5.3 Medians and Altitudes of a Triangle

What you should learn

GOAL 1 Use properties of medians of a triangle.

GOAL 2 Use properties of altitudes of a triangle.

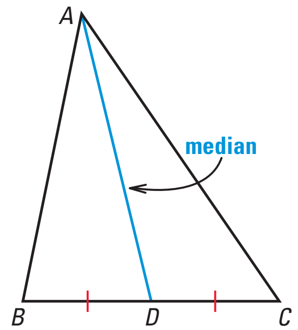
Why you should learn it

▼ To solve **real-life** problems, such as locating points in a triangle used to measure a person's heart fitness as in **Exs. 30–33**.

REAL LIFE

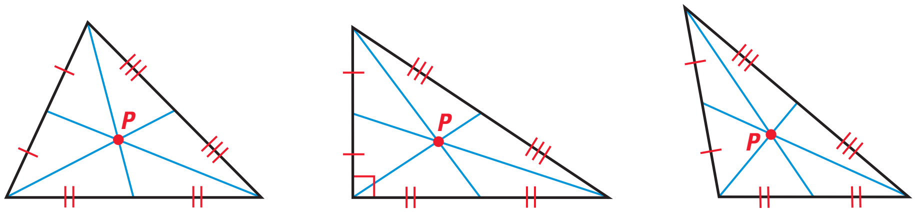
GOAL 1 USING MEDIANS OF A TRIANGLE

In Lesson 5.2, you studied two special types of segments of a triangle: perpendicular bisectors of the sides and angle bisectors. In this lesson, you will study two other special types of segments of a triangle: *medians* and *altitudes*.



A **median of a triangle** is a segment whose endpoints are a vertex of the triangle and the midpoint of the opposite side. For instance, in $\triangle ABC$ shown at the right, D is the midpoint of side $\overline{BC}$. So, $\overline{AD}$ is a median of the triangle.

The three medians of a triangle are concurrent. The point of concurrency is called the **centroid of the triangle.** The centroid, labeled P in the diagrams below, is always inside the triangle.



acute triangle | right triangle | obtuse triangle

The medians of a triangle have a special concurrency property, as described in Theorem 5.7. Exercises 13–16 ask you to use paper folding to demonstrate the relationships in this theorem. A proof appears on pages 836–837.

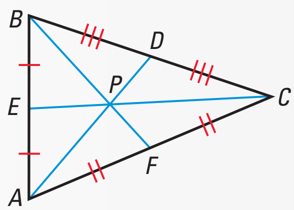
THEOREM

THEOREM 5.7 ***Concurrency of Medians of a Triangle***

The medians of a triangle intersect at a point that is two thirds of the distance from each vertex to the midpoint of the opposite side.

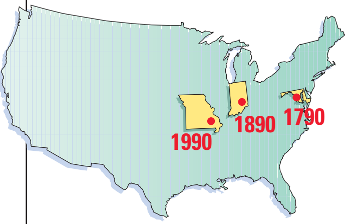
If P is the centroid of $\triangle ABC$, then

$AP = \frac{2}{3}AD$, $BP = \frac{2}{3}BF$, and $CP = \frac{2}{3}CE$.



The centroid of a triangle can be used as its balancing point, as shown on the next page.

FOCUS ON APPLICATIONS

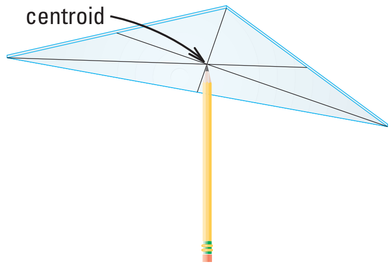


REAL LIFE

CENTER OF POPULATION

Suppose the location of each person counted in a census is identified by a weight placed on a flat, weightless map of the United States. The map would balance at a point that is the center of the population. This center has been moving westward over time.

A triangular model of uniform thickness and density will balance at the centroid of the triangle. For instance, in the diagram shown at the right, the triangular model will balance if the tip of a pencil is placed at its centroid.



EXAMPLE 1 *Using the Centroid of a Triangle*

P is the centroid of $\triangle QRS$ shown below and $PT = 5$. Find RT and RP.

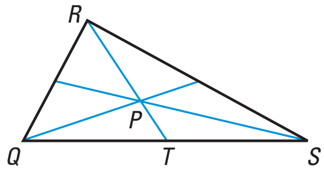
SOLUTION

Because P is the centroid, $RP = \frac{2}{3}RT$.

Then $PT = RT - RP = \frac{1}{3}RT$.

Substituting 5 for PT, $5 = \frac{1}{3}RT$, so $RT = 15$.

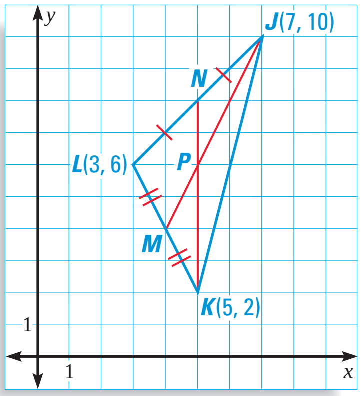
Then $RP = \frac{2}{3}RT = \frac{2}{3}(15) = 10$.



▶ So, $RP = 10$ and $RT = 15$.

EXAMPLE 2 *Finding the Centroid of a Triangle*

Find the coordinates of the centroid of $\triangle JKL$.



SOLUTION

You know that the centroid is two thirds of the distance from each vertex to the midpoint of the opposite side.

Choose the median $\overline{KN}$. Find the coordinates of N, the midpoint of $\overline{JL}$. The coordinates of N are

$$\left(\frac{3+7}{2}, \frac{6+10}{2}\right) = \left(\frac{10}{2}, \frac{16}{2}\right) = (5, 8).$$

Find the distance from vertex K to midpoint N. The distance from $K(5, 2)$ to $N(5, 8)$ is $8 - 2$, or 6 units.

Determine the coordinates of the centroid, which is $\frac{2}{3} \cdot 6$, or **4** units up from vertex K along the median $\overline{KN}$.

▶ The coordinates of centroid P are $(5, 2 + \mathbf{4})$, or $(5, 6)$.

....

Exercises 21–23 ask you to use the Distance Formula to confirm that the distance from vertex J to the centroid P in Example 2 is two thirds of the distance from J to M, the midpoint of the opposite side.

STUDENT HELP

INTERNET **HOMEWORK HELP**
Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 USING ALTITUDES OF A TRIANGLE

An **altitude of a triangle** is the perpendicular segment from a vertex to the opposite side or to the line that contains the opposite side. An altitude can lie inside, on, or outside the triangle.

Every triangle has three altitudes. The lines containing the altitudes are concurrent and intersect at a point called the **orthocenter of the triangle**.

Logical Reasoning

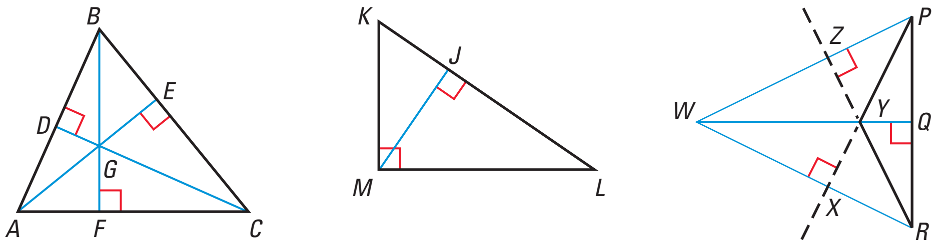
EXAMPLE 3 *Drawing Altitudes and Orthocenters*

Where is the orthocenter located in each type of triangle?

a. Acute triangle **b.** Right triangle **c.** Obtuse triangle

SOLUTION

Draw an example of each type of triangle and locate its orthocenter.



a. $\triangle ABC$ is an acute triangle. The three altitudes intersect at G, a point *inside* the triangle.

b. $\triangle KLM$ is a right triangle. The two legs, $\overline{LM}$ and $\overline{KM}$, are also altitudes. They intersect at the triangle's right angle. This implies that the orthocenter is *on* the triangle at M, the vertex of the right angle of the triangle.

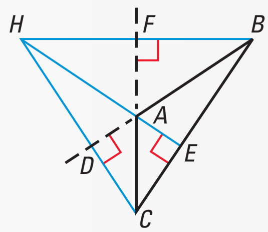
c. $\triangle YPR$ is an obtuse triangle. The three lines that contain the altitudes intersect at W, a point that is *outside* the triangle.

THEOREM

THEOREM 5.8 ***Concurrency of Altitudes of a Triangle***

The lines containing the altitudes of a triangle are concurrent.

If $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ are the altitudes of $\triangle ABC$, then the lines $\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$, and $\overleftrightarrow{CD}$ intersect at some point H.



Exercises 24–26 ask you to use construction to verify Theorem 5.8. A proof appears on page 838.

GUIDED PRACTICE

Vocabulary Check ✓

1. The *centroid* of a triangle is the point where the three __?__ intersect.

Concept Check ✓

2. In Example 3 on page 281, explain why the two legs of the right triangle in part (b) are also altitudes of the triangle.

Skill Check ✓

Use the diagram shown and the given information to decide in each case whether $\overline{EG}$ is a *perpendicular bisector*, an *angle bisector*, a *median*, or an *altitude* of $\triangle DEF$.

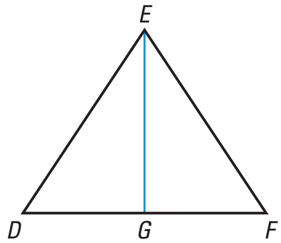
3. $\overline{DG} \cong \overline{FG}$

4. $\overline{EG} \perp \overline{DF}$

5. $\angle DEG \cong \angle FEG$

6. $\overline{EG} \perp \overline{DF}$ and $\overline{DG} \cong \overline{FG}$

7. $\triangle DGE \cong \triangle FGE$



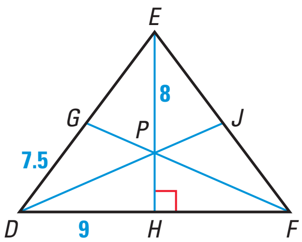
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 811.

USING MEDIANS OF A TRIANGLE **In Exercises 8–12, use the figure below and the given information.**

P is the centroid of $\triangle DEF$, $\overline{EH} \perp \overline{DF}$, $DH = 9$, $DG = 7.5$, $EP = 8$, and $DE = FE$.



8. Find the length of $\overline{FH}$.

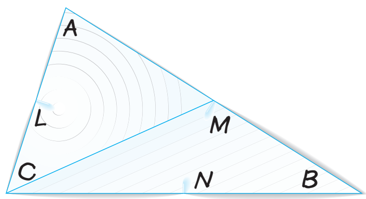
9. Find the length of $\overline{EH}$.

10. Find the length of $\overline{PH}$.

11. Find the perimeter of $\triangle DEF$.

12. **LOGICAL REASONING** In the diagram of $\triangle DEF$ above, $\frac{EP}{EH} = \frac{2}{3}$. Find $\frac{PH}{EH}$ and $\frac{PH}{EP}$.

PAPER FOLDING **Cut out a large acute, right, or obtuse triangle. Label the vertices. Follow the steps in Exercises 13–16 to verify Theorem 5.7.**



13. Fold the sides to locate the midpoint of each side. Label the midpoints.

14. Fold to form the median from each vertex to the midpoint of the opposite side.

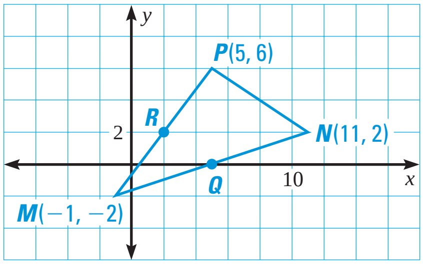
15. Did your medians meet at about the same point? If so, label this centroid point.

16. Verify that the distance from the centroid to a vertex is two thirds of the distance from that vertex to the midpoint of the opposite side.

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 8–11, 13–16
Example 2: Exs. 17–23
Example 3: Exs. 24–26

USING ALGEBRA **Use the graph shown.**



17. Find the coordinates of Q, the midpoint of $\overline{MN}$.

18. Find the length of the median $\overline{PQ}$.

19. Find the coordinates of the centroid. Label this point as T.

20. Find the coordinates of R, the midpoint of $\overline{MP}$. Show that the quotient $\frac{NT}{NR}$ is $\frac{2}{3}$.

USING ALGEBRA **Refer back to Example 2 on page 280.**

21. Find the coordinates of M, the midpoint of $\overline{KL}$.

22. Use the Distance Formula to find the lengths of $\overline{JP}$ and $\overline{JM}$.

23. Verify that $JP = \frac{2}{3}JM$.

STUDENT HELP

Look Back
To construct an altitude, use the construction of a perpendicular to a line through a point not on the line, as shown on p. 130.

CONSTRUCTION **Draw and label a large scalene triangle of the given type and construct the altitudes. Verify Theorem 5.8 by showing that the lines containing the altitudes are concurrent, and label the orthocenter.**

24. an acute $\triangle ABC$

25. a right $\triangle EFG$ with right angle at G

26. an obtuse $\triangle KLM$

TECHNOLOGY **Use geometry software to draw a triangle. Label the vertices as *A*, *B*, and *C*.**

27. Construct the altitudes of $\triangle ABC$ by drawing perpendicular lines through each side to the opposite vertex. Label them $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.

28. Find and label G and H, the intersections of $\overline{AD}$ and $\overline{BE}$ and of $\overline{BE}$ and $\overline{CF}$.

29. Prove that the altitudes are concurrent by showing that $GH = 0$.

FOCUS ON CAREERS

CARDIOLOGY TECHNICIAN
Technicians use equipment like electrocardiographs to test, monitor, and evaluate heart function.

CAREER LINK
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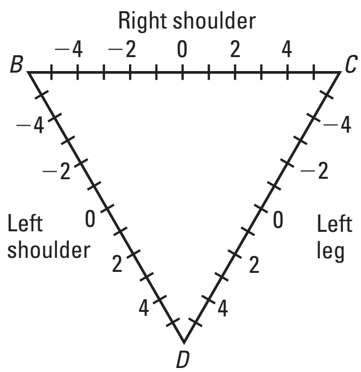
ELECTROCARDIOGRAPH **In Exercises 30–33, use the following information about electrocardiographs.**
The equilateral triangle $\triangle BCD$ is used to plot electrocardiograph readings. Consider a person who has a left shoulder reading (S) of -1, a right shoulder reading (R) of 2, and a left leg reading (L) of 3.



30. On a large copy of $\triangle BCD$, plot the reading to form the vertices of $\triangle SRL$. (This triangle is an *Einthoven's Triangle*, named for the inventor of the electrocardiograph.)

31. Construct the circumcenter M of $\triangle SRL$.

32. Construct the centroid P of $\triangle SRL$. Draw line r through P parallel to $\overline{BC}$.

33. Estimate the measure of the acute angle between line r and $\overline{MP}$. Cardiologists call this the angle of a person's heart.

Test Preparation

34. MULTI-STEP PROBLEM Recall the formula for the area of a triangle, $A = \frac{1}{2}bh$, where b is the length of the base and h is the height. The height of a triangle is the length of an altitude.

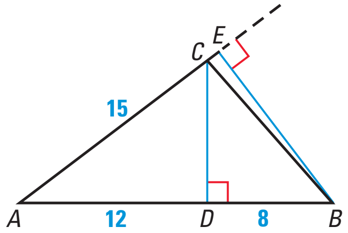
a. Make a sketch of $\triangle ABC$. Find CD, the height of the triangle (the length of the altitude to side $\overline{AB}$).

b. Use CD and AB to find the area of $\triangle ABC$.

c. Draw $\overline{BE}$, the altitude to the line containing side $\overline{AC}$.

d. Use the results of part (b) to find the length of $\overline{BE}$.

e. *Writing* Write a formula for the length of an altitude in terms of the base and the area of the triangle. Explain.

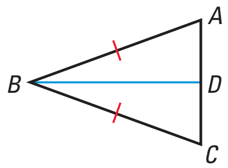


★ Challenge

SPECIAL TRIANGLES Use the diagram at the right.

35. GIVEN ▶ $\triangle ABC$ is isosceles.
$\overline{BD}$ is a median to base $\overline{AC}$.

PROVE ▶ $\overline{BD}$ is also an altitude.



36. Are the medians to the *legs* of an isosceles triangle also altitudes? Explain your reasoning.

37. Are the medians of an *equilateral* triangle also altitudes? Are they contained in the angle bisectors? Are they contained in the perpendicular bisectors?

EXTRA CHALLENGE
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38. LOGICAL REASONING In a proof, if you are given a median of an equilateral triangle, what else can you conclude about the segment?

MIXED REVIEW

USING ALGEBRA Write an equation of the line that passes through point *P* and is parallel to the line with the given equation. (Review 3.6 for 5.4)

39. $P(1, 7), y = -x + 3$

40. $P(-3, -8), y = -2x - 3$

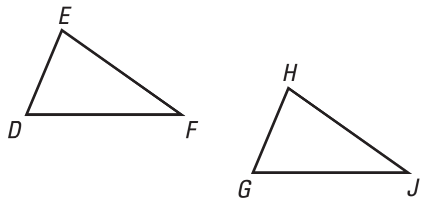
41. $P(4, -9), y = 3x + 5$

42. $P(4, -2), y = -\frac{1}{2}x - 1$

DEVELOPING PROOF In Exercises 43 and 44, state the third congruence that must be given to prove that $\triangle DEF \cong \triangle GHJ$ using the indicated postulate or theorem. (Review 4.4)

43. GIVEN ▶ $\angle D \cong \angle G$, $\overline{DF} \cong \overline{GJ}$
AAS Congruence Theorem

44. GIVEN ▶ $\angle E \cong \angle H$, $\overline{EF} \cong \overline{HJ}$
ASA Congruence Postulate



45. USING THE DISTANCE FORMULA Place a right triangle with legs of length 9 units and 13 units in a coordinate plane and use the Distance Formula to find the length of the hypotenuse. (Review 4.7)

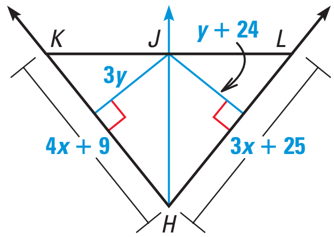
QUIZ 1

Self-Test for Lessons 5.1–5.3

Use the diagram shown and the given information. (Lesson 5.1)

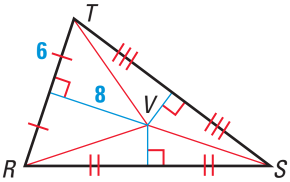
$\overline{HJ}$ is the perpendicular bisector of $\overline{KL}$.

$\overrightarrow{HJ}$ bisects $\angle KHL$.

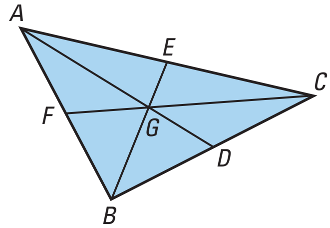


1. Find the value of x.
2. Find the value of y.

In the diagram shown, the perpendicular bisectors of $\triangle RST$ meet at V. (Lesson 5.2)



3. Find the length of $\overline{VT}$.
4. What is the length of $\overline{VS}$? Explain.
5. **BUILDING A MOBILE** Suppose you want to attach the items in a mobile so that they hang horizontally. You would want to find the balancing point of each item. For the triangular metal plate shown, describe where the balancing point would be located. (Lesson 5.3)



$\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are medians. $CF = 12$ in.

MATH & History — Optimization

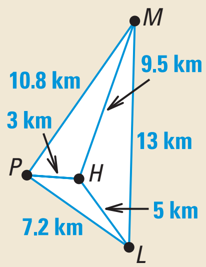
APPLICATION LINK www.mcdougallittell.com

THEN

THROUGHOUT HISTORY, people have faced problems involving minimizing resources or maximizing output, a process called optimization. The use of mathematics in solving these types of problems has increased greatly since World War II, when mathematicians found the optimal shape for naval convoys to avoid enemy fire.

NOW

TODAY, with the help of computers, optimization techniques are used in many industries, including manufacturing, economics, and architecture.



1. Your house is located at point H in the diagram. You need to do errands at the post office (P), the market (M), and the library (L). In what order should you do your errands to minimize the distance traveled?
2. Look back at Exercise 34 on page 270. Explain why the goalie's position on the angle bisector optimizes the chances of blocking a scoring shot.