An-Najah National University
Faculty of Graduate Studies

Numerical Treatment of The Fredholm Integral Equations of the Second Kind

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Dedication

I dedicate this thesis to my parents, my husband Jafar and my daughter Shayma'a, withouttheir patience, understanding, support and most of all love, this workwouldnot have been possible.

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الإقرار

أنا الموقعة أدناه مقدمة الرسالة التي تحمل العنوان:

Numerical Treatment of The Fredholm Integral Equations of the Second Kind

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص، باستثناء ما تمت الإشارة اليه حيثما ورد، وأن هذه الرسالة ككل، أو أي جزء منها لم يقدم من قبل لنيل أية درجة علمية أو بحث علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis , unless otherwise referenced , is the researcher's own work , and has not been submitted elsewhere for any other degree or qualification.

إسم الطالبة: **Student's name**

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التاريخ : **Date**

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Numerical Treatment of The Fredholm Integral Equations of the Second Kind

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Abstract

In this thesis we focus on the mathematical and numerical aspects of the Fredholm integral equation of the second kinddue to their wide range of physical application such as heat conducting radiation, elasticity, potential theory and electrostatics. After the classification of these integral equations we will investigate some analytical and numerical methods for solving the Fredholm integral equation of the second kind. Such analytical methods include: the degenerate kernel methods, converting Fredholm integral equation to ODE, the Adomain decomposition method, the modified decomposition method andthe method of successive approximations.

The numerical methods that will be presented here are: Projection methods including collocation method and Galerkin method, Degenerate kernel approximation methods and Nyström methods.

The mathematical framework of these numerical methods together with their convergence properties will be analyzed.

Some numerical examples implementing these numerical methods have been obtained for solving a Fredholm integral equation of the second kind.

The numerical results show a closed agreement with the exact solution.

Introduction

The subject of integral equations is one of the most important mathematical tools in both pure and applied mathematics. Integral equations play a very important role in modern science such as numerous problems in engineering and mechanics, for more details see [4] and [25]. In fact, many physical problems are modeled in the form of Fredholm integral equations, such problems as potential theory and Dirichlet problems which discussed in [4] and [37], electrostatics [34], mathematical problems of radiative equilibrium [23], the particle transport problems of astrophysics and reactor theory [29], and radiative heat transfer problems which discussed in [40], [41], [42], and [49].

Many initial and boundary value problems associated with ordinary differential equations (ODEs) and partial differential equations (PDEs) can be solved more effectively by integral equations methods. Integral equations also form one of the most useful tools in many branches of pure analysis, such as the theories of functional analysis and stochastic processes, see [27] and [32].

Historical background of the integral equation

An integral equation is an equation in which an unknown function appears under one or more integral signs.

There is a close connection between differential and integral equations and some problems may be formulated either way. The most basic type of integral equation is a Fredholm equation of the second kind

$$\lambda f(x) - \int_D G(x,y)f(y)dy = g(x) \qquad x \in D\,, \lambda \neq 0 \qquad (1)$$

where D is a closed bounded set in R^m, for some $m \geq 1$.

G is a function called the kernel of the integral equation and is assumed to be absolutely integrable, and satisfy other properties that are sufficient for the Fredholm Alternative Theorem, for more details see [4]. For $g \neq 0$, we have λ which is a non zero real or complex parameter and g given, and we seek f, this is the nonhomogeneos problem. For $g = 0$, equation (1) becomes an eigenvalue problem, and we seek both the eigenvalue λ and the eigenfunction f.

The integral equation (1) can be written abstractly as

$$(I - \lambda K)f = g \qquad (2)$$

with K is an integral operator on a Banach space X to the same Banach space X, e.g. $C\,[a,b]$ or $L^2\,[a,b]$.

At the time in the early 1960's, researchers were interested principally in one-dimensional case. It was for a kernel function G that was at least continuous; and then it was assumed that $G(x,y)$ was several times

continuously differentiable. This was the type of equation studied by Ivar Fredholm, and in his honor such equation is called Fredholm integral equation of the second kind. Today the work is with multi-dimensional Fredholm integral equations of the second kind in which the integral operator is completely continuous and the integration region is commonly a surface in R^3, in addition, the kernel function G is often singular. The Fredholm theory is still valid for such equations, and this theory is critical for the convergence and stability analysis of associated numerical methods. For more details see [4] and [14].

There are many analytical methods which are developed for solving Fredholm integral equations such methods as the degenerate kernel methods, converting Fredholm integral equation to ODE, the Adomain decomposition method, the modified decomposition method, the method of successive approximations and others. For more details see [1], [14], [28], [30], [44] and [50].

The numerical methods for solving Fredholm integral equations may be subdivided into the following classes: Degenerate kernel approximation methods, Projection methods, Nyström methods. For more details see [2], [5], [11], [13], [21], [36], [38] and [53]. All of these methods have iterative variants. There are other numerical methods, but the above methods and their variants include the most popular general methods.

There are only a few books on the numerical solutions of integral equations as compared to the much larger number that have been published on the numerical solution of ordinary and partial differential equations. General books on the numerical solution of integral equations include, in historical order, [10], and [16], and [19]. More specialized treatments of numerical methods for integral equations are given in [4], [7], [31] and [33].

Chapter 1

Mathematical Preliminaries

Chapter 1

Mathematical Preliminaries

Definition 1.1

An integral equation is an equation in which the unknown function f appears under the integral sign. A standard integral equation is of the form

$$f(x) = g(x) + \lambda \int_{v(x)}^{u(x)} G(x,y)\, f(y) dy, \qquad (1.1)$$

where $u(x)$ and $v(x)$ are limits of integration, λ is a constant parameter, and $G(x,y)$ is a function of two variables x and y called the kernel or the *nucleus* of the integral equation. The function f that will be determined appears under the integral sign, and sometimes outside the integral sign. The functions $g(x)$ and $G(x,y)$ are given. The limits of integration $u(x)$ and $v(x)$ may be both variables, constants, or mixed, and they may be in one dimension or two or more.

1.1 Classification of integral equations

1.1.1 Types of integral equations

There are four major types of integral equations, the first two are of main classes and the other two are related types of integral equations.

1. Fredholm integral equations

The most standard form of Fredholm integral equations is given by the form

$$h(x)\, f(x) \;=\; g(x) \;+\; \lambda \int_D G(x,y) f(y) dy\,, \qquad (1.2)$$

with D a closed bounded set in R^m, for some $m \;\geq\; 1$.

(*i*) If the function $h(x) \;=\; 1$, then (1.2) becomes simply

$$f(x) \;=\; g(x) \;+\; \lambda \int_D G(x,y) f(y) dy\,, \qquad (1.3)$$

and this equation is called *Fredholm integral equation of the second kind.*

(*ii*) If the function $h(x) \;=\; 0$, then (1.2) yields

$$g(x) \;+\; \lambda \int_D G(x,y) f(y) dy = 0, \qquad (1.4)$$

which is called *Fredholm integral equation of the first kind.*

(*iii*) If h(x) is neither 0 nor 1 then (1.2) called *Fredholm integral equation of the third kind*

2. Volterra integral equations

The most standard form of Volterra integral equation is of the form

$$h(x)\,f(x) \;=\; g(x) \;+\; \lambda \int_a^x G(x,y)f(y)dy\,, \qquad (1.5)$$

where the upper limit of integration is a variable and the unknown function f appears linearly or nonlinearly under the integral sign.

(i) If the function $h(x) \;=\; 1$, then equation (1.5) simply becomes

$$f(x) \;=\; g(x) \;+\; \lambda \int_a^x G(x,y)f(y)dy\,, \qquad (1.6)$$

and this equation is known as the *Volterra integral equation of the second kind.*

(ii) If the function $h(x) \;=\; 0$, then equation (1.5) becomes

$$g(x) \;+\; \lambda \int_a^x G(x,y)f(y)dy = 0, \qquad (1.7)$$

which is known as the *Volterra integral equation of the first kind.*

(*iii*) If $h(x)$ is neither 0 nor 1 then (1.5) called *Volterra* integral equation of the third kind.

3. Singular integral equations

When one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is called *singular*. For example, the integral equation

$$f(x) \;=\; g(x) + \lambda \int_{-\infty}^{\infty} (exp - |x-y|)f(y)dy, \qquad (1.8)$$

is a singular integral equation of the second kind.

(*i*) *Weakly singular integral equation*: The kernel is of the form

$$G(x,y) = \frac{H(x,y)}{|x-y|^{\alpha}}$$

or

$$G(x,y) = H(x,y)\, ln|x-y|$$

where $H(x,y)$ is bounded (that is, several times continuously differentiable) $a \leq x \leq b \; and \; a \leq y \leq b$ with $H(x,y) \neq 0$, and α is a constant such that $0 < \alpha < 1$. For example, the equation

$$g(x) = \lambda \int_0^x \frac{1}{(x-y)^{\alpha}} f(y)dy, \quad 0 < \alpha < 1 \tag{1.9}$$

is a singular integral equation with a weakly singular kernel. For more details see [9] and [17].

(*ii*) *Singular integral equation*: Here the kernel is of the form

$$G(x,y) = \frac{H(x,y)}{x - y}$$

where $H(x,y)$ *is* a differentiable function of (x,y) with $H(x,y) \neq 0$, then the integral equation is said to be a *singular equation with Cauchy* kernel where the integral $\int_a^b \frac{H(x,y)}{x-y} f(y)dy$ is understood in the sense of Cauchy Principal Value (CPV) and the notation $P.V. \int_a^b \frac{H(x,y)}{x-y} dy$, is usually used to denote this. Thus

$$P.V.\int_{\mathrm{a}}^{\mathrm{b}} \frac{H(x,y)}{x-y}\,dy = \lim_{\epsilon\to 0}\left\{\int_{a}^{x-\epsilon} \frac{H(x,y)}{x-y}\,\mathrm{dy} + \int_{x+\epsilon}^{b} \frac{H(x,y)}{x-y}\,\mathrm{dy}\right\}$$

(*iii*) *Strongly singular integral equations*: if the kernel $G(x, y)$ is of the form

$$G(x,y) = \frac{H(x,y)}{(x-y)^2}$$

where $H(x,y)$ is a differentiable function of (x, y) with $H(x,y) \neq 0$, then the integral equation is said to be a strongly singular integral equation. For more details see [22].

4. Integro-differential equations

In this type of equations, the unknown function f appears as a combination of both ordinary derivative and under the integral sign. In the electrical engineering problem, the current $I(t)$ flowing in a closed circuit containing resistance, inductance and capacitance is governed by the following integro-differential equation,

$$L\frac{dI}{dt} + RI + \frac{I}{C}\int_0^t I(\tau)\,d\tau = E(t) \qquad (1.10)$$

where L is the inductance, R the resistance, C the capacitance, and $E(t)$ the applied voltage. Similar examples can be cited as follows

$$f'(x) = 1 - \frac{1}{3}x + \int_0^1 xyf(y)dy, \qquad \mathrm{f}(0) = 1 \qquad (1.11)$$

$$f''(x) = g(x) + \lambda \int_0^x (x-y)\, f(y) d(y),$$

$$f(0) = 0, f'(0) = 1, \quad (1.12)$$

Equations (1.10) and (1.12) are of Volterra type integro-differential equations, whereas equation (1.11) is Fredholm type integro-differential equations.

1.1.2. Linearity of integral equations

There are two kinds of integral equations according to linearity and this depends on the unknown function under the integral sign.

(*i*) *Linear integral equations*

They are of the form

$$f(x) = g(x) + \lambda \int_{v(x)}^{u(x)} G(x,y)\, f(y) dy, \quad (1.13)$$

where only linear operations are performed upon the unknown function inside the integral sign, that is the exponent of the unknown inside the integral sign is one, for example

$$f(x) = \frac{3}{2}x - \frac{1}{3} + \int_0^1 (x-y) f(y) dy. \quad (1.14)$$

here the unknown function *f* appears in the linear form.

(*ii*)*Nonlinear integral equations*

They are of the form

$$f(x) = g(x) + \lambda \int_{v(x)}^{u(x)} G(x, y, f(y))\, dy, \qquad (1.15)$$

the unknown function f under the integral sign has exponent other than one, or the equation contains nonlinear functions of f, such as e^f , $sinh\, f$, $ln(1 + f)$, for example

$$f(x) = 1 + \int_0^x (1 + x - y)\, f^4(y) dy \qquad (1.16)$$

1.1.3 Homogeneity of integral equations

Integral equations of the second kind are classified as homogeneous or non-homogeneous.

(i)Homogeneous integral equation

if the function g in the second kind of Volterra or Fredholm integral equations is identically zero, the equation is called homogeneous, for example,

$$f(x) = \lambda \int_D G(x, y)\, f(y) dy \qquad (1.17)$$

and this kind of equations becomes an *eigenvalue* problem, and we seek both the *eigenvalue* λ and the *eigenfunction f,* where by an *eigenvalue* (or *characteristic value*)we mean that the value of the constant λ, for which the homogeneous Fredholm equation has a solution $f = f(x)$ which is *not* identically zero on $[a, b]$, the non-zero solution $f = f(x)$ is called an *eigenfunction*, or *characteristic function*.

(*ii*) *Non-homogeneous integral equation*

if the function g in the second kind of Volterra or Fredholm integral equation is not equal zero, the equation is called non-homogeneous, for example,

$$f(x) = \mathrm{g}(x) + \int_0^1 G(x,y)\, f(y)dy \qquad (1.18)$$

where $g(x)$ is not equal zero.

Notice that this property of classification holds for equations of the second kind only since . For more details see [4] and [50].

1.2 Kinds of kernels

1. *Separable or degenerate kernel*

A kernel $G(x,y)$ is called separable or degenerate if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x only and a function of y only, (some authors say $G(x,y)$is degenerate if it is of finite rank) that means,

$$\mathrm{G}(x,\mathrm{y}) = \sum_{i=1}^{n} u_i\,(x)v_i(y). \qquad (1.19)$$

The functions $u_i(x)$ and the functions $v_i(y)$ are linearly independent.

2. *Symmetric (or Hermitian) kernel*

A complex-valued function $G(x,y)$ is called symmetric if

$$G(x,y) = G^*(y,x), \tag{1.20}$$

where the asterisk denotes the complex conjugate. For a real kernel, this coincides with definition

$$G(x,y) = G(y,x). \tag{1.21}$$

3. Hilbert-Schmidt kernel

If the kernel $G(x, y)$, for each sets of values of x, y in the square $a \leq x \leq b \text{ and } a \leq y \leq b$, is such that

$$\int_a^b \int_a^b |G(x,y)|^2 dxdy < \infty,$$

also for each value of x in $a \leq x \leq b$, is

$$\int_a^b |G(x,y)|^2 \, dy < \infty,$$

And for each value of y in $a \leq y \leq b$, is

$$\int_a^b |G(x,y)|^2 \, dx < \infty, \tag{1.22}$$

has a finite value, then we call the kernel a regular kernel and the corresponding integral equation is called a *regular integral equation*.

4. Cauchy kernel

If the kernel $G(x,y)$ is of the form

$$G(x,y) = \frac{H(x,y)}{x - y} \tag{1.23}$$

where $H(x,y)$ is a differentiable function of (x, y) with $H(x,y) \neq 0$, then the integral equation is said to be a *singular equation with Cauchy kernel.*

5. Abel's kernels

If the kernel $G(x,y)$ is of the form

$$G(x,y) = \frac{H(x,y)}{|x-y|^{\alpha}} \tag{1.24}$$

where $0 < \alpha < 1$ and the function $H(x,y)$ is assumed to be several times continuously differentiable such integral equations contain this kernel are called *Abel integral equation.*

6. Hilbert kernel

The kernel is of the form

$$G(x,y) = cot\left[\frac{y-x}{2}\right], \tag{1.25}$$

where x and y are real variables, is called the *Hilbert kernel* and is closely connected with the Cauchy kernel, since in the unit circle

$$\frac{dt}{t-\tau} = \frac{1}{2}\left(cot\frac{y-x}{2} + i\right)dy,$$

where $t = e^{iy}, \tau = e^{ix}$.

7. ***Skew – symmetric kernel***

The kernel is of the form

$$G(x,y) = -G(y,x) \tag{1.26}$$

For more details see [51].

1.3 Review of spaces and operators

Definition 1.2 ***Vector spaces***

A vector space (or linear space) consists of the following:

1. a field F of scalors.
2. a set X of objects, called vectors.
3. a rule (or operation), called vector addition, which associates with each pair of vectors x, y in X a vector $x + y$ in X, called the sum of x and y, in such a way that

   (i) addition is commutative, $x + y = y + x$.

   (ii) addition is associative, $x + (y + z) = (x + y) + z$.

   (iii) there is a unique vector 0 in V, called the zero vector, such that $a + 0 = a$ for all a in X.

   (iv) for each vector x in X there is a unique vector $-x$ in X such that $x + (-x) = 0$.
4. A rule (or operation) called scalar multiplication, which associates with each scalar c in F and vector x in X a vector cx in X , called the product of c and x, in such a way that

   (i) $1x = x$, for every x in X.

   (ii) $(c_1 c_2)\, x = c_1\, (c_2)\, x$.

   (iii) $c(x + y) = cx + cy$.

(iv) $(c_1 + c_2)\, x = c_1 x + c_2 x.$

Definition 1.3 ***Vector norm on*** *X*

A vector norm on *X* is a function $\|.\|$ from *X* into *F*, (where the notation $\|.\|$ denotes the norm**,** *X* is a set of vectors and *F* is a scalar field) whose value at an x ϵ *X* is denoted by $\|x\|$ with the following properties:

(i) $\|x\| \geq 0$ for all $\boldsymbol{x} \epsilon X$

(ii) $\|x\| = 0$ iff $\boldsymbol{x} = 0$

(iii) $\|\propto x\| = |\propto| \|\boldsymbol{x}\|$ for all $\propto \epsilon F$ and $\boldsymbol{x} \epsilon X$

(iv) $\|x + y\| \leq \|x\| + \|y\|.$ (triangular inequality)

Examples of the vector norms from R^n into *R* (where *R* denotes the set of all real numbers) are: the maximum norm

$$\|x\|_\infty = max\{|x_i|: 1 \leq i \leq n\}$$

and the Euclidean norm

$$\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$$

for the vectors $x = (x_1, \ldots, x_n)$

Definition 1.4 ***Normed space***

A normed space *X* is a vector space with a norm defined on it**.** The normed space is denoted by $(X, \|.\|)$.

Definition 1.5 ***Cauchy sequence***

A Cauchy sequence is a sequence whose elements become arbitrary close to each other as the sequence progresses.

In the other words a sequence (x_n) is said to be a Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer N such that in the case of real numbers

$$for\ all\ m,n\ \geq N\ \rightarrow |x_m - x_n| < \epsilon. \qquad (1.27).$$

To define Cauchy sequences in any metric space X, the absolute value $|x_m - x_n|$ is replaced by the *distance* $d(x_m, x_n)$, where

$d : X \times X \rightarrow \mathrm{R}$.

Definition 1.6 ***Complete space***

X is complete if every Cauchy sequence of points in X has a limit that is also in X or if every Cauchy sequence in X converges in X.

Definition 1.7 ***Banach space***

Banach space is a complete normed vector space.

an example for Banach spaces is the finite-dimensional vector spaces R^n with the maximum norm

$$\|x\|_\infty = max\{|x_i|: 1 \leq i \leq n\}$$

and the Euclidean norm $\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$ for the vectors

$x = (x_1, \dots, x_n)$

Definition 1.8: Let X be a *Banach* space, for $x_0 \epsilon X$ and $r > 0$, the set $B(x_0, r) := \{x \epsilon X : \|x - x_0\| \leq r\}$ is called ***(closed) ball*** of X with the centre x_0 and radius r. A set S $\subset X$ is called:

bounded if it is contained in a ball of X;

open if for any $x_0 \epsilon S$ there is an $r > 0$ such that $B(x_0, r) \subset S$;

closed if $(x_n) \subset S, x_n \to x$ implies $x \epsilon S$;

relatively compact if every sequence $(x_n) \subset S$ contains a convergent subsequence (with a limit in X not necessarily belonging to S).

compact if S is closed and relatively compact.

The closure $\bar{S}$ of a set $S \subset X$ is the smallest closed set containing S. A set $S \subset X$ is said to be ***dense*** in X if $\bar{S} = X$.

Theorem 1.1: The sequence of vectors $\{x^k\}$converges to x in R^n with respect to $\|.\|_\infty$if

$$\lim_{k\to\infty} x_n^k = x_n \qquad \text{for each } n = 1, 2, \dots n$$

Definition 1.9 ***Inner product and Inner product space***

Let X be a vector space over F (either R *or* C) An inner product on X is a function

$$\langle ., . \rangle : X \times X \to F$$

That assigns to each pair $\langle x, y\rangle \epsilon\, X^2$ a number in F denoted $\langle x, y\rangle$ satisfying the following properties.

1. Positivity: $\langle x, x\rangle \geq 0$, moreover $\langle x, x\rangle = 0$ if and only if $x = 0$
2. Conjugate symmetry :$\langle x, y\rangle = \overline{\langle y, x\rangle}$ if $F = R$ then $\langle x, y\rangle = \langle y, x\rangle$
3. Linearity: if the vector $y \in X$ is fixed and with respect to the first variable for all $a, b \in F$,

$$\langle ax_1 + bx_2, y\rangle = a\langle x_1, y\rangle + b\langle x_2, y\rangle,$$

The pair $(X, \langle .,. \rangle)$is an ***inner product space*** over F. If $F = \boldsymbol{C}$ it is a complex inner product space, while if $F = R$ it is a real inner product space.

In particular the L^2 inner product on $L^2([a, b])$ is defined as

$$(f, g)_{L^2} = \int_a^b f(x)\overline{g(x)}\, dx \quad for\ f, g \in L^2([a, b]). \qquad (1.28)$$

Definition 1.10 ***Hilbert space*** (H)

It is a complete inner product space.

Definition 1.11 ***Linearly independent functions.***

By linear independence of set of functions $a_i's$ it is meant that, if

$$c_1a_1 + c_2a_2 + \ldots + c_na_n = 0,$$

where $c_i's$ are arbitrary constants, then $c_1 = c_2 = \ldots = c_n = 0$.

Definition 1.12 ***L^2-functions and L^2-spaces***.

L^2-function is a complex-valued function $f(x)$ of a real variable x on an interval (a, b), and such that in the Lebesgue sense

$$\int_a^b |f(x)|^2 \, dx < \infty \tag{1.29}$$

The set of all such functions is referred to as the ***function space L^2 [a, b]***; in other words by Lebesgue sense

$$L^2([a,b]) = \left\{ f : [a,b] \to C; \int_a^b |f(x)|^2 \, dx < \infty \right\} \tag{1.30}$$

Where C is the complex numbers.

Two L^2-functions f and g which are equal for 'almost all' values of x, that is, except for values of x being Lebesgue measure zero, are 'equivalent'. Thus, f and g are equivalent if

$$\int_a^b [f(x) - g(x)]^2 \, dx = 0 \tag{1.31}$$

while a function $h(x)$ (a 'null function') which is zero almost everywhere will not be distinguished from the zero function:

$$h(x) = 0 \leftrightarrow \int_a^b h^2(x) dx = 0 \tag{1.32}$$

With this convention, the set of L^2-functions forms a ***complete inner product space*** with respect to the inner product (1.28). Furthermore, the

space L^2, with an appropriate norm and inner product, is an example of a ***Hilbert space***.

We define the $\boldsymbol{L^2}$ ***norm*** of an L^2 function as

$$\|f\|_2 = \left\{\int_a^b |f(x)|^2\, dx\right\}^{\frac{1}{2}} \tag{1.33}$$

Definition 1.13 ***regularity conditions***

For the two-dimensional kernel function G(x, y). It is an L^2-function if the following conditions are satisfied

(i) For each set of values of x, y in the rectangle a ≤ x ≤ b, a ≤ y ≤b,

$$\int_a^b \int_a^b |G(x,y)|^2 dxdy < \infty, \tag{1.34}$$

(ii) For each set of value of x in a ≤ x ≤ b,

$$\int_a^b |G(x,y)|^2\, dy < \infty \tag{1.35}$$

(iii) For each set of value of y in a ≤ y ≤ b,

$$\int_a^b |G(x,y)|^2\, dx < \infty, \tag{1.36}$$

and this is called the regularity conditions on the kernel $G(x,y)$.

For more details see [16].

Definition 1.14 ***Measurabl functions***

They are structure-preserving functions between measurable spaces; as such, they form a natural context for the theory of integration. Specifically,

a function between measurable spaces is said to be measurable if the preimage of each measurable set is measurable.

Definition 1.15 ***L ^{p}-space***

The set of L^p-functions (where $p \geq 1$) generalizes L^2-space. Instead of square integrable, the measurable function f must be p-integrable, for f to be in L^p.

On a measure space X, the L^p norm of a function f is

$$\|f\|_{L^p} = \left(\int_X |f(x)|^p dx \right)^{\frac{1}{p}} \tag{1.37}$$

The L^p-functions are the functions for which this integral converges. For $p \neq 2$, the space of L^p-functions is ***a Banach space*** which is not a Hilbert space.

In the case where $p = \infty$, we have $\boldsymbol{L^{\infty}(D)}$ defined as

$$\{f : \text{measurable in D and } \|f\|_{\infty} < \infty\},$$

where

$$\|f\|_{\infty} = \inf\{\sup\{|f(x)| : x \epsilon S\}, S \subset \text{D}\} \tag{1.38}$$

with Lebesgue measure of the set S equals zero.

Definition 1.16 ***The vector space C(R)***

is a vector space consisting of all continuous functions $f : R \rightarrow F$,

where F stand for R or **C**.

$C[0,1]$ consists of all continuous functions $f : [0,1] \rightarrow F$,

$$\|f\|_{C[0,1]} = \|f\|_\infty = \max_{0 \le x \le 1} |f(x)| \tag{1.39}$$

Theorem 1.2 ***(Arzela-Ascoli)***

A set $S \subset C[0,1]$ is relatively compact in $C[0,1]$ if and only if the following two conditions are fulfilled:

(i) the functions $f \in S$ are ***uniformly bounded***, in the other words, there is a constant c such that $| f(x) | \le c$ for all $x \in [0,1], f \in S$.

(ii) the functions $f \in S$ are ***equicontinuous***, in the other words, for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x_1, x_2 \in [0,1], | x_1 - x_2 | \le \delta$$

implies

$$| f(x_1) - f(x_2) | \le \varepsilon \, for \, all \, f \in S.$$

Definition 1.17 ***The operators*** An operator $A : X \rightarrow Y$ assigns to every function $f \in X$ a function $Af \in Y$. It is therefore a mapping between two function spaces. If the range is on the real line or in the complex plane, the mapping is usually called a functional instead.

***There are many kinds of operators such as*:**

Differential Operator, Integral Operator, Binary Operator, Convective Operator, Delta Operator, Hermitian Operator, Identity Operator

Definition 1.18 ***Linear operator***

Let X and Y be two vector spaces, then $A: X \to Y$ which is a function defined on X and with values in Y is said to be a linear operator if;

$$A(f + g) = Af + Ag, \qquad A(\alpha f) = \alpha Af$$

for all $f, g \in X$ and α ∈ R or **C**.

Assume now that X and Y are normed spaces. An operator $A: X \to Y$ is said to be ***continuous*** if

$$\|f_n - f\|_X \to 0 .$$

Implies

$$\|Af_n - Af\|_Y \to 0 .$$

A linear operator $A: X \to Y$ occurs to be continuous if and only if it is bounded, in other words, if there is a constant c such that

$$\|Af\|_Y \leq c\|f\|_X \tag{1.40}$$

for all $f \in X$. The smallest constant c in this inequality is called the norm of A,

$$\|A\|_{X\to Y} = sup\{\|Af\|_Y : f \in X, \|f\|_X = 1\} \tag{1.41}$$

A sequence of linear bounded operators $A_n : X \to Y$ is said to be *point wise convergent* (or strongly convergent) if the sequence $(A_n f)$ is convergent in Y for any $f \in X$.

Definition 1.19 ***Inverse operator***

Let X and Y be Banach spaces and $A : X \to Y$ a linear operator. Introduce the subspaces

$N(A) = \{f \in X : Af = 0\} \subset X$ (the null space of A),

$R(A) = \{u \in Y : u = Af, f \in X\} \subset Y$ (the range of A).

If $N(A) = \{0\}$ then the inverse operator

$$A^{-1}: R(A) \subset Y \to X \text{ exists on } R(A) ?$$

that's mean

$$A^{-1}Af = f \quad \forall f \in X, AA^{-1}u = u \ \forall u \in R(A);$$

$If\ N(A) = \{0\}\ and\ R(A) = Y$ (that means A is onto) then A is invertible and the inverse operator $A^{-1}: Y \to X$ is defined on whole Y and linear by the theorem that says if A is a linear operator and invertible then $A^{-1}$is linear.

Definition 1.20 ***Compact operator***

Let X and Y be normed vector spaces, and let $K: X \to Y$ be linear. Then K is compact if the set

$$\{K\,x \mid \|\boldsymbol{x}\|_X \leq 1\} \tag{1.42}$$

has compact closure in Y. This is equivalent to saying that for every bounded sequence $\{x_n\} \subset X$, the sequence $\{K\,x_n\}$ has a subsequence that is convergent to some point in Y. Compact operators are also called completely continuous operators. (By a set S having compact closure in Y, we mean its closure $\bar{S}$ is a compact set in Y).

Definition 1.21 ***Compact integral operators on C(D)***

Let D be a bounded set in R^n, for some $n \geq 1$, then the compact integral operator on $C(D)$ is defined as

$$Kf(x) = \int_D G(x,y)f(y)dy, \qquad x \in D, \quad f \in C(D) \tag{1.43}$$

together with $\|.\|_\infty$. where $C(D)$ is the vector space of all continuous functions on D.

Definition 1.22

Let X and Y be vector spaces. The linear operator $K: X \rightarrow Y$ is a finite rank operator if Range (K) is finite dimensional.

Lemma 1.3

Let X and Y be normed linear spaces, and let $K: X \rightarrow Y$ be a bounded finite rank operator. Then K is a compact operator.

Proof: Let $R = Range(K)$. Then R is a normed finite-dimensional space, and therefore it is complete. Consider the set

$$S = \{ K\, x \mid \|x\| \leq 1\} \qquad (1.44)$$

The set S is bounded by $\|K\|$. Also S $\subset$ R. Then S has compact closure, since all bounded closed sets in a finite dimensional space are compact. This shows K is compact.

Lemma 1.4

Let $K \in L\,[X,Y]\ and\ L \in L\,[Y,Z]$,where $L[X,Y]$ denotes the set of linear transformations from X to Y and L[Y, Z] denotes the set of linear transformations from Y to Z, and let K or L (or both) be compact. Then $L\,K$ is compact on X to Z.

Lemma 1.5

Let X and Y be normed linear spaces, with Y complete. Let $K \in L\,[X,Y]$, let $\{K_n\}$ be a sequence of compact operators in $L[X,Y]$, and assume $K_n \rightarrow K\ in\ L\,[X,Y]$, which means

$\|K_n - K\| \rightarrow 0$. Then K is compact.

Proof: Let $\{x_n\}$ be a sequence in X satisfying $\|x_n\| \leq 1, n \geq 1$. We must show that $\{K\,x_n\}$ contains a convergent subsequence. Since K_1 is compact, the sequence $\{K_1 x_n\}$ contains a convergent subsequence. Denote the convergent subsequence by $\{K_1 x_n^{(1)} \mid n \geq 1\}$, and let its limit be denoted by $y_1 \in Y$. For $k \geq 2$, inductively pick a subsequence $\{x_n^{(k)} \mid n \geq 1\} \subset \{x_n^{(k-1)}\}$ such that $\{K_k x_n^{(k)}\}$ converges to a point $y_k \in Y$ Thus,

$$lim_{n\to\infty} K_k\, x_n^{(k)} = yk \quad \text{and } \{x_n{}^{(k)}\} \subset \{x_n{}^{(k-1)}\},\ k \geq 1 \qquad (1.45)$$

We will now choose a special subsequence $\{z_k\} \subset \{x_n\}$ for which $\{K\, z_k\}$ is convergent in Y. Let $z_1 = z_j{}^{(1)}$ for some j, such that $\left\|K_1 x_n^{(1)} - y_1\right\| \leq 1$ for all $n \geq j$. Inductively, for $k \geq 2$, pick $z_k = z_j{}^{(k)}$ for some j, such that z_k is further along in the sequence $\{x_n\}$ than is z_{k-1} and such that

$$\left\|K_k x_n^{(k)} - y_k\right\| \leq \frac{1}{k}, \quad n \geq j \qquad (1.46)$$

The sequence $\{K\, z_k\}$ is a Cauchy sequence in Y. To show this, consider

$$\begin{aligned}\|Kz_{k+p} - Kz_k\| &\leq \|Kz_{k+p} - K_k z_{k+p}\| + \|K_k z_{k+p} - K_k z_k\| \\ &\quad + \|K_k z_k - Kz_k\| \\ &\leq 2\|K - K_k\| + \|K_k z_{k+p} - y_k\| + \|y_k - K_k z_k\| \\ &\leq 2\|K - K_k\| + \frac{2}{k}, \quad p \geq 1 \qquad (147)\end{aligned}$$

noting that $z_{k+p} \in \{x_n{}^{(k)}\}$ for all $p \geq 1$. Use the assumption that $\|K - K_k\| \to 0$ to conclude the proof that $\{K\, z_k\}$ is a Cauchy sequence in Y. Since Y is complete, $\{K\, z_k\}$ is convergent in Y, and this shows that K is compact.

For more details see [4], [16], [19] and [32].

Chapter 2

Analytical methods for solving Fredholm integral equations of the second kind

Chapter 2

Analytical methods for solving Fredholm integral equations of the second kind

In this chapter we will present some important analytical methods for solving the Fredholm integral equations of the second kind, but first we state some theorems about the existence and uniqueness of the solution.

2.1 The existence and uniqueness

Some integral equations has a solution and some other has no solution or that it has an infinite number of solutions, the following theorems state the existence and uniqueness among the solution of Fredholm integral equation of the second kind.

Note: It is important to say that we will discuss the analytical methods in the space $X = [a, b]$ with $\|.\|_\infty$.

Theorem 2.1 (***Fredholm Alternative Theorem***)

If the homogeneous Fredholm integral equation

$$f(x) = \lambda \int_a^b G(x, y) f(y) dy \tag{2.1}$$

has only the trivial solution $f(x) = 0$, then the corresponding nonhomogeneous Fredholm equation

$$f(x) = g(x) + \lambda \int_a^b G(x, y) f(y) dy \tag{2.2}$$

has always a unique solution. This theorem is known by the Fredholm alternative theorem.

Theorem 2.2 (Unique Solution) If the kernel $G\ (x, y)$ in Fredholm integral equation (2.2) is continuous, real valued function, bounded in the square $a \leq x \leq b\ and\ a \leq y \leq b$, and if $g(x)$ is a continuous real valued function, then a necessary condition for the existence of a unique solution for Fredholm integral equation (2.2) is given by

$$|\boldsymbol{\lambda}|\ M\ (b\ -\ a)\ <\ 1, \tag{2.3}$$

where

$$|G(x, y)| \leq\ \mathrm{M}\ \in\ R. \tag{2.4}$$

On the contrary, if the necessary condition (2.3) does not hold, then a continuous solution may exist for Fredholm integral equation.To illustrate this, we consider the Fredholm integral equation

$$f(x) = 2 - 3x + \int_0^1 (3x + y) f(y) dy. \tag{2.5}$$

It is clear that λ = 1, $|G(x, y)|\ \leq\ 4$ and (b − a) = 1. This gives

$$|\boldsymbol{\lambda}|\ M\ (b\ -\ a)\ =\ 4\ \not<\ 1. \tag{2.6}$$

However, the Fredholm equation (2.5) has an exact solution given by

$$f(x)\ =\ 6x. \tag{2.7}$$

2.2 Some analytical methods for solving Fredholm integral equations of the second kind

2.2.1 The degenerate kernel method

In this section, the ***degenerate kernel method*** will be applied to solve the Fredholm integral equations with separable kernels. The method approaches Fredholm integral equations in a direct manner and gives the solution in an exact form and not in a series form, this method will be applied for the degenerate or separable kernels of the form

$$G(x,y) = \sum_{i=1}^{n} u_i\,(x)v_i(y). \tag{2.9}$$

where the functions $u_1(x)$, … , $u_n(x)$ and the functions $v_1(y)$, … , $v_n(y)$ are linearly independent. With such a kernel, the Fredholm integral of the second kind,

$$f(x) = \; g(x) + \lambda \int_a^b G(x,y)f(y)dy \tag{2.10}$$

becomes

$$f(x) = \; g(x) + \lambda \sum_{i=1}^{n} u_i\,(x) \int_a^b v_i(y)f(y)dy. \tag{2.11}$$

The technique of solving this equation is essentially dependent on the choice of the complex parameter λ and on the definition of

$$\alpha_i = \int_a^b v_i(y)f(y)dy. \tag{2.12}$$

each integral at the right side depends only on the variable *y* with constant limits of integration for *y*. This means that each integral is equivalent to a constant. Based on this, substituting (2.12) in (2.11) gives

$$f(x) = g(x) + \lambda \sum_{i=1}^{n} \alpha_i \, u_i(x), \tag{2.13}$$

and the problem reduces to finding the quantities α_i. To do this, we put the value of $f(x)$ as given by (2.13) in (2.11) and get

$$\sum_{i=1}^{n} u_i(x) \left\{ \alpha_i - \int_a^b v_i(y) \left[g(y) + \lambda \sum_{\mathrm{k}=1}^{\mathrm{n}} \alpha_{\mathrm{k}} \, u_{\mathrm{k}}(\mathrm{y}) \right] dy \right\} = 0. \tag{2.14}$$

But the functions $u_{\mathrm{i}}(x)$ are linearly independent, therefore

$$\alpha_i - \int_a^b v_i(y) \left[g(y) + \lambda \sum_{\mathrm{k}=1}^{\mathrm{n}} \alpha_{\mathrm{k}} \, u_{\mathrm{k}}(\mathrm{y}) \right] dy = 0, \qquad i = 1, \dots, n \tag{2.15}$$

Using the simplified notation

$$\int_a^b v_i(y) g(y) dy = h_{i,} \qquad \int_a^b v_i(y) u_k(y) dy = c_{ik}, \tag{2.16}$$

where h_i and c_{ik} are known constants, equation (2.15) becomes

$$\alpha_i - \lambda \sum_{k=1}^{n} c_{ik} \alpha_k = h_i \qquad i = 1, \dots, n \tag{2.17}$$

that is, a system of *n* algebraic equations for the unknowns α_i. The determinant D(λ) of this system is

$$D(\lambda) = \begin{vmatrix} 1-\lambda c_{11} & -\lambda c_{12} & \cdots & -\lambda c_{1n} \\ -\lambda c_{21} & 1-\lambda c_{22} & \cdots & -\lambda c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ -\lambda c_{n1} & -\lambda c_{n2} & & 1-\lambda c_{nn} \end{vmatrix} \qquad (2.18)$$

Which is a polynomial in λ of degree at most *n*. Moreover, it is not identically zero, since, when λ = 0, it reduces to unity.

For all values of λ for which D(λ) ≠ 0, the algebraic system (2.17), and thereby the integral equation (2.10), has a unique solution. On the other hand, for all values of λ for which D(λ) becomes equal to zero, the algebraic system (2.17), and with it the integral equation (2.10), either is insoluble or has an infinite number of solutions. Note that we have considered only the integral equation of the second kind, where alone this method is applicable.

Examples of separable kernels are $x - y, xy, x^2 - y^2, xy^2 + x^2y$, etc.

Example 2.1

To illustrate the above method we consider the following integral equation in the degenerate kernel

$$f(x) = -\frac{2}{\pi}\cos(x) + \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos(x-y)\, f(y)dy. \qquad (2.19)$$

now the kernel $G(x,y) = \cos(x-y)$ can be written as $G(x,y) = cos(x)\, cos(y) + sin(x)\, sin(y)$ which is a separable kernel such that

$$G(x,y) = \sum_{i=0}^{n} u_i(x) v_i(y) \qquad (2.20)$$

Where

$$u_1(x) = \cos(x) \qquad u_2(x) = \sin(x)$$

$$v_1(y) = \cos(y) \qquad v_2(y) = \sin(y) \qquad (2.21)$$

Now using the techniques in section (2.2.1) in one dimension $[a, b]$ and the relations

$$\int_a^b v_i(y)u_k(y)dy = c_{ik} \qquad \int_a^b v_i(y)g(y)dy. = h_{i,} \qquad (2.22)$$

we have

$$\begin{aligned} c_{11} &= \int_0^{\frac{\pi}{2}} v_1(y)u_1(y)dy = \int_0^{\frac{\pi}{2}} \cos(y)\cos(y)dy \\ &= \int_0^{\frac{\pi}{2}} \cos^2(y)\,dy \quad = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \cos(2y)\,dy \qquad (2.23) \\ &= \frac{1}{2}\left[y + \frac{1}{2}\sin(2y)\right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{2}\left(\frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right) - \frac{1}{2}\left(0 + \frac{1}{2}\sin(0)\right) = \frac{\pi}{4} \end{aligned}$$

$$c_{12} = \int_0^{\frac{\pi}{2}} v_1(y)u_2(y)dy = \int_0^{\frac{\pi}{2}} \cos(y)\sin(y)\,dy = c_{21} \qquad (2.24)$$

Using the integration by parts we let $w = sin(y) \rightarrow dw = \cos(y)dy$

Substituting *w* in (2.24) we have

$$\int_0^{\frac{\pi}{2}} wdw \rightarrow \left[\frac{1}{2}u^2\right]_0^{\frac{\pi}{2}} = \left[\frac{1}{2}sin^2(y)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(sin^2\left(\frac{\pi}{2}\right) - sin^2(0)\right) = \frac{1}{2}$$

$$c_{22} = \int_0^{\frac{\pi}{2}} v_2(y)u_2(y)dy = \int_0^{\frac{\pi}{2}} \sin(y)\sin(y)dy$$

$$= \int_0^{\frac{\pi}{2}} \sin^2(y)\, dy$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} -\cos(2y)\, dy \qquad (2.25)$$

$$= \frac{1}{2}\left[y - \frac{1}{2}\sin(2y)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}\sin(\pi)\right) - \frac{1}{2}\left(0 - \frac{1}{2}\sin(0)\right)$$

$$= \frac{\pi}{4}$$

$$h_1 = \int_0^{\frac{\pi}{2}} v_1(y)g(y)dy = \frac{-2}{\pi}\int_0^{\frac{\pi}{2}} cos(y)cos(y)dy$$

Using the relation (2.23)

$$h_1 = \frac{-2}{\pi}.\left(\frac{\pi}{4}\right) = \frac{-1}{2}$$

$$h_2 = \int_0^{\frac{\pi}{2}} v_2(y)g(y)dy = \frac{-2}{\pi}\int_0^{\frac{\pi}{2}} sin(y)cos(y)dy \qquad (2.26)$$

Using the relation (2.24)

$$h_2 = \frac{-2}{\pi}.\left(\frac{1}{2}\right) = \frac{-1}{\pi}$$

Now to find α_i in the relation

$$\alpha_i - \lambda\sum_{k=1}^{n} c_{ik}\alpha_k = h_i \qquad i = 1, \dots, n \qquad (2.27)$$

Where this can be written in the matrix form as

$$\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\lambda\begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{bmatrix}\right)\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\begin{bmatrix}h_1\\ h_2\end{bmatrix}$$

$$\rightarrow\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\frac{4}{\pi}\begin{bmatrix}\frac{\pi}{4} & \frac{1}{2}\\ \frac{1}{2} & \frac{\pi}{4}\end{bmatrix}\right)\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{-1}{\pi}\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}0 & \frac{-2}{\pi}\\ \frac{-2}{\pi} & 0\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{-1}{\pi}\end{bmatrix}\rightarrow\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\frac{\pi^2}{-4}\begin{bmatrix}0 & \frac{2}{\pi}\\ \frac{2}{\pi} & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ \frac{-1}{\pi}\end{bmatrix}\rightarrow\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{\pi}{4}\end{bmatrix}$$

Now using the relation

$$f(x)=g(x)+\lambda\sum_{i=1}^{n}\alpha_i\,u_i(x), \qquad (2.28)$$

$$f(x)=\frac{-2}{\pi}\,cos(x)+\frac{4}{\pi}\left(\frac{1}{2}\right)cos(x)+\frac{4}{\pi}\left(\frac{\pi}{4}\right)\sin(x)$$

then

$f(x)=\sin(x)$ which is the exact solution of the example (2.19).

For more examples see [14], [4], and [28].

2.2.2 Converting Fredholm integral equation to ODE

Before starting the discussion of this method we state the

Leibnitz rule for differentiation of integrals.

Definition 2.1

Let $f(x,y)$ be continuous and $\frac{\partial f}{\partial y}$ be continuous in a domain of the $x-y$ plane that includes the rectangle $a\leq x\leq b, y_0\leq y\leq y_1$, let

$$F(x) = \int_{v(x)}^{u(x)} f(x,y)dy, \tag{2.29}$$

then differentiation of this integral exists and is given by

$$F'(x) = \frac{dF}{dx} = f\big(x,u(x)\big)\frac{du(x)}{dx} - f\big(x,v(x)\big)\frac{dv(x)}{dx} + \int_{v(x)}^{u(x)} \frac{\partial f(x,\ y)}{\partial x}dy. \tag{2.30}$$

If $v(x) = a\ and\ u(x) = b$ where a and b are constants, then the Leibniz rule reduces to

$$F'(x) = \frac{dF}{dx} = \int_a^b \frac{\partial f(x,y)}{\partial x}dy.$$

which means that differentiation and integration can be interchanged such as

$$\frac{d}{dx}\int_a^b e^{xy}dy = \int_a^b ye^{xy}dy. \tag{2.31}$$

It is interested to notice that Leibnitz rule is not applicable for the Abel's singular integral equation:

$$F(x) = \int_0^1 \frac{f(y)}{(x-y)^{\propto}}dy, \qquad 0 <\propto< 1. \tag{2.32}$$

The integrand in this equation does not satisfy the conditions that $f(x,y)$ be continuous and $\frac{\partial f}{\partial y}$ be continuous, because it is unbounded at $x = y$.

Now a well-known method for solving Fredholm integral equations

$$f(x) = g(x) + \lambda \int_a^b G(x,y) f(y) dy \qquad (2.33)$$

Is converting these equations to equivalent boundary value problems. The method is achieved simply by differentiating both sides of Fredholm equations with respect to *x* as many times as we need to get rid of the integral sign and come out with a differential equation. The conversion of Fredholm equations requires the use of Leibnitz rule for differentiating the integral at the right hand side.

The boundary conditions can be obtained by substituting $x = a$, and $x = b$ into $f(x)$. The resulting boundary value problems can be solved easily by using ODEs methods. The conversion process will be illustrated by discussing the following two types of problems:

TypeI: We first consider the Fredholm integral equation given by

$$f(x) = g(x) + \int_0^1 G(x,y)\, f(y) dy, \qquad (2.34)$$

where *g*(*x*) is a given function, and the kernel *G*(*x, y*) is given by

$$G(x,y) = \begin{cases} y(1-x)h(x) \ \ for\ 0 \le y \le x, \\ x(1-y)h(x) \ \ for\ x \le y \le 1. \end{cases} \qquad (2.35)$$

For simplicity reasons, we may consider $h(x) = \lambda$ where λ is constant. Equation (2.34) can be written as

$$f(x) = g(x) + \lambda \int_0^x y(1-x)\, f(y)dy + \lambda \int_x^1 x(1-y)\, f(y)dy, \quad (2.36)$$

or equivalently

$$f(x) = g(x) + \lambda(1-x) \int_0^x y\, f(y)dy + \lambda x \int_x^1 (1-y)\, f(y)dy, \quad (2.37)$$

Each term of the last two terms at the right side of (2.37) is a product of two functions of *x*. differentiating both sides of (2.37), using the product rule of differentiation and using Leibnitz rule we obtain

$$f'(x) = g'(x) + \lambda x(1-x)f(x) - \lambda \int_0^x yf(y)d(y) - \lambda x(1-x)f(x) + \lambda \int_x^1 (1-y)f(y)d(y),$$

$$= g'(x) - \lambda \int_0^x yf(y)d(y) + \lambda \int_x^1 (1-y)f(y)d(y), \quad (2.38)$$

To get rid of integral signs, we differentiate both sides of (2.38) again with respect to *x* to find that

$$f''(x) = g''(x) - \lambda x f(x) - \lambda(1-x)f(x) \quad (2.39)$$

that gives the ordinary differential equations

$$f''(x) + \lambda f(x) = g''(x). \quad (2.40)$$

The related boundary conditions can be obtained by substituting $x = 0$ and $x = 1$ in (2.37) to find that

$$f(0) = g(0), f(1) = g(1). \quad (2.41)$$

Combining (2.40) and (2.41) gives the boundary value problem equivalent to the Fredholm equation (2.34). Moreover, if $h(x)$ is not a constant, we can proceed in a manner similar to the discussion presented above to obtain the boundary value problem. The technique above for *type I* will be explained by studying the following examples.

Example 2.2

To illustrate the above method we consider the following integral equation

$$f(x) = e^x + \int_0^1 G(x,y)\, f(y)dy, \tag{2.42}$$

where the kernel $\mathrm{G}(x,\mathrm{y})$ given by

$$\mathrm{G}(x,\mathrm{y}) = \begin{cases} 9y(1-x) & for\ 0 \le y \le x, \\ 9x(1-y) & for\ x \le y \le 1. \end{cases} \tag{2.43}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$f(x) = e^x + 9(1-x)\int_0^x y\, f(y)dy + 9x\int_x^1 (1-y)\, f(y)dy, \tag{2.44}$$

differentiating (2.44) twice with respect to *x* gives

$$f'(x) = e^x - 9\int_0^x yf(y)d(y) + 9\int_x^1 (1-y)f(y)d(y) \tag{2.45}$$

and

$$f''(x) = e^x - 9f(x) \tag{2.46}$$

this in turn gives the ODE

$$f''(x) + 9f(x) = e^x, \tag{2.47}$$

the related boundary conditions are given by

$$f(0) = g(0) = 1, \quad f(1) = g(1) = e, \tag{2.48}$$

obtained upon substituting $x = 0$ and $x = 1$ into (2.35).

Type II:

We next consider the Fredholm integral equation given by

$$f(x) = g(x) + \int_0^1 G(x,y)\, f(y)dy, \tag{2.49}$$

where $g(x)$ is a given function, and the kernel $G(x,y)$ is given by

$$G(x,y) = \begin{cases} yh(x) & for\ 0 \le y \le x, \\ xh(x) & for\ x \le y \le 1. \end{cases} \tag{2.50}$$

For simplicity reasons, we may consider $h(x) = \lambda$ where λ is constant. Equation (2.49) can be written as

$$f(x) = g(x) + \lambda \int_0^x y\, f(y)dy + \lambda x \int_x^1 f(y)\, dy, \tag{2.51}$$

each term of the last term at the right side of (2.51) is a product of two functions of x. differentiating both sides of (2.51), using the product rule of differentiation and using Leibnitz rule we obtain

$$f'(x) = g'(x) + \lambda \int_x^1 f(y)d(y), \tag{2.52}$$

to get rid of integral signs, we differentiate again with respect to *x* to find that

$$f''(x) = g''(x) - \lambda f(x), \tag{2.53}$$

that gives the ordinary differential equations

$$f''(x) + \lambda f(x) = g''(x). \tag{2.54}$$

Notice that the boundary condition $f(1)$ in this case cannot be obtained from (2.51). Therefore, the related boundary conditions can be obtained by substituting $x = 0$ and $x = 1$ in (2.51) and (2.52) respectively to find that

$$f(0) = g(0), f'(1) = g'(1), \tag{2.55}$$

combining (2.54) and (2.55) gives the boundary value problem equivalent to the Fredholm equation (2.49). Moreover, if $h(x)$ is not a constant, we can proceed in a manner similar to the discussion presented above to obtain the boundary value problem. The approach presented above for *type II* will be illustrated by studying the following example.

Example 2.3

Consider the following Fredholm integral equation

$$f(x) = e^x + \int_0^1 G(x,y)\, f(y)dy, \tag{2.56}$$

where the kernel $G(x,y)$ given by

$$G(x,y) = \begin{cases} 4y & for\ 0 \le y \le x, \\ 4x & for\ x \le y \le 1. \end{cases} \tag{2.57}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$f(x) = e^x + 4\int_0^x y\, f(y)dy + 4x\int_x^1 f(y)\, dy, \qquad (2.58)$$

differentiating (2.58) twice with respect to *x* gives

$$f'(x) = e^x + 4\int_x^1 f(y)d(y), \qquad (2.59)$$

and

$$f''(x) = e^x - 4f(x), \qquad (2.60)$$

that gives the ordinary differential equations

$$f''(x) + 4f(x) = e^x, \qquad (2.61)$$

the related boundary conditions are given by

$$f(0) = g(0) = 1, \qquad f'(1) = g'(1) = e, \qquad (2.62)$$

obtained upon substituting $x = 0$ and $x = 1$ into (2.58) and (2.59) respectively. Recall that the boundary condition $f(1)$ cannot obtained in this case. For more details see [50].

2.2.3 The Adomain decomposition method, [50], section 4.2.1, page 121.

The Adomian decomposition method (ADM) was introduced and developed by George Adomian [1]. It consists of decomposing the

unknown function $f(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$f(x) = \sum_{n=0}^{\infty} f_n(x), \tag{2.63}$$

or equivalently

$$f(x) = f_0(x) + f_1(x) + f_2(x) + \cdots, \tag{2.64}$$

where the components $f_n(x), n \geq 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components $f_0, f_1, f_2, \ldots$ individually. The determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated. To establish the recurrence relation, we substitute (2.63) into the Fredholm integral equation

$$f(x) = g(x) + \lambda \int_a^b G(x,y) f(y) dy, \tag{2.65}$$

to obtain

$$\sum_{n=0}^{\infty} f_n(x) = g(x) + \lambda \int_a^b G(x,y) \left(\sum_{n=0}^{\infty} f_n(y) \right) dy, \tag{2.66}$$

or equivalently

$$f_0(x) + f_1(x) + f_2(x) + \cdots = g(x) + \lambda \int_a^b G(x,y)[f_0(y) + f_1(y) + f_2(y) + \cdots] dy \tag{2.67}$$

The zeroth component $f_0(x)$ is identified by all terms that are not included under the integral sign. This means that the components $f_j(x), j \geq 0$ of the unknown function $f(x)$ are completely determined by setting the recurrence relation

$$f_0(x) = g(x), \qquad f_{n+1}(x) = \lambda \int_a^b G(x,y) f_n(y) dy, \quad n \geq 0, \qquad (2.68)$$

or equivalently

$$\begin{aligned} f_0(x) &= g(x), \\ f_1(x) &= \lambda \int_a^b G(x,y) f_0(y)\, dy, \\ f_2(x) &= \lambda \int_a^b G(x,y) f_1(y)\, dy, \\ f_3(x) &= \lambda \int_a^b G(x,y) f_2(y)\, dy, \end{aligned} \qquad (2.69)$$

and so on for other components. As a result the components $f_0(x), f_1(x), f_2(x), f_3(x), \ldots$ are completely determined. As a result, the solution $f(x)$ of the Fredholm integral equation (2.65) is readily obtained in a series form by using the series assumption in (2.63).

The decomposition method converted the integral equation into an elegant determination of computable components, if an exact solution exists for the problem, then the obtained series converges very rapidly to that exact solution. However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for

numerical purposes. The more components we use the higher accuracy we obtain.

Example 2.4

To illustrate the above method we consider the following integral equation which is shown in [50]

$$f(x) = e^x - x + x\int_0^1 yf(y)dy. \tag{2.70}$$

The Adomian decomposition method assumes that the solution $f(x)$ has a series form given in (2.63). Substituting the decomposition series (2.63) into both sides of (2.70) gives

$$\sum_{n=0}^{\infty} f_n(x) = e^x - x + x\int_0^1 y\sum_{n=0}^{\infty} f_n(y)dy. \tag{2.71}$$

or equivalently

$$f_0(x) + f_1(x) + f_2(x) + \cdots = e^x - x + x\int_a^b y[f_0(y) + f_1(y) + f_2(y) + \cdots]\,dy \tag{2.72}$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation

$$f_0(x) = e^x - x, \qquad f_{k+1}(x) = x\int_0^1 yf_k(y)dy, \qquad k \geq 0 \tag{2.73}$$

Consequently, we obtain

$$f_0(x) = e^x - x,$$

$$f_1(x) = x\int_0^1 yf_0(y)dy = x\int_0^1 y(e^y - y)dy = \frac{2}{3}x,$$

$$f_2(x) = x\int_0^1 yf_1(y)dy = x\int_0^1 \frac{2}{3}y^2dy = \frac{2}{9}x,$$

$$f_3(x) = x\int_0^1 yf_2(y)dy = x\int_0^1 \frac{2}{9}y^2dy = \frac{2}{27}x,$$

$$f_4(x) = x\int_0^1 yf_3(y)dy = x\int_0^1 \frac{2}{27}y^2dy = \frac{2}{81}x, \qquad (2.74)$$

and so on. Using (2.63) gives the series solution

$$f(x) = e^x - x + \frac{2}{3}x\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right) \qquad (2.75)$$

Notice that the infinite geometric series at the right side has $a_1 = 1$, and the ratio $r = \frac{1}{3}$. The sum of the infinite series is therefore given by

$$\mathrm{S} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}. \qquad (2.76)$$

The series solution (2.75) converges to the closed form solution

$$f(x) = e^x, \qquad (2.77)$$

obtained upon using (2.76) into (2.75).

2.2.4 The Modified Decomposition Method, [50], section 4.2.2, page 128.

As shown before, the Adomian decomposition method provides the solution in an infinite series of components. The components $f_j, j \geq 0$ are

easily computed if the inhomogeneous term $g(x)$ in the Fredholm integral equation:

$$f(x) = g(x) + \lambda \int_a^b G(x,y)\, f(y)dy, \qquad (2.78)$$

consists of a polynomial. However, if the function $g(x)$ consists of a combination of two or more of polynomials, trigonometric functions, hyperbolic functions, and others, the evaluation of the components $f_j, j \geq 0$ requires cumbersome work. A reliable modification of the Adomian decomposition method was developed by Wazwaz [51]. The modified decomposition method will facilitate the computational process and further accelerate the convergence of the series solution. The modified decomposition method will be applied, wherever it is appropriate, to all integral equations and differential equations of any order. It is interesting to note that the modified decomposition method depends mainly on splitting the function $g(x)$ into two parts; therefore it cannot be used if the function $g(x)$ consists of only one term.

To give a clear description of the technique, we recall that the standard Adomian decomposition method admits the use of the recurrence relation:

$$f_0(x) = g(x),$$

$$f_{k+1}(x) = \lambda \int_a^b G(x,y) f_k(y)dy, \quad k \geq 0, \qquad (2.79)$$

where the solution $f(x)$ is expressed by an infinite sum of components defined by

$$f(x) = \sum_{n=0}^{\infty} f_n(x), \tag{2.80}$$

in view of (2.79), the components $f_n(x), n \geq 0$ can be easily evaluated.

The modified decomposition method introduces a slight variation to the recurrence relation (2.79) that will lead to the determination of the components of $f(x)$ in an easier and faster manner. For many cases, the function $g(x)$ can be set as the sum of two partial functions, namely $g_1(x)$ and $g_2(x)$. In other words, we can set

$$g(x) = g_1(x) + g_2(x), \tag{2.81}$$

in view of (2.81), we introduce a qualitative change in the formation of the recurrence relation (2.79). To minimize the size of calculations, we identify the zeroth component $f_0(x)$ by one part of $g(x)$, namely $g_1(x)$ or $g_2(x)$. The other part of $g(x)$ can be added to the component $f_1(x)$ that exists in the standard recurrence relation (2.79). In other words, the modified decomposition method introduces the modified recurrence relation:

$$f_0(x) = g_1(x),$$

$$f_1(x) = g_2(x) + \lambda \int_a^b G(x,y) f_0(y)\, dy,$$

$$f_{k+1}(x) = \lambda \int_a^b G(x,y) f_k(y)\, dy, k \geq 1. \tag{2.82}$$

This shows that the difference between the standard recurrence relation (2.79) and the modified recurrence relation (2.82) rests only in the

formation of the first two components $f_0(x)$ and $f_1(x)$ only. The other components $f_j, j \geq 2$ remain the same in the two recurrence relations. Although this variation in the formation of $f_0(x)$ and $f_1(x)$ is slight, however it plays a major role in accelerating the convergence of the solution and in minimizing the size of computational work. Moreover, reducing the number of terms in $g_1(x)$ affects not only the component $f_1(x)$, but also the other components as well. This result was confirmed by several research works as in [52].

Two important remarks related to the modified method can be made

(*i*) by proper selection of the functions $g_1(x)$ and $g_2(x)$, the exact solution $f(x)$ may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this modification depends only on the proper choice of $g_1(x)$ and $g_2(x)$and this can be made through trials only.

(*ii*) if $g(x)$ consists of one term only, the standard decomposition method can be used in this case.

It is worth mentioning that the modified decomposition method will be used for Volterra and Fredholm integral equations, linear and nonlinear equations. The modified decomposition method will be illustrated by discussing the following example.

Example 2.5

Consider solving the Fredholm integral equation by using the modified decomposition method

$$f(x) = \frac{1}{1+x^2} - 2sinh\frac{\pi}{4} + \int_{-1}^{1} e^{\arctan y} f(y)dy. \qquad (2.83)$$

We first decompose $g(x)$ given by

$$g(x) = \frac{1}{1+x^2} - 2sinh\frac{\pi}{4}, \qquad (2.84)$$

into two parts, namely

$$g_1(x) = \frac{1}{1+x^2}, \qquad g_2(x) = -2sinh\frac{\pi}{4}. \qquad (2.85)$$

We next use the modified recurrence formula (2.82) to obtain

$$f_0(x) = g_1(x) = \frac{1}{1+x^2},$$

$$f_1(x) = -2sinh\frac{\pi}{4} + \int_{-1}^{1} e^{\arctan y} f_0(y)dy = 0,$$

$$f_{k+1}(x) = \int_{-1}^{1} e^{\arctan y} f_k(y)dy = 0, \quad k \geq 1 \qquad (2.86)$$

It is obvious that each component of $f_j, j \geq 1$ is zero. This in turn gives the exact solution by

$$f(x) = \frac{1}{1+x^2}. \qquad (2.87)$$

For more details see [44], and [50].

2.2.5 The method of successive approximations

The successive approximation method provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess as $f_0(x)$, called the zeroth approximation which can be any real valued function $f_0(x)$, that will be used in a recurrence relation to determine the other approximations.

Given the Fredholm integral equations of the second kind

$$f(x) = g(x) + \lambda \int_a^b G(x,y) f(y) dy \qquad (2.88)$$

and according to the choice of $f_0(x)$ there is two methods of successive approximations:

i) ***The Picard's method:*** is obtained when $f_0(x) = 0, 1, or\ x$, or any real-valued function, where $a \leq x \leq b$. Accordingly, the first approximation $f_1(x)$ of the solution of $f(x)$ is defined by

$$f_1(x) = g(x) + \lambda \int_a^b G(x,y) f_0(y) dy, \qquad (2.89)$$

The second approximation $f_2(x)$ of the solution $f(x)$ can be obtained by replacing $f_0(x)$ in equation (2.88) by the previously obtained $f_1(x)$; hence we find

$$f_2(x) = g(x) + \lambda \int_a^b G(x,y) f_1(y) dy, \qquad (2.90)$$

This process can be continued in the same manner to obtain the *nth* approximation. In other words, the various approximations can be put in a recursive scheme given by

$$f_0(x) = \text{any selective real valued function}$$

$$f_n(x) = g(x) + \lambda \int_a^b G(x,y) f_{n-1}(y) dy, \quad n \geq 1 \qquad (2.91)$$

Even though we can select any real-valued function for the zeroth approximation $f_0(x)$, the most commonly selected functions for $f_0(x)$are $f_0(x) = 0, 1, or\ x$. Notice that with the selection of $f_0(x) = 0$, the first approximation $f_1(x) = g(x)$.

The final solution *f*(*x*) is obtained by

$$f(x) = \lim_{n \to \infty} f_n(x) \qquad (2.92)$$

so that the resulting solution $f(x)$ is independent of the choice of $f_0(x)$.

Example 2.6

We consider solving the Fredholm integral equation by using the successive approximations method

$$f(x) = x + e^x - \int_0^1 xyf(y) dy. \qquad (2.93)$$

for the zeroth approximation $f_0(x)$, we can select

$$f_0(x) = 0, \tag{2.94}$$

the method of successive approximations admits the use of the iteration formula

$$f_{n+1}(x) = x + e^x - \int_0^1 xyf_n(y)dy, \quad n \geq 0. \tag{2.95}$$

substituting (2.94) into (2.95) we obtain

$$f_1(x) = x + e^x - \int_0^1 xyf_0(y)dy = e^x + x,$$

$$f_2(x) = x + e^x - \int_0^1 xyf_1(y)dy = e^x - \frac{1}{3}x,$$

$$f_3(x) = x + e^x - \int_0^1 xyf_2(y)dy = e^x + \frac{1}{9}x,$$

$$f_{n+1}(x) = x + e^x - \int_0^1 xyf_n(y)dy = e^x + \frac{(-1)^n}{3^n}x.$$

consequently, the solution $f(x)$ of (2.93) is given by

$$f(x) = \lim_{n\to\infty} f_{n+1}(x) = e^x. \tag{2.96}$$

For more details and examples see [14].

ii)***The Neumann series method:*** is obtained when $f_0(x) = g(x)$, in the other words all terms that are not included under the integral sign such that

$$f_1(x) = g(x) + \lambda \int_a^b G(x,y)f_0(y)dy \tag{2.97}$$

$$= \ g(x) + \lambda \int_a^b G(x,y)g(y)dy$$

$$= g(x) + \lambda\varphi_1(x) \tag{2.98}$$

where

$$\varphi_1(x) = \int_a^b G(x,y)g(y)dy \tag{2.99}$$

The second approximation $f_2(x)$ can be obtained as

$$f_2(x) = \ g(x) + \lambda \int_a^b G(x,y)f_1(y)dy$$

$$= g(x) + \ \lambda \int_a^b G(x,y)\{g(y) + \lambda\varphi_1(y)\}dy$$

$$= g(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) \tag{2.100}$$

where

$$\varphi_2(x) = \int_a^b G(x,y)\varphi_1\,(y)dy \tag{2.101}$$

Proceeding in this manner, the final solution *f*(*x*) can be obtained

$$f(x) = g(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots + \lambda^n\varphi_n(x) + \cdots$$

$$= g(x) + \sum_{n=1}^{\infty} \lambda^n\varphi_n(x), \tag{2.102}$$

where

$$\varphi_n(x) = \int_a^b G(x,y)\varphi_{n-1}\,dy \qquad n \geq 1 \tag{2.103}$$

Series (2.102) is known as Neumann series. This infinite series is absolutely and uniformly convergent, since

$$|\lambda| < \frac{1}{B}, \qquad B = \sqrt{\int_a^b \int_a^b G^2(x,y)dxdy},$$

if in addition we have

$$\int_a^b G^2(x,y)dy \leq A, \qquad a \leq x \leq b,$$

where A is a constant, then the Neumann series converges absolutely and uniformly on $[a,b]$.

The final solution $f(x)$ is obtained by

$$f(x) = g(x) + \lim_{n\to\infty} \sum_{k=1}^{n} \lambda^k \varphi_k(x). \qquad (2.104)$$

For more details see [16], [44] and [50].

Example 2.7

Consider solving the Fredholm integral equation

$$f(x) = 1 + \int_0^1 xf(y)dy$$

by using the successive approximation method (***The Neumann series method***).

For solution let us consider the zeroth approximation is $f_0(x) = g(x) = 1$, and then the first approximation can be computed as

$$f_1(x) = 1 + \int_0^1 x f_0(y)dy$$

$$= 1 + \int_0^1 x dy$$

$$= 1 + x$$

Proceeding in this manner, we find

$$f_2(x) = 1 + \int_0^1 x f_1(y)dy$$

$$= 1 + \int_0^1 x(1 + y)dy$$

$$= 1 + x\left(1 + \frac{1}{2}\right)$$

In the same way, the third approximation is

$$f_3(x) = 1 + x\int_0^1 \left(1 + \frac{3y}{2}\right)dy$$

$$= 1 + x\left(1 + \frac{1}{2} + \frac{1}{4}\right)$$

Thus, we get

$$f_n(x) = 1 + x\left\{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}\right\}$$

and hence

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= 1 + \lim_{n\to\infty} x \sum_{k=0}^{n} \frac{1}{2^k}$$

$$= 1 + x\left(1 - \frac{1}{2}\right)^{-1}$$

$$= 1 + 2x$$

This is the desired solution.

Chapter 3

Numerical methods for solving Fredholm integral equations of the second kind

Chapter 3

Numerical Methods for Solving Fredholm Integral Equations of the Second Kind

There are many methods for solving integral equations numerically. Here we are interested with the following numerical methods:

(*i*) ***Degenerate kernel approximation methods***

(*ii*) ***Projection methods***

(*iii*) ***Nyström methods (also called quadrature methods)***

All of these methods have iterative variants. There are other numerical methods, but these methods and their variants include the most popular general methods.

3.1 Degenerate kernel approximation methods

We discussed the degenerate kernel method as an analytical method in chapter two (2.2.1) for solving the Fredholm integral equation

$$f(x) = g(x) + \lambda \int_D G(x,y) f(y) dy, \quad x \in \mathrm{D} \qquad (3.1)$$

with $\lambda \neq 0$ *and* $D \subset \mathbf{R}^{\mathrm{m}}$, for some $\mathrm{m} \geq 1$. where D is a closed and bounded set.

We said that the kernel $G(x,y)$ is degenerate (or separable) if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of *x* only and a function of y only such that

$$G(x,y) = \sum_{i=1}^{n} u_i\,(x)v_i(y). \tag{3.2}$$

but most kernel functions $G(x,y)$ are not degenerate, so that in this chapter we seek to approximate them by degenerate kernels.

3.1.1 The solution of the integral equation by the degenerate kernel method

In the view of the integral equation (3.1), the kernel function $G(x,y)$ is to be approximated by a sequence of degenerate kernel functions,

$$G_n(x,y) = \sum_{i=1}^{n} u_{i,n}(x)v_{i,n}(y), \quad n \geq 1 \tag{3.3}$$

in such a way that the associated integral operators K_n satisfy

$$\lim_{n\to\infty} \|K - K_n\| = 0 \tag{3.4}$$

where the associated integral operator is defined as

$$K_n f(x) = \int_D G_n(x,y)f(y)dy, \;\; x \in D, \;\; f \in C(D), \quad n \geq 1 \tag{3.5}$$

where D is a closed bounded set in R^m, for some $m \geq 1$, and using $X = C(D)$ with $\|.\|_\infty$, such that $K: C(D) \to C(D)$ is compact.

We can write the integral equation (3.1) in the operator form as

$$(I - \lambda K)f = g \tag{3.6}$$

then (3.6)can be written using (3.5) as

$$(I - \lambda K_n)f_n = g \tag{3.7}$$

Where f_n is the solution of the approximating equation. Using the formula (3.3) for $G_n(x, y)$, the integral equation (3.7) becomes

$$f_n(x) = g(x) + \lambda \sum_{i=1}^{n} u_{i,n}(x) \int_D v_{i,n}(y) f_n(y) dy,$$

And using the technique discussed in section (2.2.1) we have

$$f_n(x) = g(x) + \lambda \sum_{i=1}^{n} \alpha_i u_i(x), \tag{3.9}$$

where

$$\alpha_i - \lambda \sum_{k=1}^{n} c_{ik}\alpha_k = h_i \qquad i = 1, \dots, n; \tag{3.10}$$

such that

$$h_i = \int v_i(y) g(y) dy, \tag{3.11}$$

and

$$c_{ik} = \int v_i(y) u_k(y) dy, \tag{3.12}$$

are known constants. Again as we stated in section 2.2.1 equation (3.10) represents a system of *n* algebraic equations for the unknowns α_i whose determinant $D(\lambda)$ is given by

$$D(\lambda) = \begin{vmatrix} 1-\lambda c_{11} & -\lambda c_{12} & \cdots & -\lambda c_{1n} \\ -\lambda c_{21} & 1-\lambda c_{22} & \cdots & -\lambda c_{2n} \\ \vdots & \vdots & & \vdots \\ -\lambda c_{n1} & -\lambda c_{n2} & \cdots & 1-\lambda c_{nn} \end{vmatrix} \qquad (3.13)$$

which is a polynomial in λ of degree at most n , that is not identically zero.

To analyze the solution of (3.1) by the degenerate kernel method the following situations arise:

Situation I : when at least one right member of the system (3.9) $h_1, h_2, \ldots, h_n$ is non zero, the following two cases arise under this situation

(i) if $D(\lambda) \neq 0$, then a unique non zero solution of system (3.10) exists and so (3.1) has unique non zero solution given by (3.9).

(ii) if $D(\lambda) = 0$,then the system (3.10) have either no solution or they possess infinite solution and hence (3.1) has either no solution or infinite solution.

Situation II: when $g(x) = 0$, then (3.11) shows that $h_i = 0$ for $i = 1,2, \ldots, n$. Hence the system (3.10) reduces to a system of homogenous linear equation .The following two cases arises under this situation

(i) if $D(\lambda) \neq 0$,then a unique zero solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ of the system (3.10) exists and so we see that (3.1) has unique zero solution $f_n(x) = 0$.

(ii) if $D(\lambda) = 0$,then the system (3.10) posses infinite non zero solutions and so (3.1) has infinite non zero solutions , those value of λ for which $D(\lambda) = 0$ are known as the eigenvalues and any nonzero solution of the homogenous Fredholm integral equation $f(x) = \int_D G(x,y)f(y)dy,$ is known as a corresponding eigenfunction of integral equation .

Situation III: when $g(x) \neq 0$ but

$$\int_D g(y)v_1(y) = 0, \int_D g(y)v_2(y) = 0, \dots, \int_D g(y)v_n(y) = 0 \qquad (3.14)$$

that is $g(x)$ is orthogonal to all the functions

$$v_1(y), v_2(y), \dots, v_n(y), \qquad (3.15)$$

then

$h_1, h_2, \dots, h_n$ are zeros and reduces (3.11) to a system of homogenous linear equations. The following two cases arise under this situation

(i) If $D(\lambda) \neq 0$,then a unique zero solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, and hence (3.1) has only unique solution $f_n(x) = 0$.

(ii) If $D(\lambda) = 0$ then the system (3.10) possess infinite nonzero solutions and he (3.1) has infinite nonzero solutions.

For more details, see [15], [20] and [39]

By returning to the approximating of the kernel which is not degenerate so as to have degenerate one, We use different approximations to approximate the solution of the integral equation (3.1) such as

Taylor series approximation

- ***Interpolatory degenerate kernel approximations***
- ***Orthonormal expansions***

Here we will discuss Taylor series approximation only.

3.1.2 Taylor series approximation, [4], section 2.2, page 29.

Let $f(x, y)$ is a continuous function of two variables *x* and y, then the Taylor series expansion of function *f* at the neighborhood of any real number *a* with respect to the variable y is :

$$Taylor\ (f, y, a) = \sum_{n=0}^{\infty} \frac{(y-a)^n}{n!} \frac{\partial^n}{\partial y^n} f(x, a) \qquad (3.16)$$

and

$$Taylor\ (f, y, m, a) = \sum_{n=0}^{m} \frac{(y-a)^n}{n!} \frac{\partial^n}{\partial y^n} f(x, y = a) \qquad (3.17)$$

that mean the mth terms of Taylor expansion to the function at the neighborhood a with respect to the variable y.

Consider the one-dimensional integral equation

$$f(x) = g(x) + \lambda \int_a^b G(x,y) f(y) dy, \quad a \le x \le b \tag{3.18}$$

we can write $G(x,y)$ as a power series in y using Taylor (G, y, a), then

$$G(x,y) = \sum_{i=0}^{\infty} q_i(x)(y-a)^i \tag{3.19}$$

or a power series in *x* using Taylor (G, x, a), then

$$G(x,y) = \sum_{i=0}^{\infty} q_i(y)(x-a)^i \tag{3.20}$$

Assume $G_n(x,y)$ denote the partial sum of the first *n* terms on the right side of (3.19),

$$G_n(x,y) = \sum_{i=0}^{n-1} q_i(x)(y-a)^i \tag{3.21}$$

Using the notation of (3.2), $G_n(x,y)$ is a degenerate kernel with

$$u_i(x) = q_{i-1}(x), \qquad v_i(y) = (y-a)^{i-1}, \qquad i = 1, 2, \dots, n \tag{3.22}$$

The linear system (3.14) with (3.13) becomes

$$\propto_i - \lambda \sum_{k=1}^{n} \propto_k \int_a^b (y-a)^{i-1} q_{k-1}(y) dy = \int_a^b g(y)(y-a)^{i-1} dy, \qquad i = 1, \dots, n; \tag{3.23}$$

and the solution f_n is given by

$$f_n(x) = g(x) + \lambda \sum_{i=0}^{n-1} \propto_{i+1} g_i(x) \tag{3.24}$$

The integrals in (3.23) are calculated numerically, However, the following remarks are necessary:

(i) The integrals involve the entire interval [a, b].

(ii) Most of the integrands will be zero or quite small, in the neighborhood of $y = a$, the left end of the interval.

For more details see [4], [6], [20] and [46].

3.2 Projection methods

With all projection methods, we consider solving (3.1) within the framework of some complete function space , usually $C(D)$ or $L^2(D)$.We choose a sequence of finite-dimensional approximating subspaces $X_n \subseteq X, n \geq 1$, with X_n having dimension k_n. Let X_n have a basis $\{\varphi_1, \ldots, \varphi_k\}$, with $k \equiv k_n$ for notational simplicity. We seek a function $f_n \in X_n$, which can be written as

$$f_n(x) = \sum_{j=1}^{k_n} c_j \varphi_j(x), \qquad x \in D. \tag{3.25}$$

This is substituted into (3.1), and the coefficients $\{c_1, \ldots, c_k\}$ are determined by forcing the equation to be almost exact in some sense. For later use, introduce

$$r_n(x) = f_n(x) - \lambda \int_D G(x,y) f_n(y) dy - g(x)$$

$$= \sum_{j=1}^{k} c_j \left\{ \varphi_j(x) - \lambda \int_D G(x,y) \varphi_j(y) dy \right\} - g(x), \tag{3.26}$$

for $x \in D$. This is called the residual in the approximation of the equation when using $f \approx f_n$. Now, we write (3.1) in operator notation as

$$(I - \lambda K)f = g. \tag{3.27}$$

Then the residual can be written as

$$r_n = (I - \lambda K)f_n - g.$$

The coefficients $\{c_1, \ldots, c_k\}$ are chosen by forcing $r_n(x)$ to be approximately zero in some sense. The hope, and expectation, is that the resulting function $f_n(x)$ will be a good approximation of the true solution $f(x)$. For more details see [4], [26] and [35].

We have different types of projection methods. The most popular of these are

- ***collocation methods***.
- ***Galerkin methods***.

Before discussing these methods we illustrate this theoretical framework.

3.2.1 Theoretical framework

3.2.1.1 Lagrange polynomial interpolation

Let f be a continuous function defined on a finite closed interval $[a, b]$. Let

$$\Delta: a \leq x_0 < x_1 < \cdots < x_n \leq b$$

be a partition of the interval $[a, b]$. Choose $X = C[a, b]$, the space of continuous functions $f : [a, b] \to F$; (where F is real or complex) and choose X_{n+1} to be P_n, the space of the polynomials of degree less than or equal to n. Then the Lagrange interpolant of degree n of f is defined by the conditions

$$p_n(x_i) = f(x_i), \qquad 0 \le i \le n, \qquad p_n \in P_n. \tag{3.28}$$

Here the interpolation linear functionals are

$$L_i f = f(x_i) \qquad 0 \le i \le n. \tag{3.29}$$

If we choose the regular basis $v_j(x) = x_j$ ($0 \le j \le$ n) for P_n, then it can be shown that

$$\det\left(L_i v_j\right)_{(\mathrm{n}+1)\times(\mathrm{n}+1)} = \prod_{j>i}\left(x_j - x_i\right) \neq 0. \tag{3.30}$$

Thus there exists a unique Lagrange interpolation polynomial.

Furthermore, we have the representation formula

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\varphi_i(x), \qquad \varphi_i(x) \equiv \prod_{j\neq i}\frac{x - x_j}{x_i - x_j}, \tag{3.31}$$

called Lagrange's formula for the interpolation polynomial. The functions φ_i satisfy the special interpolation conditions

$$\varphi_i(x_j) = \delta_{ij} = \begin{cases} 0, i \neq j, \\ 1, i = j, \end{cases} \tag{3.32}$$

The functions $\{\varphi_i\}_{i=0}^{n}$ form a basis for P_n, and they are often called Lagrange basis functions.

Theorem 3.1 The following statements are equivalent:

1. The interpolation problem has a unique solution.

2. The functionals $L_1, \ldots, L_n$ are linearly independent over X_n.

3. The only element $f_n \in X_n$ satisfying

$$L_i f_n = 0, \qquad 1 \le i \le n,$$

is $f_n = 0$.

4. For any data $\{b_i\}_{i=1}^{n}$, there exists one $f_n \in X_n$ such that

$$L_i f_n = b_i, \qquad 1 \le i \le n. \tag{3.33}$$

Outside of the framework of Theorem 3.1, the formula (3.31) shows directly the existence of a solution to the Lagrange interpolation problem (3.28). The uniqueness result can also be proved by showing that the interpolant corresponding to the homogeneous data is zero.

let $p_n \in Pn$ with $p_n(x_i) = 0, \ 0 \le i \le n$. Then the polynomial p_n must contain the factors $(x - x_i), 1 \le i \le n$. Since deg $(p_n) \le n$ and

$$deg \prod_{i=1}^{n} (x - x_i) = n$$

we have

$$p_n(x) = c\prod_{i=1}^{n}(x - x_i) \tag{3.34}$$

for some constant c. Using the condition $p_n(x_0) = 0$, we see that c = 0 and therefore, $p_n \equiv 0$. We note that by Theorem 3.1, this result on the uniqueness of the solvability of the homogeneous problem also implies the existence of a solution.

In the above, we have indicated three methods for showing the existence and uniqueness of a solution to the interpolation problem (3.28). The method based on showing the determinant of the coefficient is nonzero, as in (3.30), this can be done easily only in simple situations such as Lagrange polynomial interpolation. Usually it is simpler to show that the interpolant corresponding to the homogeneous data is zero, even for complicated interpolation conditions. For practical calculations, it is also useful to have a representation formula that is the analogue of (3.31), but such a formula is sometimes difficult to find. For more details see [6].

3.2.1.2 Projection operators

Definition 3.1 Let X be a linear space, X_1 and X_2 subspaces of X. We say X is the direct sum of X_1 and X_2 and write $X = X_1 \oplus X_2$, if any element $v \in X$ can be uniquely decomposed as

$$v = v_1 + v_2, \qquad v_1 \in X_1, \qquad v_2 \in X_2. \tag{3.35}$$

Furthermore, if X is an inner product space, and $\langle v_1, v_2 \rangle = 0$ for any $v_1 \in X_1$ and any $v_2 \in X_2$, then X is called the orthogonal direct sum of X_1 and X_2.

There exists a one-to-one correspondence between direct sums and linear operators P satisfying $P^2 = P$.

Proposition 3.2 Let V be a linear space. Then $X = X_1 \oplus X_2$ if and only if there is a linear operator $P : X \to X$ with $P^2 = P$ such that in the decomposition (3.35), $v_1 = Pv$ and $v_2 = (I - P)v$, and also $X_1 = P(X)$ and $X_2 = (I - P)(X)$.

Proof

Let $X = X_1 \oplus X_2$.Then $Pv = v_1$ defines an operator from $to\ X$.

It is easy to verify that P is linear and maps $X\ onto\ X_1$ $(Pv_1 = v_1, \forall\ v_1 \in X_1)$, and so $X_1 = P(X)$. Obviously $v_2 = (I - P)v$

and $(I - P)v_2 = v_2, \forall\ v_2 \in X$. Conversely, with the operator P, for any $v \in X$ we have the decomposition

$$v = Pv + (I - P)v.$$

We must show this decomposition is unique. Suppose

$$v = v_1 + v_2, \qquad v_1 \in X_1,\ v_2 \in X_2.$$

Then $v_1 = Pw$, for some $w \in X$.This implies

$$Pv_1 = P^2w = Pw = v_1.$$

Similarly,

$Pv_2 = 0.$ Hence, $Pv = v_1$, then $v_2 = v - v_1 = (I - P)v.$

Definition 3.2 Let v_1 and v_2 be vectors in an inner product space X. Then v_1 is orthogonal to v_2 if $\langle v_1, v_2 \rangle = 0$; since this implies that v_2 is orthogonal to v_1 we often simplify say that v_1 and v_2 is orthogonal. If W is a set of vectors in X, then W is called an orthogonal set provided all pairs of distinct vectors in W are orthogonal. An ***orthonormal*** set is an orthogonal set W with the additional property that $\|v\| = 1$ for every v in W.

Definition 3.3 Let X be a *Banach space*. An operator $P \in L(X)$, where $L(X)$ is the set of linear operators from X to X, with the property $P^2 = P$ is called a projection operator. The subspace $P(X)$ is called the corresponding projection space. The direct sum

$$X = P(X) \oplus (I - P)(X) \tag{3.36}$$

is called a topological direct sum.

If X is a *Hilbert space*, P is a projection operator, and $X = P(X) \oplus (I - P)(X)$ is an orthogonal direct sum, then we call P an orthogonal projection operator.

It is easy to see that a projection operator P is orthogonal if and only if

$$(Pv, (I - P)w) = 0 \quad \forall\, v, w \in X. \tag{3.37}$$

Proposition 3.3 ***(Orthogonal projection)***

Let X_1 be a closed linear subspace of the Hilbert space , with its orthogonal complement $X_1^{\perp}$. Let $P : X \rightarrow X_1$. Then

(a) The operator P is an orthogonal projection if and only if it is a ***selfadjoint*** projection. (We mean by ***selfadjoint*** that $T^* = T$ where T is any linear operator on a finite dimension inner product space X such that for any T there is a unique linear operator T^* on X such that

$$\langle Tv_1, v_2\rangle = \langle v_1, T^*v_2\rangle, \qquad \forall\, v_1, v_2 \in X$$

(b) $X = X_1 \oplus X_1^{\perp}$.

(c) There exists exactly one orthogonal projection operator P from X onto X_1. We have

$$\|v - Pv\| = inf_{w\in X_1}\|v - w\| \qquad \forall\, v \in X.$$

The operator I − P is the orthogonal projection onto $X_1^{\perp}$.

(d) If $P : X \rightarrow X$ is an orthogonal projection operator, then $P(X)$ is a closed subspace of X, and we have the orthogonal direct sum

$$X = P(X) \oplus (I - P)(X).$$

Example 3.3 (Lagrange interpolation)

Let $X = C[a, b]$, $X_1 = P_n$ the space of the polynomials of degree less than or equal to n, and let $\Delta : a = x_0 < x_1 < \cdots < x_n = b$ be a

partition of the interval $[a, b]$. $For\ v \in C[a, b]$, we define $Pv \in P_n$ to be the Lagrange interpolant of *v* corresponding to the partition Δ; i.e., *Pv* satisfies the interpolation conditions*:* $Pv(x_i) = v(x_i), 0 \leq i \leq n$. From the discussion of Section 3.2.1.1, the interpolant *Pv* is uniquely determined. The uniqueness of the interpolant implies that *P* is a projection operator. Explicitly,

$$Pv(x) = \sum_{i=0}^{n} \left(\prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \right) v(x_i),$$

For more details see [6].

3.2.2 Collocation method, [4], section 3.1.1, page 50

Pick distinct node points $x_1, \ldots, x_k \in$ D, and require

$$r_n(x_i) = 0, \qquad i = 1, \ldots, k_n. \tag{3.38}$$

This leads to determine $\{c_1, \ldots, c_k\}$ as the solution of the linear system

$$\sum_{j=1}^{k} c_j \left\{ \varphi_j(x_i) - \lambda \int_D G(x_i, y) \varphi_j(y) dy \right\} = g(x_i), \quad i = 1, \ldots, k. \tag{3.39}$$

An immediate question is whether this system has a solution and whether it is unique. If so, does f_n converge to f ? This what we will answer later.

We should have written the node points as $\{x_{1,n}, \ldots, x_{k,n}\}$, but for notational simplicity, the explicit dependence on n has been suppressed, to be understood only implicitly.

The function space framework for collocation methods is often $C(D)$, which is what we use here.

As a part of writing (3.39) in a more abstract form, we introduce a projection operator P_n that maps $X = C(D)$ onto X_n. Given $f \in C(D)$, define $P_n f$ to be that element of X_n that interpolates f at the nodes $\{x_1, \ldots, x_k\}$. This means writing

$$P_n f(x) = \sum_{j=1}^{k_n} \alpha_j \varphi_j(x) \tag{3.40}$$

with the coefficients $\{\alpha_j\}$ determined by solving the linear system

$$\sum_{j=1}^{k_n} \alpha_j \varphi_j(x_i) = f(x_i), \qquad i = 1, \ldots, k_n . \tag{3.41}$$

This linear system has a unique solution if

$$det\left[\varphi_j(x_i)\right] \neq 0. \tag{3.42}$$

Then in this chapter, we assume this is true whenever the collocation method is being discussed. By a simple argument, this condition also implies that the functions $\{\varphi_1 \ldots, \varphi_k\}$ are a linearly independent set over D.

In the case of polynomial interpolation for functions of one variable and monomials $\{1, x, \ldots, x^n\}$ as the basis functions, the determinant in (3.42) is referred to as the *Vandermonde* determinant. To see more clearly that P_n is linear, and to give a more explicit formula, we introduce a new set of basis

functions. For each i, $1 \leq i \leq k_n$, let $L_i \in X_n$ be that element that satisfies the interpolation conditions

$$L_i(x_j) = \delta_{ij}, \quad j = 1, \dots, k_n \tag{3.43}$$

By (3.42), there is a unique such L_i; and the set $\{L_1, \dots, L_k\}$ is a new basis for X_n. With polynomial interpolation, such functions L_i are called Lagrange basis functions; and we use this name with all types of approximating subspaces X_n. With this new basis, we can write

$$P_n f(x) = \sum_{j=1}^{k_n} f(x_j) L_j(x), \qquad x \in D. \tag{3.44}$$

In the view of Lagrange polynomial interpolation (which is illustrated above) Clearly, P_n is linear and finite rank. In addition, as an operator on $C(D)$ $onto$ $C(D)$,

$$\|P_n\| = \max_{x \in D} \sum_{j=1}^{k_n} |l_j(x)|. \tag{3.45}$$

Example 3.4 Let $X_n = span\{1, x, \dots, x^n\}$. Then

the Lagrange basis functions are given of

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right), \qquad i = 0,1, \dots, n \tag{3.46}$$

In this case, formula (3.44) is called Lagrange's form of the interpolation polynomial .

Note that

$$P_n f = 0 \quad if\ and\ only\ if \quad f(x_j) = 0, \quad j = 1, \dots, k_n. \tag{3.47}$$

Thus the condition (3.39) can now be rewritten as

$$P_n r_n = 0$$

or equivalently,

$$P_n(I - \lambda K) f_n = P_n g, \qquad f_n \in \mathrm{X}_n. \tag{3.48}$$

For more details see [8], [24] and [45].

There are two main types of collocation method

- Decompose the integration region D into elements $\Delta_1, \cdots, \Delta_n$, and then approximate a function $f \in C(D)$ by a low degree polynomial over each of the elements Δ_i. These are referred to as ***piecewise polynomial collocation methods***, but when D is the boundary of a region, these methods are called ***boundary element methods***.
- Approximate an $f \in C(D)$ by using a family of functions that are defined over all of D, such as, polynomials, trigonometric polynomials, or spherical polynomials. These approximating functions in general are also infinitely differentiable. Sometimes these types of collocation methods are referred to as ***spectral methods***, especially when trigonometric polynomials are used.

For more details see [4].

Here we will study the first type of collocation method.

3.2.2.1 Piecewise linear interpolation

Let $D = [a, b], n > 0,\ h = (b - a)/n, and\ x_i = a + ih,\ i = 0, \cdots, n.$

The subspace X_n we take to be the set of all functions that are piecewise linear on $[a, b]$, with breakpoints $\{x_0, \ldots, x_n]$, so that its dimension is n + 1.

Introduce the Lagrange basis functions for piecewise linear interpolation:

$$l_i(x) = \begin{cases} 1 - \dfrac{|x - x_i|}{h}, & x_{i-1} \le x \le x_{i+1} \\ 0, & otherwise \end{cases} \qquad (3.49)$$

With the obvious adjustment of the definition for $l_0(x) and\ l_n(x)$.

The projection operator is defined by

$$P_n f(x) = \sum_{i=0}^{n} f(x_i) l_i(x) \qquad (3.50)$$

Now the linear system (3.39) takes the simpler form

$$f_n(x_i) - \lambda \sum_{j=0}^{n} f_n(x_j) \int_a^b G(x_i, y) l_j(y)\, dy = g(x_i), i = 0, \cdots, n, \qquad (3.51)$$

and we can simplify the integral for $j = 1, \ldots, n - 1,$

$$\begin{aligned} \int_a^b G(x_i, y) l_j(y) dy \\ &= \frac{1}{h} \int_{x_{j-1}}^{x_j} G(x_i, y)(y - x_{j-1}) dy \\ &+ \frac{1}{h} \int_{x_j}^{x_{j+1}} G(x_i, y)(x_j - y) dy \end{aligned} \qquad (3.52)$$

The integrals for $j = 0$ and $j = n$ are modified accordingly. These must usually be calculated numerically, and we want to use the quadrature method with the trapezoidal rule.

3.2.3 Galerkin methods, [4], section 3.1.2, page 53.

Let $X = L^2(D)$ or some other Hilbert function space, and let $\langle .,. \rangle$ denote the inner product for X. Require the residual r_n to satisfy

$$\langle r_n, \varphi_i \rangle = 0, \quad i = 1, \dots, k_n. \tag{3.53}$$

The left side is the Fourier coefficient of r_n associated with φ_i. If

$\{\varphi_1, \dots, \varphi_k\}$ consists of the leading members of an orthonormal family $\Phi \equiv \{\varphi_i\}_i \geq 1$ which spans X, then (3.53) requires the leading terms to be zero in the Fourier expansion of r_n with respect to Φ.

To find f_n, apply (3.53) to (3.1) written as $(\lambda - K) f = g$. This yields the linear system

$$\sum_{j=1}^{k_n} c_j \{ \langle \varphi_j, \varphi_i \rangle - \lambda \langle K\varphi_j, \varphi_i \rangle \} = \langle g, \varphi_i \rangle, \quad i = 1, \dots, k_n. \tag{3.54}$$

This is Galerkin's method for obtaining an approximate solution to (3.1) or (3.27). Does the system have a solution? If so, is it unique? Does the resulting sequence of approximate solutions f_n converge to f in X? Does the sequence converge in $C(D)$, that means, does f_n converge uniformly to f ? Note also that the above formulation contains double integrals $\langle K\varphi_j, \varphi_i \rangle$. These must often be computed numerically.

As a part of writing (3.54) in a more abstract form, we recall the

orthogonal projection operator P_n of Proposition (3.3) of Section 3.2.1.2, which maps X onto X_n. Recall that

$$P_n h = 0 \quad \text{if and only if} \quad \langle h, \varphi_i \rangle = 0, \quad i = 1, \dots, k_n. \tag{3.55}$$

Using the orthogonal projection P_n, we can rewrite (3.53) as

$$P_n \, r_n \; = \; 0,$$

or equivalently,

$$P_n(I - \lambda K) f_n = P_n g, \qquad f_n \in \mathrm{X}_n. \tag{3.56}$$

which is similar to (3.48).

Bernstein polynomials are used as trial functions in the basis. For this, we give a short introduction of Bernstein polynomials first. Then we derive a matrix formulation by the technique of Galerkin method.

For more details see [12].

3.2.3.1 Bernstein polynomials

The general form of the Bernstein polynomials of nth degree over the interval $[a, b]$ is defined by

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, a \le x \le b, \quad i = 0,1, \dots, n \tag{3.57}$$

Note that each of these n+1 polynomials having degree n satisfies the following properties:

$$i)\ B_{i,n}(x) = 0,\ \ if\ i < 0\ or\ i > n,$$

$$ii)\ \sum_{i=0}^{n} B_{i,n}(x) = 1,$$

$$iii)\ B_{i,n}(a) = B_{i,n}(b) = 0, \qquad 1 \le i \le n-1 \tag{3.58}$$

2.3.2 Formulation of integral equation in matrix form

Consider a general linear Fredholm integral equation of second kind which is is given in (3.1), and using the technique of Galerkin method mentioned above to find an approximate solution $f_n(x)$ in (3.26). For this we assume that

$$f_n(x) = \sum_{i=0}^{n} c_i B_{i,n}(x) \tag{3.59}$$

where $B_{i,n(x)}$ are Bernstein polynomials (basis) of degree i defined in eqn. (3.57), and c_i are unknown parameters, to be determined. Substituting (3.59) into (3.1), we obtain

$$\sum_{i=0}^{n} c_i B_{i,n}(x) + \lambda \int_a^b \left[G(x,y) \sum_{i=0}^{n} c_i B_{i,n}(y) \right] dy = g(x), \tag{3.60}$$

or ,

$$\sum_{i=0}^{n} c_i \left[B_{i,n}(x) + \lambda \int_a^b G(x,y)\, B_{i,n}(y) dy \right] = g(x), \tag{3.61}$$

Then the Galerkin equations are obtained by multiplying both sides of (3.61) by $B_{i,n}(x)$ and then integrating with respect to x from a to b, we have

$$\sum_{i=0}^{n} c_i \left[\int_a^b \left[B_{i,n}(x) + \lambda \int_a^b G(x,y)\, B_{i,n}(y)dy\right] B_{j,n}(x)dx\right] = \int_a^b B_{j,n}(x)g(x)dx\,, \qquad j = 0,1,\dots,n \tag{3.62}$$

Since in each equation, there are three integrals, the inner integrand of the left side is a function of x, and y, and is integrated with respect to y from a to b, as a result the outer integrand becomes a function of x only and integration with respect to x yields a constant. Thus for each $j = 0,1,\dots,n$ we have a linear equation with n+1 unknowns c_i $(i = 0,1,\dots,n)$. Finally (3.63) represents the system of $n+1$, linear equations in $n+1$ unknowns,

$$\sum_{i=0}^{n} c_j A_{i,j} = D_j, \quad j = 0,1,2,\dots,n \tag{3.63}$$

where

$$A_{i,j} = \int_a^b \left[B_{i,n}(x) + \lambda \int_a^b G(x,y)\, B_{i,n}(y)dy\right] B_{j,n}(x)dx \tag{3.64}$$

$$i,j = 0,1,2,\dots,n$$

$$D_j = \int_a^b B_{i,n}(x)g(x)dx\,, \qquad j = 0,1,2,\dots,n \tag{3.65}$$

Now the unknown parameters c_j are determined by solving the system of equations (3.63) and substituting these values of parameters in (3.59), we

get the approximate solution $f_n(x)$ of integral equation (3.1). For more details see [39] and [47].

3.2.4 The convergence of the projection methods, [4]

Let X be a Banach space, and let $\{X_n | n \geq 1\}$ be a sequence of finite dimensional subspaces of dimension n. Let $P_n : X \rightarrow X_n$ be a bounded projection operator. This means that P_n is a bounded linear operator with

$$P_n f = f, \quad f \in X_n$$

This implies $P_n^2 = P_n$, and thus

$$\|P_n\| = \left\|P_n{}^2\right\| \leq \|P_n\|^2$$

$$\|P_n\| \geq 1 \tag{3.66}$$

we approximate (3.1) by attempting to solve the problem

$$P_n(I - \lambda K)f_n = P_n g, \qquad f_n \in X_n \tag{3.67}$$

This is the form in which the method is implemented, as it leads directly to equivalent finite linear systems such as (3.39) and (3.54). For the error analysis, we write (3.67) in an equivalent form such that if f_n is a solution of (3.67), then by using $P_n f_n = f_n$, the equation can be written as

$$(I - \lambda P_n K)f_n = P_n g, \qquad f_n \in X_n \tag{3.68}$$

For the error analysis, we compare (3.68) with the original equation

$$(I - \lambda K)f = g. \tag{3.69}$$

The theoretical analysis is based on the approximation of $I - \lambda P_n K$ by $I - \lambda K$, since both equations are defined on the original space X, we have

$$I - \lambda P_n K = (I - \lambda K) + (\lambda K - \lambda P_n K)$$

$$= (I - \lambda K)[I + (I - \lambda K)^{-1}(\lambda K - \lambda P_n K)] \quad (3.70)$$

Now we use this in the following theorem.

Theorem 3.4 [4], page 55,

Assume $K : X \to X$ is bounded, with X a Banach space, and assume $\lambda - K: X \xrightarrow{1-1} X$. Further assume

onto

$$\|K - P_n K\| \to 0 \quad as \quad n \to \infty \quad (3.71)$$

Then for all sufficiently large n, say $n \geq N$, the operator $(\lambda - P_n K)^{-1}$ exists as a bounded operator from $X\ to\ X$. Moreover, it is uniformly bounded:

$$\sup_{n \geq N} \|(\lambda - P_n K)^{-1}\| < \infty \quad (3.72)$$

For the solution of (3.68) and (3.69),

$$f - f_n = \lambda(\lambda - P_n K)^{-1}(f - P_n f) \quad (3.73)$$

$$\frac{|\lambda|}{\|\lambda - P_n K\|}\|f - P_n f\| \leq \|f - f_n\|$$

$$\leq |\lambda|\|(\lambda - P_n K)^{-1}\|\|f - P_n fx\| \quad (3.74)$$

This leads to $\|f - f_n\|$ converging to zero at exactly the same speed as $\|f - P_n f\|$.

Proof

(a) Pick N such that

$$\epsilon_N \equiv \sup_{n \geq N} \|K - P_n K\| < \frac{1}{\|(\lambda - K)^{-1}\|}$$

Then the inverse $[I + (\lambda - K)^{-1}(K - P_n K)]^{-1}$ exist and is uniformly bounded by the geometric series theorem.

$$\|[I + (\lambda - K)^{-1}(K - P_n K)]^{-1}\| \leq \frac{1}{1 - \epsilon_N \|(\lambda - K)^{-1}\|}$$

Using (3.70), $(\lambda - P_n K)^{-1}$ exists,

$$(\lambda - P_n K)^{-1} = [I + (\lambda - K)^{-1}(K - P_n K)]^{-1}(\lambda - K)^{-1}$$

$$\|(\lambda - P_n K)^{-1}\| \leq \frac{\|(\lambda - K)^{-1}\|}{1 - \epsilon_N \|(\lambda - K)^{-1}\|} \equiv M \qquad (3.75)$$

This show (3.72).

(b) For the error formula (3.73), multiply $(\lambda - K)f = g$ by P_n, and then we rearrange to obtain

$$(\lambda - P_n K)f = P_n g + \lambda(f - P_n f)$$

Subtract $(\lambda - P_n K)f_n = P_n g$ to get

$$(\lambda - P_n K)(f - f_n) = \lambda(f - P_n f) \qquad (3.76)$$

$$f - f_n = \lambda(\lambda - P_n K)^{-1}(f - P_n f)$$

Which is (3.73). Taking norms and using (3.75),

$$\|f - f_n\| \leq |\lambda| M \|f - P_n f\| \tag{3.77}$$

Thus if $P_n f \to f$, then $f_n \to f \; as \; n \to \infty$.

(c) The upper bound in (3.74) follows directly from (3.33), as we have just seen. The lower bound follows by taking bounds in (3.76), to obtain

$$|\lambda| \|f - P_n f\| \leq \|\lambda - P_n K\| \|f - f_n\|$$

This is equivalent to the lower bound in (3.74).

Now to obtain a lower bound which is uniform in n, note that for $n \geq N$,

$$\|\lambda - P_n K\| \leq \|\lambda - K\| + \|K - P_n K\|$$

$$\leq \|\lambda - K\| + \epsilon_N$$

The lower bound in (3.74), can now be replaced by

$$\frac{|\lambda|}{\|\lambda - K\| + \epsilon_N} \|f - P_n f\| \leq \|f - f_n\|$$

Combining this and (3.77), we have

$$\frac{|\lambda|}{\|\lambda - K\| + \epsilon_N} \|f - P_n f\| \leq \|f - f_n\| \leq |\lambda| M \|f - P_n f\| \tag{3.78}$$

This shows that f_n, converges to f if and only if $P_n f$ converges to f.

Moreover, if convergence does occur, then $\|f - P_n f\|$ and $\|f - f_n\|$ tend to zero with exactly the same speed.

To apply the above theorem, we need to know whether $\|K - P_n K\| \to 0$ as $n \to \infty$. The following two lemmas address this question,

Lemma 3.5 Let X, Y be Banach spaces, and let $A_n : X \to Y, n \geq 1$ be a sequence of bounded linear operators. Assume $\{A_n f\}$ converges for all $f \in X$. Then the convergence is uniform on compact subsets of X.

Lemma 3.6 Let X be a Banach space, and let $\{P_n\}$ be a family of bounded projections on X with

$$P_n f \to f \quad as \quad n \to \infty, \ f \in X \tag{3.79}$$

Let $K : X \to X$ be compact. Then

$$\|K - P_n K\| \to 0 \quad as \quad n \to \infty$$

Proof

From the definition of operator norm,

$$\|K - P_n K\| = \sup_{\|f\| \leq 1} \|Kf - P_n Kf\| = \sup_{v \in K(U)} \|v - P_n v\|$$

with $K(U) = \{Kf \mid \|f\| \leq 1\}$. The set $K(U)$ is compact. Therefore, by the preceding Lemma 3.5 and the assumption (3.79),

$$\sup_{v \in K(U)} \|v - P_n v\| \to 0 \ \ as \ \ n \to \infty$$

This proves the lemma.

For more details see [3] and [4].

3.3 Nyström (Quadrature) method, [4], section 4.1, page 100,

The Nyström method was found to handle approximations based on numerical integration of the integral operator in the equation (3.1). The solution is found first at the set of quadrature node points, and then it is extended to all points in D by means of a special interpolation formula. The numerical method is much simpler to implement on a computer, but the error analysis is more sophisticated than for the methods of the preceding two sections.

For solving the Fredholm integral equation in (3.1) by this method we use the numerical integration scheme

$$\int_D h(y)dy \approx \sum_{j=1}^{k_n} w_{n,j} h(x_{n,j}), \qquad h \in C(D) \qquad (3.80)$$

with an increasing sequence of values of n. Assuming that the numerical integrals for every $h \in D$ converge to the true integral as n$\rightarrow \infty$.

To simplify the notation, we omit the subscript n so that $w_{n,j} \equiv w_j$, $x_{n,j} \equiv x_j$ and sometimes $k_n \equiv k$, but we understand the presence of n implicitly.

Let the kernel function be continuous for all x, y $\in D$ where D is a closed and bounded set in R^m for some $m \geq 1$. By approximating the integral in (3.1) using the quadrature scheme in (3.80) we obtain a new equation

$$f_n(x) - \lambda \sum_{j=1}^{k_n} w_j \, G(x, x_j) \, f_n(x_j) = g(x), \qquad x \in D \tag{3.81}$$

where its solution $f_n(x)$ is an approximation of the exact solution $f(x)$ to (3.1). A solution to a functional equation (3.81) may be obtained if we assign $x_i's$ to x in which $i = 1, \dots, k_n$ and $x_i \in D$. In this way, (3.81) is reduced to the system of equations

$$f_n(x_i) - \lambda \sum_{j=1}^{k_n} w_j \, G(x_i, x_j) \, f_n(x_j) = g(x_i), \qquad i = 1, \dots, k_n \tag{3.82}$$

which is a linear system of order k_n. The unknown is a vector

$$f_n \equiv [f_n(x_1), \dots, f_n(x_q)]^t$$

Each solution $f_n(x)$ of (3.81) furnishes a solution to (3.82): merely evaluate $f_n(x)$ at the node points. The converse is also true. To each solution $\underline{u} \equiv [u_1, \dots, u_k]^t$ of (3.82), there is a unique solution of (3.81) that agrees with u at the node points. If one solves for $f_n(x)$ in (3.81), then $f_n(x)$ is determined by its values at the node points $\{x_j\}$. Therefore, when given a solution $\underline{u}$ to (3.82), define

$$u(x) = \lambda \sum_{j=1}^{k_n} w_j \, G(x, x_j) \, u_j + g(x), \qquad x \in D \tag{3.83}$$

This is an interpolation formula. In fact,

$$u(x_i) = \lambda \sum_{j=1}^{k_n} w_j \, G(x_i, x_j) \, u_j + g(x_i),$$

$$= u_i \qquad \text{for } i = 1, \ldots, k_n.$$

This formula (3.83) is called the Nyström interpolation formula. In the original paper of Nyström, he uses a highly accurate Gaussian quadrature formula with a very small number of quadrature nodes (for example, $k = 3$). He then uses (3.83) to extend the solution to all other $x \in D$ while retaining the accuracy found in the solution at the node points. The formula (3.83) is usually a very good interpolation formula.

For more details see [4].

The last step follows from u being a solution to (3.82). Using this interpolation result in (3.83), we have that $u(x)$ solves (3.81). The uniqueness of the relationship between $\underline{u}$ and $u(x)$ follows from the solutions $f_n(x)$. Moreover, (3.82) can be represented by

$$(I - \lambda KD) f_n = g, \tag{3.84}$$

where

$$f_n = [f_n(x_i)]^t, \qquad g = [g(x_i)]^t, \qquad K = \left[G(x_i, x_j)\right],$$

and $D = diag(w_1, \ w_2, \ldots, w_k)$.

It is worth noting that $I - \lambda KD$ may be singular for a chosen quadrature rule (3.80). However, under suitable restrictions, we can preserve the non-singularity of $I - \lambda KD$ if we decide on a sufficiently accurate (3.80). In addition, whether quadrature rule is sufficiently accurate or not itself depends on λ, $G(x, y)$, and $g(x)$.

For more details see [43] and [48].

Chapter 4

Numerical Examples and Results

Chapter Four

Numerical Examples and Results

In this chapter we try to apply some of the numerical methods illustrated in chapter three to approximate the solution of the Fredholm integral equation

$$f(x) = -\frac{2}{\pi}\cos(x) + \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos(x-y)\, f(y)dy. \qquad (4.1)$$

These methods include: ***the degenerate kernel method***, ***the collocation method*** and ***the Nyström method,*** we will use suitable algorithms and Matlab software, then we will compare the exact solution with the approximate one using suitable number of ***n*** points.

Note: the exact solution $f(x) = \sin(x)$ of the above integral equation (4.1) is done in chapter two section 2.2.1

4.1 The numerical realization of equation (4.1) using *the degenerate kernel method*

First we expand the kernel $G(x, y)$, with respect to y using the Taylor series such that

$$Taylor\ (G, y, a) = \sum_{n=0}^{m} \frac{(y-a)^n}{n!} \frac{\partial^n}{\partial y^n} G(x, y = a) \qquad (4.2)$$

where m is the number of Taylor series terms, by this expansion, the kernel can be written as the sum of tow separated functions one with respect to x, and the other with respect to y, such that

$$G_m(x,y) = \sum_{i=0}^{m-1} u_i(x) v_i(y) \tag{4.3}$$

where

$$u_{i-1}(x) = \left(\frac{1}{i!}\right) \frac{\partial^{i-1}}{\partial y^{i-1}} G(x,a) \tag{4.4}$$

and

$$v_{i-1}(y) = (y-a)^{i-1}, \forall i = 1,2,\dots,m, \tag{4.5}$$

then we calculate the values c_{ij}, and h_i, such that

$$c_{ij} = \int_a^b v_i(y) u_j(y) dy \,, h_i = \int_a^b v_i(y) g(y) dy \,, \quad i = 1,2,\dots,m \tag{4.6}$$

using the relations in section 2.2.1, and the above relations, we have

$$\alpha_i - \lambda \sum_{j=1}^{n} c_{ij} \alpha_j = h_i \qquad i = 1,\dots,n \tag{4.7}$$

now putting this relations in the matrix form we have,

$$A[\alpha_i] = H,$$

where

$$A = I - \lambda C$$

such that I is the identity matrix,

$$C = [c_{ij}], \forall i,j = 1,2,\dots,m,$$

$$H = [h_i], \forall i = 1,2,\dots,m.$$

And the matrix

$$[\alpha_i] = A^{-1}H.$$

the solution f_m is given by

$$f_m(x) = g(x) + \lambda \sum_{i=0}^{m-1} \alpha_{i+1}\, g_i(x) \qquad (4.8)$$

The following algorithm implements the ***degenerate kernel method*** using the ***Matlab*** software.

Algorithm 1

1. *Input* $a, b,\ \lambda, g(x), G(x,y)$
2. *input the number of Taylor series' terms m*
3. *calculate the Taylor expansion of* $G(x,y)$ *with respect to y ,*

Tay

from f find $u_i(x)$ *and* $v_i(y)$, $i = 1,2,\dots,m$

4. *calculate* $c_{ik} = \int_a^b v_i(y)u_k(y)dy\ \ i,k = 1,2,\dots,m$
5. *calculate* $h_i = \int_a^b v_i(y)g(y)dy\ \ i = 1,2,\dots,m$
6. *calculate the matrix*

$$A = \begin{vmatrix} 1-\lambda c_{11} & -\lambda c_{12} & \cdots & -\lambda c_{1m} \\ -\lambda c_{21} & 1-\lambda c_{22} & \cdots & -\lambda c_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ -\lambda c_{m1} & -\lambda c_{m2} & & 1-\lambda c_{mm} \end{vmatrix}$$

7. *calculate the determinate D(A) of matrix A*

8. *if* $g(x) \neq 0$ *go to step 12*
9. *if D(A) = 0 the system has infinite number of solutions ,go to step 16*
10. *the system has unique solution* $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$,*go to step 16*
11. *if* $h_i \neq 0$ *go to step 15*
12. *if D(A) = 0, the system has infinite number of solutions, go to step 16, the system has unique solution* $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$
13. *if D(A) = 0,the system has no real solution, go to step 16*
14. *the solution of system is* $[\alpha_i] = [A_{ik}]^{-1}[h_i]^T$

    then

$$f_m(x) = g(x) + \lambda \sum_{i=1}^{m} \alpha_i \, u_i(x)$$

15. *end*

For more details see [20].

By returning to the integral equation (3.85), and using algorithm 1, the kernel of this integral equation $G(x, y) = cos(x - y)$, can be expaneded using Taylor series for 5 tems as

$$Taylor(\cos(x - y), y, 5) =$$

$$\cos(x) + ysin(x) - \frac{y^2}{2}\cos(x) - \frac{y^3}{6}sin(x) + \frac{y^4}{24}\cos(x) \quad (4.10)$$

implies

$$u_1(x) = \cos(x), u_2(x) = sin(x), u_3(x) = \frac{-1}{2}\cos(x), u_4(x) = \frac{-1}{6} sin(x), u_5(x) = \frac{1}{24}\cos(x), \quad (4.11)$$

and

$$v_1(y) = 1,\ v_2(y) = y,\ v_3(y) = y^2,\ v_4(y) = y^3,\ v_5(y) = y^4\ . \quad (4.12)$$

The related Matlab program gives the following results

The matrix C =

1.0000	1.0000	-0.5000	-0.1667	0.0417
0.5708	1.0000	-0.2854	-0.1667	0.0238
0.4674	1.1416	-0.2337	-0.1903	0.0195
0.4510	1.4022	-0.2255	-0.2337	-0.1895
0.4793	1.8040	-0.2396	-0.3007	0.0200

The matrix $A = I - \lambda C =$

-0.2732	-1.2732	0.6366	0.2122	-0.0531
-0.7268	-0.2732	0.3634	0.2122	-0.0303
-0.5951	-1.4535	1.2976	0.2423	-0.0248
-0.5742	-1.7853	0.2871	1.2976	0.2413
-0.6102	-2.2970	0.3051	0.3828	0.9746

The matrix $[\alpha_i] = A^{-1}H =$

$$0.8752$$

$$0.9251$$

$$1.0775$$

$$0.8782$$

$$1.7330$$

then

$$f_m(x_j) = \frac{-2}{\pi}\, cos(x_j) + \frac{4}{\pi}[\alpha_i][u_i(x_j)], \qquad i, j = 1,2, \dots, m \qquad (4.13)$$

where

$$x_{j+1} = x_j + \frac{(b-a)}{m-1}, \qquad and\ x_1 = a, \qquad (4.14)$$

Table 4.1 shows the exact and numerical results when $m = 5$, and showing the error resulting of using the numerical solution,

Table 4.1: The exact and numerical solution of applying Algorithm 1 for equation (4.1).

x	Analytical solution $y_1 = \sin(x)$	Approximate solution y_2	Error= $\lvert y_1 - y_2 \rvert$
0	0	-0.116299822082018	0.116299822082018
0.3927	0.382683432365090	0.271988984127792	0.110694448237297
0.7854	0.707106781186547	0.618869933090427	0.088236848096121
1.1781	0.923879532511287	0.871533544809957	0.052345987701329
1.5708	1.000000000000000	0.991514074803429	0.008485925196571

Figure 4.1 shows the exact solution $f(x) = \sin(x)$ and the approximate one when $m = 5$.

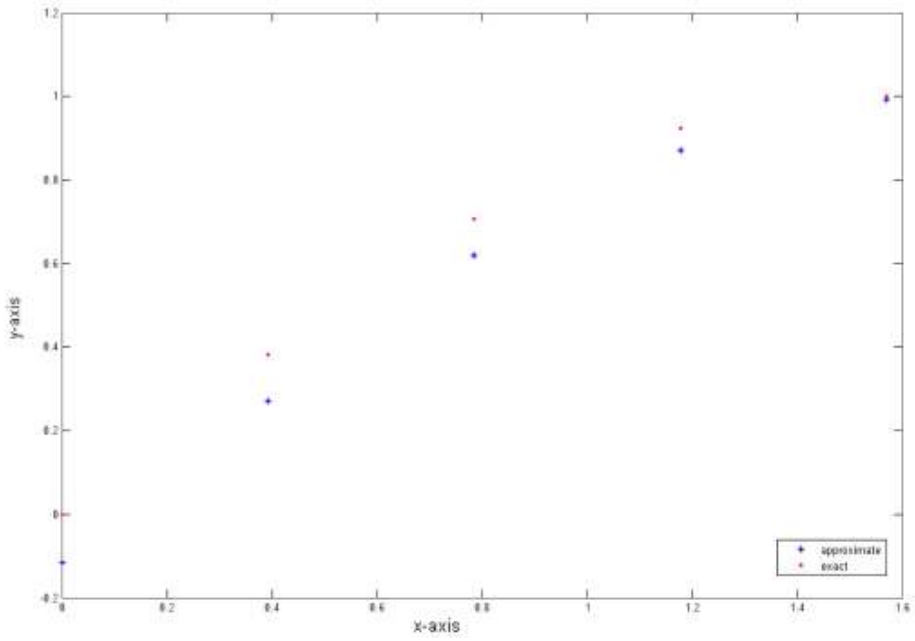

Figure 4.1: The exact and numerical solution of applying Algorithm 1 for equation (4.1).

While Figure 4.2 shows the absolute error which approaches zero .

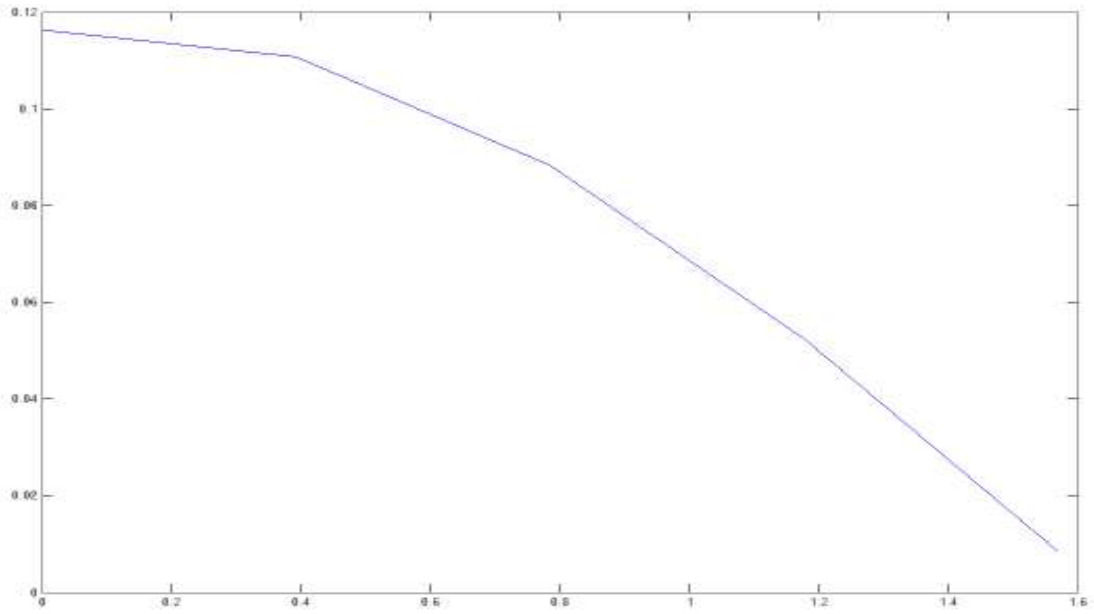

Figure 4.2: The resulting error of applying algorithm 1 to equation (4.1).

4.2 The numerical realization of equation (4.1) using *the collocation method*

First we expand the function $f_n(x)$ as a sum of basis $\{\varphi_1, \ldots, \varphi_k\}$ such that

$$f_n(x) = \sum_{j=1}^{k_n} c_j \varphi_j(x), \qquad x \in [0, \frac{\pi}{2}]. \tag{4.15}$$

Since the residual $r_n(x)$ can be written as

$$r_n(x) = f_n(x) - \lambda \int_D G(x,y) f_n(y) dy - g(x) \tag{4.16}$$

then by substituting (4.15) into the equation (4.16) so as to determine the values of the coefficients $\{c_1, \ldots, c_k\}$, such that

$$r_n(x) = \sum_{j=1}^{k} c_j \left\{ \varphi_j(x) - \lambda \int_D G(x,y)\varphi_j(y)dy \right\} - g(x), \qquad (4.17)$$

but we pick distinct node points $x_1, \dots, x_n \in D$, such that

$$r_n(x_i) = 0, \quad i = 1, \dots, n \qquad (4.18)$$

then (4.17) can be rewritten as

$$\sum_{j=1}^{k} c_j \left\{ \varphi_j(x) - \lambda \int_D G(x,y)\varphi_j(y)dy \right\} = g(x), \qquad (4.19)$$

In this example we have $D = [a,b]$, $h = (b-a)/n$. Hence we take the node points are,

$$x_i = a + ih, \quad i = 0, 1, \dots, n$$

we introduce the Lagrange basis functions for piecewise linear interpolation as

$$l_i(x) = \begin{cases} 1 - \dfrac{|x - x_i|}{h}, & x_{i-1} \le x \le x_{i+1} \\ 0, & otherwise \end{cases} \qquad (4.20)$$

where the subspace X_n is the set of all functions that are piecewise linear on $[a,b]$, with breakpoints $\{x_0, \dots, x_n\}$. Its dimension is $n+1$.

The projection operator is defined by

$$P_n f(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x) \qquad (4.21)$$

now for convergence of $P_n(x)$

$$\|f - P_n f\|_\infty \leq \begin{cases} \omega(f,h), & f \in C[a,b] \\ \frac{h^2}{8}\|f''\|_\infty, & f \in C^2[a,b] \end{cases} \tag{4.22}$$

where the function ω is defined by

$$\omega(f,h) = \sup_{\substack{a\leq x,y\leq b \\ |x-y|\leq h}} |f(x) - f(y)| \tag{4.23}$$

and it is called the modulus of the function f. This shows that

$P_n f \to f$ for all $f \in C[a,b]$.

Now for any compact operator $K : C[a,b] \to C[a,b]$, Lemma (3.6) implies $\|K - P_n K\| \to 0 \ as \ n \to \infty$. Therefore the results of Theorem (3.4) can be applied directly to the numerical solution of the integral equation $(\lambda - K)f = g$. For sufficiently large n, say $n \geq N$, the equation $(\lambda - P_n K)f_n = P_n g$ has a unique solution f_n for each $g \in C[a,b]$; and we can write

$$\|f - f_n\|_\infty \leq |\lambda| M \|f - P_n f\|_\infty$$

for $f \in C^2[a,b]$,

$$\|f - f_n\|_\infty \leq |\lambda| M \frac{h^2}{8}\|f''\|_\infty \tag{4.24}$$

The linear system (4.19) takes the simpler form

$$f_n(x_i) - \lambda \sum_{j=0}^{n} f_n(x_j) \int_a^b G(x_i, y) l_j(y)\, dy = g(x_i), i = 0, \cdots, n \tag{4.25}$$

And we can simplify the integral for $j = 1, \ldots, n-1$,

$$\int_a^b G(x_i,y)l_j(y)dy$$
$$= \frac{1}{h}\int_{x_{j-1}}^{x_j} G(x_i,y)\big(y-x_{j-1}\big)dy$$
$$+ \frac{1}{h}\int_{x_j}^{x_{j+1}} G(x_i,y)\big(x_j-y\big)dy \qquad (4.26)$$

we have calculated the integrals above numerically using quadrature rules specifically ***Trapezoidal Rule*** which is of the form,

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\left[\frac{1}{2}f(x_0)+\sum_{i=1}^{n-1} f(x_i)+\frac{1}{2}f(x_n)\right] \qquad (4.27)$$

Now substituting (4.26) in (4.25) and putting this relation in the matrix form we have

$$F-\frac{\lambda}{h}F(KDU+KDV)=G \rightarrow \left(I-\frac{\lambda}{h}(KDU+KDV)\right)F=G \qquad (4.28)$$

Where

$F=[f_n(x_i)]^T,\ G=[g(x_i)]^T,\ K=\big[G(x_i,x_j)\big],\ D=diag(w_1,w_2,\dots,w_n),$
$U=\big[x_i-\ x_{j-1}\big],\qquad V=\big[x_j-x_i\big].$

The following algorithm implements the ***collocation method*** using the ***Matlab*** software

Algorithm 2

Input $a, b, n, \lambda, g(x), G(x)$

$h \rightarrow \dfrac{b-a}{n}$

$x_1 = a,\ x_{n+1} = b$

$for\ i = 2\ to\ n$

$x_i = a + h * i$

end

$for\ i = 1\ to\ n + 1$

$G_i = g(x_i)$

$S_i = x_i$

$D_{ii} = h \rightarrow D\ is\ diagonal\ matrix$

$for\ j = 1\ to\ n + 1$

$K_{ij} = k(x_i, x_j)$

end

end

$I \rightarrow identity\ matrix$

$for\ i = 1\ to\ n + 1$

$for\ j = 2\ to\ n + 1$

$U_{i1} = x_i - x_{j-1}$

$V_{i1} = x_j - x_i$

$U_{ij} = x_i - x_{j-1}$

$V_{ij} = x_j - x_i$

$lhs \rightarrow I - \frac{\lambda}{h}(DKU + DKV)$

$F \rightarrow the\ answer\ of\ lhs * f = G$

$p(f) \rightarrow the\ interpolating\ polynomial\ at\ [S_i, f_i]$

Table 4.2 compare the exact solution $f(x) = \sin(x)$ with the approximate one when $n = 50$, and showing the error resulting of using the numerical solution.

Note: The table shows the first 10 values and the last 10 values only

Table 4.2: The exact and numerical solution of applying Algorithm 2 for equation (4.1).

x	Analytical solution $y_1 = \sin(x)$	Approximate *solution* y_2	Error $= \lvert y_1 - y_2 \rvert$
0	0	-0.031467686762045	0.031467686762045
0.0314	0.031410759078128	-0.000000000000004	0.031410759078132
0.0628	0.062790519529313	0.031467686762042	0.031322832767271
0.0942	0.094108313318514	0.062904318716399	0.031203994602115
0.1257	0.125333233564304	0.094278871702702	0.031054361861602
0.1571	0.156434465040231	0.125560382825064	0.030874082215167
0.1885	0.187381314585725	0.156717981008673	0.030663333577051
0.2199	0.218143241396543	0.187720917465807	0.030422323930735
0.2513	0.248689887164855	0.218538596041232	0.030151291123623
0.2827	0.278991106039229	0.249140603406845	0.029850502632384
1.2881	0.960293685676943	0.952780175523255	0.007513510153688
1.3195	0.968583161128631	0.962034086005045	0.006549075123586
1.3509	0.975916761938747	0.970338584991732	0.005578176947016
1.3823	0.982287250728689	0.977685476945429	0.004601773783260
1.4137	0.987688340595138	0.984067511370779	0.003620829224359
1.4451	0.992114701314478	0.989478389970310	0.002636311344168
1.4765	0.995561964603080	0.993912772860129	0.001649191742951
1.5080	0.998026728428272	0.997366283839667	0.000660444588605
1.5394	0.999506560365732	0.999835514710550	0.000328954344819
1.5708	1.000000000000000	1.001318028640015	0.001318028640015

Figure 4.3 shows the exact solution $f(x) = \sin(x)$ with the approximate one when $n = 50$.

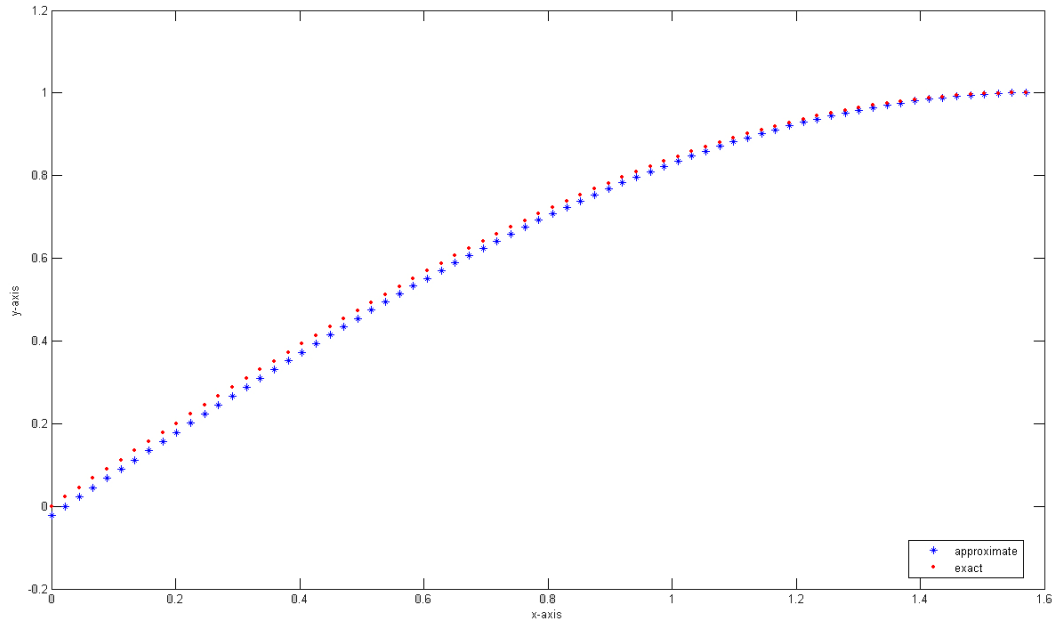

Figure 4.3: : The exact and numerical solution of applying Algorithm 2 for equation (4.1).

The CPU time is 0.066202 seconds.

These results show that the algorithm yields acceptable results since the maximum absolute error which is 0.03 is less than or equal O(h).

While figure 4.4 shows the absolute error resulting of applying algorithm 2 on equation (4.1), and show how it approaches zero.

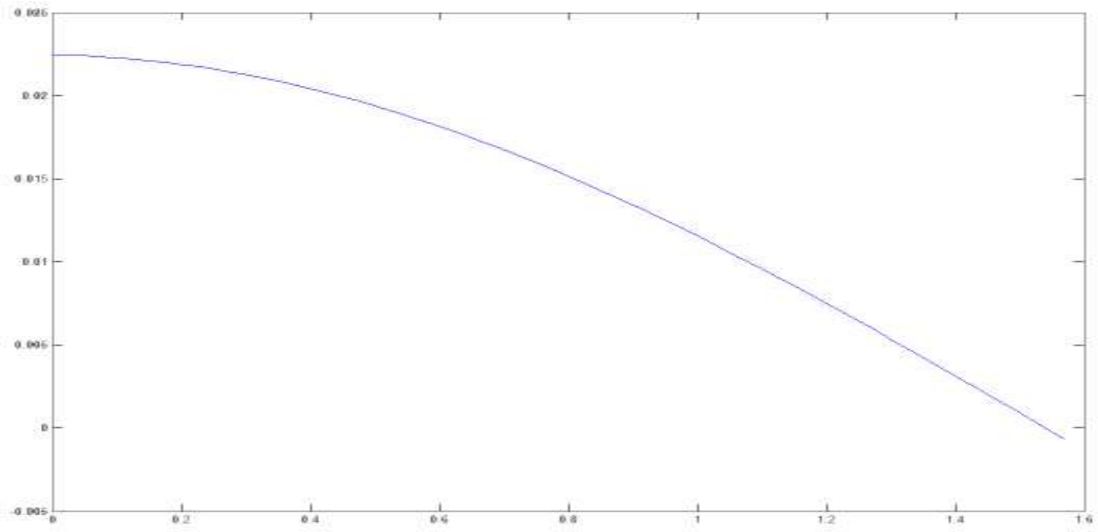

Figure 4.4: the error resulting of applying algorithm 2 on equation (4.1)

4.3 The numerical realization of equation (4.1) using *the Nyström method*

To solve the Fredholm integral equation of the second kind which is given by

$$f(x) = -\frac{2}{\pi}\cos(x) + \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos(x-y)\, f(y)dy.$$

by ***Nyström method***, first we should remember that the kernel $\cos(x-y)$ and the function $-\frac{2}{\pi}\cos(x)$ must be continuous, secondly, we should know that we can approximate the integral $\int_a^b \emptyset(y)dy$ using quadrature rule by $\sum_{j=0}^{n} w_j \emptyset(y_j)$. By such approximation, for $a \le x \le b$, the Fredholm integral equation

$$f(x) = g(x) + \lambda \int_D G(x,y)f(y)dy, \quad x \in \mathrm{D} \qquad (4.29)$$

can be reduced to

$$f_n(x) = \lambda \sum_{j=1}^{n} w_j \, G(x, x_j) \, f_n(x_j) + g(x), \tag{4.30}$$

where its solution $f_n(x)$ is an approximation of the exact solution $f(x)$ to (4.29). A solution to a functional equation (4.30) can be obtained if we assign x_i's to x in which $i = 1,2, \dots, n$ and $a \le x_i \le b$. In this way, (4.30) is reduced to a system of equations

$$f_n(x_i) = \lambda \sum_{j=1}^{n} w_j \, G(x_i, x_j) \, f_n(x_j) + g(x_i), \tag{4.31}$$

Next, writing the equation (4.31) in the matrix form

$$F = \lambda KDF + G \quad \rightarrow \quad F - \lambda KDF = G \quad \rightarrow \quad (I - \lambda KD) \, F = G \tag{4.32}$$

where

$$F = [f_n(x_i)]^t, \qquad G = [g(x_i)]^t, \qquad K = \left[G(x_i, x_j)\right],$$
$$D = diag(w_1, w_2, \dots, w_n)$$

It's worth to mention that in order to approximate the integral, we will use the ***Trapezoidal Rule.***

Here, we implement it in the form such that

$$\int_a^b G(x, y) dy = \sum_{\substack{j=1 \\ i=1}}^{n} w_j \, G(x_i, x_j) = DK \tag{4.33}$$

where D is a diagonal matrix such that the elements of its diagonal equal $\boldsymbol{h}$ where $\boldsymbol{h}$ depends on the initial and the end points of the interval $[a, b]$, and the number of the approximations $\boldsymbol{n}$ such that $h = \frac{b-a}{n}$. The elements of the matrix K consist of the entries $k(x_i, x_j)$, where $i, j = 1,2, \dots, n$,such that the approximations $x_i's$ obtained as $x_i = a + h * i$, where $i = 2,3, \dots, n$, and $x_1 = a$.

The following algorithm implements the ***Nyström method*** using the ***Matlab*** software.

Algorithm 3

Input $a, b, n, \lambda, g(x), G(x)$

$h \rightarrow \frac{b-a}{n}$

$x_1 = a,\ x_n = b$

$for\ i = 2\ to\ n-1$

$x_i = a + h * i$

end

$for\ i = 1\ to\ n$

$G_i = g(x_i)$

$S_i = x_i$

$D_{ii} = h \rightarrow D\ is\ diagonal\ matrix$

$for\ j = 1\ to\ n$

$K_{ij} = k(x_i, x_j)$

end

end

$I \rightarrow identity\ matrix$

$lhs \rightarrow I - \lambda DK$

$F \rightarrow the\ answer\ of\ lhs * f = G$

$p(f) \rightarrow the\ interpolating\ polynomial\ at\ [S_i, f_i]$

Table 4.3 shows the exact solution $f(x) = \sin(x)$ and the approximate one when $n = 50$, and showing the error resulting of using the numerical solution.

Note: The table shows the first 10 values and the last 10 values only

Table 4.3: The exact and numerical solution of applying Algorithm 3 for equation (4.1).

x	Analytical solution $y_1 = \sin(x)$	Approximate solution y_2	Error $= \lvert y_1 - y_2 \rvert$
0	0	0.031405592470328	0.031405592470328
0.0314	0.031410759078128	0.062780191412531	0.031369432334402
0.0628	0.062790519529313	0.094092833885359	0.031302314356046
0.0942	0.094108313318514	0.125312618091103	0.031204304772588
0.1257	0.125333233564304	0.156408733871965	0.031075500307661
0.1571	0.156434465040231	0.187350493115954	0.030916028075723
0.1885	0.187381314585725	0.218107360042338	0.030726045456613
0.2199	0.218143241396543	0.248648981336784	0.030505739940241
0.2513	0.248689887164855	0.278945216106394	0.030255328941540
0.2827	0.278991106039229	0.308966165625180	0.029975059585951
1.2881	0.960293685676943	0.968423843447016	0.008130157770073
1.3195	0.968583161128631	0.975756237987680	0.007173076859049
1.3509	0.975916761938747	0.982125678925927	0.006208916987179
1.3823	0.982287250728689	0.987525880392547	0.005238629663858
1.4137	0.987688340595138	0.991951513040665	0.004263172445527
1.4451	0.992114701314478	0.995398209305166	0.003283507990688
1.4765	0.995561964603080	0.997862567712965	0.002300603109885
1.5080	0.998026728428272	0.999342156239842	0.001315427811571
1.5394	0.999506560365732	0.999835514710546	0.000328954344814

Figure 4.5 compare the exact solution $f(x) = \sin(x)$ with the approximate one when $n = 50$, while Figure 4.6 shows the error resulting of applying Algorithm 3 on the equation (4.1), and how it approaches zero.

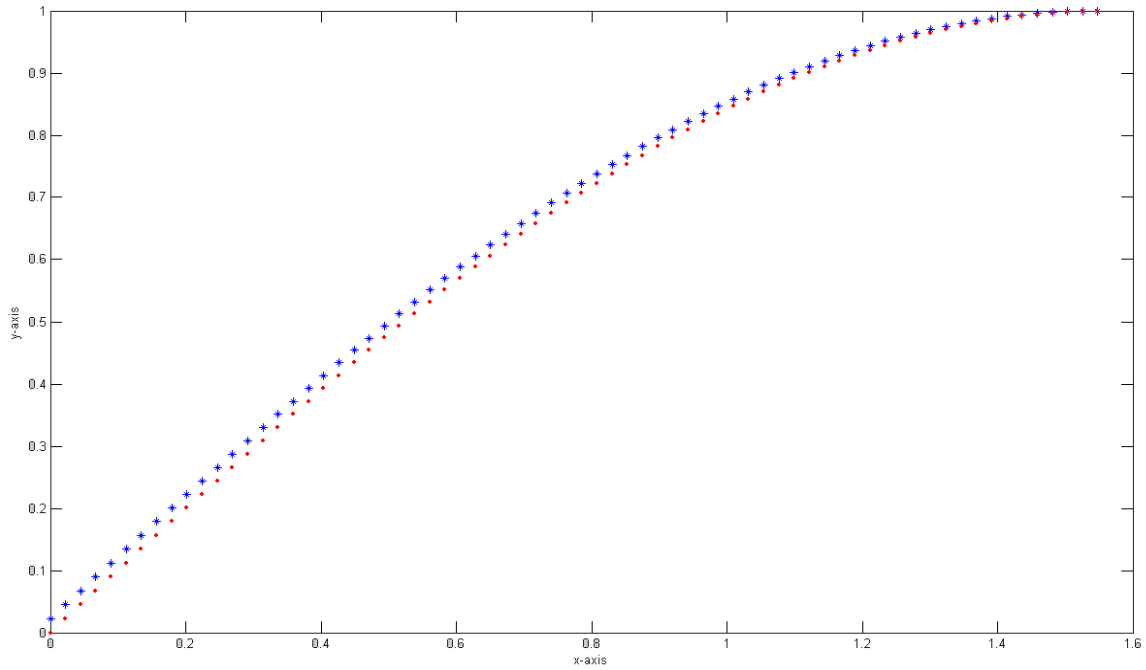

Figure 4.5: The exact and numerical solution of applying Algorithm 2 for equation (4.1).

The CPU time is 0.064010 seconds.

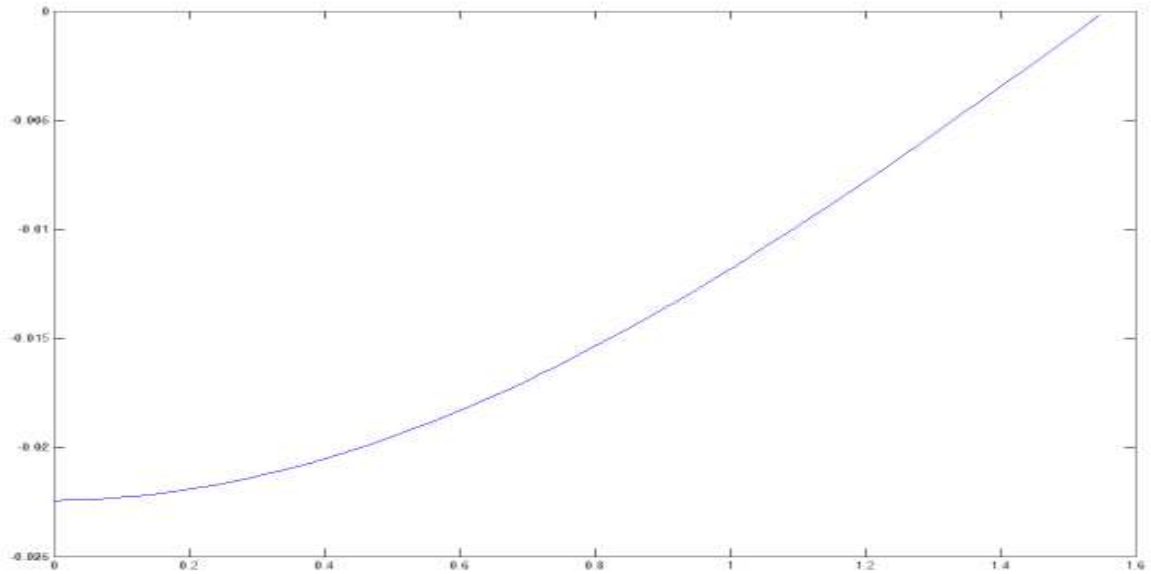

Figure 4.6: The resulting error of applying algorithm 3 to (4.1).

4.4 The error analysis of the *Nyström method*

If we consider the trapezoidal numerical integration rule

$$\int_a^b f(y)dy \approx h \sum_{i=0}^{n} {}''f(x_i) \qquad (4.34)$$

with $h = \frac{b-a}{n}$ and $x_i = a + ih$ for $i = 0, \dots, n$. The notation $\sum$" means the first and last terms are to be halved before summing. For the error,

$$\int_a^b f(y)dy - h \sum_{i=0}^{n} {}''f(x_i) = -\frac{h^2(b-a)}{12} f''(\varepsilon_n), \quad f \in C^2[a,b],$$

$$n \geq 1 \qquad (4.35)$$

with ε_n some point in $[a, b]$. There is also the asymptotic error formula

$$\int_a^b f(y)dy - h \sum_{i=0}^{n} {}''f(x_i) = -\frac{h^2}{12}[f'(b) - f'(a)] + O(h^4),$$

$$f \in C^4[a,b], \quad (4.36)$$

When this is applied to the integral equation

$$f(x) = g(x) + \lambda \int_a^b G(x,y)f(y)dy, \quad a \leq x \leq b \qquad (4.37)$$

we obtain the approximating linear system

$$f_n(x_i) = g(x_i) + \lambda h \sum_{j=0}^{n} {}''G(x_i, x_j) f_n(x_j), \qquad i = 0,1, \dots, n \qquad (4.38)$$

which is of order $q_n = n + 1$.

The Nystrom interpolation formula is given by

$$f_n(x) = g(x) + \lambda h \sum_{j=0}^{n} {}''G(x, x_j) f_n(x_j), \quad a \le x \le b \tag{4.39}$$

The speed of convergence is based on the numerical integration error

$$(K - K_n) f(y) = -\frac{h^2 (b-a)}{12} \left[\frac{\partial^2 G(x,y) f(y)}{\partial y^2} \right]_{y=\varepsilon_n(x)} \tag{4.40}$$

with $\varepsilon_n(x) \in [a, b]$. From (4.36), the asymptotic integration error is

$$(K - K_n) f(y) = -\frac{h^2}{12} \left[\frac{\partial G(x,y) f(y)}{\partial y} \right]_{y=a}^{y=b} + O(h^4) \tag{4.41}$$

From (4.40), we see the ***Nyström method*** converges with an order of $O(h^2)$, provided $G(x,y)f(y)$ is twice continuously differentiable with respect to y, uniformly in x. For more details see [18].

These results show that the algorithm 3 yields acceptable results since the maximum absolute error which is $0.0003 \le O(h^2)$.

Conclusions

In this thesis we have presented some analytical and numerical methods for solving a fredholm integral equation of the second kind. The analytical methods are the degenerate kernel methods, converting Fredholm integral equation to ODE, the Adomain decomposition method, the modified decomposition method and the method of successive approximations.

Moreover, we have used the following numerical methods: Projection methods including collocation method and Galerkin method, Degenerate kernel approximation methods and Nyström methods, for approximating the solution of the Fredholm integral equations.

We have presented each numerical method as algorithm and applied these algorithms on the same Freedholm integral equation using Matlab Software; we have found that the numerical solution was approximately as the exact solution. The absolute error has approached zero which was shown that numerical results were acceptable.

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Appendix

Matlab Code for Collocation Method:

```
 tic

%The collocation method to approximate the Fredholm
integral equation of the

%second kind.

%the problem is f(x)=(-
2/pi)*cos(x)+(4/pi)*int(0,pi/2)cos(x-y)f(y)dy

clc

clear

format long

a=0;

b=pi/2;

lambda=4/pi;

n=70   ;

h=(b-a)/n;

x(1)=0;

for i=1:n

    x(i+1)=a+h*i;

end

G=zeros(1,n+1);

S=zeros(1,n+1);

z=zeros(1,n+1);

K=zeros(n+1,n+1);

u=zeros(n+1,n+1);

v=zeros(n+1,n+1);

for i=1:n+1
```

```
    G(i)=g(x(i));
    S(i)=x(i);
    for j=1:n+1
    K(i,j)=k(x(i),x(j));
    end
end
% we approximate the integrals using Trapezoid rule.
for i=1:n+1
    D(i,i)=h;
    end
for j=1:n+1
    for l=2:n+1
        u(j,1)=x(j)-x(l-1);
        v(j,1)=x(l)-x(j);
        u(j,l)=x(j)-x(l-1);
        v(j,l)=x(l)-x(j);
    end
end
   I=diag(ones(n+1,1),0);
for l=1:n+1
    for j=1:n+1
    c(l,j)=D(l,1:n+1)*K(1:n+1,j);
    end
end
   for l=1:n+1
    for j=1:n+1
    e(l,j)=(lambda/h)*(c(l,j)*u(j,l));
```

```
    q(l,j)=(lambda/h)*(c(l,j)*v(j,l));
    end
    end
    for l=1:n+1
    for j=1:n+1
    lhs(l,j)=I(l,j)-(e(l,j)+q(l,j));
    end
end
F=inv(lhs)*G';
 Fe=sin(S);
 y=Fe'-F;
 m=[S',F, Fe',y ]
%The exact solution is f(x)=sin(x).
 plot(S,F,'*',S,Fe,'r.',S,y)
plot(S,y)
%legend('approximate','exact','error',4)
%the nested functions are
% #1 to approximate the kernel
%function ker=k(x,y)
% ker=cos(x-y);
% #2 to approximate the known function g(x)
%function Ge=g(x)
%Ge=(-2/pi)*cos(x);
toc
```

Matlab Code for Nystrom Method:

```
tic
```

```
%The Nystrum method to approximate the Fredholm integral
equation of the

%second kind.

%the problem is f(x)=(-
2/pi)*cos(x)+(4/pi)*int(0,pi/2)cos(x-y)f(y)dy

clc

clear

format long

a=0 ;

b=pi/2;

lambda=4/pi;

n=70;

h=(b-a)/n;

x(1)=a;

x(n)=b;

for l=1:n-1

    x(l+1)=a+h*l;

end

G=zeros(1,n);

S=zeros(1,n);

F=zeros(1,n);

K=zeros(n,n);

for i=1:n

    G(i)=g(x(i));

    S(i)=x(i);

for j=1:n

    K(i,j)=k(x(i),x(j));

 end
```

```
end
 % we approximate the integrals using Trapezoid rule.
for i=1:n
    D(i,i)=h;
end
I=diag(ones(n,1),0);
lhs=I-lambda*D*K;
F=inv(lhs)*G';
 %The exact solution is f(x)=sin(x).
  Fe=sin(S);
y=[Fe'-F];
 plot(S,F,'*',S,Fe,'r.',S,y)
 plot(S,y)
 %plot(S,y)
%legend('approximate','exact','error',4)
   disp('  S  Fe    F   y')
 [S',Fe', F, y]
%the nested functions are
% #1 to approximate the kernel
%function ker=k(x,y)
% ker=cos(x-y);
% #2 to approximate the known function g(x)
%function Ge=g(x)
%Ge=(-2/pi)*cos(x);
toc
```

Matlab Code for degenerate kernel Method:

```
%Degenerate kernel method using taylor series
```

```
%the problem is: f(x)=1+ int(0,1)sin(x+y)dy

clc

clear

format long

a=0;b=pi/2;lambda=4/pi;

%The five terms of taylor series s.t
G(x,y)=sum(i=1:m)ui(x)*vi(y)

m=5; h=(b-a)/(m-1);

u=zeros(m,m);

v=zeros(m,m);

c=zeros(m,m);

x(1)=0

for i=1:m

    v(i,1:m)=[x(i)^0; x(i); x(i)^2; x(i)^3; x(i)^4];

u1(i)=k1(x(i));

u2(i)=k2(x(i));

u3(i)=k3(x(i));

u4(i)=k4(x(i));

u5(i)=k5(x(i));

x(i+1)=x(i)+h;

end

%WE USE THE TRAPOZOIDAL RULE TO APPROXIMATE THE INTEGRALS

for i=1:m

    G(i)=g(x(i));

    S(i)=x(i);

  x(i+1)=x(i)+h;

end
```

```
D(1,1)=h/2;

D(m,m)=h/2;

for i=2:m-1

    D(i,i)=h;

        f(i)=h;

end

for i=1:m

    for j=1:m

c(i,1:m)=[v(i,j)'*u1(j)'; v(i,j)'*u2(j)'; v(i,j)'*u3(j)';
v(i,j)'*u4(j)'; v(i,j)'*u5(j)'];

r(i)=v(i,j)*G(j)';

     end

end

 e=D*c;

n=r*D;

I=diag(ones(m,1),0);

lhs=I-lambda*e;

z=inv(lhs)*n';

p=G'+lambda*[u1;u2;u3;u4;u5]*z;

[u1;u2;u3;u4;u5]

k=[S',p]

pe=sin(S);

plot(S,p,'*',S,pe,'r.')

%legend('approximate','exact',4)

%THE NESTED FUNCTIONS which related to g(x) and ui's(x)

% #1 function g=g1(x)

%g=1;
```

```
%#2 function ker=k1(x,y)
%ker=sin(x+y);
%#3function ker=k2(x,y)
%ker=cos(x+y);
%#4function ker=k3(x,y)
%ker=(-1/2)*sin(x+y);
%#4function ker=k4(x,y)
%ker=(-1/6)*cos(x+y);
%#5function ker=k5(x,y)
%ker=(1/24)*sin(x+y);
```

جامعة النجاح الوطنية

كلية الدراسات العليا

المعالجة العددية لحل معادلات فريدهولم التكاملية من الدرجة الثانية

إعداد

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قدمت هذه الأطروحة استكمالا لمتطلبات درجة الماجستير في الرياضيات المحوسبة بكلية الدراسات العليا في جامعة النجاح الوطنية، نابلس- فلسطين

2013

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الملخص

في هذه الأطروحة ركزنا على المفاهيم الرياضية والعددية الخاصة بمعادلات فريدهولم التكاملية من النوع الثاني. بعد تصنيف هذه المعادلات التكاملية قمنا باستقصاء بعض الطرق التحليلية والعددية لحل معادلات فريدهولم التكاملية من النوع الثاني.

هذه الطرق التحليلية شملت: طريقة النواة القابلة للفصل، تحويل معادلة فريدهولم التكاملية إلى معادلة تفاضلية عادية، طريقة أدومين التحليلية، طريقة أدومين التحليلية المعدلة، و طريقة التقريبات المتتالية.

الطرق العددية التي تناولناها هي : طريقة المساقط العمودية بنوعيها : طريقة التجميع وطريقة جاليركين، وطريقة النواة القابلة للفصل التقريبية وطريقة نيسترون.

بعض الأمثلة نفذت باستخدام هذه الطرق العددية لحل معادلة فريدهولم التكاملية من النوع الثاني.

النتائج العددية التقريبية كانت قريبة من النتائج التحليلية.