Limiting behavior of global attractors for singularly perturbed beam equations with strong damping

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Abstract. The limiting behavior of global attractors $\mathcal{A}_\varepsilon$ for singularly perturbed beam equations

$$\varepsilon^2\frac{\partial^2 u}{\partial t^2}+\varepsilon\delta\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial t}+\alpha Au+g(\|u\|^2_{1/4})A^{1/2}u=0$$

is investigated. It is shown that for any neighborhood $\mathcal{U}$ of $\mathcal{A}_0$ the set $\mathcal{A}_\varepsilon$ is included in $\mathcal{U}$ for ε small.

Keywords: strongly damped beam equation, compact attractor, upper semicontinuity of global attractors

Classification: 35B40, 35Q20

1. Introduction.

Consider the following problems

$$(1.1)_\varepsilon \qquad \begin{cases} \varepsilon^2\frac{\partial^2 u}{\partial t^2}+\varepsilon\delta\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial t}+\alpha Au+g(\|u\|^2_{1/4})A^{1/2}u=0\\ u(0)=u_0\\ \frac{\partial u}{\partial t}(0)=v_0\end{cases}$$

and

$$(1.1)_0 \qquad \begin{cases} \frac{\partial u}{\partial t}+\alpha u+g(\|u\|^2_{1/4})A^{-1/2}u=0\\ u(0)=u_0\end{cases}$$

where g is an increased C^1 function, $\varepsilon>0$ is a small parameter, $\alpha<0$ and δ is a real unrestricted on the sign. Here A is a sectorial operator in $\mathcal{L}_2(0,l)$ defined by a differential operator $\partial^4/\partial x^4$ and the boundary conditions corresponding either to hinged ends, when

$$(1.2)_H \qquad u(x)=u_{xx}(x)=0 \qquad \text{at } x=0,l$$

or to clamped ends, when

$$(1.2)_C \qquad u(x)=u_x(x)=0 \qquad \text{at } x=0,l.$$

Let $\{S(t); t \geq 0\}$ be a semidynamical system in a Banach space $\mathcal{X}$ (for definition, see, for example, [H, Chapter 4]). A set $J \subseteq \mathcal{X}$ is called *invariant* if $S(t)J = J$ for all $t \geq 0$. An invariant set $\mathcal{U} \subseteq \mathcal{X}$ is called *a global compact attractor* for the semidynamical system $S(t)$ if it is a compact set in $\mathcal{X}$ and $\lim_{t\to\infty} \operatorname{dist}(S(t)B, \mathcal{U}) = 0$ for any bounded set $B \subseteq \mathcal{X}$, where

$$\operatorname{dist}(\mathcal{A}, \mathcal{B}) = \sup_{x\in\mathcal{A}} \inf_{y\in\mathcal{B}} \|x - y\|.$$

It is shown (Theorem 3.1) that, for small ε, there is a compact global attractor $\mathcal{A}_\varepsilon \subseteq W^{2,2}(0,l) \times \mathcal{L}_2(0,l)$ for a semidynamical system generated by $(1.1)_\varepsilon$. For $\varepsilon = 0$, the problem $(1.1)_0$ also has a compact attractor which can be naturally embedded into compact set $\mathcal{A}_0 \subseteq W^{2,2} \times \mathcal{L}_2(0,l)$.

Let us note that under the assumptions $g \geq 0$ and $\delta \geq 0$, the dynamics of $(1.1)_\varepsilon, \varepsilon \geq 0$, is simple—every trajectory approaches a zero equilibrium state (see Remark 3.2). On the other hand, if $g(0) < 0$ is sufficiently small, then the attractor $\mathcal{A}_\varepsilon, \varepsilon \geq 0$, contains $2n - 1$ distinct equilibrium states (Remark 3.1) for some $n \in \mathbb{N}$. In this case the attractor $\mathcal{A}_\varepsilon$ is a union of unstable manifolds for equilibrium states (see, for example, [BV, Theorem 10.1]).

The purpose of this paper is to obtain some relationships between the attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ for small ε. It is given in terms of upper semicontinuity of $\mathcal{A}_0$ at $\varepsilon = 0$ with respect to the sets $\{\mathcal{A}_\varepsilon; \varepsilon > 0\}$.

In this paper, the following hypotheses are needed:

(H1) $$g \in C^1(\mathbb{R}^+, \mathbb{R}); g'(r) > 0 \ \text{ for } \ r \geq 0 \ \text{ and } \ \int_0^\infty g(s)\, ds > -\infty$$

(H2) $$\alpha > 0, \delta \in \mathbb{R}.$$

We can now state our main result.

Theorem 1.1. *Suppose that the hypotheses* (H1)-(H2) *are satisfied. Then the attractor $\mathcal{A}_0$ is upper semicontinuous at zero with respect to the sets $\mathcal{A}_\varepsilon; \varepsilon > 0$, i.e.*

$$\lim_{\varepsilon \longrightarrow 0+} \operatorname{dist}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0.$$

In other words, for any neighborhood $\mathcal{U}$ of $\mathcal{A}_0$, the set $\mathcal{A}_\varepsilon$ is included in $\mathcal{U}$ for ε small.

As an example for $(1.1)_\varepsilon$ one can consider a problem of a transverse motion, at a small strain, in the $x - y$ plane, of a viscoelastic beam in a viscous medium whose resistance is proportional to the velocity. The ends of the beam are fixed at the points $x = 0$ and $x = l + d$, where d is a load (positive or negative) of the beam and a stress-free state of the beam occupies the interval $[0, l]$. Shear deformations are neglected in this model. Then the equation of the motion in y-direction is

$$(1.3)\quad \frac{\partial^2 u}{\partial t^2} + \delta \cdot \frac{\partial u}{\partial t} + \frac{\xi I}{\varrho} \cdot A\frac{\partial u}{\partial t} + \frac{EI}{\varrho} Au + \left(\frac{ESd}{l\varrho} + \frac{ES}{2l\varrho} \cdot \int_0^l u_x^2\, dx \right) A^{1/2} u = 0$$

where E is the Young's modulus, S the cross-sectional area, ξ the effective viscosity, I the cross-sectional second moment of area, ϱ the mass per unit length and δ the coefficient of external damping. For details see [F], [B1], [B2] and references therein.

Put $\varepsilon = \frac{\varrho}{\xi I} > 0$. Then the equation $(1.1)_\varepsilon$ follows from (1.3) by a suitably rescaling the time. The limit $\varepsilon \longrightarrow 0^+$ corresponds to the case in which the effective viscosity tends to $+\infty$.

In recent years, many authors have studied the attractors for a singularly perturbed hyperbolic equation

$$(1.4)_\varepsilon \qquad \varepsilon^2 \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u = f(u).$$

See, for example, [GT], [ChL] and other references in [HR1] and [HR2]. Hale and Rougel have shown that the attractors of $(1.4)_\varepsilon$ converge in the Hausdorff topology towards the one corresponding to $\varepsilon = 0$

$$(1.4)_0 \qquad \frac{\partial u}{\partial t} - \Delta u = f(u).$$

Clearly, the main difference between $(1.1)_\varepsilon$-$(1.1)_0$ and $(1.4)_\varepsilon$-$(1.4)_0$ is that $(1.4)_0$ is the quasilinear parabolic equation with an unbounded linear operator $-\Delta$, while the problem $(1.1)_0$ is the quasilinear differential equation in a Hilbert space with a bounded operator $\alpha \cdot Id$.

The paper is organized as follows. Definitions and notations are recalled in Section 2. Following the style of Henry's lecture notes [H, Chapter 3, 4], one can obtain a local and global existence of solutions of $(1.1)_\varepsilon$. Section 3 deals with the existence and uniform boundedness of attractors $\mathcal{A}_\varepsilon$. Section 4 is devoted to the singular equation $(1.1)_0$. The proof of the existence of $\mathcal{A}_0$ is given. In Section 5 we prove Theorem 1.1.

2. Preliminaries.

Let $X = L_2(0, l)$ be a real Hilbert space equipped with its usual scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Define $A : X \longrightarrow X; Au = \partial^4 u/\partial x^4$ for each $u \in C_B^\infty(0, l)$, where

$$C_B^\infty(0, l) = \{\Phi \in C^\infty(0, l);\ \Phi \text{ satisfies b.c. } B\},$$

for $B = H$ or $B = C$. Let A be the self-adjoint closure in X of its restriction to $C_B^\infty(0, l)$. It is well known that A is a sectorial operator in X (see [H, p. 19]). Therefore the fractional powers A^β can be defined. Let X^β be a Hilbert space consisting of the domain of fractional power A^β with the graph norm, i.e. $\|u\|_\beta = \|A^\beta u\|$ for all $u \in X^\beta$. Let us note that $X^\beta \hookrightarrow W^{4\beta,2}(0,l)$ for $\beta \ge 0$. We also have $\|u\|_\beta \le \lambda_1^{\beta-\sigma} \|u\|_\sigma$ for any $0 \le \beta \le \sigma$ and $u \in X^\sigma$. Recall that A has a compact resolvent A^{-1}. Therefore the imbedding $X^\sigma \hookrightarrow\hookrightarrow X^\beta$ is compact, whenever $0 \le \beta < \sigma$.

Let $\Phi_n, j \in \mathbb{N}$, denote the orthonormal basis of X consisting of eigenvectors of the operators A:

$$A\Phi_n = \lambda_n \Phi_n; \qquad 0 < \lambda_1 < \lambda_2 < \dots; \ \lambda_n \longrightarrow +\infty \text{ as } n \longrightarrow +\infty.$$

Denote by $\mathbb{P}_m$ the projector in X onto the space spanned by $\{\Phi_1, \dots, \Phi_m\}$. Clearly,

$$\|\mathbb{P}_m u\|_\beta \le \lambda_m^{\beta-\sigma} \|\mathbb{P}_m u\|_\sigma \le \lambda_m^{\beta-\sigma} \|u\|_\sigma \text{ for each } u \in X^\sigma \text{ and } \beta, \sigma \ge 0.$$

Let $S(t)$ be a semidynamical system in a Banach space $\mathcal{X}$.

A set B *dissipates* a set J if there exists $T = T(J) > 0$ such that $t \ge T$ implies $S(t)J \subseteq B$. A semidynamical system $S(t)$ is called *bounded dissipative* if there exists a bounded set B which dissipates all bounded sets.

The *omega-limit set* is defined by

$$\Omega(B) = \bigcap_{t\ge 0} \mathrm{cl}\,(\bigcup_{s \ge t} S(s)B) \quad \text{(the closure is taken in } \mathcal{X}).$$

In this paper, the time derivatives will be denoted by

$$\frac{\partial}{\partial t}(\cdot) = (\cdot)'.$$

In order to obtain a local and global existence we rewrite $(1.1)_\varepsilon$ as a first order ordinary differential equation in the Hilbert space $\mathcal{X} = X^{1/2} \times X$. This is to do by letting $v = u'$. Then we can rewrite $(1.1)_\varepsilon$ as

$$\frac{d}{dt}\phi(t) + \mathcal{L}_\varepsilon \phi(t) + \mathcal{F}_\varepsilon(\phi(t)) = 0; \quad \phi(0) = \phi_0 \tag{2.1}$$

where

$$\phi(t) = [u(t), v(t)]; \quad \mathcal{L}_\varepsilon[u,v] = [-v, \varepsilon^{-2}A(\alpha u + v) + \varepsilon^{-1}\delta v]$$
$$\text{and} \quad \mathcal{F}_\varepsilon([u,v]) = [0, -\varepsilon^{-2} g(\|u\|_{1/4}^2) A^{1/2} u].$$

It is known [M1, Theorem 1.1] that the operator $\mathcal{L}([u,v]) = [-v, A(\alpha u + v)]$ is sectorial in $X^{1/2} \times X$. Then Theorem 1.3.2 of [H] demonstrates that the operator $\mathcal{L}_\varepsilon$ is sectorial in $\mathcal{X}$. The domain of $\mathcal{L}_\varepsilon$ is

$$D(\mathcal{L}_\varepsilon) = \{[u,v] \in X^{1/2} \times X^{1/2};\ \alpha u + v \in D(A)\}.$$

From now on we restrict ε_0 by

$$\text{(H3)} \qquad \lambda_1 - 2 \cdot \varepsilon_0 |\delta| > 0.$$

Since Re $\sigma(A) \geq \lambda_1$, then, by looking at the spectrum $\sigma(\mathcal{L}_\varepsilon)$, we see that

$$\text{(2.2)} \qquad \operatorname{Re} \sigma(\mathcal{L}_\varepsilon) > \frac{\alpha}{2} \quad \text{for each } \varepsilon \in (0, \varepsilon_0].$$

Since $\mathcal{L}_\varepsilon$ is the sectorial operator, then $-\mathcal{L}_\varepsilon$ generates an analytic semigroup $\exp(-\mathcal{L}_\varepsilon)$. Let $\omega \in (0, \alpha/2)$. Due to the estimate (2.2), it follows that there is $M(\varepsilon) > 0$ such that

$$\text{(2.3)} \qquad \|\exp(-\mathcal{L}_\varepsilon t)\|_{\mathcal{X}} \leq M(\varepsilon) \cdot e^{-\omega t} \qquad \text{for each } t \geq 0.$$

According to [H, Theorems 3.3.3, 3.3.4, 3.4.1 and 3.5.2], the local existence, uniqueness, continuous dependence on initial conditions and continuation of solutions od (2.1) immediately follow. More precisely, for each $\Phi_0 \in \mathcal{X}$ there exists $T = T(\Phi_0) > 0$ and a unique function $\Phi = \Phi(t, \Phi_0)$ such that

$$\Phi \in C([0, t_1) : \mathcal{X}) \cap C_1((t_0, t_1) : \mathcal{X}) \quad \text{for each} \quad 0 < t_0 < t_1 < T,$$

$\Phi(0) = \Phi_0, \Phi(t) \in D(L)$ for each $t \in (0, T)$ and $\Phi(t)$ is the solution of (2.1) on the interval of existence $(0, T)$.

If we take the scalar product in X of $(1.1)_\varepsilon$ with u', we conclude that

$$\text{(2.4)} \qquad \frac{1}{2}\frac{d}{dt}\left\{\alpha\|u\|_{1/2}^2 + \varepsilon^2\|u'\|^2 + \mathcal{G}(\|u\|_{1/4}^2)\right\} + \|u'\|_{1/2}^2 + \varepsilon\delta\|u'\|^2 = 0$$

where $\mathcal{G}$ is the primitive of g, i.e.

$$\mathcal{G}(r) = \int_0^r g(s)\, ds \qquad \text{for } \; r \geq 0.$$

Thanks to (H1) we infer the existence of $C_0 > 0$ such that

$$\text{(2.5)} \qquad g(r) \cdot r \geq \int_0^r g(s)\, ds \geq -C_0 \qquad \text{for each } r \geq 0.$$

From (2.4) we observe that

$$\text{(2.6)} \qquad \begin{aligned} &\int_0^r \|u'(s)\|_{1/2}^2\, ds + \varepsilon^2\|u'(t)\|^2 + \alpha \cdot \|u(t)\|_{1/2}^2 \leq \\ &\leq \varepsilon^2\|u'(0)\|^2 + \alpha \cdot \|u(0)\|_{1/2}^2 + \mathcal{G}(\|u(0)\|_{1/4}^2) + C_0 \\ &\qquad \text{for each } t \geq 0. \end{aligned}$$

Thus the solutions of $(1.1)_\varepsilon$ and (2.1) exist globally on $\mathbb{R}^+$. Hence the initial value problem (2.1) generates a semidynamical system $\{S_\varepsilon(t);\, t \geq 0\}$ in $\mathcal{X}$, where $S_\varepsilon(t)\Phi(0) = \Phi_\varepsilon(t, \Phi(0))$ for $t \geq 0$.

Since there are many estimates in this paper, we will let $C_0, C_1, C_2, \ldots$ be generic positive constants always assumed to be independent of ε.

3. The existence and uniform regularity of global attractors.

Lemma 3.1. *The semidynamical system S_ε is bounded dissipative in $\mathcal{X}$. More precisely, there exists a constant $C_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon]$ and any bounded set $B \subseteq X^{1/2} \times X$ there is $T(\varepsilon, B) > 0$ with the property*

$$t \geq T(\varepsilon, B) \text{ implies}$$
$$\varepsilon^2 \|v\|^2 + \alpha \|u\|^2_{1/2} \leq C_1 \text{ for each } (u, v) \in S_\varepsilon(t)B.$$

Proof: Define a functional $V_\varepsilon : \mathcal{X} \longrightarrow \mathbb{R}$ by

$$V_\varepsilon(\Phi, \Psi) = \frac{1}{2}\left\{\alpha\|\Phi\|^2_{1/2} + \varepsilon^2\|\Psi\|^2 + \mathcal{G}(\|\Phi\|^2_{1/4})\right\} + b\varepsilon^2(\Phi, \Psi)$$

where b is a positive real satisfying

$$0 < b < \min\left\{\alpha, \frac{\sqrt{\alpha\lambda_1}}{2\varepsilon_0}; (\lambda_1 - \varepsilon_0|\delta|)\left(\frac{\lambda_1}{\alpha} + \varepsilon_0^2 + \frac{\varepsilon_0^2\delta^2}{\alpha\lambda_1}\right)^{-1}\right\}.$$

From (2.4) we obtain

$$\begin{aligned}\frac{d}{dt}V_\varepsilon(u_\varepsilon, u_\varepsilon') = -\|u_\varepsilon'\|^2_{1/2} - \varepsilon\delta\|u_\varepsilon'\|^2 + b\varepsilon^2\|u_\varepsilon'\|^2 - b\cdot(Au_\varepsilon', u_\varepsilon) - \\ -b\alpha\cdot(Au_\varepsilon, u_\varepsilon) - b\varepsilon\delta\cdot(u_\varepsilon', u_\varepsilon) - b\cdot g(\|u_\varepsilon\|^2_{1/4})\cdot\|u_\varepsilon\|^2_{1/4} \leq \\ \leq -\|u_\varepsilon'\|^2_{1/2} - (\varepsilon\delta - b\varepsilon^2)\cdot\|u_\varepsilon'\|^2 - b\alpha\cdot\|u_\varepsilon\|^2_{1/2} - b\cdot(A^{1/2}u_\varepsilon', A^{1/2}u_\varepsilon) - \\ -b\varepsilon\delta\cdot(u_\varepsilon', u_\varepsilon) + bC_0.\end{aligned}$$

Then we deduce from the Young's inequality

$$|(\Phi, \Psi)| \leq (r^2\|\Phi\|^2 + r^{-2}\|\Psi\|^2)/2$$

that

$$\begin{aligned}\frac{d}{dt}V_\varepsilon(u_\varepsilon, u_\varepsilon') \leq -\|u_\varepsilon'\|^2_{1/2} - (\varepsilon\delta - b\varepsilon^2)\cdot\|u_\varepsilon'\|^2 - b\alpha\cdot\|u_\varepsilon\|^2_{1/2} + bC_0 + \\ +b\cdot(r^2\|u_\varepsilon'\|^2_{1/2} + r^{-2}\|u_\varepsilon\|^2_{1/2})/2 + b\varepsilon|\delta|\cdot(s^2\|u_\varepsilon'\|^2 + s^{-2}\|u_\varepsilon\|^2)/2.\end{aligned}$$

Put $r^2 = 2/\alpha$ and $s^2 = \frac{2\varepsilon|\delta|}{\alpha\cdot\lambda_1}$. Then

$$\begin{aligned}\frac{d}{dt}V_\varepsilon(u_\varepsilon, u_\varepsilon') \leq -(1 - \frac{b}{\alpha})\cdot\|u_\varepsilon'\|^2_{1/2} - (\varepsilon\delta - b\varepsilon^2 - b\frac{\varepsilon^2\delta^2}{\alpha\cdot\lambda_1})\cdot\|u_\varepsilon'\|^2 - \\ -b(\alpha - \alpha/4 - \alpha/4)\cdot\|u_\varepsilon\|^2_{1/2} + bC_0 \leq \\ \leq -\left(\lambda_1(1 - \frac{b}{\alpha}) + \varepsilon\delta - b\varepsilon^2 - b\frac{\varepsilon^2\delta^2}{\alpha\cdot\lambda_1}\right)\cdot\|u_\varepsilon'\|^2 - b\cdot\frac{\alpha}{2}\cdot\|u_\varepsilon\|^2_{1/2} + bC_0.\end{aligned}$$

Since $b \cdot \left(\frac{\lambda_1}{\alpha} + \varepsilon_0^2 + \frac{\varepsilon_0^2\delta^2}{\alpha\lambda_1}\right) < \lambda_1 - \varepsilon_0|\delta|$ and $b < \alpha$, one can easily show that there are constants $C_2, C_3 > 0$ such that

$$\frac{d}{dt}V_\varepsilon(u_\varepsilon, u_\varepsilon') \leq -C_2(\|u_\varepsilon'\|^2 + \|u_\varepsilon\|_{1/2}^2) + C_3\,. \tag{3.1}$$

Let us introduce a function

$$y_\varepsilon(t) = V_\varepsilon(u_\varepsilon(t), u_\varepsilon'(t)) + C_3\,.$$

Thanks to the inequality

$$b\varepsilon^2(u_\varepsilon', u_\varepsilon) \leq \frac{\varepsilon^2}{2}\cdot\|u_\varepsilon'\|^2 + \frac{\varepsilon^2 b^2}{2\cdot\lambda_1}\cdot\|u_\varepsilon\|_{1/2}^2$$

we have

$$0 \leq y_\varepsilon(t) \leq \alpha\cdot\|u_\varepsilon(t)\|_{1/2}^2 + \varepsilon^2\|u_\varepsilon'(t)\|^2 + \frac{1}{2}\mathcal{G}(\|u_\varepsilon(t)\|_{1/4}^2) + C_3\,.$$

Since g increases on $\mathbb{R}^+$, there exists an increasing function $\vartheta \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$0 \leq y_\varepsilon(t) \leq \vartheta(\|u_\varepsilon(t)\|_{1/2}^2 + \|u_\varepsilon'(t)\|^2)$$

and $\vartheta'(r) \geq \sigma > 0$ for each $r \geq 0$.

Then we can rewrite (3.1) as an ordinary differential inequality

$$\frac{d}{dt}y_\varepsilon \leq -C_2\vartheta^{-1}(y_\varepsilon) + C_3\,.$$

An obvious contradiction argument gives us either $0 \leq y_\varepsilon(t) \leq \vartheta(C_3/C_2)$ for each $t \geq 0$ or there is $T(\varepsilon, y_\varepsilon(0)) > 0$ such that $0 \leq y_\varepsilon(t) \leq \vartheta(C_3/C_2) + 1$ for each $t \geq T(\varepsilon, y_\varepsilon(0))$. Due to the assumption on b, it follows that

$$y_\varepsilon(t) \geq \frac{1}{4}(\alpha\|u_\varepsilon(t)\|_{1/2}^2 + \varepsilon^2\|u_\varepsilon'(t)\|^2) + C_3 - C_0/2\,.$$

Thus Lemma 3.1 is proved. □

Consider a solution w_ε of the following linear strongly damped evolution equation

$$\varepsilon^2 w_\varepsilon'' + Aw_\varepsilon' + \alpha\cdot Aw_\varepsilon + \varepsilon\delta\cdot w_\varepsilon' + h_\varepsilon = 0$$

where

$$h_\varepsilon \in \mathcal{L}_p(\mathbb{R}^+; X) \quad \text{for} \quad p = 2 \quad \text{or} \quad p = \infty\,. \tag{3.2}$$

Lemma 3.2. *Assume $p = 2$ or $p = \infty$. Then there are constants $C_4, C_5, a > 0$ such that*

$$\varepsilon^2 \|\mathbb{P}_m w_\varepsilon'(t)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m w_\varepsilon(t)\|_1^2 \leq$$
$$\leq C_4(\varepsilon^2 \|\mathbb{P}_m w_\varepsilon'(0)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m w_\varepsilon(0)\|_1^2) \cdot e^{-2at} + C_5 \|h_\varepsilon\|_{\mathcal{L}_p(\mathbb{R}^+;X)}^2$$
$$\textit{for each } \ t \geq 0; \ \varepsilon \in (0, \varepsilon_0] \textit{ and } \ m \in \mathbb{N}.$$

Proof: Put $y(t) = \mathbb{P}_m w_\varepsilon(t)$. Clearly, $y(t), y'(t) \in D(A)$ for each $t \geq 0$. Let us introduce a substitution

$$z = y' + a \cdot y$$

where a is a positive real satisfying

$$0 < a < \min \left\{ \frac{\alpha}{2}; \frac{\lambda_1 - 2|\delta|\varepsilon_0}{4\varepsilon_0^2}; \ \frac{\alpha\lambda_1}{4\varepsilon_0} \left(\frac{\varepsilon_0 \alpha}{2} + |\delta| \right)^{-1} \right\}.$$

Then

$$\varepsilon^2 z' + (A - a\varepsilon^2 + \delta\varepsilon) z + ((\alpha - a)A + a^2\varepsilon^2 - a\delta\varepsilon) y + \mathbb{P}_m h_\varepsilon = 0 \tag{3.3}$$

Take the scalar product in X of (3.3) with Az to obtain

$$\frac{1}{2} \frac{d}{dt} \left\{ \varepsilon^2 \|z\|_{1/2}^2 + (\alpha - a)\|y\|_1^2 + (a^2\varepsilon^2 - a\delta\varepsilon)\|y\|_{1/2}^2 \right\} +$$
$$+ \|z\|_1^2 + (\delta\varepsilon - a\varepsilon^2)\|z\|_{1/2}^2 + a \cdot \left\{ (\alpha - a)\|y\|_1^2 + (a^2\varepsilon^2 - a\delta\varepsilon)\|y\|_{1/2}^2 \right\} =$$
$$= -(\mathbb{P}_m h_\varepsilon, Az) \leq \frac{1}{2} \cdot \|\mathbb{P}_m h_\varepsilon\|^2 + \frac{1}{2} \cdot \|z\|_1^2 .$$

From the assumption $a < \frac{\lambda_1 - 2|\delta|\varepsilon_0}{4\varepsilon_0^2}$ we have

$$\theta'(t) + 2a\theta(t) \leq \|h_\varepsilon(t)\|^2 \quad \text{ for } \ t \geq 0$$

where
$$\theta(t) = \varepsilon^2 \|z\|_{1/2}^2 + (\alpha - a)\|y\|_1^2 + (a^2\varepsilon^2 - a\delta\varepsilon)\|y\|_{1/2}^2 .$$

Therefore

$$\theta(t) \leq \theta(0) \cdot e^{-2at} + \int_0^t e^{-2a(t-s)} \|h_\varepsilon(s)\|^2 \, ds \leq$$
$$\leq \theta(0) \cdot e^{-2at} + C_5' \|h_\varepsilon\|_{\mathcal{L}_p(\mathbb{R}^+;X)}^2 .$$

Since $a < \frac{\alpha}{2}$ and $\frac{a\varepsilon_0}{\lambda_1} \left(\frac{\varepsilon_0 \alpha}{2} + |\delta| \right) < \frac{\alpha}{4}$, then

$$(\alpha - a)\|y\|_1^2 + (a^2\varepsilon^2 - a\delta\varepsilon)\|y\|_{1/2}^2 \geq$$
$$\geq \frac{\alpha}{2} \cdot \|y\|_1^2 - a\varepsilon_0 \left(\frac{\varepsilon_0 \alpha}{2} + |\delta| \right) \cdot \|y\|_{1/2}^2 \geq \frac{\alpha}{4} \cdot \|y\|_1^2 .$$

Then one can easily show that there are $C_4, C_5 > 0$ such that

$$\varepsilon^2 \|y'(t)\|_{1/2}^2 + \alpha \cdot \|y(t)\|_1^2 \leq$$
$$\leq C_4(\varepsilon^2 \|y'(0)\|_{1/2}^2 + \alpha \cdot \|y(0)\|_1^2) \cdot e^{-2at} + C_5 \|h_\varepsilon\|_{\mathcal{L}_p(\mathbb{R}^+;X)}^2$$

as claimed. □

The solution of (2.1) is given by the variation of constants by the formula

$$S_\varepsilon(t)\Phi_0 = \exp(-\mathcal{L}_\varepsilon t)\Phi_0 + \mathcal{U}_\varepsilon(t)\Phi_0$$

$$\text{where } \mathcal{U}_\varepsilon(t)\Phi_0 = \int_0^t \exp(-\mathcal{L}_\varepsilon(t-s)) \left[0, -\varepsilon^{-2} g(\|u_\varepsilon(s)\|_{1/4}^2) A^{1/2} u_\varepsilon(s)\right] ds\,.$$

Put $\left[w_\varepsilon(t), w_\varepsilon'(t)\right] = \mathcal{U}_\varepsilon(t)[u_0, v_0]$. Clearly, w_ε is a solution of the linear strongly damped evolution equation

$$\varepsilon^2 w_\varepsilon''(t) + Aw_\varepsilon'(t) + \alpha A w_\varepsilon(t) + \varepsilon\delta w_\varepsilon'(t) + h_\varepsilon(t) = 0$$
$$w_\varepsilon'(0) = w_\varepsilon(0) = 0$$

where $h_\varepsilon(t) = g(\|u_\varepsilon(t)\|_{1/4}^2) A^{1/2} u_\varepsilon(t)$ and u_ε is a solution of $(1.1)_\varepsilon$ satisfying the initial conditions

$$u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = v_0\,.$$

Lemma 3.3. *Let $\varepsilon \in (0, \varepsilon_0]$ be fixed. Then the set $K_\varepsilon = \bigcup_{t\geq 0} \mathcal{U}_\varepsilon(t) B$ is bounded in $X^1 \times X^{1/2}$ for any bounded set $B \subseteq X^{1/2} \times X$.*

Proof: Let B be a bounded set in $X^{1/2} \times X$, i.e. there is $M_1 > 0$ such that

$$\varepsilon^2 \|v\|^2 + \alpha \|u\|_{1/2}^2 + \mathcal{G}(\|u\|_{1/4}^2) \leq M_1 \text{ for each } (u,v) \in B\,.$$

Let $(u_0, v_0) \in B$ and u_ε be a solution of $(1.1)_\varepsilon$ which satisfies the initial data $u_\varepsilon(0) = u_0, u_\varepsilon'(0) = v_0$. From (2.6) we have

$$\varepsilon^2 \|u_\varepsilon'(t)\|^2 + \alpha \|u(t)\|_{1/2}^2 \leq M_1 + C_0 = M_1' \quad \text{for each } t \geq 0.$$

Therefore there exists $M_2 > 0$ such that

$$\|h_\varepsilon\|_{\mathcal{L}_\infty(\mathbb{R}^+;X)}^2 \leq M_2\,.$$

Thanks to Lemma 3.2 (with $p = \infty$) we have

$$\varepsilon^2 \|\mathbb{P}_m w_\varepsilon'(t)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m w_\varepsilon(t)\|_1^2 \leq C_5 M_2 \quad \text{for each } t \geq 0 \text{ and } m \in \mathbb{N}.$$

Letting $m \longrightarrow \infty$, we conclude that

$$\varepsilon^2 \|w_\varepsilon'(t)\|_{1/2}^2 + \alpha \cdot \|w_\varepsilon(t)\|_1^2 \leq C_5 M_2 = M_3 \quad \text{for each } t \geq 0.$$

Then the arbitrariness of $(u_0, v_0) \in B$ implies the assertion of Lemma 3.3. □

Theorem 3.1. *Let $\varepsilon \in (0, \varepsilon_0]$ be fixed. Then there exists a compact global attractor $\mathcal{A}_\varepsilon$ for S_ε. Moreover, $\mathcal{A}_\varepsilon$ is bounded in $X^1 \times X^{1/2}$.*

Proof: In order to exploit the general results of [GT], we have to show that S_ε is bounded dissipative and for any bounded set $B \subseteq X^{1/2} \times X$ there is a compact set K_ε^B which attracts B, i.e.

$$\lim_{t \to \infty} \operatorname{dist}(S_\varepsilon(t)B,\, K_\varepsilon^B) = 0.$$

Clearly, by Lemma 3.1, S_ε is bounded dissipative, i.e. there exists a bounded set B_ε which dissipates all bounded sets of $X^{1/2} \times X$.

Let B be any bounded set in $X^{1/2} \times X$. From Lemma 3.3 we have that

$$K_\varepsilon^B = \bigcup_{t \geq 0} \mathcal{U}_\varepsilon(t)B \quad \text{is bounded in} \quad X^1 \times X^{1/2}.$$

Therefore K_ε^B is compact in $X^{1/2} \times X$. Since

$$\operatorname{dist}(S_\varepsilon(t)B,\, K_\varepsilon^B) \leq \sup_{\Phi \in B} \|\exp(-\mathcal{L}_\varepsilon t)\Phi\|_{\mathcal{X}} \leq M(\varepsilon)\exp(-\omega t) \cdot \sup_{\Phi \in B} \|\Phi\|_{\mathcal{X}}$$

$$\text{where} \quad \omega \in (0, \frac{\alpha}{2}),$$

then

$$\lim_{t \to \infty} \operatorname{dist}(S_\varepsilon(t)B,\, K_\varepsilon^B) = 0.$$

According to [GT, Proposition 3.1] $\mathcal{A}_\varepsilon = \Omega(B_\varepsilon)$ is a compact global attractor for S_ε. Furthermore, since $\Omega(B_\varepsilon)$ is the bounded and invariant set then we see that

$$\operatorname{dist}(\Omega(B_\varepsilon),\, K_\varepsilon^{\Omega(B_\varepsilon)}) = 0.$$

Thus $\mathcal{A}_\varepsilon = \Omega(B_\varepsilon) \subseteq K_\varepsilon^{\Omega(B_\varepsilon)}$. Hence $\mathcal{A}_\varepsilon$ is bounded in $X^1 \times X^{1/2}$. □

Remark 3.1. In the general case (under the hypotheses H1-H3) the attractor $\mathcal{A}_\varepsilon$, $\varepsilon > 0$, does not reduce to a single point. Indeed, one can consider the case in which

$$-\alpha\sqrt{\lambda_{n+1}} < g(0) \leq -\alpha\sqrt{\lambda_n}$$

where $0 < \lambda_1 < \lambda_2 < \dots$ are eigenvalues of A and Φ_k, $k \geq 1$, are corresponding orthonormal eigenvectors. Since we assume

$$\int_0^\infty g(s)\, ds > -\infty \qquad \text{and } g \text{ is an increasing function,}$$

the domain of g^{-1} (the inverse function of g) contains a subinterval $[g(0), 0)$. Hence

$$w_k^\pm = \left[\pm\left(g^{-1}(-\alpha \cdot (\lambda_k)^{1/2})/\lambda_k^{1/2}\right)^{1/2} \cdot \Phi_k, 0\right] \quad k = 1, 2, \dots, n$$

are non-zero equilibrium states for (2.1), $\varepsilon > 0$, which are contained in $\mathcal{A}_\varepsilon$.

Remark 3.2. If we restrict g, δ by $\delta > -\lambda_1$ and $g(s) = \beta + k \cdot s$, where $k > 0$ and $\beta > -\alpha\sqrt{\lambda_1}$ then it is known ([B2, Theorem 6]) that every solution of $(1.1)_\varepsilon$, $\varepsilon > 0$, and its time derivative decay to zero, as $t \longrightarrow +\infty$. Due to (4.1) it follows that every solution of $(1.1)_0$ also decays to zero. Hence, under the above assumption on δ and g, the dynamics of (2.1), $\varepsilon > 0$ is very simple—each trajectory approaches a zero equilibrium state.

From the invariance property of $\mathcal{A}_\varepsilon$ and Lemma 3.1, we infer the following

Corollary 3.1.

$$\varepsilon^2\|v\|^2 + \alpha \cdot \|u\|^2_{1/2} \le C_1 \quad \textit{for each} \;\; \varepsilon \in (0, \varepsilon_0] \;\; \textit{and} \;\; (u, v) \in \mathcal{A}_\varepsilon \,.$$

The following lemma gives us the uniform estimate of $X^1 \times X^{1/2}$—norm of $\mathcal{A}_\varepsilon$, for $\varepsilon \in (0, \varepsilon_0]$.

Lemma 3.4. *There is $C_6 > 0$ such that*

$$\varepsilon^2\|u''_\varepsilon(t)\|^2_{1/2} + \|u'_\varepsilon(t)\|^2_1 + \|u_\varepsilon(t)\|^2_1 \le C_6$$
$$\textit{for each} \;\; \varepsilon \in (0, \varepsilon_0], \; t \in \mathbb{R} \;\; \textit{and any orbit}$$
$$\{(u_\varepsilon(t), u'_\varepsilon(t));\; t \in \mathbb{R}\} \subseteq \mathcal{A}_\varepsilon \,.$$

Proof: Let $m \in \mathbb{N}$ be an arbitrary integer. We take the projection $\mathbb{P}_m$ of $(1.1)_\varepsilon$ to obtain

$$\varepsilon^2\mathbb{P}_m u''_\varepsilon + \varepsilon\delta\mathbb{P}_m u'_\varepsilon + A\mathbb{P}_m u'_\varepsilon + \alpha A\mathbb{P}_m u_\varepsilon + g(\|u_\varepsilon\|^2_{1/4})A^{1/2}\mathbb{P}_m u_\varepsilon = 0.$$

Put $w_\varepsilon(t) = \mathbb{P}_m u'_\varepsilon(t)$. Then w_ε satisfies the linear strongly damped equation

$$\varepsilon^2 w''_\varepsilon + \varepsilon\delta w'_\varepsilon + Aw'_\varepsilon + \alpha A w_\varepsilon + h_\varepsilon = 0$$

where

$$h_\varepsilon(t) = 2g'(\|u_\varepsilon(t)\|^2_{1/4}) \cdot (A^{1/2}u'_\varepsilon(t)), u_\varepsilon(t))A^{1/2}\mathbb{P}_m u_\varepsilon(t) +$$
$$+ g(\|u_\varepsilon(t)\|^2_{1/4})A^{1/2}\mathbb{P}_m u'_\varepsilon(t).$$

From Corollary 3.1 and (2.6) we infer the existence of $C_7 > 0$ such that

$$\|h_\varepsilon\|^2_{\mathcal{L}_2(\mathbb{R}^+;X)} \le C_7 \quad \text{for each} \;\; \varepsilon \in (0, \varepsilon_0] \,.$$

Obviously, we can choose C_7 to be independent of ε and $m \in \mathbb{N}$.

Recall that $\mathbb{P}_m w_\varepsilon = w_\varepsilon$. Then by Lemma 3.2, we have

$$\varepsilon^2\|w'_\varepsilon(t)\|^2_{1/2} + \alpha \cdot \|w_\varepsilon(t)\|^2_1 \le$$
$$\le C_4(\varepsilon^2\|w'_\varepsilon(0)\|^2_{1/2} + \alpha \cdot \|w_\varepsilon(0)\|^2_1) \cdot e^{-2at} + C_5 \cdot C_7 \,.$$

Clearly,

$$\|w_\varepsilon(0)\|_1^2 = \|\mathbb{P}_m u_\varepsilon'(0)\|_1^2 \le \lambda_m^2 \cdot \|u_\varepsilon'(0)\|^2$$

and

$$\|w_\varepsilon'(0)\|_{1/2} = \|\mathbb{P}_m u_\varepsilon''(0)\|_{1/2} =$$
$$= \varepsilon^{-2}\|\mathbb{P}_m(\varepsilon\delta u_\varepsilon'(0) + Au_\varepsilon'(0) + \alpha A u_\varepsilon(0) + g(\|u_\varepsilon(0)\|_{1/4}^2)A^{1/2}u_\varepsilon(0))\|_{1/2} \le$$
$$\le \varepsilon^{-2}\{\lambda_m^{3/2}\|u_\varepsilon'(0)\| + \alpha \cdot \lambda_m \|u_\varepsilon(0)\|_{1/2} + \varepsilon|\delta|\lambda_m^{1/2}\|u_\varepsilon'(0)\| +$$
$$+\lambda_m^{1/2}|g(\|u_\varepsilon(0)\|_{1/4}^2)| \cdot \|u_\varepsilon(0)\|_{1/2}\}\,.$$

Therefore there exists $M(m) > 0$ and an increasing function $\rho : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, which is independent of ε, such that

$$\begin{aligned} &\varepsilon^2\|w_\varepsilon'(t)\|_{1/2}^2 + \alpha \cdot \|w_\varepsilon(t)\|_1^2 \le \\ &\le \varepsilon^{-4} \cdot M(m) \cdot \rho(\varepsilon^2\|u_\varepsilon'(0)\|^2 + \alpha \cdot \|u_\varepsilon(0)\|_{1/2}^2) \cdot e^{-2at} + C_5 \cdot C_7\,. \end{aligned} \tag{3.4}$$

Let $T \ge 0$. We set $(\bar u_\varepsilon(t), \bar u_\varepsilon'(t)) = (u_\varepsilon(t-T), u_\varepsilon'(t-T))$ for each $t \in \mathbb{R}$. Using the invariance property of $\mathcal{A}_\varepsilon$, we have

$$((\bar u_\varepsilon(t), \bar u_\varepsilon'(t));\ t \in \mathbb{R}) \subseteq \mathcal{A}_\varepsilon\,.$$

Then, from (3.4), we obtain

$$\varepsilon^2\|\mathbb{P}_m u_\varepsilon''(t)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m u_\varepsilon'(t)\|_1^2 =$$
$$= \varepsilon^2\|\mathbb{P}_m \bar u_\varepsilon''(t+T)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m \bar u_\varepsilon'(t+T)\|_1^2 \le$$
$$\le \varepsilon^{-4}M(m)\rho(\varepsilon^2\|\bar u_\varepsilon'(0)\|^2 + \alpha \cdot \|\bar u_\varepsilon(0)\|_{1/2}^2) \cdot e^{-2a(t+T)} + C_5 \cdot C_7 \le$$
$$\le \varepsilon^{-4} \cdot M(m) \cdot \rho(C_1) \cdot e^{-2a(t+T)} + C_5 \cdot C_7\,.$$

Then, by letting $T \longrightarrow \infty$, we obtain

$$\varepsilon^2\|\mathbb{P}_m u_\varepsilon''(t)\|_{1/2}^2 + \alpha \cdot \|\mathbb{P}_m u_\varepsilon'(t)\|_1^2 \le 1 + C_5 \cdot C_7\,.$$

Since $m \in \mathbb{N}$ was an arbitrary integer then

$$\varepsilon^2\|u_\varepsilon''(t)\|_{1/2}^2 + \alpha \cdot \|u_\varepsilon'(t)\|_1^2 \le 1 + C_5 \cdot C_7 \quad \text{for each} \quad t \in \mathbb{R}\,.$$

According to the equation $(1.1)_\varepsilon$ we have

$$\alpha \cdot \|u_\varepsilon(t)\|_1 \le \|u_\varepsilon'(t)\|_1 + \varepsilon^2\|u_\varepsilon''(t)\| + \varepsilon|\delta| \cdot \|u_\varepsilon'(t)\| +$$
$$+|g(\|u_\varepsilon(t)\|_{1/4}^2)| \cdot \|u_\varepsilon(t)\|_{1/2}\,.$$

Then, with regard to Corollary 3.1, one can easily find the constant $C_6 > 0$, as claimed. □

4. Existence of a global attractor for the equation $(1.1)_0$.

We now turn our attention to the limiting equation $(1.1)_0$.

$$Au' + \alpha Au + g(\|u\|_{1/4}^2)A^{1/2}u = 0$$

which is equivalent ($0 \in \rho(A)$) to the differential equation in $X^{1/2}$

$$u' + \alpha u + g(\|u\|_{1/4}^2)A^{-1/2}u = 0.$$

According to the assumption on g, a local existence uniqueness and continuation of solutions of $(1.1)_0$ immediately follow from the theory of semilinear abstract evolution equations. See, for example, [H, Theorem 3.3.3, 3.3.4, 3.4.1 and 3.5.2].

We first give some a priori estimates of solutions of $(1.1)_0$. Take the scalar product in $X^{1/2}$ with u to obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{1/2}^2 + \alpha \cdot \|u(t)\|_{1/2}^2 + g(\|u(t)\|_{1/4}^2) \cdot \|u(t)\|_{1/4}^2 = 0. \tag{4.1}$$

Thanks to (2.5) we have

$$\|u(t)\|_{1/2}^2 \le e^{-2\alpha t}\|u(0)\|_{1/2}^2 + \frac{C_0}{\alpha} \cdot (1 - e^{-2\alpha t}). \tag{4.2}$$

Hence the solution $u(t)$ exists on $\mathbb{R}^+$. We set $S_0(t)u_0 = u(t)$, where $u(t)$ is a solution of $(1.1)_0$ with $u(0) = u_0$. Then, from (4.2), we have that S_0 is the bounded dissipative semidynamical system in $X^{1/2}$. Recall that the variation of constants formula gives

$$S_0(t)u_0 = e^{-\alpha t}u_0 + \mathcal{U}_0(t)u_0$$

where

$$\mathcal{U}_0(t)u_0 = \int_0^t e^{-\alpha(t-s)}g(\|u(s)\|_{1/4}^2)A^{-1/2}u(s)\,ds.$$

From (4.2) one can show that

$$\bigcup_{t\ge 0} \mathcal{U}_0(t)B \ \text{ is bounded in } X^1,$$
$$\text{whenever } B \text{ is bounded in } X^{1/2}.$$

Again, by [GT, Proposition 3.1], there exists a compact global attractor $\tilde{\mathcal{A}}_0$ for S_0 which is bounded in X^1.

Finally, the attractor $\tilde{\mathcal{A}}_0$ can be naturally embedded into a compact set $\mathcal{A}_0$ in $X^{1/2} \times X$. The set $\mathcal{A}_0$ is defined by

$$\mathcal{A}_0 = \left\{(\Phi, \Psi) \in X^{1/2} \times X;\ \Phi \in \tilde{\mathcal{A}}_0 \ \text{and} \ \Psi = -\alpha\Phi - g(\|\Phi\|_{1/4}^2)A^{-1/2}\Phi\right\}.$$

Obviously, $\mathcal{A}_0$ is bounded in $X^1 \times X^{1/2}$.

5. Upper semicontinuity of attractors $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$.

Recall that we are going to prove the property

$$\lim_{\varepsilon \longrightarrow 0^+} \operatorname{dist}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0.$$

In Lemma 3.4, we have shown that there exists $C_6 > 0$ such that

$$\begin{gathered} \varepsilon^2 \|u''_\varepsilon(t)\|^2_{1/2} + \|u'_\varepsilon(t)\|^2_1 + \|u_\varepsilon(t)\|^2_1 \le C_6 \\ \text{for each } \varepsilon \in (0, \varepsilon_0],\ t \in \mathbb{R} \text{ and any orbit} \\ \{(u_\varepsilon(t), u'_\varepsilon(t));\ t \in \mathbb{R}\} \subseteq \mathcal{A}_\varepsilon\,. \end{gathered} \tag{5.1}$$

Concerning the attractor $\mathcal{A}_0$, we have shown that there is $C_7 > 0$ with the property

$$\|u'_0(t)\|^2_{1/2} + \|u_0(t)\|^2_1 \le C_7$$

for any orbit

$$\{(u_0(t), u'_0(t));\ t \in \mathbb{R}\} \subseteq \mathcal{A}_0\,.$$

The idea of the proof is essentially the same as of [HR1]. Let us consider a sequence $\varepsilon_n \longrightarrow 0^+$ and an orbit

$$\{(u_n(t), u'_n(t));\ t \in \mathbb{R}\} \subseteq \mathcal{A}_{\varepsilon_n}\,.$$

Since the set $\bigcup_{t\in\mathbb{R}} \bigcup_{n\in\mathbb{N}} u_n(t)$ is bounded in X^1 and

$$\|u'_n(t)\| \le C_6 \quad \text{for each } n \in \mathbb{N} \text{ and } t \in \mathbb{R}.$$

By the Ascoli–Arzelào's theorem we may thus extract a subsequence $\{u_{n_1}\}$ of $\{u_n\}$ which converges to $\bar{u}$ in the space $C(\langle -1, 1\rangle; X^{1/2})$. Again, there is a subsequence $\{u_{n_2}\}$ which converges to $\bar{u}$ in $C(\langle -2, 2\rangle; X^{1/2})$. Thanks to the Cantor's diagonalization process, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \longrightarrow \bar{u}$ in $C(J; X^{1/2})$ for any compact interval $J \subseteq \mathbb{R}$. Since

$$\sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{R}} \|u_n(t)\|^2_{1/2} < +\infty\,,$$

then

$$\sup_{t\in\mathbb{R}} \|\bar{u}(t)\|^2_{1/2} < +\infty\,.$$

On the one hand $\frac{\partial u_{n_k}}{\partial t} \longrightarrow \frac{\partial \bar{u}}{\partial t}$ in $\mathcal{D}'(I; X^{1/2})$ (in the sense of distributions) for any bounded open interval $I \subseteq \mathbb{R}$.

On the other hand

$$\begin{aligned} u'_{n_k}(t) = -A^{-1}\left\{\varepsilon^2_{n_k} \cdot u''_{n_k}(t) + \varepsilon_{n_k}\delta \cdot u'_{n_k}(t)\right\} - \alpha \cdot u_{n_k}(t) - \\ -g(\|u_{n_k}(t)\|^2_{1/4}) A^{-1/2} u_{n_k}(t)\,. \end{aligned}$$

From (5.1) we observe that

$$\varepsilon_{n_k}^2 \|u_{n_k}''(t)\|_{1/2} \longrightarrow 0 \quad \text{and} \quad \varepsilon_{n_k}|\delta| \cdot \|u_{n_k}'(t)\| \longrightarrow 0, \quad \text{as } \varepsilon_{n_k} \longrightarrow 0^+ .$$

Therefore

$$\frac{\partial \bar{u}}{\partial t} = -\alpha \bar{u} - g(\|\bar{u}\|_{1/4}^2) A^{-1/2} \bar{u} .$$

Hence $\bar{u}(t)$ is the solution of $(1.1)_0$ which exists and is bounded on $\mathbb{R}$. Therefore

$$\{(\bar{u}(t), \bar{u}'(t));\ t \in \mathbb{R}\} \subseteq \mathcal{A}_0 .$$

Since $(u_{n_k}(\cdot), u_{n_k}'(\cdot)) \longrightarrow (\bar{u}(\cdot), \bar{u}'(\cdot))$ in $C(J; X^{1/2})$ for any compact interval $J \in \mathbb{R}$ then we have

$$(u_{n_k}(0), u_{n_k}'(0)) \longrightarrow (\bar{u}(0), \bar{u}'(0)) \in \mathcal{A}_0 \quad \text{in} \quad X^{1/2} \times X.$$

It means that

$$\lim_{\varepsilon \longrightarrow 0^+} \operatorname{dist}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0.$$

Indeed, suppose to the contrary that there exists $\varepsilon_n \longrightarrow 0^+, \sigma > 0$ and a sequence $(u_{n0}, u_{n0}') \in \mathcal{A}_{\varepsilon_n}$ such that

$$\operatorname{dist}((u_{n0}, u_{n0}'), \mathcal{A}_0) \geq \sigma .$$

Obviously, there are orbits $\{(u_{\varepsilon_n}(t), u_{\varepsilon_n}'(t));\ t \in \mathbb{R}\} \subseteq \mathcal{A}_{\varepsilon_n}$, for $n \in \mathbb{N}$, such that $u_{\varepsilon_n}(0) = u_{n0}$ and $u_{\varepsilon_n}'(0) = u_{n0}'$. Then there exists a subsequence ε_{n_k} with the property

$$(u_{n_k}(0), u_{n_k}'(0)) \longrightarrow (\bar{u}(0), \bar{u}'(0)) \in \mathcal{A}_0 ,$$

a contradiction. Hence Theorem 1.1 is proved. □

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