_1^{(2)} = 50 + 0.25x_2^{(1)} + 0.25x_3^{(1)} = 93.75$$

$$x_2^{(2)} = 50 + 0.25x_1^{(2)} + 0.25x_4^{(1)} = 90.62 \qquad x_3^{(2)} = 50 + 0.25x_1^{(2)} + 0.25x_4^{(1)} = 65.62$$

$$x_4^{(2)} = 25 + 0.25x_2^{(2)} + 0.25x_3^{(2)} = 64.06$$&

E`eV eYRe eYV ViRTe d`]feZ`_ e` eYV djdeV^ Zd

$$x_1 = x_2 = 87.5, \quad x_3 = x_4 = 62.5$$

Example 14 KdZ_X >Rfdd lZVUV] ZeVcReZ`_ d`]gV eYV W`]]`hZ_X djdeV^ `W VbfReZ`_d$ Z_ eYcVV deVad deRceZ_X Wc`^)$)$)&

$$10x + y + z = 6$$

$$x + 10y + z = 6$$

$$x + y + 10z = 6$$

Solution

$$x = 0.6 - 0.1y - 0.1z$$

$$y = 0.6 - 0.1x - 0.1z$$

$$z = 0.6 - 0.1x - 0.1y$$

Step 1 KdZ_X U(! 5 V(! 5 W(! 5)$ hV YRgV

U)! - (&. – (&) V(! J (&) W(! 5 (&. – (&) – (&) 5 (&,

V)! - (&. – (&) U)! J (&) W(! 5 (&. – (&)×(&, – (&) 5 (&,.

W)! - (&. – (&) U)! J (&) V)! 5 (&. – (&)×(&, – (&)×(&,. 5 (&-),

Step 2 KdZ_X U)! 5 (&,$ V)!5 (&,.$ W)! 5 (&-),$ hV YRgV

U*! - (&. – (&) V)! J (&) W)! 5 (&. – (&) × (&,. – (&)×(&-), 5 (&-(*.

V*! - (&. – (&) U*! J (&) W)! 5 (&. – (&)×(&-(*. – (&)×(&-), 5 (&,10+,

W*! - (&. – (&) U*! J (&) V*!

5 (&. – (&)×(&-(*.– (&)×(&,10+, 5 (&,111(.

Step 3 KdZ_X U*! 5 (&-(*.$ V*!5 (&,10+,$ W*! 5 (&,111(.$ hV YRgV

U+! - (&. – (&) V*! J (&) W*! 5 (&. – (&) × (&,10+,– (&)×(&,111(.5 (&-(()/-,

V+! - (&. – (&) U+! J (&) W*!

5 (&. – (&)×(&-(()/-,– (&)×(&,111(.5 (&,1111)0.

W+! - (&. – (&) U+! J (&) V+!

5 (&. – (&)×(&-(()/-,– (&)×(&-(()/-,5 (&,111.,1*

MVeR\V U ù -$ V ù -$ W ù - Rd eYV d`]feZ`_ `W eYV XZgV_ djdeV^ `W VbfReZ`_d&

OUBO@FPBP

1. 8aa]j >Rfdd lVZUV] ZeVcReZ`_ ^VeY`U e` d`]gV2

$$10x+2y+z=9$$

$$2x+20y-2z=-44$$

$$-2x+3y+10z=22$$

2. 8aa]j >Rfdd lVZUV] ZeVcReZ`_ ^VeY`U e` d`]gV2

$$1.2x+2.1y+4.2z=9.9$$

$$5.3x+6.1y+4.7z=21.6$$

$$9.2x+8.3y+z=15.2$$

3. 8aa]j ART`SZqd ZeVcReZ`_ ^VeY`U e` d`]gV2

$$5x-y+z=10$$

$$2x-y+z=10$$

$$x+y+5z=-1$$

4. 8aa]j ART`SZqd ZeVcReZ`_ ^VeY`U e` d`]gV2

$$5x+2y+z=12$$

$$x+4y+2z=15$$

$$x+2y+5z=20$$

Answers

1. $x=1.013, y=-1.996, z=3.001$

2. $x=2, y=3, z=4$ 8aac`iZ^ReV]j!

3. $x=-13.223,\ y=16.766,\ z=-2.306$

4. $x=2.556, y=1.722,\ z=-1.005$

5. $x=1.08,\ y=1.95,\ z=3.16$

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EIGEN VALUES

Eigen Values

Definitions I f aa`dV X SV R_ Z_UVeVc ^Z_ReV& : `_dZUVc eYV K m K ^RecZi

$$A=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & \ldots & . \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}=[a_{ij}]_{n\times n} \qquad \text{\& \& \&)!}$$

JYV_ eYV ^RecZi . Y X2 $ hYVcV 2 Zd eYV ZUV_eZej ^RecZi `W `cUVc K$ Zd TR]]VU eYV **characteristic matrix of** . R_U Zd XZgV_ Sj

$$A-ª I=\begin{bmatrix} a_{11}-ª & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22}-ª & . & . & . & a_{2n} \\ & . & & . & & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn}-ª \end{bmatrix}. \qquad \text{\& \& \& *!}$$

JYV UVeVc^Z_R_e |. J X2 | `W eYV TYRcRTeVcZdeZT ^RecZi `W . XZgV_ Z_ *! TR_ SV W`f_U `fe e` SV

?(! ?) X ! ?* X* ! $ $ $! ?K –)XK –) ! ?K XK &&& +!

hYVcV ? F RcV dTR]Rcd& E`h +! Zd R _`_%kVc` a`]j_`^ZR] `W UVXcVV K Z_ eYV Z_UVeVc^Z_ReV X& JYZd a`]j_`^ZR] Zd TR]]VU eYV **characteristic polynomial** `W .& JYRe Zd eYV @E>O>@QBOFPQF@ MLIVKLJF>I `W eYV ^RecZi . Zd XZgV_ Sj

|. J X2 |& &&& +′!

JYV Vbf ReZ`_

|. J X2 | 5 (&&& ,!

Z&V&$ eYV Vbf ReZ`_

$$\begin{vmatrix} a_{11}-ª & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22}-ª & . & . & . & a_{2n} \\ & . & & . & & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn}-ª \end{vmatrix}=0 \qquad \text{t ,′!}$$

Zd TR]]VU eYV **characteristic equation of the matrix** .$

JYV c``ed `W eYV TYRcRTeVcZdeZT VbfReZ`_ ,! RcV TR]]VU eYV **characteristic roots** `c **latent roots** `c **eigen values** `W eYV ^RecZi .& @W ª Zd R_ VZXV_ gR]fV$ eYV_ T`]f^_ gVTe`c = dfTY eYRe $AX = \lambda X$ Zd TR]]VU R_ **eigen vector** Rdd`TZReVU hZeY eYV VZXV_ gR]fV ª&

Example =Z_U eYV VZXV_ gR]fVd R_U eYV T`ccVda`_UZ_X VZXV_ gVTe`cd `W eYV ^RecZi

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

9LIRQFLK

JYV TYRcRTeVcZdeZT VbfReZ`_ `W . Zd | . J X2 | 5 (&

Z&V&$
$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

F_dZ^a]ZWZTReZ`_ hV XVe

%X + #)0 X* – ,- X 5 ($

hYZTY XZgVd eYV VZXV_ gR]fVd n 5 (3 n 5 +3 n 5)-&

/BQBO J FK>QFLK LC BFDBK SB@QLO @LOOBPMLKAFKD QL QEB BFDBK S>IRB $\lambda = 0$!

CVe $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ SV eYV VZXV_ gVTe`c T`ccVda`_UZ_X e` $\lambda = 0$ Zd `SeRZ_VU Sj d`]gZ_X $AX = 0X$

Z&V&$ Sj d`]gZ_X

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Z&V&$ Sj d`]gZ_X

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

JYV T`ccVda`_UZ_X djdeV^ `W ]Z_VRc VbfReZ`_d Zd

$$\begin{aligned} 8x_1 - 6x_2 + 2x_3 &= 0 \quad \cdots(1) \\ -6x_1 + 7x_2 - 4x_3 &= 0 \quad \cdots(2) \\ 2x_1 - 4x_2 + 3x_3 &= 0 \quad \cdots(3) \end{aligned}$$

E`h)! R_U +! TR_ SV hcZeeV_ Rd

$$4x_1 - 3x_2 + x_3 = 0$$

R_U $\qquad 2x_1 - 4x_2 + 3x_3 = 0.$

E`h Sj eYV ^VeY`U `W Tc`dd ^f]eZa]ZTReZ`_

$$\frac{x_1}{-3\cdot 3-1\cdot(-4)} = \frac{x_2}{1\cdot 2-4\cdot 3} = \frac{x_3}{4\cdot(-4)-3\cdot 2}$$

`c $\qquad \frac{x_1}{-5} = \frac{x_2}{-10} = \frac{x_3}{-10}$&

`c $\qquad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$&

?V_TV $\qquad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = k,$

hYVcV H Zd RcSZecRcj&

$\therefore \qquad x_1 = k, \;\; x_2 = 2k, \;\; x_3 = 2k.$ &&& ,!

JYV d`]feZ`_ XZgV_ Z_ ,! R]d` dReZdWZVd eYV VbfReZ`_ *!&

$\therefore$ VZXV_ gVTe`c T`ccVda`_UZ_X e` X5(Zd XZgV_ Sj $X = \begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix}$&

8 aRceZTf]Rc VZXV_ gR]fV Zd hZeY $k = 1$! Zd $X = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$

/BQBO JFK>QFLK LC BFDBK SB@QLO @LOOBPMLKAFKD QL QEB BFDBK S>IRB n5+!

JYV VZXV_ gVTe`c = T`ccVda`_UZ_X e` n5+ Zd `SeRZ_VU Sj d`]gZ_X $AX = 3X$ `c Sj d`]gZ_X .Y+2 = -&

Z&V&$ Sj d`]gZ_X

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

9j V]V^V_eRcj c`h ecR_dW`c^ReZ`_d$ eYV RS`gV ^RecZi VbfReZ`_ Zd VbfZgR]V_e e` eYV ^RecZi VbfReZ`_

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

:Y``dZ_X $x_3 = k$, RcSZecRcj$ hV YRgV $x_1 + x_3 = 0, \;\; x_2 + \frac{1}{2}x_3 = 0$&

?V_TV $X = \begin{bmatrix} -k \\ -\frac{1}{2}k \\ k \end{bmatrix}$

Zd R_ VZXV_ gVTe`c T`ccVda`_UZ_X e` eYV VZXV_ gR]fV n 5 +&

8 aRceZT f]Rc VZXV_ gR]fV Zd hZeY $k = 2$! Zd $X = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$.

/ BQBO J FK>QFLK LC BFDBK SB@QLO @LOOBPMLKAFKD QL QEB BFDBK S>IRB n 5)- !

JYV VZXV_ gVTe`c = T`ccVda`_UZ_X e` n 5)- Zd `SeRZ_VU Sj d`]gZ_X $AX = 15X$

Z&V&$ Sj d`]gZ_X . Y)- 2 = - &

Z&V&$ Sj d`]gZ_X

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

?V_TV $X = \begin{bmatrix} 2a \\ -2a \\ a \end{bmatrix}$

Zd R_ VZXV_ gVTe`c T`ccVda`_UZ_X e` eYV VZXV_ gR]fV n 5)-&

Example=Z_U eYV VZXV_ gR]fVd R_U eYV VZXV_ gVTe`c T`ccVda`_UZ_X e` eYV]RcXVde VZXV_ gR]fV `W eYV ^RecZi

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

9LIRQFLK

@e TR_ SV dVV_ eYRe eYV VZXV_ gR]fVd RcV *$ * R_U 0&

E`h hV UVeVc^Z_V eYV VZXV_ gVTe`c T`ccVda`_UZ_X e` eYV]RcXVde VZXV_ gR]fV 02

JYV VZXV_ gVTe`c = T`ccVda`_UZ_X e` X 5 0 Zd `SeRZ_VU Sj d`]gZ_X $AX = 8X$ Z&V&$ Sj d`]gZ_X P. Y 0 2 Q = - &

Z&V&$ Sj d`]gZ_X

$$\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Z&V&$ Sj d`]gZ_X

$$\begin{bmatrix} 2 & -2 & 2 \\ -2 & -4 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

JYV T`ccVda`_UZ_X djdeV^ `W ]Z_VRc Vbf ReZ`_d Zd

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 &= 0 \quad \cdots(1) \\ -2x_1 - 5x_2 - x_3 &= 0 \quad \cdots(2) \\ 2x_1 - x_2 - 5x_3 &= 0 \quad \cdots(3) \end{aligned}$$

E`h)! R_U +! TR_ SV hcZeeV_ Rd

$$x_1 - x_2 + x_3 = 0$$

R_U $$2x_1 - x_2 - 5x_3 = 0.$$

E`h Sj eYV ^VeY`U `W Tc`dd ^f]eZa]ZTReZ`_

$$\frac{x_1}{-1\cdot(-5)-1\cdot(-1)} = \frac{x_2}{1\cdot 2-(1)\cdot(-5)} = \frac{x_3}{(-1)\cdot(-1)-(-1)\cdot 2}$$

`c $$\frac{x_1}{6} = \frac{x_2}{-3} = \frac{x_3}{3}$$ &

`c $$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$ &

?V_TV $$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} = k.$$

$\therefore$ $x_1 = 2k, \quad x_2 = -k, \quad x_3 = k.$ &&& ,!

JYV d`]feZ`_ XZgV_ Z_ ,! R]d` dReZdWZVd eYV Vbf ReZ`_ *!&

$\therefore$ eYV VZXV_ gVTe`c T`ccVda`_UZ_X e` X 5 0 Zd

$$X = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix}.$$

8 aRceZTf]Rc VZXV_ gR]fV Zd hZeY $k=1$! Zd $X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$

Example =Z_U eYV VZXV_gR]f Vd R_U VZXV_gVTe`cd `W eYV ^ReZi2

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

JYV TYRcRTeVcZdeZT Vbf ReZ`_ `W eYV ^RecZi Zd XZgV_ Sj

$$\begin{vmatrix} 5-\lambda & 0 & 1 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & 5-\lambda \end{vmatrix} = 0$$

hYZTY XZgVd $\lambda_1 = -2, \lambda_2 = 4$ R_U $\lambda_3 = 6$&

; VeVc^Z_ReZ`_ `W VZXV_gVTe`cd T`ccVda`_UZ_X e` $\lambda_1 = -2$& CVe eYV VZXV_gVTe`c SV

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot$$

JYV_ hV YRgV2

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \$$$

hYZTY XZgVd eYV Vbf ReZ`_d

$$7x_1 + x_3 = 0$$

and $x_1 + 7x_3 = 0$

JYV d`]feZ`_ Zd $x_1 = x_3 = 0$ hZeY x_2 RcSZecRcj&@_ aRceZTf]Rc$ hV eR\V $x_2 = 1$ R_U R_ VZXV_gVTe`c Zd

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

; VeVc^Z_ReZ`_ `W VZXV_gVTe`cd T`ccVda`_UZ_X e` $\lambda_2 = 4$&@W

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Zd R_ VZXV_gVTe`c$ eYV Vbf ReZ`_d RcV

$$x_1 + x_3 = 0$$

R_U $-6x_2 = 0$

Wc`^ hYZTY hV `SeRZ_

$x_1 = -x_3$ R_U $x_2 = 0.$

MV TY``dV$ Z_ aRceZT f]Rc$ $x_1 = 1/\sqrt{2}$ R_U $x_3 = -1\sqrt{2}$ d` eYRe $x_1^{2} + x_2^{2} + x_3^{2} = 1$& JYV VZXV_gVTe`c TY`dV_ Z_ eYZd hRj Zd dRZU SV _`c^R]ZkVU& MV eYVcVW`cV YRgV $X_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.

; VeVc^Z_ReZ`_ `W VZXV_gVTe`cd T`ccVda`_UZ_X e` $a_3 = 6$& @W

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Zd eYV cVbfZcVU VZXV_gVTe`c$ eYV_ eYV Vbf ReZ`_d RcV

$$-x_1 + x_3 = 0$$

$$-8x_2 = 0$$

$$x_1 - x_3 = 0$$

hYZTY XZgV $x_1 = x_3$ R_U $x_2 = 0$&

: Y``dZ_X $x_1 = x_3 = 1/\sqrt{2}$$ eYV _`c^R]ZkVU VZXV_gVTe`c Zd XZgV_ Sj

$$X_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Example ; VeVc^Z_V eYV]RcXVde VZXV_ gR]fV R_U eYV T`ccVda`_UZ_X VZXV_gVTe`c `W eYV ^RecZi

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

CVe eYV Z_ZeZR] VZXV_gVTe`c SV

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = X^{(0)}$$&

JYV_ hV YRgV

$$AX^{(0)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

CVe $X^{(1)} = \begin{bmatrix}1\\1\\0\end{bmatrix}$. JYV_ hV YRgV $AX^{(0)} = X^{(1)}$ R_U hV YRgV R_ Raac`iZ^ReV VZXV_gR]fV Zd ) R_U R_ Raac`iZ^ReV VZXV_gVTe`c Zd $X^{(1)}$& ?V_TV hV YRgV

$$AX^{(1)} = \begin{bmatrix}1 & 6 & 1\\1 & 2 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}7\\3\\0\end{bmatrix} = 3\begin{bmatrix}2.3\\1\\0\end{bmatrix}$$

Wc`^ hYZTY hV dVV eYRe

$$X^{(2)} = \begin{bmatrix}2.3\\1\\0\end{bmatrix}$$

R_U eYRe R_ Raac`iZ^ReV VZXV_gR]fV Zd +&

HVaVReZ_X eYV RS`gV ac`TVUfcV$ hV dfTTVddZgV]j `SeRZ_

$$4\begin{bmatrix}2.1\\1.1\\0\end{bmatrix};\quad 4\begin{bmatrix}2.2\\1.1\\0\end{bmatrix};\quad 4.4\begin{bmatrix}2\\1\\0\end{bmatrix};\quad 4\begin{bmatrix}2\\1\\0\end{bmatrix};\quad 4\begin{bmatrix}2\\1\\0\end{bmatrix}.$$

@e W`]]`hd eYRe eYV]RcXVde VZXV_gR]fV Zd , R_U eYV T`ccVda`_UZ_X VZXV_gVTe`c Zd

$$\begin{bmatrix}2\\1\\0\end{bmatrix}&$$

Eigenvalues of a Symmetric Tridiagonal Matrix

IZ_TV dj^^VecZT ^RecZTVd TR_ SV cVUfTVU e` dj^^VecZT ecZUZRX`_R] ^RecZTVd$ eYV UVeVc^Z_RZ`_ `W VZXV_gR]fVd `W R dj^^VecZT ecZUZRX`_R] ^RecZi Zd `W aRceZTf]Rc Z_eVcVde& :`_dZUVc eYV **tridiagonal matrix**

$$A_1 = \begin{bmatrix}a_{11} & a_{12} & 0\\a_{12} & a_{22} & a_{23}\\0 & a_{23} & a_{33}\end{bmatrix}&$$

J` `SeRZ_ eYV VZXV_gR]fVd `W $A_1$$ hV W`c^ eYV UVeVc^Z_R_e Vbf ReZ`_

$$|A_1 - ª| = \begin{vmatrix}a_{11} - ª & a_{12} & 0\\a_{12} & a_{22} - ª & a_{23}\\0 & a_{23} & a_{33} - ª\end{vmatrix} = 0.$$

I faa`dV eYRe eYV RS`gV Vbf ReZ`_ Zd hcZeeV_ Z_ eYV W`c^

$$¤_3(ª) = 0 \qquad \&\&\& \ )!$$

<iaR_UZ_X eYV UVeVc^Z_R_ed Z_ eVc^d `W eYV eYZcU c`h$ hV `SeRZ_

$$¤_3(ª) = (a_{33} - ª)\begin{vmatrix} a_{11} - ª & a_{12} \\ a_{12} & a_{22} - ª \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} - ª & 0 \\ a_{12} & a_{23} \end{vmatrix}$$

$$= (a_{33} - ª)¤_2(ª) - a_{23}(a_{11} - ª)a_{23}\$$$

hYVcV $¤_2(ª) = \begin{vmatrix} a_{11} - ª & a_{12} \\ a_{12} & a_{22} - ª \end{vmatrix}$

$$= (a_{33} - ª)¤_2(ª) - a_{23}{}^2¤_1(ª)\$$$ hYVcV $¤_1(ª) = (a_{11} - ª)$

?V_TV)! Z^a]ZVd$

$$(a_{33} - ª)¤_2(ª) \; - \; a_{23}^2¤_1(ª) = 0\&$$

MV eYfd `SeRZ_ eYV cVTfcdZ`_ W`c^f]R

$$¤_0(ª) = 1$$

$$¤_1(ª) = a_{11} - ª$$

$$= (a_{11} - ª)¤_0(ª)$$

$$¤_2(ª) = \begin{vmatrix} a_{11} - ª & a_{12} \\ a_{12} & a_{22} - ª \end{vmatrix}$$

$$= (a_{11} - ª)(a_{22} - ª) - a_{12}^2$$

$$= ¤_1(ª)(a_{22} - ª) - a_{12}^2¤_0(ª)$$

$$¤_3(ª) = ¤_2(ª)(a_{33} - ª) - a_{23}^2¤_1(ª)\&$$

@_ XV_VcR]$ ZW

$$¤_k(ª) = \begin{vmatrix} a_{11} - ª & a_{12} & 0 & \dots & 0 \\ a_{12} & a_{22} - ª & a_{23} & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & a_{k-1}, k & a_{kk} - ª \end{vmatrix}\$ \quad (2 \le k \le n),$$

eYV_ eYV cVTfcdZ`_ W`c^f]R Zd

$$¤_k(ª) = (a_{kk} - ª)¤_{k-1}(ª) - a_{k-1,k}^2¤_{k-2}(ª) \;\$\; (2 \le k \le n)$$

JYV Vbf ReZ`_ $¤_k(ª) = 0$ Zd eYV TYRcRTeVcZdeZT Vbf ReZ`_ R_U TR_ SV d`]gVU fdZ_X eYV ^VeY`Ud UZdTfddVU Z_ :YRaeVc *& MYV_ eYV VZXV_ gR]fVd RcV _`h_ Zed VZXV_ gVTe`cd TR_ SV TR]Tf]ReVU&

Exercises

1. =Z_U eYV VZXV_ gR]fVd R_U eYV T`ccVda`_UZ_X VZXV_ gVTe`cd `W eYV W`]]`hZ_X ^RecZTVd2

R! $\begin{bmatrix} -3 & 0 \\ 5 & -1 \end{bmatrix}$ S! $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

T! $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ U! $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

V! $\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$ W! $\begin{bmatrix} 5 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$(g)\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ $(h)\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

$(i)\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ $(j)\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

$(k)\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$ $(l)\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

2. =Z_U eYV VZXV_ gR] fVd R_U VZXV_ gVTe`cd `W $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ =Z_U eYV TYRcRTeVcZdeZT c``ed `W

$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$&

3. =Z_U R]d` eYV T`ccVda`_UZ_X TYRcRTeVcZdeZT gVTe`cd&

4. =Z_U eYV VZXV_ gR] fVd R_U eYV VZXV_ gVTe`c T`ccVda`_UZ_X e` eYV]RcXVde VZXV_ gR] fV `W eYV ^RecZi $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & -1 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

5. FSeRZ_ eYV VZXV_ gR] fVd R_U eYV T`ccVda`_UZ_X VZXV_ gVTe`c `W ^RecZi

$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

6. KdV eYV ZeVcReZgV ^VeY`U e` WZ_U eYV]RcXVde VZXV_ gR] fV R_U eYV T`ccVda`_UZ_X VZXV_ gVTe`c `W eYV ^RecZi

$$A = \begin{bmatrix} 5 & 2 & 1 & -2 \\ 2 & 6 & 3 & -4 \\ 1 & 3 & 19 & 2 \\ -2 & -4 & 2 & 1 \end{bmatrix}.$$

14

TAYLOR SERIES METHOD

METHODS FOR NUMERCIAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

JYVcV RcV UZWWVcV_eZR] Vbf ReZ`_d eYRe TR___`e SV d`]gVU fdZ_X eYV deR_URcU ^VeY`Ud VgV_ eY`fXY eYVj a`ddVdd d`]feZ`_d& @_ dfTY dZefReZ`_d$ hV Raa]j _f^VcZTR] ^VeY`Ud W`c `SeRZ_Z_X Raac`iZ^ReV d`]feZ`_d$ hYVcV eYV RTTfcRTj Zd dfWWZTZV_e& JYVdV ^VeY`Ud jZV]U eYV d`]feZ`_ Z_ `_V `W eYV W`]]`hZ_X W`c^d2

F **Single-step method**2 8 dVcZVd W`c $y$ Z_ eVc^d `W a`hVcd `W x, Wc`^ hYZTY eYV gR]fV `W y Re R aRceZTf]Rc gR]fV `W i TR_ SV `SeRZ_VU Sj UZcVTe dfSdeZefeZ`_&

FF **Multi-step method**2 @_ ^f]eZ deVa ^VeY`Ud$ eYV d`]feZ`_ Re R_j a`Z_e U Zd `SeRZ_VU fdZ_X eYV d`]feZ`_ Re R _f^SVc `W acVgZ`fd a`Z_ed&

JRj]`cqd$ GZTRcUqd$ <f]Vcqd R_U D`UZWZVU <f]Vcqd ^VeY`Ud RcV T`^Z_X f_UVc dZ_X]V% deVa ^VeY`U `W d`]gZ_X R_ `cUZ_Rcj UZWWVcV_eZR] Vbf ReZ`_&

JYV _VVU W`c WZ_UZ_X eYV d`]feZ`_ `W eYV Z_ZeZR] gR]fV ac`S]V^d `TTfc WcVbfV_e]j Z_ <_XZ_VVcZ_X R_U GYjdZTd& JYVcV RcV d`^V WZcde `cUVc UZWWVcV_eZR] Vbf ReZ`_d eYRe TR___`e SV d`]gVU fdZ_X eYV deR_URcU ^VeY`Ud& @_ dfTY dZefReZ`_d hV Raa]j _f^VcZTR] ^VeY`Ud& JYVdV ^VeY`Ud jZV]U eYV d`]feZ`_ Z_ `_V `W eYV eh` W`c^d2

FFF 8 dVcZVd W`c $y$ Z_ eVc^d `W a`hVcd `W x, Wc`^ hYZTY eYV gR]fV `W y TR_ SV `SeRZ_VU Sj UZcVTe dfSdeZefeZ`_&

FS 8 dVe `W eRSf]ReVU gR]fVd `W x R_U y.

JYV ^VeY`Ud `W JRj]`c R_U GZTRcU SV]`_X e` T]Rdd F!$ hYVcVRd eY`dV `W <f]Vc$ Hf_XV% BfeeR$ VeT&$ SV]`_X e` eYV T]Rdd FF!& @_ eYZd TYRaeVc hV T`_dZUVc JRj]`c dVcZVd ^VeY`U&

JRj]`c IVcZVd

MV cVTR]] eYV W`]]`hZ_X HVW& =`fceY IV^VdeVc :`cV JVie!2

The Taylor series generated by f **at** $x = a$ **is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$

$$+\frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(a) + \ldots$$

In most of the cases, the Taylor's series converges to $f(x)$ **at every** x **and we often write the** JRj]`cqd dVcZVd **at** $x = a$ **as**

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$ t)!

Instead of $f(x)$ and a, we prefer $y(x)$ and x_0, and in that case (1) becomes

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \ldots$$ **... (2)**

Solution of First Order IVP by Taylor Series Method

E`h T`_dZUVc eYV Z_ZeZR] gR]fV ac`S]V^

$$y' = f(x, y), \quad y(x_0) = y_0.$$ t +!

@W $y(x)$ Zd eYV ViRTe d`]feZ`_ `W +!$ eYV_ fdZ_X *! hZeY $y(x_0) = y_0$$ $y'(x_0) = y'_0,\ y''(x_0) = y''_0,$

R_U d` `_$ hV `SeRZ\_ eYV JRj]`cqd dVcZVd W`c $y(x)$ Rc`f_U $x = x_0$ Rd

$$y(x) = y_0 + (x-x_0)y'_0 + \frac{(x-x_0)^2}{2!} y''_0 + \ldots$$ **... (4)**

@W eYV gR]fVd `W $y'_0,\ y''_0, \ldots$ RcV \\_`h_$ eYV_ ,! XZgVd R a`hVc dVcZVd W`c y. =c`^ +! hV

YRgV $y' = f$, hYZTY `\_ UZWWVcV\_eZReZ`_ hZeY cVdaVTe e` U fdZ_X TYRZ_ cf]V! XZgVd

$$y'' = f' = \frac{df}{dx} = \frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial x}\right) y'$$ t -!

IZ^Z]Rc]j$ YZXYVc UVcZgReZgVd `W V TR\_ SV V iacVddVU Z\_ eVc^d `W C&

Example KdZ_X JRj]`c dVcZVd$ d`]gV $y' = x - y^2,\ \ y(0) = 1.$ 8]d` WZ\_U $y(0.1)$ T`ccVTe e` W`fc UVTZ^R] a]RTVd&

?VcV $x_0 = 0;\ y_0 = y(0) = 1.$?V_TV ,! eR\\Vd eYV W`c^

$$y(x) = y_0 + \frac{x}{1!} y'_0 + \frac{x^3}{2!} y''_0 + \frac{x^3}{3!} y'''_0 + \frac{x^4}{4!} y_0^{(4)} + \frac{x^5}{5!} y_0^{(5)} + \cdots$$ t .!

MV YRgV

$y' = x - y^2, \qquad y'_0 = y'(x = x_0,\ y = y_0) = x_0 - y_0^2 = 0 - 1^2 = -1.$

$y'' = 1 - 2yy', \qquad y''_0 = y''(x = x_0,\ y = y_0) = 1 - 2y_0 y'_0 = 1 - 2(1)(-1) = 3.$

$y''' = -2yy' - 2(y')^2, \quad y'''_0 = y'''(x = x_0,\ y = y_0) = -2y_0 y'_0 - 2\left(y'_0\right)^2 = -8.$

$y^{(4)} = -2yy''' - 6y'y'',$

$$y_0^{(4)} = y^{(4)}(x = x_0,\ y = y_0) = -2y_0 y'''_0 - 6y'_0 y''_0 = 34.$$

$y^{(5)} = -2yy^{(4)} - 8y'y''' - 6(y')^2,$

$$y_0^{(5)} = y^{(5)}(x = x_0,\ y = y_0) = -2y_0 y_0^{(4)} - 8y'_0 y'''_0 - 6(y'_0)^2 = -186.$$

I f SdeZe f eZ_X eYVdV gR]fVd Z_ .!$ h V `SeRZ_

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 - \frac{31}{20}x^5 + \ . \ . \ . \qquad \text{t /!}$$

J` `SeRZ_ $y(0.1)$ T`ccVTe e` W`fc UVTZ^R] a]RTVd$ h V T`_dZUVc eYV eVc^d f ae` $x^4$ R_U a feeZ_X $x = 0.1,$ h V `SeRZ_

$$y(0.1) = 0.9138.$$

Remark to the Example :ORK@>QFLK >KA O>KDB LC U I f aa`dV eYRe h V hZdY e` WZ_U eYV cR_XV `W gR]fVd `W U W`c hYZTY eYV RS`gV dVcZVd$ ec f_TReVU RWeVc eYV eVc^ T`_eRZ_Z_X $x^4$, TR_ SV fdVU e` T`^afeV eYV gR]fVd `W V T`ccVTe e` W`fc UVTZ^R] a]RTVd& MV _VVU `_]j e` hcZeV

$$\frac{31}{20}x^5 \le 0.00005,$$

d` eYRe $\qquad x \le 0.126.$

Example I`]gV fdZ_X JRj]`c dVcZVd ^VeY`U $\frac{dy}{dx} = x + y$ _f^VcZTR]]j deRceZ_X hZeY $x = 1, \ y = 0$ & 8]d` WZ_U V Re $x = 1.1.$

?VcV $x_0 = 1;\ y_0 = y(1) = 0.$?V_TV ,! eR\Vd eYV W`c^

$$y(x) = y_0 + (x-1)y_0' + \frac{(x-1)^2}{2!}y_0'' + \frac{(x-1)^3}{3!}y_0''' + \frac{(x-1)^4}{4!}y_0^{(4)} + \ldots \qquad \text{t /!}$$

?VcV

$y' = x + y$ 3 $\qquad y_0' = y'(x = x_0,\ y = y_0) = x_0 + y_0 = 1 + 0 = 1 \quad y'' = \frac{d}{dx}(x+y) = 1 + y'$ 3

$y_0'' = y''(x = x_0,\ y = y_0) = 1 + y_0' = 1 + 1 = 2$

$y''' = y''$ 3 $\qquad y_0''' = y''(x = x_0,\ y = y_0) = y_0'' = 2.$

$y^{(4)} = y'''$ $\qquad$ 3 $y_0^{(4)} = y'''(x = x_0,\ y = y_0) = y_0''' = 2.$

I f SdeZe f eZ_X eYVdV gR]fVd Z_ /!$ h V `SeRZ_

$$y(x) = (x-1) + (x-1)^2 + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{12} + \ \ldots$$

E`h e` WZ_U $y(1.1),$ h V a f e $x = 1.1$ Z_ eYV RS`gV dVcZVd T`_dZUVcZ_X eVc^d f ae` ,eY a`hVc `W U! h V XVe

$$y(1.1) = 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12} \quad 5 \ (&))&$$

<i RTe d`]feZ`_ `W eYV RS`gV Z_ZeZR] gR]fV ac`S]V^ Zd

$$y = -x - 1 + 2e^{x-1}$$

R_U YV_TV eYV ViRTe gR]fV `W y Re $x = 1.1$ Zd

V)&)! 5 (&))(+,&

Example KdZ_X JRj]`c dVcZVd$ d`]gV

$$5xy' + y^2 - 2 = 0,\ \ y(4) = 1.$$

8]d`$ WZ_U $y(4.1)$.

?VcV $x_0 = 4;\ y_0 = y(4) = 1.$?V_TV ,! eR\Vd eYV W`c^

$$y(x) = y_0 + (x-4)y_0' + \frac{(x-4)^2}{2!}y_0'' + \frac{(x-4)^3}{3!}y_0''' + \frac{(x-4)^4}{4!}y_0^{(4)} + \ldots \qquad t\ 0!$$

?VcV $y_0',\ y_0'', \ldots$ RcV VgR]fReVU Rd W`]]`hd2

: `_dZUVc eYV UZWWVcV_eZR] Vbf ReZ`_

$$5xy' + y^2 - 2 = 0 \qquad t\ 1!$$

; ZWWVcV_eZReZ_X 1! hZeY cVdaVTe e` U$ hV XVe

$$5xy'' + 5y' + 2yy' = 0\ \& \qquad t\ )(!$$

; ZWWVcV_eZReZ_X dfTTVddZgV]j hZeY cVdaVTe e` U$ hV `SeRZ_

- UVààà !)(Vàà ! * VVàà ! * Và !* 5 (t))!

- UVàààà !)- Vààà ! * VVààà ! . Và Vàà 5 (t)*!

- UVààààà ! *(Vàààà ! * VVàààà ! 0 Và Vààà ! . Vàà !* 5 (t)+!

KdZ_X $x_0 = 4;\ y_0 = 1,$ 1! XZgVd $5x_0y_0' + y_0^{\,2} - 2 = 0$ `c $5\cdot 4\cdot y_0' + 1^2 - 2 = 0$ hYZTY XZgVd $y_0' = 0.05$
&)(! XZgVd

$$5x_0y_0'' + 5y_0' + 2y_0y_0' = 0 \quad \text{`c} \quad 5\times 4y_0'' \times 5\times 0.05 + 2\times 1\times 0.05 = 0$$

R_U XZgVd $y_0'' = -0.0175.$

lZ^Z]Rc]j$ $y_0''' =$ (&()(*-$ $y_0^{(4)} = -$ (&((0,-$ $y_0^{(5)} =$ (&((0110)*-"

?V_TV 0! XZgVd

$$y(x) = 1 + (x-4)(0.05) + \frac{(x-4)^2}{2!}(-0.0175) + \frac{(x-4)^3}{3!}(0.01025)$$

$$+\frac{(x-4)^4}{4!}(-0.00845) + \frac{(x-4)^5}{5!}(0.008998125)$$

GfeeZ_X U5 ,&)$ hV XVe

$$y(4.1) = 1 + (0.1)(0.05) + \frac{(0.1)^2}{2!}(-0.0175) + \frac{(0.1)^3}{3!}(0.01025)$$

$$+\frac{(0.1)^4}{4!}(-0.00845) + \frac{(0.1)^5}{5!}(0.008998125)$$

5)&((,1

I`]feZ`_ `W IVT`_U FcUVc @LG Sj JRj]`c IVcZVd DVeY`U

: `_dZUVc eYV dVT`_U `cUVc Z_ZeZR] gR]fV ac`S]V^

$$y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = l_0. \qquad t\ ),!$$

IVeeZ_X $y' = p$, hV XVe $y'' = p'$, R_U eYV UZWWVcV_eZR] VbfReZ`_ Z_ ),! SVT`^Vd

$$p' = f(x, y, p) \qquad t\ )-!$$

hZeY eYV Z_ZeZR] T`_UZeZ`_d

$$y(x_0) = y_0 \qquad t\ ).!$$

R_U

$$p(x_0) = p_0 = l_0. \qquad t\ )/!$$

E`h JRj]`c dVcZVd Zd XZgV_ Sj

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots \qquad \textbf{.....(18)}$$

hYVcV y_0' , y_0'', … RcV UVeVc^Z_VU fdZ_X).! R_U)/! R_U dfTTVddZgV UZWWVcV_eZReZ`_& JYV ^VeY`U Zd Z]]fdecReVU Z_ eYV W`]]`hZ_X VjR^a]V&

Example KdZ_X JRj]`c dVcZVd ^VeY`U$ ac`gV eYRe eYV d`]feZ`_ `W

$$\frac{d^2y}{dx^2} + xy = 0$$

hZeY eYV Z_ZeZR] T`_UZeZ`_d $y(0) = d$ R_U $y'(0) = 0$ Zd XZgV_ Sj

$$y(x) = d\left[1 - \frac{1}{3!}x^3 + \frac{4}{6!}x^6 - \frac{28}{9!}x^9 + \ \ldots\right] \qquad \&\&\&\&\&\&\&\ )1!$$

IVe $y' = p.$

JYV_$ $y'' = p',$

R_U eYV XZgV_ UZWWVcV_eZR] Vb f ReZ`_ SVT`^Vd

$$p' + xy = 0. \qquad \text{t *(!}$$

E`h hV YRgV e` UVeVc^Z_V eYV T`VWWZTZV_ed `W eYV JRj]`c dVcZVd2

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots \qquad \textbf{... (21)}$$

Here $x_0 = 0, \quad y_0 = y(x_0) = y(0) = d, \quad y_0' = y'(x_0) = y'(0) = 0.$

From (20), $\qquad p' = -xy,$

so $\quad y'' = p' = -xy, \qquad y_0'' = -x_0 y_0 = 0;$

$$y''' = p'' = -y - xy', \qquad y_0''' = -y_0 - x_0 y_0' = -d;$$

$$y^{(4)} = -2y' - xy'', \qquad y_0{}^{(4)} = -2y_0' - x_0 y_0'' = 0;$$

$$y^{(5)} = -3y'' - xy''', \qquad y_0^{(5)} = -3y_0'' - x_0 y_0''' = 0;$$

$$y^{(6)} = -4y''' - xy^{(4)}, \qquad y_0^{(6)} = -4y_0''' - x_0 y_0{}^{(4)} = -4d;$$

$$y^{(7)} = -5y^{(4)} - xy^{(5)}, \qquad y^{(7)} = -5y_0^{(4)} - x_0 y_0^{(5)} = 0;$$

$$y^{(8)} = -6y^{(5)} - xy^{(6)}, \qquad y_0^{(8)} = -6y_0^{(5)} - x_0 y_0^{(6)} = 0;$$

$$y^{(9)} = -7y^{(6)} - xy^{(7)}, \qquad y_0^{(9)} = -7y_0^{(6)} - x_0 y_0^{(7)} = -7 \times 4d = -28d.$$

G f eeZ_X eYVdV gR] f Vd Z_ *)!$ hV `SeRZ_)1!&

***Example* 9** <gR] f ReV $y(0.1)$, f dZ_X JRj]`c dVcZVd ^VeY`U$ XZgV_

$$y'' - x(y')^2 + y^2 = 0, \quad y(0) = 1, \quad y'(0) = 0$$

9LIRQFLK

I Ve $\qquad y' = p.$

JYV_$ $\qquad y'' = p',$

R_U eYV XZgV_ UZWWVcV_eZR] Vb f ReZ`_ SVT`^Vd

$$p' - xp^2 + y^2 = 0. \qquad \text{t **!}$$

E`h hV YRgV e` UVeVc^Z_V eYV T`VWWZTZV_ed `W eYV JRj]`c dVcZVd2

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots \qquad \textbf{... (23)}$$

Here $x_0 = 0, \quad y_0 = y(x_0) = y(0) = 1, \quad p_0 = y_0' = y'(x_0) = y'(0) = 0.$

From (22), $\qquad p' = xp^2 - y^2,$

so $\quad y'' = p' = xp^2 - y^2, \qquad y_0'' = x_0 p_0{}^2 - y_0{}^2 = 0 - 1 = -1;$

$$y''' = p'' = p^2 + 2xpp' - 2yy', \qquad y_0''' = p_0{}^2 + 2x_0 p_0 p_0' - 2y_0 y_0' = 0;$$

$y''' = p'' = p^2 + 2xpp' - 2yy',$ $\qquad y_0''' = p_0{}^2 + 2x_0 p_0 p_0' - 2y_0 y_0' = 0;$

Putting these values in (23), we obtain

$$y(x) = 1 - \frac{x^2}{2!} + \ldots \qquad \textbf{... (24)}$$

Putting $x = 0.1$ in (24), neglecting higher powers of x, we obtain

$$y(0.1) \approx 1 - \frac{(0.1)^2}{2!} = 1 - 0.005 = 0.995.$$

Exercises

@_ < i VcTZdVd) –) * $ d`]gV eYV XZgV_ Z_ZeZR] gR] f V ac`S]V^ f dZ_X JR j]`c dVcZVd ^VeY`U& 8]d` WZ_U eYV gR] f V `W V W`c eYV XZgV_ U &

1. $\frac{dy}{dx} - 1 = xy, \quad y(0) = 1.$ 8]d` WZ_U $y(0.1).$

2. $\frac{dy}{dx} = x^2 + y^2 - 2, \quad y = 1$ Re $x = 0.$ 8]d` WZ_U $y(0.1).$

3. $\frac{dy}{dx} = y^2 + 1, \quad y(0) = 0.$ 8]d` WZ_U $y(0.1)$ R_U $y(0.2).$

4. $\frac{dy}{dx} = x - y^2$ $ $y(0) = 1.$ FSeRZ_ _f ^VcZTR] gR] f Vd W`c $x = 0.2(0.2)0.6.$

5. $y' = x + y^2,\ y(0) = 0.$ FSeRZ_ _f ^VcZTR] gR] f Vd W`c

   U 5 (&((&*!(&,&

6. $y' = x^2 + y^2,\ y(1) = 0.$ =Z_U V)&+!&

7. I`]gV $y' = x + y,\ y(1) = 0.$ FSeRZ_ _f ^VcZTR] gR] f Vd W`c $x = 1.0(0.1)1.2.$

8. I`]gV $\frac{dy}{dx} = \frac{1}{x^2 + y},\ y(4) = 4.$ 8]d` WZ_U $y(4.1)$ R_U $y(4.2).$

9. I`]gV $\frac{dy}{dx} = 1 - 2xy,\ y(0) = 0.$ 8]d` WZ_U $y(0.2)$ R_U $y(0.4).$

10. I`]gV $\frac{dy}{dx} = xy^{1/3},\ y(1) = 1.$ 8]d` WZ_U $y(1.1)$ R_U $y(1.2).$

11. I`]gV $\frac{dy}{dx} = x^2 - y,\ y(0) = 1.$ 8]d` WZ_U y Re $x = 0.1(0.1)0.4.$

12. I`]gV $\frac{dy}{dx}-2y=3e^{x}$, $y(0)=0$. 8]d` WZ_U $y(0.1)$ R_U $y(0.2)$.

@_ <iVcTZdVd)+–)-$ d`]gV eYV XZgV_ dVT`_U `cUVc Z_ZeZR] gR]fV ac`S]V^ fdZ_X JRj]`c dVcZVd ^VeY`U& 8]d` WZ_U eYV gR]fV `W V W`c eYV XZgV_ U&

13. $\frac{d^2y}{dx^2}=y+x\frac{dy}{dx}$, $y(0)=1$, $y'(0)=0$. 8]d` WZ_U $y(0.1)$.

14. $\frac{d^2y}{dx^2}+xy=0$, $y(0)=1$, $y'(0)=0.5$. 8]d` WZ_U $y(0.1)$ R_U $y(0.2)$.

15. $\frac{d^2y}{dx^2}=x^2-xy$, $y(0)=1$, $y'(0)=0$. 8]d` WZ_U $y(0.1)$ R_U $y(0.2)$.

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PICARDS ITERATION METHOD

: `_dZUVc eYV Z_ZeZR] gR]fV ac`S]V^

$$y' = f(x, y) \$ \quad y(x_0) = y_0.$$ t)"!

8]d`$ Rddf^V )"! YRgV R f_Zbf V d`]feZ`_ `_ d`^V Z_eVcgR] T`_eRZ_Z_X x_0& 9j dVaRcReZ_X gRcZRS]Vd$ eYV UZWWVcV_eZR] VbfReZ`_ Z_ )! SVT`^Vd

$$dy = f(x, y)dx.$$ &&&)""!

@_eVXcReZ_X)""! Wc`^ U( e` U hZeY cVdaVTe e` U$ Re eYV dR^V eZ^V V TYR_XVd Wc`^ y_0 e` y! hV XVe

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)dx$$

`c
$$y(x) - y_0 = \int_{x_0}^{x} f(x, y)dx$$

`c
$$y(x) = y_0 + \int_{x_0}^{x} f(x, y)dx$$ t *!

@e TR_ SV gVcZWZVU$ Sj dfSdeZefeZ_X $x = x_0$ R_U $y = y_0$ Z_ *!$ eYRe *! dReZdWZVd eYV Z_ZeZR] T`_UZeZ`_ Z_)!&

J` WZ_U eYV Raac`iZ^ReZ`_d e` eYV d`]feZ`_ $y(x)$ `W *! hV ac`TVVU Rd W`]]`hd2

MV dfSdeZefeV eYV WZcde Raac`iZ^ReZ`_ $y = y_0$ `_ eYV cZXYe dZUV `W *!$ R_U `SeRZ_ eYV SVeeVc Raac`iZ^ReZ`_

$$y^{(1)}(x) = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$ t +!

@_ eYV _Vie deVa hV dfSdeZefeV eYV Wf_TeZ`_ $y^{(1)}(x)$ `_ eYV cZXYe dZUV `W *! R_U `SeRZ_

$$y^{(2)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(1)}(x))dx \$$$ t ,!

JYV K[eY] deVa `W eYZd ZeVcReZ`_ XZgVd R_ Raac`iZ^ReZ_X Wf_TeZ`_

$$y^{(n)}(x) = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)}(x))dx$$ t -!

@_ eYZd hRj hV `SeRZ_ R dVbfV_TV `W Raac`iZ^ReZ`_d

$$y^{(1)}(x),\ y^{(2)}(x),\ \dots,\ y^{(n)}(x),\ \dots$$

Working Rule

: `_dZUVc eYV Z_ZeZR] gR] fV ac`S]V^

$$y' = f(x, y)\ \$\ \ y(x_0) = y_0.$$

JYV_ GZTRcUqd ZeVcReZgV W`c^ f]R Zd

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx \qquad (n = 1, 2, 3, \dots) \qquad \&\&\&\ .!$$

hZeY $y^{(0)} = y_0$.

Example =Z_U Raac`iZ^ReV d`]feZ`_d Sj GZTRcUqd ZeVcReZ`_ ^VeY`U e` eYV Z_ZeZR] gR] fV ac`S]V^ $y' = 1 + y^2$ hZeY eYV Z_ZeZR] T`_UZeZ`_ $y(0) = 0$. ?V_TV WZ_U eYV Raac`iZ^ReV gR] fV `W y Re $x = 0.1$ R_U $x = 0.2$&

GZTRcUqd ZeVcReZ`_qd KeY deVa Zd XZgV_ Sj .!&

@_ eYZd ac`S]V^

$$f(x, y) = 1 + y^2\ 3\ \ x_0 = 0,\ \ y^{(0)} = y_0 = y(x_0) = y(0) = 0,$$

R_U YV_TV

$$f(x, y^{(n-1)}) = 1 + \left(y^{(n-1)}\right)^2.$$

I fSdeZefeZ_X eYVdV gR] fVd Z_ .!$

$$y^{(n)} = 0 + \int_0^x \left[1 + \left(y^{(n-1)}\right)^2\right] dx \quad (n = 1, 2, 3, \dots)$$

Z&V&$

$$y^{(n)} = x + \int_0^x \left(y^{(n-1)}\right)^2 dx \quad (n = 1, 2, 3, \dots)$$

$$y^{(1)} = x + \int_0^x \left(y^{(0)}\right)^2 dx$$

G feeZ_X $y^{(0)} = 0$,

$$y^{(1)} = x + \int_0^x 0^2 dx = x.$$

$$y^{(2)} = x + \int_0^x \left(y^{(1)}\right)^2 dx$$

G feeZ_X $y^{(1)} = x$,

$$y^{(2)} = x + \int_0^x x^2 dx = x + \frac{1}{3}x^3.$$

$$y^{(3)} = x + \int_0^x \left(y^{(2)}\right)^2 dx$$

GfeeZ_X $y^{(2)} = x + \frac{1}{3}x^3,$

$$y^{(3)} = x + \int_0^x \left(x + \frac{1}{3}x^3\right)^2 dx$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{1}{63}x^7.$$

MV TR_ T`_eZ_fV eYV ac`TVdd& 9fe hV eR\V eYV RS`gV Rd R_ Raac`iZ^ReV d`]feZ`_ e` eYV XZgV_ Z_ZeZR] gR]fV ac`S]V^& JYRe Zd$

$$y = y(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{1}{63}x^7. \qquad t \ /!$$

I fSdeZefeZ_X U - (&)$ R_U U - (&*$ Z_ /!$ hV `SeRZ_

$y(0.1) =$ (&)((++,

R_U $y(0.2) =$ (&*(*/(1&

JYV RS`gV RcV _`e ViRTe gR]fVd W`c $y$ Re eYV XZgV_ $x$ a`Z_ed$ Sfe eYV Raac`iZ^ReV gR]fVd&

Example >ZgV_ $\frac{dy}{dx} = x + y$ hZeY eYV Z_ZeZR] T`_UZeZ`_ $y(0) = 1.$ =Z_U Raac`iZ^ReV]j eYV gR]fV `W y W`c $x = 0.2$ R_U $x = 1.$

?VcV $f(x, y) = x + y$ 3 $x_0 = 0,\ \ y^{(0)} = y_0 = y(x_0) = y(0) = 1,$ R_U YV_TV fdZ_X .!

$$y^{(n)} = 1 + \int_0^x \left(x + y^{(n-1)}\right) dx$$

Z&V&$

$$y^{(n)} = 1 + \frac{x^2}{2} + \int_0^x y^{(n-1)} dx$$

$$y^{(1)} = 1 + \frac{x^2}{2} + \int_0^x y^{(0)} dx$$

GfeeZ_X $y^{(0)}$ 5)$ hV `SeRZ_

$$y^{(1)} = 1 + \frac{x^2}{2} + \int_0^x dx = 1 + x + \frac{x^2}{2}.$$

$$y^{(2)} = 1 + \frac{x^2}{2} + \int_0^x y^{(1)} dx$$

GfeeZ_X $y^{(1)} = 1 + x + \frac{x^2}{2},$ hV `SeRZ_

$$y^{(2)} = 1 + \frac{x^2}{2} + \int_0^x \left(1 + x + \frac{x^2}{2}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{6}$$

$$y^{(3)} = 1 + \frac{x^2}{2} + \int_0^x y^{(2)} dx$$

Gf eeZ_X $y^{(2)} = 1 + x + x^2 + \frac{x^3}{6},$ hV `SeRZ_

$$y^{(3)} = 1 + \frac{x^2}{2} + \int_0^x \left(1 + x + x^2 + \frac{x^3}{6}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

MV RTTVae

$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Rd R_ Raac`iZ^ReV d`]feZ`_&

MYV_ U 5 (&*\$ hV YRgV

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24} = 1.2427.$$

MYV_ U 5)&(\$ hV YRgV

$$y(0.2) = 1 + 1 + 1 + \frac{1}{3} + \frac{1}{24} = 3.3751.$$

Example I`]gV Sj GZTRcUqd ^VeY`U

$$y' - xy = 1, \text{ XZgV\_ } y = 0 \text{\$ hYV\_ } x = 2.$$

8]d` WZ_U $y(2.05)$ T`ccVTe e` W`fc a]RTVd `W UVTZ^R]&

?VcV $y' = 1 + xy.$

?V_TV $f(x, y) = 1 + xy$ 3 $x_0 = 2, \quad y^{(0)} = y_0 = y(x_0) = y(2) = 0,$

R_U YV_TV

$$f(x, y^{(n-1)}) = 1 + xy^{(n-1)}.$$

If SdeZefeZ_X eYVdV gR]fVd Z_ -!\$ hV `SeRZ_

$$y^{(n)} = 0 + \int_2^x \left(1 + xy^{(n-1)}\right) dx \quad (n = 1, 2, 3, \dots)$$

Z&V&$ $$y^{(n)} = x - 2 + \int_2^x xy^{(n-1)}dx \quad (n = 1,\ 2,\ 3,\ \dots\ )$$

$$y^{(1)} = x - 2 + \int_2^x xy^{(0)}dx$$

G f eeZ_X $y^{(0)} = 0,$ h V `SeRZ_

$$y^{(1)} = x - 2 + \int_2^x x \cdot 0dx$$

Z&V&$ $$y^{(1)} = x - 2.$$

$$y^{(2)} = x - 2 + \int_2^x xy^{(1)}dx$$

G f eeZ_X $y^{(1)} = x - 2,$ h V `SeRZ_

$$y^{(2)} = x - 2 + \int_2^x x(x-2)dx$$

$$= -\frac{2}{3} + x - x^2 + \frac{x^3}{3}.$$

$$y^{(3)} = x - 2 + \int_2^x xy^{(2)}dx$$

G f eeZ_X $y^{(2)} = -\frac{2}{3} + x - x^2 + \frac{x^3}{3},$ h V `SeRZ_

$$y^{(3)} = x - 2 + \int_2^x x\left(-\frac{2}{3} + x - x^2 + \frac{x^3}{3}\right)dx$$

$$= -\frac{22}{15} + x - \frac{x^2}{3} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{15}.$$

M V T`_dZUVc

$$y = -\frac{22}{15} + x - \frac{x^2}{3} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{15}$$

Rd R_ Raac` i Z ^ ReV d`] f eZ`_& I f SdeZe f eZ_X U 5 *&(-$ h V XVe

V *&(-! ≈ (&(-*.&

Example I `]gV eYV $\frac{dy}{dx} = \frac{y-x}{y+x}$ " V (! 5) f dZ_X GZTRcUqd ^ VeY`U& =Z_U eYV gR] f V `W V Re U 5 (&) Raac` i Z ^ ReV] j &

?VcV $f(x,\ y) = \frac{y-x}{y+x}$ 3 $x_0 = 0,\ \ y^{(0)} = y_0 = y(x_0) = y(0) = 1,$ R_U YV_TV S j .!$

$$y^{(n)} = 1 + \int_0^x \frac{y^{(n-1)} - x}{y^{(n-1)} + x} dx$$

$$y^{(1)} = 1 + \int_0^x \frac{y^{(0)} - x}{y^{(0)} + x} dx$$

G f eeZ_X $y^{(0)}$ 5)$ hV `SeRZ_

$$y^{(1)} = 1 + \int_0^x \frac{1-x}{1+x} dx$$

9 j RTe f R] UZgZdZ`_$

$$\frac{1-x}{1+x} = -1 + \frac{2}{1+x}$$

R_U YV_TV eYV RS`gV TR_ SV hcZeeV_ Rd

$$y^{(1)} = 1 + \int_0^x \left(-1 + \frac{2}{1+x} \right) dx$$

$$= 1 - x + 2\ln(1+x).$$

MV eR\V $y = 1 - x + 2\ln(1+x)$ Rd R_ Raac`iZ^ReV d`] f eZ`_ R_U YV_TV eYV gR] f V `W V Re U 5 (&) hZeY]_)&) 5 _Re f cR]]`XRcZeY^ `W)&) 5 (&(1-+! Zd XZgV_ Sj

$$y(0.1) \approx 1 - 0.1 + 2\ln(1+0.1) = 0.9 + 2\ln 1.1 \stackrel{*}{=} 1.0906.$$

Example >ZgV_ eYV UZWWVcV_eZR] Vb f ReZ`_

$$\frac{dy}{dx} = \frac{x^2}{y^2 + 1}$$

hZeY eYV Z_ZeZR] T`_UZeZ`_ $y = 0$ hYV_ $x = 0,$ f dV GZTRcUqd ^VeY`U e` `SeRZ_ VW`c $x = 0.25$, 0.5 R_U)&(T`ccVTe e` eYcVV UVTZ^R] a]RTVd&

?VcV $f(x, y) = \frac{x^2}{y^2 + 1}$ 3 $x_0 = 0,\ y^{(0)} = y_0 = y(x_0) = y(0) = 0,$ R_U YV_TV Sj .!$

$$y^{(n)} = \int_0^x \frac{x^2}{\left(y^{(n-1)}\right)^2 + 1} dx$$

$$y^{(1)} = \int_0^x \frac{x^2}{\left(y^{(0)}\right)^2 + 1} dx$$

G f eeZ_X $y^{(0)} = 0,$ hV `SeRZ_

$$y^{(1)} = \int_0^x x^2 dx = \frac{1}{3}x^3$$

$$y^{(2)} = \int_0^x \frac{x^2}{\left(y^{(1)}\right)^2 + 1} dx$$

GfeeZ_X $y^{(1)} = \frac{1}{3}x^3$, hV `SeRZ_

$$y^{(2)} = \int_0^x \frac{x^2}{(1/9)x^6 + 1} dx = \int_0^x \frac{d(\frac{1}{3}x^3)}{(\frac{1}{3}x^3)^2 + 1} dx$$

$$= \tan^{-1}\left(\frac{1}{3}x^3\right) = \frac{1}{3}x^3 - \frac{1}{81}x^9 + \cdots$$

d` eYRe $y^{(1)}$ R_U $y^{(2)}$ RXcVV e` eYV WZcde eVc^$ gZk&$ $(1/3)x^3$. J` WZ_U eYV cR_XV `W gR]fVd `W U d` eYRe eYV dVcZVd hZeY eYV eVc^ $(1/3)x^3$ R]`_V hZ]] XZgV eYV cVdf]e T`ccVTe e` eYcVV UVTZ^R] a]RTVd$ hV afe

$$\frac{1}{81}x^9 \le 0.0005$$

hYZTY jZV]Ud

$$x \le 0.7$$

?V_TV

$$y(0.25) = \frac{1}{3}(0.25)^3 = 0.005$$

$$y(0.5) = \frac{1}{3}(0.5)^3 = 0.042$$

MYV_ $x = 1.0$ $x \le 0.7$ Zd _`e ecfV! d` hV YRgV e` T`_dZUVc eYV dVT`_U eVc^ $-\frac{1}{81}x^9$ R]d` Z_e` T`_dZUVcReZ`_ R_U XVe

$$y(1.0) = \frac{1}{3} - \frac{1}{81} = 0.321.$$

Exercises

@_ <iVcTZdVd)%/$ d`]gV eYV Z_ZeZR] gR]fV ac`S]V^ Sj GZRTcUqd ZeVcReZ`_ ^VeY`U ; ` eYcVV deVad!&

)& $y' = y,\ y(0) = 1.$

*& $y' = x + y,\ y(0) = -1.$

+& $y' = xy + 2x - x^3,\ y(0) = 0.$

,& $y' = y - y^2,\ y(0) = \frac{1}{2}.$

-& $y' = y^2,\ y(0) = 1.$

.& $y' = 2\sqrt{y},\ y(1) = 0.$

/& $y' = \frac{3y}{x}, \; y(1) = 1.$

@_ <iVcTZdVd 0%).$ d`]gV eYV Z_ZeZR] gR]fV ac`S]V^ Sj GZRTcUqd ZeVcReZ`_ ^VeY`U ;` W`fc deVad!& 8]d` WZ_U eYV gR]fV `W y Re eYV XZgV_ a`Z_ed `W x.

0& $y' = 2x - y, \; y(1) = 3.$ 8]d` WZ_U $y(1.1).$

1& $y' = x - y, \; y(0) = 1.$ 8]d` WZ_U $y(0.2).$

)(& $y' = x^2 y, \; y(1) = 2.$ 8]d` WZ_U $y(1.2).$

))& $y' = 3x + y^2, \; y(0) = 1.$ 8]d` WZ_U $y(0.1).$

)*& $y' = 2x + 3y, \; y(0) = 1.$ 8]d` WZ_U $y(0.25).$

)+& $2\frac{dy}{dx} = x + y, \; y(0) = 2.$ 8]d` WZ_U $y(0.1).$

),& $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \; y(1) = 1.$ 8]d` WZ_U $y(1.1).$

)-& $\frac{dy}{dx} - 1 = xy, \; y(0) = 1.$ 8]d` WZ_U $y(0.1).$

).& $\frac{dy}{dx} = x(1 + x^3 y), \; y(0) = 3.$ 8]d` WZ_U $y(0.1)$ R_U $y(0.2).$

)/& FSeRZ_ eYV Raac`iZ^ReV d`]feZ`_ `W

$$\frac{dy}{dx} = x + x^4 y, \; y(0) = 3$$

Sj GZTRcUqd ZeVcReZ`_ ^VeY`U& JRSf]ReV eYV gR]fVd `W $y$, W`c $x = 0.1(0.1)0.5, 3D.$

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EULER METHODS

: `_dZUVc eYV Z_ZeZR] gR]fV ac`S]V^ `W WZcde `cUVc

$$y' = f(x, y), \quad y(x_0) = y_0. \qquad \text{t)!}$$

I eRceZ_X hZeY XZgV_ x_0 R_U eYV gR]fV `W $h$ Zd TY`dV_ d` d^R]]$ hV dfaa`dV $x_0,\ x_1,\ x_2, \ldots$ SV Vbf R]]j daRTVU x gR]fVd TR]]VU JBPE MLFKQP! hZeY Z_eVcgR] h.

Z&V&$ $\quad x_1 = x_0 + h, \quad x_2 = x_1 + h, \ldots$

8]d` UV_`eV $\quad y_0 = y(x_0), \ y_1 = y(x_1), \ y_2 = y(x_2), \ \ldots$

9j dVaRcReZ_X gRcZRS]Vd$ eYV UZWWVcV_eZR] Vbf ReZ`_ Z_ )! SVT`^Vd

$$dy = f(x, y)dx. \qquad \text{\&\&\&)8!}$$

@_eVXcReZ_X)8! Wc`^ $x_0$ e` x_1 hZeY cVdaVTe e` U$ Re eYV dR^V eZ^V V TYR_XVd Wc`^ y_0 e` y_1! hV XVe

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x, y)dx$$

`c $\quad y_1 - y_0 = \int_{x_0}^{x_1} f(x, y)dx$

`c $\quad y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)dx \qquad$ t *!

8ddf^Z_X eYRe $f(x, y) \approx f(x_0, y_0)$ Z_ $x_0 \le x \le x_1$, *! XZgVd

$$y_1 \approx y_0 + f(x_0, y_0)(x_1 - x_0)$$

`c $\quad y_1 \approx y_0 + hf(x_0, y_0).$

IZ^Z]Rc]j$ W`c eYV cR_XV $x_1 \le x \le x_2$, hV YRgV

$$y_2 = y_1 + \int_{x_1}^{x_2} f(x, y)dx \qquad \text{t +!}$$

8ddf^Z_X eYRe $f(x, y) \approx f(x_1, y_1)$ Z_ $x_1 \le x \le x_2$, +! XZgVd

$$y_2 \approx y_1 + hf(x_1, y_1).$$

Gc`TVVUZ_X Z_ eYZd hRj$ hV `SeRZ_ eYV XV_VcR] W`c^ f]R

$$y_{n+1} = y_n + hf(x_n, y_n) \qquad (n = 0, 1, \cdots) \qquad \text{t ,!}$$

JYV RS`gV Zd TR]]VU eYV **Euler method** `c **Euler-Cauchy method.**

Working Rule (Euler method)

>ZgV_ eYV Z_ZeZR] gR]fV ac`S]V^ )!& I faa`dV $x_0,\ x_1,\ x_2, \ldots$ SV VbfR]]j daRTVU x gR]fVd hZeY Z_eVcgR] h. Z&V&$ $x_1 = x_0 + h,\quad x_2 = x_1 + h, \ldots$ 8]d` UV_`eV $y_0 = y(x_0),\quad y_1 = y(x_1),\quad y_2 = y(x_2), \ldots$

JYV_ eYV ZeVcReZgV W`c^ f]R `W **Euler method** Zd2

$$y_{n+1} = y_n + hf(x_n, y_n) \qquad (n = 0, 1, \cdots) \qquad \text{t -!}$$

Example KdV <f]Vcqd ^VeY`U hZeY E 5 (&) e` d`]gV eYV Z_ZeZR] gR]fV ac`S]V^ $\frac{dy}{dx} = x^2 + y^2$ hZeY $y(0) = 0$ Z_ eYV cR_XV $0 \le x \le 0.5.$

?VcV $f(x,\ y) = x^2 + y^2,\ x_0 = 0,\ y_0 = 0,\ h = 0.1.$

?V_TV

$$x_1 = x_0 + h = 0.2, \quad x_2 = x_1 + h = 0.2, \quad x_3 = x_2 + h = 0.3, \quad x_4 = x_3 + h = 0.4, \quad x_5 = x_4 + h = 0.5.$$

MV UVeVc^Z_V $y_1,\ y_2,\ y_3,\ y_4,\ y_5$ fdZ_X eYV <f]Vc W`c^ f]R -!& I fSdeZefeZ_X eYV XZgV_ gR]fV Z_

$$y_{n+1} = y_n + hf(x_n, y_n)$$

hV `SeRZ_

$$y_{n+1} = y_n + 0.1(x_n^2 + y_n^2) \qquad (n = 0, 1, \cdots)$$

$$y_1 = y_0 + 0.1(x_0^2 + y_0^2) = 0 + 0.1(0 + 0) = 0.$$

$$y_2 = y_1 + 0.1(x_1^2 + y_1^2) = 0 + 0.1\left[(0.1)^2 + 0^2\right] = 0.001.$$

$$y_3 = y_2 + 0.1(x_2^2 + y_2^2) = 0.001 + 0.1\left[(0.2)^2 + (0.001)^2\right] = 0.005.$$

$$y_4 = y_3 + 0.1(x_3^2 + y_3^2) = 0.005 + 0.1\left[(0.3)^2 + (0.005)^2\right] = 0.014.$$

$$y_5 = y_4 + 0.1(x_4^2 + y_4^2) = 0.014 + 0.1\left[(0.4)^2 + (0.014)^2\right] = 0.0300196.$$

?V_TV

$y(0) = 0 \qquad y(0.1) = 0 \qquad y(0.2) = 0.001$

$y(0.3) = 0.005 \qquad y(0.4) = 0.014 \qquad y(0.5) = 0.0300196.$

Example KdZ_X <f]Vc ^VeY`U d`]gV eYV VbfReZ`_ $y' = 2xy + 1$ hZeY $y(0) = 0,\ h = 0.02$ W`c $x = 0.1.$

?VcV $f(x,\ y) = 2xy + 1,\ x_0 = 0,\ \ y_0 = 0,\ \ h = 0.02.$?V_TV

$$x_1 = x_0 + h = 0.02,\quad x_2 = x_1 + h = 0.04,\quad x_3 = x_2 + h = 0.06,\qquad x_4 = x_3 + h = 0.08,\quad x_5 = x_4 + h = 0.1.$$

MV UVeVc^Z_V $y_1,\ y_2,\ y_3, y_4, y_5$ fdZ_X eYV <f]Vc W`c^f]R -!& I fSdeZefeZ_X eYV XZgV_ gR]fVZ_

$$y_{n+1} = y_n + hf(x_n, y_n)$$

hV`SeRZ_

$$y_{n+1} = y_n + 0.02(2x_n y_n + 1) \qquad (n = 0, 1, \cdots\ )$$

$$y_1 = y_0 + 0.02(2x_0 y_0 + 1) = 0 + 0.02(0+1) = 0.02.$$

$$y_2 = y_1 + 0.02(2x_1 y_1 + 1) = 0.02 + 0.02(2\times 0.02\times 0.02 + 1) = 0.04\$$$

Raac`iZ^ReV e` * a]RTVd `W UVTZ^R]d

$$y_3 = y_2 + 0.02(2x_2 y_2 + 1) = 0.04 + 0.02(2\times 0.04\times 0.04 + 1) = 0.06$$

$$y_4 = y_3 + 0.02(2x_3 y_3 + 1) = 0.06 + 0.02(2\times 0.06\times 0.06 + 1) = 0.08$$

$$y_5 = y_4 + 0.02(2x_4 y_4 + 1) = 0.08 + 0.02(2\times 0.08\times 0.08 + 1) = 0.1$$

?V_TV

$y(0) = 0 \qquad y(0.02) = 0.02 \qquad y(0.04) = 0.04$

$y(0.06) = 0.06 \qquad y(0.08) = 0.08 \qquad y(0.1) = 0.1.$

JYRe Zd eYV Raac`iZ^ReV gR]fV `W $y(0.1)$ Zd $0.1.$

Example >ZgV_ eYV Z_ZeZR] gR]fV ac`S]V^ $y' = x + y,\ y(0) = 0.$ =Z_U eYV gR]fV `W y Raac`iZ^ReV]j W`c $x = 1$ Sj <f]Vc ^VeY`U Z_ WZgV deVad& :`^aRcV eYV cVdf]e hZeY eYV ViRTe gR]fV&

?VcV $f(x,\ y) = x + y,\quad x_0 = 0,\ \ y_0 = y(x_0)y(0) = 0.$ 8d hV YRgV e` TR]Tf]ReV eYV gR]fV `W y Z_ WZgV deVad$ hV YRgV e` eR\V $h = \dfrac{x_n - x_0}{n} = \dfrac{1-0}{5} = 0.2.$?V_TV

$$x_1 = x_0 + h = 0.2,\quad x_2 = x_1 + h = 0.4,\quad x_3 = x_2 + h = 0.6,\qquad x_4 = x_3 + h = 0.8,\quad x_5 = x_4 + h = 1.0.$$

MV UVeVc^Z_V $y_1,\ y_2,\ y_3, y_4, y_5$ fdZ_X eYV <f]Vc W`c^f]R -!& I fSdeZefeZ_X eYV XZgV_ gR]fV Z_ -!\$ hV`SeRZ_

$$y_{n+1} = y_n + 0.2(x_n + y_n) \qquad (n = 0, 1, \cdots\ )$$

JYV deVad RcV XZgV_ Z_ eYV W`]]`hZ_X JRS]V&

8]d` eYV ViRTe d`]feZ`_ e` eYV]Z_VRc UZWWVcV_eZR] VbfReZ`_ $y' = x + y$ hZeY eYV Z_ZeZR] T`_UZeZ`_ $y(0) = 0$ TR_ SV W`f_U `fe e` SV

$$y = e^x - x - 1. \qquad \text{t .!}$$

JYV ViRTe gR]fVd `W $y$ TR_ SV VgR]fReVU Wc`^ .! Sj dfSdeZefeZ_X eYV T`ccVda`_UZ_X x gR]fVd$ Z_ aRceZTf]Rc$

$y_1 = y(x_1) = e^{x_1} - x_1 - 1 = e^{0.2} - 0.2 - 1 = 0.000,$ Raac`iZ^ReV]j&

JYV `eYVc ViRTe gR]fVd RcV R]d` dY`h_ Z_ eYV W`]]`hZ_X eRS]V&

n	x_n	Raac`iZ ^ReV gR]fV `W y_n	$0.2(x_n + y_n)$	<iRTe gR]fVd	8Sd`]feV gR]fV `W <cc`c
(	(&(	(&(((	(&(((	(&(((	(&(((
)	(&*	(&(((	(&(,(	(&(*)	(&(*)
*	(&,	(&(,(	(&(00	(&(1*	(&(-*
+	(&.	(&)*0	(&),.	(&***	(&(1,
,	(&0	(&*/,	(&*)-	(&,*.	(&)-*
-	)&(	(&,01		(&/)0	(&**1

JYV Raac`iZ^ReV gR]fV `W $y(1.0)$ Sj <f]Vcqd ^VeY`U Zd (&,01$ hYZ]V ViRTe gR]fV Zd (&/)0&

Exercises

@_ <iVcTZdVd)%))$ d`]gV eYV Z_ZeZR] gR]fV ac`S]V^ fdZ_X <f]Vcqd ^VeY`U W`c gR]fV `W $y$ Re eYV XZgV_ a`Z_e `W x hZeY XZgV_ h Zd XZgV_ Z_ ScRT\Ved!

)& $\frac{dy}{dx} = 1 - y,\ y(0) = 0$ Re eYV a`Z_e $x = 0.2$ $h = 0.1).$

*& $\frac{dy}{dx} = \frac{y - x}{1 + x},\ y(0) = 1$ Re eYV a`Z_e $x = 0.1$ $h = 0.02).$

+& $yy' = x,\ y(0) = 1.5$ Re eYV a`Z_e $x = 0.2$ $h = 0.1).$

,& $\frac{dy}{dx} = 3x + \frac{1}{2}y,\ y(0) = 1$ Re eYV a`Z_e $x = 0.2$ $h = 0.05).$

-& $y' = x + y + xy,\ y(0) = 1$ Re eYV a`Z_e $x = 0.1$ $h = 0.02).$

.& $\frac{dy}{dx} = 1 + y^2,\ y(0) = 0$ Re eYV a`Z_e $x = 0.4$ $h = 0.2).$

/& $\frac{dy}{dx} = xy,\ y(0) = 1$ Re eYV a`Z_e $x = 0.4$ $h = 0.2).$

0& $\frac{dy}{dx} = 1 + \ln(x + y),\ y(0) = 1$ Re eYV a`Z_e $x = 0.2$ $h = 0.1).$

1& $y' = x^2 + y,\ y(0) = 1$ Re eYV a`Z_e $x = 0.1$ $h = 0.05).$

)(& $y' = 2xy,\ y(0) = 1$ Re eYV a`Z_e $x = 0.5$ $h = 0.1).$

))& $y' = -y,\ y(0) = 1$ Re eYV a`Z_e $x = 0.04$ $h = 0.01).$

@_ <iVcTZdVd)*%)-$ Raa]j <f]Vcqd ^VeY`U& ;`)(deVad& 8]d` d`]gV eYV ac`S]V^ ViRTe]j& : `^afeV eYV Vcc`cd e` dVV eYRe eYV ^VeY`U Zd e`` Z_RTTfcReV W`c acRTeZTR] afca`dVd&

)*& $y' + 0.1y = 0,\ y(0) = 2,\ h = 0.1$

)+& $y' = \frac{1}{2}$® $\sqrt{1 - y^2},\ y(0) = 0,\ h = 0.1$

),& $y' + 5x^4y^2 = 0,\ y(0) = 1,\ h = 0.2$

)-& $y' = (y + x)^2,\ y(0) = 1,\ h = 0.1$

).& I`]gV fdZ_X <f]Vcqd ^VeY`U $y'(x + y) = y - x$ hZeY $y(0) = 2$ W`c eYV cR_XV $0.00(0.02)0.06.$

)/& I`]gV fdZ_X <f]Vcqd ^VeY`U $y' = y - \frac{2x}{y}$ hZeY $y = 1$ Re $x = 0$ W`c $h = 0.5$ `_ eYV Z_eVcgR] $[0, 1].$

)0& KdZ_X <f]Vcqd ^VeY`U WZ_U $y(0.2)$ `W eYV Z_ZeZR] gR]fV ac`S]V^ $y' = x + 2y,\ y(0) = 1,$ eR\Z_X $h = 0.1.$

)1& KdZ_X <f]Vcqd ^VeY`U WZ_U eYV gR]fV `W y Re eYV a`Z_e $x = 2$ Z_ deVad `W 0.2 `W eYV Z_ZeZR] gR]fV ac`S]V^ $\frac{dy}{dx} = 2 + \sqrt{xy},\ y(1) = 1$&

Modified Euler Method

D`UZWZVU <f]Vc ^VeY`U Zd XZgV_ Sj eYV ZeVcReZ`_ W`c^f]R

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})], \quad n = 0, 1, 2, \cdots$$

hYVcV $y_1^{(n)}$ Zd eYV KeY Raac`iZ^ReZ`_ e` $y_1$& JYV ZeVcReZ`_ W`c^f]R TR_ SV deRceVU Sj TY``dZ_X $y_1^{(0)}$ Wc`^ <f]Vcqd W`c^f]R

$$y_1^{(0)} = y_0 + hf(x_0, y_0).$$

Example KdZ_X ^`UZWZVU <f]Vcqd ^VeY`U$ UVeVc^Z_V eYV gR]fV `W y hYV_ $x = 0.1$ XZgV_ eYRe

$$y' = x^2 + y;\ \ y(0) = 1. \quad \text{JR\V } h = 0.05)$$

?VcV $f(x,\ y) = x^2 + y;\ x_0 = 0,\ \ y_0 = 1.$

$y_1^{(0)} = y_0 + hf(x_0,\ y_0) = 1 + 0.05(1) = 1.05$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,\ y_0) + f(x_1,\ y_1^{(0)})]$$

$$= 1 + \frac{0.05}{2}[f(0,1) + f(0.05,\ 1.05)]$$

$$= 1 + 0.025[1 + (0.05)^2 + 1.05]$$

$$= 1.0513$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0,\ y_0) + f(x_1,\ y_1^{(1)})]$$

$$= 1 + \frac{0.05}{2}[f(0,1) + f(0.05,\ 1.0513)]$$

$$= 1 + 0.025[1 + (0.05)^2 + 1.0513]$$

$$= 1.0513$$

?V_TV hV eR\V $y_1 = 1.0513,$ hYZTY Zd T`ccVTe e` W`fc UVTZ^R] a]RTVd&

=`c^f]R eR\Vd eYV W`c^

$$y_2^{(n+1)} = y_1 + \frac{h}{2}[f(x_1,\ y_1) + f(x_2,\ y_2^{(n)})] \quad n = 0, 1, 2, \cdots$$

hYVcV hV WZcde VgR]fReV $y_2^{(0)}$ fdZ_X eYV <f]Vc W`c^f]R

$$y_2^{(0)} = y_1 + hf(x_1,\ y_1).$$

$$= 1.0513 + 0.05\left[(0.05)^2 + 1.0513\right] = 1.1040$$

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1,\ y_1) + f(x_2,\ y_2^{(0)})]$$

$$= 1 + \frac{0.05}{2}\left\{\left[(0.05)^2 + 1.0513\right] + \left[(0.1)^2 + 1.1040\right]\right\}$$

$$= 1.1055$$

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 1 + \frac{0.05}{2}\left\{\left[(0.05)^2 + 1.0513\right] + \left[(0.1)^2 + 1.1055\right]\right\}$$

$$= 1.1055$$

?V_TV hV eR\V $y_2 = 1.1055$&

?V_TV eYV gR]fV `W y hYV_ $x = 0.1$ Zd 1.1055 T`ccVTe e` W` fc UVTZ^R] a]RTVd&

Example KdZ_X ^`UZWZVU <f]Vcqd ^VeY`U$ UVeVc^Z_V eYV gR]fV `W y hYV_ $x = 0.2$ XZgV_ eYRe

$$\frac{dy}{dx} = x + \sqrt{y};\ y(0) = 1. \qquad \text{JR\textbackslash V } h = 0.2)$$

?VcV $f(x, y) = x + \sqrt{y};\ x_0 = 0,\ \ y_0 = 1.$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.2(0 + 1) = 1.2$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.2}{2}[1 + (0.2 + \sqrt{1.2}] = 1.2295.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.2}{2}[1 + (0.2 + \sqrt{1.2295}] = 1.2309.$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.2}{2}[1 + (0.2 + \sqrt{1.2309}] = 1.2309.$$

?V_TV hV eR\V $y(0.2) = y_1 = 1.2309.$&

Exercises

@_ <iVcTZdVd)%))$ d`]gV eYV Z_ZeZR] gR]fV ac`S]V^ fdZ_X ^`UZWZVU <f]Vcqd ^VeY`U W`c gR]fV `W y Re eYV XZgV_ a`Z_e `W x hZeY XZgV_ h Zd XZgV_ Z_ ScRT\Ved!

)& $\frac{dy}{dx} = 1 - y,\ y(0) = 0$ Re eYV a`Z_e $x = 0.2$ $h = 0.1).$

*& $\frac{dy}{dx} = \frac{y - x}{1 + x},\ y(0) = 1$ Re eYV a`Z_e $x = 0.1$ $h = 0.02).$

+& $yy' = x,\ y(0) = 1.5$ Re eYV a`Z_e $x = 0.2 \quad h = 0.1).$

,& $\frac{dy}{dx} = 3x + \frac{1}{2}y,\ y(0) = 1$ Re eYV a`Z_e $x = 0.2 \quad h = 0.05).$

-& $y' = x + y + xy,\ y(0) = 1$ Re eYV a`Z_e $x = 0.1 \quad h = 0.02).$

.& $\frac{dy}{dx} = 1 + y^2,\ y(0) = 0$ Re eYV a`Z_e $x = 0.4 \quad h = 0.2).$

/& $\frac{dy}{dx} = xy,\ y(0) = 1$ Re eYV a`Z_e $x = 0.4 \quad h = 0.2).$

0& $\frac{dy}{dx} = 1 + \ln(x + y),\ y(0) = 1$ Re eYV a`Z_e $x = 0.2 \quad h = 0.1).$

1& $y' = x^2 + y,\ y(0) = 1$ Re eYV a`Z_e $x = 0.1 \quad h = 0.05).$

)(& $y' = 2xy,\ y(0) = 1$ Re eYV a`Z_e $x = 0.5 \quad h = 0.1).$

))& $y' = -y,\ y(0) = 1$ Re eYV a`Z_e $x = 0.04 \quad h = 0.01).$

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RUNGE KUTTA METHODS

JYV JRj]`c dVcZVd ^VeY`U YRd UVdZcRS]V WVRefcVd$ aRceZTf]Rc]j Z_ Zed RSZ]Zej e` \VVa eYV Vcc`cd d^R]]$ Sfe eYRe Ze R]d` YRd eYV dec`_X UZdRUgR_eRXV `W cVbfZcZ_X eYV VgR]fReZ`_ `W YZXYVc UVcZgReZgVd `W eYV Wf_TeZ`_ W i$j!& @_ eYV JRj]`c dVcZVd ^VeY`U$ VRTY `W eYVdV YZXYVc `cUVc UVcZgReZgVd Zd VgR]fReVU Re eYV a`Z_e x_i Re eYV SVXZ__Z_X `W eYV deVa$ Z_ `cUVc e` VgR]fReV $y(x_i)$ Re eYV V_U `W eYV deVa& MV `SdVcgVU eYRe eYV <f]Vc ^VeY`U T`f]U SV Z^ac`gVU Sj T`^afeZ_X eYV Wf_TeZ`_ W i$j! Re R acVUZTeVU a`Z_e Re eYV WRc V_U `W eYV deVa Z_ i& JYV Hf_XV%BfeeR Raac`RTY Zd e` RZ^ W`c eYV UVdZcRS]V WVRefcVd `W eYV JRj]`c dVcZVd ^VeY`U$ Sfe hZeY eYV cVa]RTV^V_e `W eYV cVbfZcV^V_e W`c eYV VgR]fReZ`_ `W YZXYVc `cUVc UVcZgReZgVd hZeY eYV cVbfZcV^V_e e` VgR]fReV W i$j! Re d`^V a`Z_ed hZeYZ_ eYV deVa x_i e` $x_{i+1}$ & IZ_TV Ze Zd _`e Z_ZeZR]]j _`h_ Re hYZTY a`Z_ed Z_ eYV Z_eVcgR] eYVdV VgR]fReZ`_d dY`f]U SV U`_V$ Ze Zd a`ddZS]V e` TY``dV eYVdV a`Z_ed Z_ dfTY R hRj eYRe eYV cVdf]e Zd T`_dZdeV_e hZeY eYV JRj]`c dVcZVd d`]feZ`_ e` d`^V aRceZTf]Rc$ hYZTY hV dYR]] TR]] eYV `cUVc `W eYV Hf_XV% BfeeR ^VeY`U& JYV Hf_XV%BfeeR ^VeY`U `W `cUVc E 5 , Zd ^`de a`af]Rc& @e Zd R X``U TY`ZTV W`c T`^^`_ afca`dVd SVTRfdV Ze Zd bfZeV RTTfcReV$ deRS]V$ R_U VRdj e` ac`XcR^& D`de RfeY`cZeZVd ac`T]RZ^ eYRe Ze Zd _`e _VTVddRcj e` X` e` R YZXYVc%`cUVc ^VeY`U SVTRfdV eYV Z_TcVRdVU RTTfcRTj Zd `WWdVe Sj RUUZeZ`_R] T`^afeReZ`_R] VWW`ce& @W ^`cV RTTfcRTj Zd cVbfZcVU$ eYV_ VZeYVc R d^R]]Vc deVa dZkV `c R_ RURaeZgV ^VeY`U dY`f]U SV fdVU&

<B RPB QEB C>@Q QE>Q 8RKDB#4RQQ> JBQELA LC OQE LOABO >DOBB TFQE :>VILOZP PBOFBP PLIRQFLK RM QL QEB QBO J P LC h^r.

Second Order Runge-Kutta Method

:`^afeReZ`_R]]j$ ^`de VWWZTZV_e ^VeY`Ud Z_ eVc^d `W RTTfcRTj hVcV UVgV]`aVU Sj eh` >Vc^R_ ^ReYV^ReZTZR_d$ :Rc] Hf_XV R_U MZ]YV]^ BfeeR& JYVdV ^VeY`Ud RcV hV]] _`h_ Rd Hf_XV%BfeeR ^VeY`Ud H%B ^VeY`Ud!& @_ eYZd R_U eYV T`^Z_X dVTeZ`_ hV T`_dZUVc dVT`_U R_U W`fceY `cUVc H%B ^VeY`Ud&

JYVcV RcV dVgVcR] dVT`_U `cUVc Hf_XV%BfeeR W`c^f]Rd R_U hV T`_dZUVc `_V R^`_X eYV^&

Working Method (Second Order Runge-Kutta Method)

>ZgV_ eYV Z_ZeZR] gR]fV ac`S]V^ )!& I faa`dV $x_0,\ x_1,\ x_2, \ldots$ SV VbfR]]j daRTVU x gR]fVd hZeY Z_eVcgR] h. Z&V&$

$$x_1 = x_0 + h, \quad x_2 = x_1 + h, \ldots$$

8]d` UV_`eV $y_0 = y(x_0),\ \ y_1 = y(x_1),\ \ y_2 = y(x_2),\ \ldots$

=`c $n = 0, 1, \cdots$ f_eZ] eVc^Z_ReZ`_ U`2

$$x_{n+1} = x_n + h$$

$$k_n = hf(x_n, y_n) \qquad \text{t 0!}$$

$$l_n = hf(x_{n+1}, y_n + k_n) \qquad \text{t 1!}$$

$$y_{n+1} = y_n + \frac{1}{2}(k_n + l_n) \qquad \text{t)(!}$$

Remark D`UZWZVU <f]Vc ^VeY`U Zd R daVTZR] TRdV `W dVT`_U `cUVc Hf_XV%B feeR ^VeY`U XZgV_ Sj)(!&

Example KdV dVT`_U `cUVc Hf_XV%B feeR ^VeY`U hZeY $h = 0.1$ e` WZ_U $y(0.2),$ XZgV_ $\frac{dy}{dx} = x^2 + y^2$ hZeY $y(0) = 0.$

?VcV $f(x, y) = x^2 + y^2,\ x_0 = 0,\ y_0 = 0,\ h = 0.1.$?V_TV

$$x_1 = x_0 + h = 0.1, \quad x_2 = x_1 + h = 0.2.$$

J`UVeVc^Z_V $y_1,\ y_2$ hV fdV dVT`_U `cUVc Hf_XV%B feeR ^VeY`U R_U fdZ_X 0! o)(!$

$$k_n = hf(x_n, y_n) = 0.1(x_n^2 + y_n^2)$$

$$l_n = hf(x_{n+1}, y_n + k_n) = 0.1[x_{n+1}^2 + (y_n + k_n)^2]$$

R_U $$y_{n+1} = y_n + \frac{1}{2}(k_n + l_n)$$

$$k_0 = 0.2(x_0^2 + y_0^2) = 0.1(0^2 + 0^2) = 0.$$

$$l_0 = 0.2(x_1^2 + (y_0 + k_0)^2) = 0.1\left[(0.1)^2 + (0+0)^2)\ \right] = 0.001$$

R_U $$y_1 = y_0 + \frac{1}{2}(k_0 + k_0) = 0 + \frac{1}{2}(0 + 0.001) = 0.0005.$$

$k_1 = 0.2(x_1^2 + y_1^2) = 0.1\left[(0.1)^2 + (0.0005)^2\ \right] = 0.001,$ T`ccVTe e` eYcVV a]RTVd `W UVTZ^R]d&

$$l_1 = 0.2(x_2^2 + (y_1 + k_1)^2) = 0.1\left[(0.2)^2 + (0.0015)^2)\ \right] = 0.004$$

R_U $$y_2 = y_1 + \frac{1}{2}(k_1 + l_1) = 0.0005 + \frac{1}{2}(0.001 + 0.004) = 0.003.$$

?V_TV $y(0.1)=0.0005,$ $y(0.2)=0.003.$

Example >ZgV_ eYV Z_ZeZR] gR]fV ac`S]V^ $y'=x+y,\ y(0)=0.$ =Z_U eYV gR]fV `W y Raac`iZ^ReV]j W`c $x=1$ Sj dVT`_U `cUVc Hf_XV%Bf eeR ^VeY`U Z_ WZgV deVad& :`^aRcV eYV cVdf]e hZeY eYV ViRTe gR]fV&

?VcV $f(x,y)=x+y,\ x_0=0,\ y_0=0.$ 8d hV YRgV e` TR]Tf]ReV eYV gR]fV `W y hYV_ $x=1$ Z_ WZgV deVad$ hV YRgV e` eR\V $h=\frac{x_n-x_0}{n}=\frac{1-0}{5}=0.2.$?V_TV

$$x_1=x_0+h=0.2,\quad x_2=x_1+h=0.4,\quad x_3=x_2+h=0.6,\quad x_4=x_3+h=0.8,\quad x_5=x_4+h=1.0.$$

MV UVeVc^Z_V $y_1,\ y_2,\ y_3, y_4, y_5$ hV fdV dVT`_U `cUVc Hf_XV%Bf eeR W`c^f]R2

$$k_n=hf(x_n,y_n)=0.2(x_n+y_n)$$

$$l_n=hf(x_{n+1},y_n+k_1)=0.2(x_{n+1}+(y_n+k_n))$$

$$=0.2\left[x_n+0.2+y_n+0.2(x_n+y_n)\right.$$ $Rd $x_{n+1}=x_n+h=x_n+a_2$ R_U $y_{n+1}=y_n+\frac{1}{2}(k_n+l_n)$

$$=y_n+\frac{1}{2}\left\{0.2(x_n+y_n)+0.2\left[x_n+0.2+y_n+0.2(x_n+y_n)\right.\right\}$$

$$=y_n+0.22(x_n+y_n)+0.02$$

JYV dfTTVddZgV deVad R_U TR]Tf]ReZ`_d RcV a]`eeVU Z_ eYV W`]]`hZ_X eRS]V&

n	x_n	Raac`iZ^ReV gR]fV `W y_n	x_n+y_n	$0.22(x_n+y_n)+0.02$	y_{n+1}
(	(&(	(&((((	(&((((	(&(*((	(&(*((
)	(&*	(&(*((	(&**((	(&(.0,	(&(00,
*	(&,	(&(00,	(&,00,	(&)*/,	(&*)-0
+	(&.	(&*)-0	(&0)-0	(&)11-	(&,)-+
,	(&0	(&,)-+	)&*)-+	(&*0/,	(&/(*/
-	)&(	(&/(*/			

?V_TV $y(1)=0.7027.$ @_ R_ VRc]ZVc ViR^a]V hV YRgV _`eVU eYRe eYV ViRTe gR]fV Zd (&/)0&

Exercises

In Exercises 1-10, solve the initial value problem using second order Runge-Kutta method for the value of y at the given point x with given h.

1. $\frac{dy}{dx} = 1 - y, \; y(0) = 0$ at the point $x = 0.2$ (Take $h = 0.1$).

2. $\frac{dy}{dx} = \frac{y - x}{1 + x}, \; y(0) = 1$ at the point $x = 0.1$ (Take $h = 0.02$).

3. $yy' = x, \; y(0) = 1.5$ at the point $x = 0.2$ (Take $h = 0.1$).

4. $\frac{dy}{dx} = x - y, \; y(0) = 1$ at the point $x = 0.2$ (Take $h = 0.1$).

5. $y' = x + y + xy, \; y(0) = 1$ at the point $x = 0.1$ (Take $h = 0.02$).

6. $\frac{dy}{dx} = 1 + y^2, \; y(0) = 0$ at the point $x = 0.4$ (Take $h = 0.2$).

7. $\frac{dy}{dx} = xy, \; y(0) = 1$ at the point $x = 0.4$ (Take $h = 0.2$).

8. $\frac{dy}{dx} = 1 + \ln(x + y), \; y(0) = 1$ at the point $x = 0.2$ (Take $h = 0.1$).

9. $y' = x^2 + y, \; y(0) = 1$ at the point $x = 0.1$ (Take $h = 0.05$).

10. $y' = 2xy, \; y(0) = 1$ at the point $x = 0.5$ (Take $h = 0.1$).

In Exercises 11-13, apply second order Runge-Kutta method. Do 10 steps.

11. $y' = y, \; y(0) = 1, \; h = 0.1$

12. $y' = y - y^2, \; y(0) = 0.5, \; h = 0.1$

13. $y' = 2(1 + y^2), \; y(0) = 0, \; h = 0.05$

14. $y' + 2xy^2 = 0, \; y(0) = 1, \; h = 0.2$

15. Solve using second order Runge-Kutta method $y'(x + y) = y - x$ with $y(0) = 2$ for the range $0.00(0.02)0.06$.

16. Solve using second order Runge-Kutta method $y' = y - \frac{2x}{y}$ with $y = 1$ at $x = 0$ for $h = 0.5$ on the interval $[0, 1]$.

)/& KdZ_X dVT`_U `cUVc Hf_XV%Bf eeR ^VeY`U WZ_U $y(0.2)$ `W eYV Z_ZeZR] gR]fV ac`S]V^ $y' = x + 2y,\ y(0) = 1,$ eR\Z_X $h = 0.1.$

)0& KdZ_X dVT`_U `cUVc Hf_XV%Bf eeR ^VeY`U WZ_U eYV gR]fV `W y Re eYV a`Z_e $x = 2$ Z_ deVad `W 0.2 `W eYV Z_ZeZR] gR]fV ac`S]V^ $\frac{dy}{dx} = 2 + \sqrt{xy},\ y(1) = 1$&

Fourth Order Runge-Kutta method

JYV **Runge-Kutta method**) `W W`fceY `cUVc R]d` _`h_ Rd @I>PPF@>I 8RKDB#4RQQ> JBQELA! XZgVd XcVReVc RTTfcRTj R_U Zd ^`de hZUV]j fdVU W`c WZ_UZ_X eYV Raac`iZ^ReV d`]feZ`_ `W WZcde `cUVc `cUZ_Rcj UZWWVcV_eZR] VbfReZ`_d& JYV ^VeY`U Zd hV]] dfZeVU W`c T`^afeVcd& JYV ^VeY`U Zd dY`h_ Z_ eYV W`]]`hZ_X R]X`cZeY^&

Algorithm (The Runge-Kutta method)

>ZgV_ eYV Z_ZeZR] gR]fV ac`S]V^ )!& I faa`dV $x_0,\ x_1,\ x_2, \dots$ SV Vbf R]]j daRTVU x gR]fVd hZeY Z_eVcgR] h. Z&V&$

$$x_1 = x_0 + h, \quad x_2 = x_1 + h, \dots$$

8]d` UV_`eV $y_0 = y(x_0),\ \ y_1 = y(x_1),\ \ y_2 = y(x_2),\ \dots$

=`c $n = 0, 1, \cdots$ $ f_eZ] eVc^Z_ReZ`_ U`2

$$x_{n+1} = x_n + h$$

$$A_n = hf(x_n, y_n) \qquad \text{t))!}$$

$$B_n = hf(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}A_n) \qquad \text{t)*!}$$

$$C_n = hf(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}B_n) \qquad \text{t)+!}$$

$$D_n = hf(x_n + h,\ y_n + C_n) \qquad \text{t),!}$$

$$y_{n+1} = y_n + \frac{1}{6}(A_n + 2B_n + 2C_n + D_n) \qquad \text{t)-!}$$

Example KdV Hf_XV%Bf eeR ^VeY`U hZeY $h = 0.1$ e` WZ_U $y(0.2)$ XZgV_ $\frac{dy}{dx} = x^2 + y^2$ hZeY $y(0) = 0.$

?VcV $f(x, y) = x^2 + y^2,\ x_0 = 0,\ y_0 = 0,\ h = 0.1.$?V_TV

$$x_1 = x_0 + h = 0.1, \quad x_2 = x_1 + h = 0.2.$$

J` UVeVc^Z_V $y_1,\ y_2$ hV fdV Z^ac`gVU <f]Vc W`c^f]R& KdZ_X <bd&)*!)-!$

$$x_{n+1} = x_n + h = x_n + 0.1$$

$$A_n = hf(x_n, y_n) = 0.1(x_n^2 + y_n^2)$$

$$B_n = hf(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}A_n) = 0.1\left[(x_n + 0.05)^2 + \left(y_n + \tfrac{1}{2}A_n\right)^2\right.$$

$$C_n = hf(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}B_n) = 0.1\left[(x_n + 0.05)^2 + \left(y_n + \tfrac{1}{2}B_n\right)^2\right.$$

$$D_n = hf(x_n + h,\ y_n + C_n) = 0.1\left[x_{n+1}^2 + \left(y_n + C_n\right)^2\right.$$

$$y_{n+1} = y_n + \frac{1}{6}(A_n + 2B_n + 2C_n + D_n)$$

$$x_1 = x_0 + 0.1 = 0 + 0.1 = 0.1$$

$$A_0 = 0.1(x_0^2 + y_0^2) = 0.1(0^2 + 0^2) = 0$$

$$B_0 = 0.1\left[(x_0 + 0.05)^2 + \left(y_0 + \tfrac{1}{2}A_0\right)^2\right.$$

$$= 0.1\left[(0.05)^2 + 0^2\right] = 0.00025.$$

$$C_0 = 0.1\left[(x_0 + 0.05)^2 + \left(y_0 + \tfrac{1}{2}B_0\right)^2\right.$$

$$= 0.1\left[(0.05)^2 + (0.000125)^2\right] = 0.00025.$$

$$D_0 = 0.1\left[x_1^2 + \left(y_0 + C_0\right)^2\right.$$

$$= 0.1\left[(0.1)^2 + (0.00025)^2\right] = 0.001.$$

$$y_1 = y_0 + \frac{1}{6}(A_0 + 2B_0 + 2C_0 + D_0)$$

$$= 0 + \frac{1}{6}(0 + 2 \times 0.00025 + 2 \times 0.00025 + 0.001) = 0.00033.$$

$$x_2 = x_1 + 0.1 = 0.1 + 0.1 = 0.2$$

$$A_1 = 0.1(x_1^2 + y_1^2) = 0.1\left[(0.1)^2 + (0.00033)^2\right] = 0.001$$

$$B_1 = 0.1\left[(x_1 + 0.05)^2 + \left(y_1 + \tfrac{1}{2}A_1\right)^2\right.$$

$$= 0.1\left[(0.15)^2 + (0.00083)^2\right] = 0.00225.$$

$$C_1 = 0.1\left[(x_1 + 0.05)^2 + \left(y_1 + \tfrac{1}{2}B_1\right)^2\right.$$

$$= 0.1\left[(0.15)^2 + (0.001455)^2\right] = 0.00025.$$

$$D_1 = 0.1\left[x_2^2 + \left(y_1 + C_1\right)^2\right]$$

$$= 0.1\left[(0.2)^2 + (0.0058)^2\right] = 0.004.$$

$$y_2 = y_1 + \frac{1}{6}(A_1 + 2B_1 + 2C_1 + D_1)$$

$$= 0.00033 + \frac{1}{6}(0.014) = 0.002663.$$

Example KdV Hf_XV%B feeR ^VeY`U hZeY $h = 0.2$ e` WZ_U eYV gR]fV `W y Re $x = 0.2,\ \ x = 0.4,$ R_U $x = 0.6,$ XZgV_ $\frac{dy}{dx} = 1 + y^2,\ y(0) = 0.$

?VcV $f(x, y) = 1 + y^2,\ x_0 = 0,\ y_0 = 0,\ h = 0.2.$?V_TV

$$x_1 = x_0 + h = 0.2, \quad x_2 = x_1 + h = 0.4.$$

J` UVeVc^Z_V $y_1,\ y_2$ hV fdV Z^ac`gVU <f]Vc W`c^ f]R2

$$x_{n+1} = x_n + h = x_n + 0.2$$

$$A_n = hf(x_n, y_n) = 0.2(1 + y_n^2)$$

$$B_n = hf(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}A_n) = 0.2\left[1 + \left(y_n + \tfrac{1}{2}A_n\right)^2\right]$$

$$C_n = hf(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}B_n) = 0.2\left[1 + \left(y_n + \tfrac{1}{2}B_n\right)^2\right]$$

$$D_n = hf(x_n + h,\ y_n + C_n) = 0.2\left[1 + \left(y_n + C_n\right)^2\right]$$

$$y_{n+1} = y_n + \frac{1}{6}(A_n + 2B_n + 2C_n + D_n)$$

$$x_1 = x_0 + 0.2 = 0 + 0.2 = 0.2$$

$$A_0 = 0.2(1 + y_0^2) = 0.2(1 + 0^2) = 0.2$$

$$B_0 = 0.2\left[1 + \left(y_0 + \tfrac{1}{2}A_0\right)^2\right] = 0.2\left[1 + (0.1)^2\right] = 0.202.$$

$$C_0 = 0.2\left[1 + \left(y_0 + \tfrac{1}{2}B_0\right)^2\right] = 0.2\left[1 + (0.101)^2\right] = 0.20204.$$

$$D_0 = 0.2\left[1 + \left(y_0 + C_0\right)^2\right]$$

$$= 0.2\left[1 + (0.20204)^2\right] = 0.20816.$$

$$y_1 = y_0 + \frac{1}{6}(A_0 + 2B_0 + 2C_0 + D_0)$$

$$= 0 + \frac{1}{6}(0.2 + 2\times 0.202 + 2\times 0.20204 + 0.20816) = 0.2027.$$

Z&V&$ $y(0.2) = 0.2027.$

$$x_2 = x_1 + 0.1 = 0.2 + 0.2 = 0.4$$

$$A_1 = 0.2(1 + y_1^2) = 0.2\left[1 + (0.2027)^2\right] = 0.2082$$

$$B_1 = 0.2\left[1 + \left(y_1 + \tfrac{1}{2}A_1\right)^2\right] = 0.2\left[1 + (0.3068)^2\right] = 0.2188.$$

$$C_1 = 0.2\left[1 + \left(y_1 + \tfrac{1}{2}B_1\right)^2\right] = 0.2\left[1 + (0.3121)^2\right] = 0.2195.\ D_1 = 0.2\left[1 + \left(y_1 + C_1\right)^2\right]$$

$$= 0.2\left[1 + (0.4222)^2\right] = 0.2356.$$

$$y_2 = y_1 + \frac{1}{6}(A_1 + 2B_1 + 2C_1 + D_1)$$

$$= 0.00033 + \frac{1}{6}(0.2082 + 2\times 0.2195 + 2\times 0.2195 + 0.2356)$$

$$= 0.4228.$$

Z&V&$ $y(0.4) = 0.4228,$ T`ccVTe e` W` fc UVTZ^R] a]RTVd&

$$x_3 = x_2 + 0.1 = 0.4 + 0.2 = 0.6$$

$$A_2 = 0.2(1 + y_2^2); \qquad B_2 = 0.2\left[1 + \left(y_2 + \tfrac{1}{2}A_2\right)^2\right];$$

$$C_2 = 0.2\left[1 + \left(y_2 + \tfrac{1}{2}B_2\right)^2\right]; \qquad D_2 = 0.2\left[1 + \left(y_2 + C_2\right)^2\right].$$

I fSdeZe feZ_X eYV gR] fVd$ R_U fdZ_X

$$y_3 = y_2 + \frac{1}{6}(A_2 + 2B_2 + 2C_2 + D_2)$$

hV `SeRZ_ $y(0.6) = y_3 = 0.6841,$ T`ccVTe e` W` fc UVTZ^R] a]RTVd&

Example >ZgV_ eYV Z_ZeZR] gR] fV ac`S]V^ $y' = x + y,\ y(0) = 0.$ =Z_U eYV gR] fV `W y Raac`iZ^ReV] j W`c $x = 1$ Sj Hf_XV%B feeR ^VeY`U Z_ WZgV deVad& :`^aRcV eYV cVd f]e hZeY eYV ViRTe gR] fV&

?VcV $f(x, y) = x + y,\ x_0 = 0,\ y_0 = 0.$ 8d hV YRgV e` TR]T f]ReV eYV gR] fV `W y hYV_ $x = 1$ Z_ WZgV deVad$ hV YRgV e` eR\V $h = \dfrac{x_n - x_0}{n} = \dfrac{1-0}{5} = 0.2.$?V_TV

$x_1 = x_0 + h = 0.2, \quad x_2 = x_1 + h = 0.4, \quad x_3 = x_2 + h = 0.6, \qquad x_4 = x_3 + h = 0.8, \quad x_5 = x_4 + h = 1.0.$

MV UVeVc^Z_V y_1, y_2, y_3, y_4, y_5 hV fdV Hf_XV%B feeR W`c^ f]R2

$$x_{n+1} = x_n + h = x_n + 0.2$$

$$A_n = hf(x_n, y_n) = 0.2(x_n + y_n)$$

$$B_n = hf\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}A_n\right) = 0.2\left[x_n + 0.1 + y_n + 0.1(x_n + y_n)\right]$$

$$= 0.22(x_n + y_n) + 0.02$$

$$C_n = hf\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}B_n\right)$$

$$= 0.2\left[x_n + 0.1 + y_n + 0.11(x_n + y_n) + 0.01\right]$$

$$= 0.222(x_n + y_n) + 0.022$$

$$D_n = hf(x_n + h,\ y_n + C_n)$$

$$= 0.2\left[x_n + 0.2 + y_n + 0.222(x_n + y_n) + 0.022\right]$$

$$= 0.2444(x_n + y_n) + 0.0444$$

$$y_{n+1} = y_n + \frac{1}{6}(A_n + 2B_n + 2C_n + D_n)$$

Z&V&$ $\quad y_{n+1} = y_n + 0.2214(x_n + y_n) + 0.0214.$

JYV dfTTVddZgV deVad R_U TR]Tf]ReZ`_d RcV a]`eeVU Z_ eYV W`]]`h Z_X eRS]V&

n	x_n	Raac`iZ^ReV gR]fV `W y_n	$x_n + y_n$	$0.2214(x_n + y_n)$	$0.2214(x_n + y_n) + 0.0214$
(	(&(	(&((((	(&((((	(&((((	(&(*) ,((
)	(&*	(&(*) ,((	(&**) ,((	(&(,1 ()0	(&(/(,)0
*	(&,	(&(1) 0)0	(&,1) 0)0	(&)(0 001	(&)+(*01
+	(&.	(&***)(/	(&0**)(/	(&)0* (),	(&*(+ ,),
,	(&0	(&,*- -*)	)&**- -*)	(&*/) ++(	(&*1* /+(
-	)&(	(&/)0 *-)			

Table:

: ` ^ aRcZd`_ `W eYV RTTfcRTj `W eYcVV ^ VeY`Ud UZdT f ddVU Z_ VRc]ZVc dVTeZ`_d Z_ eYV TRdV `W eYV Z_ZeZR] gR] f V ac`S]V ^ $y' = x + y,\ y(0) = 0.$

x_n	< i RTe gR] f V	8 aac` iZ ^ ReV gR] f Vd e` V `SeRZ_VU S j			8 Sd`] f eV gR] f V `W <cc`c		
		< f]Vc ^ VeY`U	H%B IVT`_U FcUVc	H%B =` f ceY FcUVc	< f]Vc ^ VeY`U	H%B IVT`_U FcUVc	H%B =` f ceY FcUVc
(&*	(&(*),(+	(&(((	(&(*((	(&(*),((	(&(*)	(&((),	(&(((((+
(&,	(&(1)0*-	(&(,(	(&(00,	(&(1)0)0	(&(-*	(&((+,	(&(((((/
(&.	(&***))1	(&)*0	(&*)-0	(&***)(/	(&(1,	(&((.+	(&(((())
(&0	(&,*--,)	(&*/,	(&,)-+	(&,*--*)	(&)-*	(&()(*	(&((((*(
)&(	(&/)0*0*	(&,01	(&/(*/	(&/)0*-)	(&**1	(&()-.	(&((((+)

Exercises

@_ < i VcTZdVd)%) ($ d`]gV eYV Z_ZeZR] gR] f V ac`S]V ^ f dZ_X W` f ceY `cUVc H f _XV%B f eeR ^ VeY`U W`c gR] f V `W $y$ Re eYV XZgV_ a`Z_e `W x hZeY h. XZgV_ Z_ ScRT\Ved!

)& $\frac{dy}{dx} = y,\ y(0) = 1$ Re eYV a`Z_e $x = 1\ \ (h = 0.5)$

*& $\frac{dy}{dx} = 1 - y,\ y(0) = 0$ Re eYV a`Z_e $x = 0.2\quad h = 0.1).$

+& $\frac{dy}{dx} = y - x,\ y(0) = 2$ Re eYV a`Z_e $x = 0.2\quad h = 0.1).$

,& $yy' = x,\ y(0) = 1.5$ Re eYV a`Z_e $x = 0.2\quad h = 0.1).$

-& $\frac{dy}{dx} = x - y,\ y(1) = 0.4$ Re eYV a`Z_e $x = 1.6$ V $h = 0.6).$

.& $y' = x + y + xy,\ y(0) = 1$ Re eYV a`Z_e $x = 0.1\quad h = 0.02).$

/& $\frac{dy}{dx} = \frac{y - x}{1 + x},\ y(0) = 1$ Re eYV a`Z_e $x = 0.1\quad h = 0.02).$

0& $\frac{dy}{dx} = xy,\ y(1) = 2$ Re eYV a`Z_e $x = 1.6\quad h = 0.2).$

1& $\frac{dy}{dx} = 1+\ln(x+y),\ y(0)=1$ Re eYV a`Z_e $x = 0.2$ $h = 0.1).$

)(& $y' = x^2 + y,\ y(0)=1$ Re eYV a`Z_e $x=0.1$ $h=0.05).$

))& $y'=2xy,\ y(0)=1$ Re eYV a`Z_e $x=0.5$ $h=0.1).$

)*& $y'=3x+\frac{1}{2},\ y(0)=1$ Re eYV a`Z_e $x=0.2$ $h=0.05).$

)+& I`]gV fdZ_X Hf_XV%B feeR ^VeY`U $y'(x+y)=y-x$ hZeY $y(0)=2$ W`c eYV cR_XV 0.00(0.02)0.06.

),& KdZ_X Hf_XV%B feeR ^VeY`U WZ_U $y(0.2)$ `W eYV Z_ZeZR] gR]fV ac`S]V^ $y'=x^2+2y,\ y(0)=0,$ eR\Z_X $h=0.2.$

)-& KdZ_X Hf_XV%B feeR ^VeY`U WZ_U eYV gR]fV `W y Re eYV a`Z_e $x=2$ Z_ deVad `W 0.2 `W eYV Z_ZeZR] gR]fV ac`S]V^ $\frac{dy}{dx}=2+\sqrt{xy},\ y(1)=1$&

).& KdZ_X Hf_XV%B feeR ^VeY`U WZ_U $y(1.3),$ XZgV_ $y'=x^2y$ R_U $y(1)=2.$ JR\V $h=0.3$&

)/& I`]gV fdZ_X Hf_XV%B feeR ^VeY`U $y'=y-\frac{2x}{y}$ hZeY $y=1$ Re $x=0$ W`c $h=0.5$ `_ eYV Z_eVcgR] [0, 1].

)0& I`]gV $y'=2x^{-1}\sqrt{y-\ln x}+x^{-1},\ y(1)=0$ W`c $1\le x\le 1.8$

R! Sj <f]Vc ^VeY`U hZeY $h=0.1$.

S! Sj Z^ac`gVU <f]Vc ^VeY`U hZeY $h=0.2$.

T! Sj Hf_XV%B feeR ^VeY`U hZeY $h=0.4$.

U! :`^aRcV eYV RS`gV cVdf]ed hZeY eYV ViRTe gR]fV& ;VeVc^Z_V eYV Vcc`cd& :`^^V_e&

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PREDICTOR CORRECTOR METHODS

Introduction

<f]Vc ^VeY`U R_U W`fceY `cUVc Hf_XV%BfeeR ^VeY`Ud RcV TR]]VU dZ_X]V%deVa ^VeY`Ud$ hYVcV hV YRgV dVV_ eYRe eYV T`^afeReZ`_ `W y_{n+1} cVbfZcVd eYV _`h]VUXV `W y_n `_]j& 9fe ^`UZWZVU <f]Vc ^VeY`U Zd R ^f]eZ%deVa ^VeY`U dZ_TV W`c eYV T`^afeReZ`_ `W y_{n+1} eYV _`h]VUXV `W y_n Zd _`e V_`fXY& @e Zd R MOBAF@QLO#@LOOB@QLO JBQELA$ Z_ hYZTY R MOBAF@QLO W`c^f]R Zd fdVU e` acVUZTe eYV gR]fV y_{n+1} `W $y$ Re $x_{n+1}$ R_U eYV_ R @LOOB@QLO W`c^f]R Zd fdVU e` Z^ac`gV eYV gR]fV `W y_{n+1}&

=`cVi R^a]V$ T`_dZUVc eYV Z_ZeZR] gR]fV ac`S]V^

$$\frac{dy}{dx} = f\left(x, y\right), \quad y(x_0) = y_0.$$

KdZ_X dZ^a]V <f]Vcqd R_U ^`UZWZVU <f]Vcqd ^VeY`U$ hV TR_ hcZeV U`h_ R dZ^a]V acVUZTe`c%T`ccVTe`c aRZc G%:!Rd

$$P: \quad y_{n+1}^{(0)} = y_n + hf(x_n, y_n).$$

$$C: \; y_{n+1}^{(1)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(0)}) \right].$$

?VcV$ $y_{n+1}^{(1)}$ Zd eYV WZcde T`ccVTeVU gR]fV `W y_{n+1}.& JYV T`ccVTe`c W`c^f]R ^Rj SV fdVU ZeVcReZgV]j Rd UVWZ_VU SV]`h2

$$y_{n+1}^{(r)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(r-1)}) \right] \quad \left(r = 1, 2, \ldots\right)$$

JYV ZeVcReZ`_ eVc^Z_ReV hYV_ eh` dfTTVddZgV ZeVcReVd RXcVV e` eYV UVdZcVU RTTfcRTj& MV YRgV T`_dZUVcVU ^`UZWZVU <f]Vc ^VeY`U Z_ eYV acVgZ`fd TYRaeVc&

@_ eYZd TYRaeVc hV T`_dZUVc eh` ^VeY`Ud2 8UR^d%D`f]e`_ R_U DZ]_Vqd DVeY`Ud& JYVj cVbfZcV Wf_TeZ`_ gR]fVd Re $x_n, x_{n-1}, x_{n-2}, \ldots$ W`c eYV T`^afeReZ`_ `W eYV Wf_TeZ`_ gR]fV Re x_{n+1}.

Adams-Moulton Method

:`_dZUVc eYV Z_ZeZR] gR]fV ac`S]V^

$$y' = f(x, y), \quad y(x_0) = y_0. \qquad \text{t)!}$$

IeRceZ_X hZeY XZgV_ x_0 R_U XZgV_ eYV deVa dZkV h\$ hV YRgV $x_1 = x_0 + h,\ x_{-1} = x_0 - h,\ x_{-2} = x_0 - 2h,$ R_U $x_{-3} = x_0 - 3h$& MV UV_`eV $f_0 = f(x_0, y_0),\quad f_1 = f(x_1, y_1),\quad f_{-1} = f(x_{-1}, y_{-1}),\quad f_{-2} = f(x_{-2}, y_{-2}),$ R_U $f_{-3} = f(x_{-3}, y_{-3}).$

@_ 8UR^d%D`f]e`_ DVeY`U\$ hV acVUZTe Sj

$$y_1^P = y_0 + \frac{h}{24}(55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}) \qquad \text{t)!}$$

R_U T`ccVTe Sj

$$y_1^C = y_0 + \frac{h}{24}(9 f_1^p + 19 f_0 - 5 f_{-1} + f_{-2}), \qquad \text{t *!}$$

hYVcV $f_1^p = f(x_1, y_1^P).$

JYV XV_VcR] W`c^d W`c W`c^ f]RV)! R_U *! RcV XZgV_ Sj

$$y_{n+1}^P = y_n + \frac{h}{24}(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}) \qquad \text{t)!}$$

hZeY T`ccVTeZ`_

$$y_{n+1}^C = y_n + \frac{h}{24}(9 f_{n+1}^p + 19 f_n - 5 f_{n-1} + f_{n-2}), \qquad \text{t *!}$$

hYVcV $f_{n+1}^p = f(x_{n+1}, y_{n+1}^P).$

JYV W`c^f]RV XZgV_ RS`gV RcV ViR^a]V `W BUMIF@FQ MOBAF@QLO Y@LOOB@QLO W`c^f]RV Rd eYVj RcV ViacVddVU Z_ `cUZ_ReV W`c^&

Example >ZgV_ $\frac{dy}{dx} = 1 + y^2;\ y(0) = 0.$:`^afeV $y(0.8)$ fdZ_X 8UR^d%D`f]e`_ DVeY`U&

?VcV $x_1 = 0.8,\ h = 0.2.$?V_TV $x_0 = x_1 - h = 0.8 - 0.2 = 0.6,$

$x_{-1} = x_0 - h = 0.4,\ x_{-2} = x_0 - 2h = 0.2,$ R_U $x_{-3} = x_0 - 3h = 0.$

JYV deRceVc gR]fVd RcV $y(0.6),\ y(0.4)$ R_U $y(0.2)$& KdZ_X W`fceY%`cUVc Hf_XV%BfeeR ^VeY`U HVW& <iR^a]V / Z_ eYV acVgZ`fd TYRaeVc!\$ eYV gR]fVd RcV W`f_U e` SV2

$$y(0.6) = 0.6841, \quad y(0.4) = 0.4228, \quad y(0.2) = 0.2027.$$

?V_TV $y_0 = y(x_0) = y(0.6) = 0.6841, \quad y_{-1} = y(x_{-1}) = y(0.4) = 0.4228,$

$y_{-2} = 0.2027$ R_U $y_{-3} = y(x_{-3}) = y(0) = 0.$

8]d`\$ $f_0 = f(x_0, y_0) = 1 + y_0^2 = 1 + (0.6841)^2$ 3

$$f_{-1} = f(x_{-1}, y_{-1}) = 1 + y_{-1}^2 = 1 + (0.4228)^2$$ 3

R_U d` `_& MV eRS f]ReV eYV^ SV]`h2

x	y	$f(x) = 1 + y^2$
$x_{-3} = 0.0$	$y_{-3} =$ (&(((	$f_{-3} =$)&((((
$x_{-2} = 0.2$	$y_{-2} =$ (&*(*/	$f_{-2} =$)&(,))
$x_{-1} = 0.4$	$y_{-1} =$ (&,**0	$f_{-1} =$)&)/0/
$x_0 = 0.6$	$y_0 =$ (&.0,)	$f_0 =$)&,.0)

I f SdeZe feZ_X eYVdV gR] fVd Z_)!$ hV `SeRZ_ eYV acVUZTeVU gR] fV `W y_1 Re $x_1 = 0.8$ Rd

$$y_1^P = 0.6841 + \frac{0.2}{24}\{55[1 + (0.6841)^2] - 59[1 + (0.4228)^2]$$
$$+37[1 + (0.4228)^2] - 9\}$$

$= 1.0233$, `_dZ^a]ZWZTReZ`_&

: `ccVTeVU gR] fV `W y_1 Re $x_1 = 0.8$ Zd `SeRZ_VU fdZ_X *! Rd SV]`h2

$$y_1^C = 0.6841 + \frac{0.2}{24}\{9[1 + (0.0233)^2] + 19[1 + (0.6841)^2]$$
$$-5[1 + (0.4228)^2] + [1 + (0.2027)^2]\}$$

$= 1.0296$, `_dZ^a]ZWZTReZ`_&

Exercises

)& KdZ_X 8UR^d%D`f]e`_ acVUZTe`c%T`ccVTe`c ^VeY`U$ WZ_U eYV gR] fV `W $y$ Re $x = 4.4$ Wc`^ eYV UZWWVcV_eZR] Vb fReZ`_

$$5x\frac{dy}{dx} + y^2 = 2,$$

XZgV_ eYRe

x	4.0	4.1	4.2	4.3
y	1.0000	1.0049	1.0097	1.0143

*& KdZ_X 8UR^d%D`f]e`_ acVUZTe`c%T`ccVTe`c ^VeY`U$ WZ_U eYV gR] fV `W $y$ Re $x = 0.8$, R_U $x = 1.0$ `W eYV Z_ZeZR] gR] fV ac`S]V^

$$\frac{dy}{dx} = y - x^2, \quad y(0) = 1$$

JR\V $h = 0.2.)$

+& KdZ_X 8UR^d%D`f]e`_ acVUZTe`c%T`ccVTe`c ^VeY`U$ WZ_U eYV gR]fV `W $y$ Re $x = 1.4$ `W eYV Z_ZeZR] gR]fV ac`S]V^

$$x^2 y' + xy = 1, \quad y(1) = 1.0$$

hZeY deRceVc gR]fVd $y(1.1) = 0.996, \quad y(1.2) = 0.986, \quad y(1.3) = 0.972.$

,& =Z_U eYV d`]feZ`_ `W eYV Z_ZeZR] gR]fV ac`S]V^

$$y' = y^2 \sin t, \quad y(0) = 1$$

fdZ_X 8UR^d%D`f]e`_ acVUZTe`c%T`ccVTe`c ^VeY`U$ Z_ eYV Z_eVcgR] $(0.2, 0.5)$ XZgV_ eYRe

$$y(0.05) = 1.00125, \quad y(0.1) = 1.00502, \quad y(0.15) = 1.01136.$$

Milne's Method

: `_dZUVc eYV Z_ZeZR] gR]fV ac`S]V^

$$y' = f(x, y), \quad y(x_0) = y_0. \qquad \text{t)!}$$

IeRceZ_X hZeY XZgV_ x_0 R_U XZgV_ eYV deVa dZkV $h$$ hV YRgV $x_1 = x_0 + h, \; x_{-1} = x_0 - h, \; x_{-2} = x_0 - 2h,$ R_U $x_{-3} = x_0 - 3h$& MV UV_`eV $f_0 = f(x_0, y_0), \; f_1 = f(x_1, y_1), \; f_{-1} = f(x_{-1}, y_{-1}), \; f_{-2} = f(x_{-2}, y_{-2}),$ R_U $f_{-3} = f(x_{-3}, y_{-3}).$

@_ DZ]_Vqd DVeY`U$ hV acVUZTe Sj

$$y_1^P = y_{-3} + \frac{4h}{3}(2f_{-2} - f_{-1} + 2f_0) \qquad \text{t)!}$$

R_U T`ccVTe Sj

$$y_1^C = y_{-1} + \frac{h}{3}(f_{-1} + 4f_0 + f_1^P), \qquad \text{t *!}$$

hYVcV $f_1^P = f(x_1, y_1^P).$

JYV XV_VcR] W`c^d W`c W`c^ f]RV)! R_U *! RcV XZgV_ Sj

$$y_{n+1}^P = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n) \qquad \text{t +!}$$

R_U T`ccVTe Sj

$$y_{n+1}^C = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1}^P), \qquad \text{t ,!}$$

hYVcV $f_{n+1}^P = f(x_{n+1},\ y_{n+1}^P).$

JYV W`c^f]RV XZgV_ RS`gV Zd R]d` BUMIF@FQ MOBAF@QLO Y@LOOB@QLO W`c^f]RV Rd eYVj RcV ViacVddVU Z_ `cUZ_ReV W`c^&

Example >ZgV_ $\frac{dy}{dx}=1+y^2;\ y(0)=0.$:`^afeV $y(0.8)$ R_U $y(1.0)$ fdZ_X DZ]_Vqd DVeY`U&

9LIRQFLK

; VeVc^Z_ReZ`_ `W $y(0.8)$:

?VcV eR\V $x_1 = 0.8,\ h = 0.2.$?V_TV

$$x_0 = x_1 - h = 0.8 - 0.2 = 0.6,\quad x_{-1} = 0.4,\quad x_{-2} = 0.2,\quad x_{-3} = 0.$$

JYV deRceVc gR]fVd RcV $y(0.6),\ y(0.4)$ R_U $y(0.2)$& KdZ_X W`fceY%`cUVc Hf_XV%BfeeR ^VeY`U$ eYV gR]fVU RcV W`f_U e` SV2

$$y(0.6) = 0.6841,\quad y(0.4) = 0.4228,\quad y(0.2) = 0.2027.$$

?V_TV

$y_0 = 0.6841,\quad y_{-1} = 0.4228,\quad y_{-2} = 0.2027$ R_U

$y_{-3} = y(x_{-3}) = y(0) = 0.$

8]d`$ $f_0 = f(x_0,\ y_0) = 1 + y_0^2 = 1 + (0.6841)^2$ 3

$f_{-1} = 1 + y_{-1}^2 = 1 + (0.4228)^2$ 3

R_U d` `_& MV eRSf]ReV eYV^ SV]`h2

x	y	$f(x) = 1 + y^2$
$x_{-3} = 0.0$	$y_{-3} =$ (&(((	$f_{-3} =$)&((((
$x_{-2} = 0.2$	$y_{-2} =$ (&*(*/	$f_{-2} =$)&(,))
$x_{-1} = 0.4$	$y_{-1} =$ (&,**0	$f_{-1} =$)&)/0/
$x_0 = 0.6$	$y_0 =$ (&.0,)	$f_0 =$)&,.0)

I fSdeZefeZ_X eYVdV gR]fVd Z_)!$ hV `SeRZ_ eYV acVUZTeVU gR]fV `W y_1 Re $x_1 = 0.8$ Rd

$$y_1^P = 0 + \frac{0.8}{3}\left[2(1.0411) - 1.1787 + 2(1.4681)\right] = 1.0239$$

?V_TV $f_1 = 1 + \left(y_1^P\right)^2 = 1 + (1.0239)^2 = 2.0480$

R_U YV_TV eYV T`ccVTeVU gR]fV `W y_1 Re $x_1 = 0.8$ Zd `SeRZ_VU fdZ_X *! Rd SV]`h2

$$y_1^C = 0.4228 + \frac{0.2}{3}\left[1.1787 + 4(1.4681) + 2.0480\right] = 1.0294.$$

?V_TV $y(0.8) = 1.0294,$ T`ccVTe e` W`fc a]RTVd `W UVTZ^R]&

; VeVc^Z_ReZ`_ `W $y(1.0)$:

?VcV eR\V $x_1 = 1.0,\ h = 0.2.$?V_TV

$$x_0 = x_1 - h = 1.0 - 0.2 = 0.8,\quad x_{-1} = 0.6,\quad x_{-2} = 0.4,\quad x_{-3} = 0.2.$$

JYV deRceVc gR]fVd RcV $y(0.8),\ y(0.6),$ R_U $y(0.4)$& MV YRgV eYV gR]fVd

$$y(0.8) = 1.0294,\quad y(0.6) = 0.6841,\quad y(0.4) = 0.4228.$$

?V_TV

$$y_0 = 1.0294,\quad y_{-1} = 0.6841,\quad y_{-2} = 0.4228 \text{ R\_U } y_{-3} = 0.$$

8]d`$ $f_0 = 1 + y_0^2 = 1 + (1.0294)^2$ 3 $f_{-1} = 1 + y_{-1}^2 = 1 + (0.6841)^2$ 3 R_U d` `_&

x	y	$f(x) = 1 + y^2$
$x_{-3} = 0.2$	$y_{-3} =$ (&*(*/	$f_{-3} =$)&(,))
$x_{-2} = 0.4$	$y_{-2} =$ (&,**0	$f_{-2} =$)&)/0/
$x_{-1} = 0.6$	$y_{-1} =$ (&.0,)	$f_{-1} =$)&,.0)
$x_0 = 0.8$	$y_0 = 1.0294$	$f_0 =$ *&(-1/

I fSdeZefeZ_X eYVdV gR]fVd Z_)!$ hV `SeRZ_ eYV acVUZTeVU gR]fV `W y_1 Re $x_1 = 1.0$ Rd

$$y_1^P = 1.5384$$

?V_TV $$f_1 = 1 + \left(y_1^P\right)^2 = 3.3667$$

: `ccVTeVU gR]fV `W y_1 Re $x_1 = 0.8$ Zd `SeRZ_VU fdZ_X *! Rd SV]`h2

$$y_1^C = 1.5557.$$

Example =Z_U$ fdZ_X DZ]_Vqd acVUZTe`c%T`ccVTe`c ^VeY`U$ $y(2.0)$ ZW $y(x)$ Zd eYV d`]feZ`_ `W

$$\frac{dy}{dx} = \frac{x + y}{2}$$

Rddf ^Z_X $y(0)=2.\quad y(0.5)=2.636,\quad y(1.0)=3.595$ R_U $y(1.5)=4.968.$

?VcV eR\V $x_1=2.0,\ h=0.5.$?V_TV

$$x_0=x_1-h=2.0-0.5=1.5,\quad x_{-1}=1,\quad x_{-2}=0.5,\quad x_{-3}=0.$$

8]d`$ Sj eYV Rddf ^aeZ`_$

$$y_0=4.968,\quad y_{-1}=3.595,\quad y_{-2}=2.636 \text{ R\_U } y_{-3}=2.$$

8d $f(x,y)=\frac{x+y}{2}$, hV YRgV

$$f_0=f(x_0,y_0)=\frac{x_0+y_0}{2}=\frac{1.5+4.968}{2}=3.2340.$$ 3

$$f_{-1}=f(x_{-1},y_{-1})=\frac{x_{-1}+y_{-1}}{2}=\frac{1.0+3.595}{2}=2.2975.$$ 3

$$f_{-2}=f(x_{-2},y_{-2})=\frac{x_{-2}+y_{-2}}{2}=\frac{0.5+2.636}{2}=1.5680.$$ 3

E`h$ fdZ_X acVUZTe`c W`c^ f]R hV T`^afeV

$$y_1^P=y_{-3}+\frac{4h}{3}\left(2f_{-2}-f_{-1}+2f_0\right)$$

$$=2+\frac{4(0.5)}{3}\left[2(1.5680)-2.2975+2(3.2340)\right]=6.8710.$$

KdZ_X eYV acVUZTeVU gR]fV$ hV dYR]] T`^afeV eYV T`ccVTeVU gR]fV `W $y_1$ Wc`^ eYV T`ccVTe`c W`c^ f]R

$$y_1^C=y_{-1}+\frac{h}{3}(f_{-1}+4f_0+f_1^P),$$ *!

hYVcV $f_1^P=f(x_1,y_1^P).$

E`h fdZ_X eYV RgRZ]RS]V acVUZTeVU gR]fV $y_1^P$$

$$f_1^P=f(x_1,y_1^P)=\frac{x_1+y_1^P}{2}=\frac{2+6.871}{2}=4.4355.$$

JYfd eYV T`ccVTeVU gR]fV Zd XZgV_ Sj

$$y_1^C=3.595+\frac{0.5}{3}\left[2.2975+4(3.234)+4.4355\right]=6.8731667.$$

?V_TV R_ Raac`iZ^ReV gR]fV `W y Re $x=2$ Zd eR\V_ Rd $y(2)=y_1^C=6.8731667.$

Example JRS f]ReV eYV d`]feZ`_ `W

$$\frac{dy}{dx} = x + y; \qquad y(0) = 1$$

Z_ eYV Z_eVcgR] (≤ U ≤ (&,$ hZeY E 5 (&)$ fdZ_X DZ]_Vqd acVUZTe`c%T`ccVTe`c ^VeY`U&

MV eR\V $x_1 = 0.4.$ MV TR__`e Z^^VUZReV]j fdV DZ]\_Vqd acVUZTe`c%T`ccVTe`c ^VeY`U Rd Ze \_VVU eYV gR]fV `W y Re eYV acVgZ`fd W`fc a`Z\_ed $x_0 = x_1 - h = 0.4 - 0.1 = 0.3,$ $x_{-1} = 0.2,$ $x_{-2} = 0.1,$ $x_{-3} = 0.$ :]VRc]j$ $y_{-3} = y(x_{-3}) = y(0) = 1.$ =`c eYV TR]Tf]ReZ`\_ `W eYV cVde eYcVV y gR]fVd hV fdV Hf_XV%Bfeer ^VeY`U `W W`fceY `cUVc R_U eYV_ dhZeTY `gVc e` DZ]_Vqd G%: ^VeY`U&

9j Hf_XV%Bfeer ^VeY`U `W W`fceY `cUVc Ze TR_ SV dVV_ eYRe h`c\ Zd]VWe Rd R_ VisVcTZdV!

$$y_0 = y(x_0) = y(0.3) = 1.3997, \quad y_{-1} = y(x_{-1}) = y(0.2) = 1.2428, \quad y_{-2} = y(x_{-2}) = y(0.1) = 1.1103.$$

=c`^ eYV XZgV\_ UZWWVcV\_eZR] VbfReZ`_ $f(x, y) = x + y$ $R_U hV YRgV

$$f_0 = f(x_0, y_0) = x_0 + y_0 = 0.3 + 1.3997 = 1.6997.$$

$$f_{-1} = f(x_{-1}, y_{-1}) = x_{-1} + y_{-1} = 0.2 + 1.2428 = 1.4428.$$

$$f_{-2} = f(x_{-2}, y_{-2}) = x_{-2} + y_{-2} = 0.1 + 1.1103 = 1.2103.$$

E`h$ fdZ\_X acVUZTe`c W`c^f]R hV T`^afeV

$$y_1^P = y_{-3} + \frac{4h}{3}\left(2f_{-2} - f_{-1} + 2f_0\right)$$

$$= 1 + \frac{4(0.5)}{3}\left[2(1.2103) \quad 1.4428 \quad 2(1.6997)\right] \; 1.58363$$

9VW`cV fdZ\_X eYV T`ccVTe`c W`c^f]R

$$y_1^C = y_{-1} + \frac{h}{3}(f_{-1} + 4f_0 + f_1^P), \qquad \text{t *!}$$

hV T`^afeV

$$f_1^P = f(x_1, y_1^P) = x_1 + y_1^P = 0.4 + 1.5836 = 1.9836.$$

?V_TV

$$y_1^C = 1.2428 + \frac{0.1}{3}\left[1.4428 \quad 4(1.6997) \quad 1.9836\right] \; 1.5836.$$

JYV cVb fZcVU d`] feZ`_ Zd eRS f]ReVU SV]`h2

i	(	(&)	(&*	(&+	(&,
j	)&((((	)&))(+	)&*,*0	)&+11/	)&-0+.

Example =Z_U$ fdZ_X DZ]_Vqd acVUZTe`c%T`ccVTe`c ^VeY`U$ $y(2.0)$ ZW $y(x)$ Zd eYV d`]feZ`_ `W $\frac{dy}{dx}=\frac{x+y}{2}$ Rddf^Z_X $y(0)=2.$ $y(0.5)=2.636,$ $y(1.0)=3.595$ R_U $y(1.5)=4.968.$

?VcV eR\V $x_1=2.0,\ h=0.5.$?V_TV

$$x_0=x_1-h=2.0-0.5=1.5,\ \ x_{-1}=1,\ \ x_{-2}=0.5,\ \ x_{-3}=0.$$

8]d`$ Sj eYV Rddf^aeZ`_$

$$y_0=4.968,\quad y_{-1}=3.595,\quad y_{-2}=2.636 \text{ R\_U } y_{-3}=2.$$

8d $f(x,y)=\frac{x+y}{2},$ hV YRgV

$$f_0=f(x_0,y_0)=\frac{x_0+y_0}{2}=\frac{1.5+4.968}{2}=3.2340.\ 3$$

$$f_{-1}=f(x_{-1},y_{-1})=\frac{x_{-1}+y_{-1}}{2}=\frac{1.0+3.595}{2}=2.2975.\ 3$$

$$f_{-2}=f(x_{-2},y_{-2})=\frac{x_{-2}+y_{-2}}{2}=\frac{0.5+2.636}{2}=1.5680.\ 3$$

E`h$ fdZ_X acVUZTe`c W`c^f]R hV T`^afeV

$$y_1^P=y_{-3}+\frac{4h}{3}\left(2f_{-2}-f_{-1}+2f_0\right)$$

$$=2+\frac{4(0.5)}{3}\left[2(1.5680)-2.2975+2(3.2340)\right]=6.8710.$$

KdZ_X eYV acVUZTeVU gR]fV$ hV dYR]] T`^afeV eYV T`ccVTeVU gR]fV `W $y_1$ Wc`^ eYV T`ccVTe`c W`c^f]R

$$y_1^C=y_{-1}+\frac{h}{3}(f_{-1}+4f_0+f_1^P),$$

hYVcV $f_1^P=f(x_1,y_1^P).$

E`h fdZ_X eYV RgRZ]RS]V acVUZTeVU gR]fV $y_1^P$$

$$f_1^P = f(x_1, y_1^P) = \frac{x_1 + y_1^P}{2} = \frac{2 + 6.871}{2} = 4.4355.$$

JYfd eYV T`ccVTeVU gR]fV Zd XZgV_ Sj

$$y_1^C = 3.595 + \frac{0.5}{3}\left[2.2975 + 4(3.234) + 4.4355\right] = 6.8731667.$$

?V_TV R_ Raac`iZ^ReV gR]fV `W y Re $x = 2$ Zd eR\V_ Rd $y(2) = y_1^C = 6.8731667.$

Exercises

)& =Z_U V (&O! fdZ_X DZ]_Vqd G%: ^VeY`U$ ZW V U! Zd eYV d`]feZ`_ `W eYV UZWWVcV_eZR] Vbf ReZ`_

$$\frac{dy}{dx} = -xy^2 \;;\quad y(0) = 2$$

Rddf ^Z_X V (&*! 5)&1*+(O$ V (&,! 5)&/*,),$ V (&.! 5)&,/(-1&

*& =Z_U eYV d`]feZ`_ `W

$$\frac{dy}{dx} = y(x+y), \quad y(0) = 1$$

fdZ_X DZ]_Vqd G%: ^VeY`U$ Re $x = 0.4$ given that

$y(0.1) = 1.11689,\ y(0.2) = 1.27739$ and $y(0.3) = 1.50412.$

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