ON FACTORIZATIONS OF MAPS BETWEEN CURVES

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ABSTRACT. We examine the different ways of writing a cover of curves $\phi\colon C \to D$ over a field K as a composition $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$, where each ϕ_i is a cover of curves over K of degree at least 2 which cannot be written as the composition of two lower-degree covers. We show that if the monodromy group $\mathrm{Mon}(\phi)$ has a transitive abelian subgroup then the sequence $(\deg \phi_i)_{1\le i\le n}$ is uniquely determined up to permutation by ϕ, so in particular the length n is uniquely determined. We prove analogous conclusions for the sequences $(\mathrm{Mon}(\phi_i))_{1\le i\le n}$ and $(\mathrm{Aut}(\phi_i))_{1\le i\le n}$. Such a transitive abelian subgroup exists in particular when ϕ is tamely and totally ramified over some point in $D(\overline{K})$, and also when ϕ is a morphism of one-dimensional algebraic groups (or a coordinate projection of such a morphism). Thus, for example, our results apply to decompositions of polynomials of degree not divisible by $\mathrm{char}(K)$, additive polynomials, elliptic curve isogenies, and Lattès maps.

1. INTRODUCTION

Let $\phi\colon C \to D$ be a cover of curves over a field K, which in this paper means a nonconstant separable morphism between nonsingular, projective, geometrically irreducible curves where ϕ, C and D are all defined over K. We will examine decompositions of ϕ, namely expressions $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$ where each $\phi_i\colon C_{i-1} \to C_i$ is a cover of curves over K with $\deg(\phi_i) \ge 2$ (so that $C_0 = C$ and $C_n = D$). Of special importance are *complete decompositions*, which are decompositions in which no ϕ_i can be written as the composition of lower-degree covers. These are the analogues of "prime factorizations" in the context of maps between curves. Based on this analogy, it is natural to ask whether a given map ϕ has essentially just one complete decomposition, but this turns out to be too much to hope for in general. As a substitute, we study properties which are shared by all complete decompositions of a given cover ϕ.

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The statements of our results involve the *monodromy group* $\mathrm{Mon}(\phi)$ of ϕ, which by definition is the Galois group of (the Galois closure of) the corresponding function field extension $K(C)/K(D)$, viewed as a permutation group. We show that if $\mathrm{Mon}(\phi)$ has a transitive abelian subgroup then the sequences $(\mathrm{Mon}(\phi_i))_{1\le i\le n}$ and $(\deg\phi_i)_{1\le i\le n}$ are uniquely determined (up to permutation) by ϕ. We prove a similar conclusion about the sequence $(\mathrm{Aut}(\phi_i))_{1\le i\le n}$ of automorphism groups of the ϕ_i's, where by definition $\mathrm{Aut}(\phi_i)$ is the group of automorphisms μ of C_{i-1} defined over K which satisfy $\phi_i\circ\mu=\phi_i$. In fact we obtain these conclusions in a slightly more general situation, as follows.

Theorem 1.1. *Let $\phi\colon C\to D$ be a cover of curves over a field K, and let $\phi=\phi_n\circ\phi_{n-1}\circ\cdots\circ\phi_1$ and $\phi=\psi_m\circ\psi_{m-1}\circ\cdots\circ\psi_1$ be complete decompositions of ϕ. If $\mathrm{Mon}(\phi)$ has a transitive quasi-Hamiltonian subgroup then $m=n$ and there is a permutation π of $\{1,2,\dots,n\}$ such that, for each i with $1\le i\le n$, we have $\deg\phi_i=\deg\psi_{\pi(i)}$ and $\mathrm{Aut}(\phi_i)\cong\mathrm{Aut}(\psi_{\pi(i)})$. If $\mathrm{Mon}(\phi)$ has a transitive Dedekind subgroup then we may choose π so that in addition $\mathrm{Mon}(\phi_i)\cong\mathrm{Mon}(\psi_{\pi(i)})$ for each i.*

Here a *Dedekind group* is a group A such that every subgroup of A is normal, and a *quasi-Hamiltonian group* is a group A such that $IJ=JI$ for all subgroups I,J of A. Note that all abelian groups are Dedekind groups, and all Dedekind groups are quasi-Hamiltonian. Dedekind [7] showed that the nonabelian finite Dedekind groups are precisely the direct products of the order-8 quaternion group with an abelian group containing no elements of order 4. Iwasawa [20] gave a similar classification of nonabelian quasi-Hamiltonian groups.

We also prove the following structural result about the automorphism group of a composition of covers.

Theorem 1.2. *Let $\theta\colon C_1\to D$ and $\rho\colon C\to C_1$ be covers of curves over a field K, and assume that the monodromy group of $\psi:=\theta\circ\rho$ has a transitive Dedekind subgroup. Then, for each $\mu\in\mathrm{Aut}(\psi)$, there is a unique $\nu\in\mathrm{Aut}(\theta)$ for which $\rho\circ\mu=\nu\circ\rho$. Moreover, the map $\mu\mapsto\nu$ defines a homomorphism $\mathrm{Aut}(\psi)\to\mathrm{Aut}(\theta)$ with kernel $\mathrm{Aut}(\rho)$.*

As a consequence, we show in Theorem 7.1 that if $\phi=\phi_n\circ\phi_{n-1}\circ\cdots\circ\phi_1$ then

$$|\mathrm{Aut}(\phi)|\ \text{ divides }\ \prod_{i=1}^{n}|\mathrm{Aut}(\phi_i)|.$$

We now present several classes of covers of curves which satisfy all the conclusions of the above results. For this, it suffices to exhibit covers whose monodromy group contains a transitive abelian subgroup. For instance, if $\phi\colon C \to D$ is totally and tamely ramified over some point $P \in D(\overline{K})$, then the inertia group at any point over P on the Galois closure of ϕ will be a transitive cyclic subgroup of $\mathrm{Mon}(\phi)$. This includes the classical case of complex polynomials, for which Ritt [36] proved in 1922 that any two complete decompositions have the same length and the same collection of degrees of the involved indecomposable polynomials. We discuss the case of polynomials further in Section 3. Covers of curves with a totally and tamely ramified point have arisen in other contexts as well, most recently as a distinguished class of "origami", meaning covers of a complex elliptic curve having a unique branch point [37].

Our results also apply to any cover $\phi\colon C \to D$ which is the projective closure of a nonconstant separable morphism $\phi_0\colon C_0 \to D_0$ of connected one-dimensional algebraic groups. The reason is that in this situation the transitive subgroup $\mathrm{Gal}(\overline{K}(C)/\overline{K}(D))$ of $\mathrm{Mon}(\phi)$ is isomorphic to the kernel of ϕ_0, and hence is abelian because every one-dimensional algebraic group is abelian. In case $C_0 \cong D_0 \cong \mathbb{G}_a$, the morphism ϕ is an *additive polynomial* $\sum_{i=0}^{r} a_i X^{p^i}$, where $a_i \in K$ and $p := \mathrm{char}(K)$. Decompositions of additive polynomials feature prominently in work on the Carlitz module and more general Drinfeld modules, see [14]. Such decompositions were originally studied in 1933 by Ore, who proved in [33, Thm. 4 of Chap. 2] that any two complete decompositions of an additive polynomial *into additive polynomials* have the same length and the same collection of degrees of the involved indecomposable polynomials. It was shown later that every decomposition of an additive polynomial into arbitrary polynomials is equivalent to a decomposition into additive polynomials [8, Thm. 4], so that Ore's result strongly resembles Ritt's. The present paper is the first to explain this resemblance, by proving a common generalization of these two results. Another class of morphisms of one-dimensional algebraic groups consists of isogenies between elliptic curves. In this case, all portions of our results are new.

Finally, our results apply to any cover $\phi\colon C \to D$ which is a coordinate projection of a morphism $\hat{\phi}\colon \hat{C} \to \hat{D}$ of one-dimensional algebraic groups. This means that there exist nonconstant morphisms $\pi_1\colon \hat{C} \to C$ and $\pi_2\colon \hat{D} \to D$ for which $\pi_2 \circ \hat{\phi} = \phi \circ \pi_1$. It was shown in [12] that $\mathrm{Mon}(\phi)$

has a transitive abelian subgroup in this situation. This case includes the *subadditive polynomials* $S(X) \in K[X]$, which are characterized by the property that there is a positive integer n for which $S(X^n) = L(X)^n$ for some additive polynomial $L(X)$. In particular this proves the assertion from [6, p. 325] that any two decompositions of a subadditive polynomial have the same length and the same degrees of the indecomposables. Our results also apply to coordinate projections of isogenies of elliptic curves, which play a prominent role in the finite fields literature [15, 32]; in case the isogeny is an endomorphism, such coordinate projections are called *Lattès maps* and play a crucial role in complex dynamics [28].

Since the transitive abelian subgroup in each of the above cases is a subgroup of the geometric monodromy group of ϕ – that is, the monodromy group of the base extension of ϕ to a morphism of curves over $\overline{K}$ – it follows that our results also apply to any cover which becomes isomorphic to one of the above covers after base extension to $\overline{K}$. For instance, this includes decompositions of Dickson polynomials [27], which are quadratic twists of Chebyshev polynomials (which in turn are coordinate projections of the multiplication-by-d endomorphism of $\mathbb{G}_m$). It also includes Rédei functions [35], which are rational functions inducing covers $\mathbb{P}^1 \to \mathbb{P}^1$ that become isomorphic to X^d over a quadratic extension of K.

In the development that follows, we also prove several other results about decompositions. In some cases we give simpler proofs (in greater generality) of results from previous papers: for instance one can compare Corollary 2.10 and Remark 3.3, or Lemma 6.3 and the last paragraph of Section 6. Also in Remark 7.4 we disprove a conjecture from [16]. These improvements on previous work are made possible in part by our generalization to the framework of covers of curves, which provides a valuable perspective even when one is only interested in questions about polynomials.

This paper is organized as follows. In the next section we explain the connection between monodromy groups and decompositions of a cover. In Section 3 we expand on this connection in the much-studied case of decompositions of polynomials. In Sections 4, 5 and 6 we prove the portions of Theorem 1.1 pertaining to degrees, monodromy groups, and automorphism groups, respectively. We conclude in Section 7 by proving Theorem 1.2.

2. Decomposition of covers via monodromy groups

In this section we translate the problem of analyzing decompositions of a cover of curves into the problem of analyzing chains of subgroups of its monodromy group, which we then reformulate as analyzing chains of certain types of subgroups of a transitive subgroup of this monodromy group.

We first introduce the terminology we will use in the paper.

Definition 2.1. A *curve* over a field K is a nonsingular, projective, geometrically irreducible one-dimensional variety defined over K.

Definition 2.2. A *cover* of curves over a field K is a nonconstant separable morphism between curves over K.

Definition 2.3. A cover of curves over K is *decomposable* if it can be written as the composition of two covers (of curves over K) which both have degree at least 2. A cover is *indecomposable* if its degree is at least 2 and it is not decomposable.

Definition 2.4. A *decomposition* of a cover $\phi\colon C \to D$ over K is an expression $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$ where each $\phi_i\colon C_{i-1} \to C_i$ is a cover (of curves over K) of degree at least 2. Such a decomposition is a *complete decomposition* if every ϕ_i is indecomposable.

Definition 2.5. Let $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1 = \psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_1$ be two decompositions of a cover $\phi\colon C \to D$, where $\phi_i\colon C_{i-1} \to C_i$ and $\psi_i\colon B_{i-1} \to B_i$ (and $C_0 = B_0 = C$ and $C_n = B_n = D$). We call these decompositions *equivalent* if $m = n$ and there are isomorphisms $\rho_i\colon C_i \to B_i$ such that $\rho_0\colon C \to C$ and $\rho_n\colon D \to D$ are the identity maps and $\psi_i \circ \rho_{i-1} = \rho_i \circ \phi_i$ for $1 \leq i \leq n$.

Definition 2.6. The *monodromy group* $\mathrm{Mon}(\phi)$ of a cover $\phi\colon C \to D$ of curves over K is the Galois group of the Galois closure of the extension of function fields $K(C)/K(D)$.

We view $\mathrm{Mon}(\phi)$ as a group of permutations of the set of embeddings of $K(C)$ into a fixed algebraic closure of $K(D)$ which restrict to the identity map on $K(D)$. The number d of such embeddings is $[K(C) : K(D)] = \deg \phi$, so that $\mathrm{Mon}(\phi)$ is a subgroup of S_d, and further $\mathrm{Mon}(\phi)$ is transitive.

Example 2.7. We illustrate the above notions in the special case of covers $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$: upon choosing coordinates on both copies of $\mathbb{P}^1$, we see that

such a cover is the same thing as a rational function $f(X) \in K(X)$ with nonzero derivative, or equivalently an element of $K(X) \setminus K(X^p)$ where $p := \mathrm{char}(K)$. Then two decompositions $f = f_n \circ f_{n-1} \circ \cdots \circ f_1$ and $f = g_m \circ g_{m-1} \circ \cdots \circ g_1$ are equivalent if $m = n$ and there are degree-one $\mu_i \in K(X)$ (for $0 \le i \le n$) such that $\mu_0 = \mu_n = X$ and $g_i \circ \mu_{i-1} = \mu_i \circ f_i$ for $1 \le i \le n$. In this case $\mathrm{Mon}(\phi)$ is the Galois group of the numerator of $f(X) - t$ over $K(t)$, where t is transcendental over K.

Having defined our terminology, we now state our first result.

Lemma 2.8. *Let K be a field and let $\phi\colon C \to D$ be a cover of curves over K. Let G be the monodromy group of ϕ, let H be a one-point stabilizer in G, and let A be a transitive subgroup of G. There are bijections between the following sets:*

(1) *the set of equivalence classes of decompositions of ϕ,*
(2) *the set of increasing chains of fields between $K(D)$ and $K(C)$,*
(3) *the set of decreasing chains of groups between G and H,*
(4) *the set of decreasing chains of groups between A and $H \cap A$ consisting of groups J for which $JH = HJ$.*

Moreover, these bijections can be chosen so that the degrees of the indecomposable covers in a decomposition in (1) *equal the indices between successive groups in the corresponding chain in each of* (3) *and* (4)*.*

Proof. Let $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$ be a decomposition of ϕ, where $\phi_i\colon C_{i-1} \to C_i$ with $C_0 = C$ and $C_n = D$. Associate to this decomposition the chain of fields $K(C_n) \subset K(C_{n-1}) \subset \cdots \subset K(C_0)$, where the inclusion $K(C_i) \hookrightarrow K(C_{i-1})$ is defined by $\psi \mapsto \psi \circ \phi_i$. Then the usual equivalence of categories between curves over K (and their covers) and finitely generated field extensions of K having transcendence degree 1 (and their separable extensions) [18, Cor. 6.12] shows that this association yields a bijection between (1) and (2), and also that $\deg \phi_i = [K(C_{i-1}) : K(C_i)]$. Next let Ω be the Galois closure of $K(C)/K(D)$, so that $G = \mathrm{Gal}(\Omega/K(D))$. Then the Galois correspondence [22, Thm. VI.1.1] yields a bijection between (2) and the set of decreasing chains of groups between G and $\tilde{H} := \mathrm{Gal}(\Omega/K(C))$, where the degree of each successive extension in the chain of fields equals the index between the corresponding groups in the chain of groups. Since $\tilde{H}$ and H are conjugate subgroups of G, this yields a bijection between (2) and (3). The following lemma yields a bijection between (3) and (4), and shows that

the indices between successive groups in a chain in (4) equal the analogous indices in the corresponding chain in (3). □

Lemma 2.9. *Let G be a permutation group, let H be a one-point stabilizer, and let A be a transitive subgroup of G. Then the map $\rho\colon U \mapsto U \cap A$ is a bijection from the set of groups between G and H to the set of groups J between A and $H \cap A$ for which $JH = HJ$. Moreover, $[G : U] = [A : U \cap A]$ and $\rho(\langle U, V\rangle) = \langle \rho(U), \rho(V)\rangle$ and $\rho(U \cap V) = \rho(U) \cap \rho(V)$ for any groups U, V between H and G.*

Proof. Transitivity of A means that $G = AH$. Every group U between H and G is a union of cosets gH with $g \in G$, and since $G = AH$ we know that every such coset equals aH for some $a \in A$, whence $U = (U \cap A)H$. Conversely, if J is a group between A and $H \cap A$ then JH is a group if and only if $JH = HJ$ (in which case $[G : JH] = [A : J]$). This proves that ρ is a bijection. The final assertion follows from bijectivity of ρ and the fact that if I, J are groups between A and $H \cap A$ which satisfy $IH = HI$ and $JH = HJ$ then also $\langle I, J\rangle H = H\langle I, J\rangle$ and $IH \cap JH = (I \cap J)H$, whence $(I \cap J)H = H(I \cap J)$. □

The utility of Lemma 2.8 stems from the fact that, for any fixed positive integer n, questions about an infinite collection of objects (namely, all degree-n covers of curves over an arbitrary field K) have been translated into questions about a finite collection of objects (namely, certain types of subgroups of S_n). One immediate consequence of this translation is as follows.

Corollary 2.10. *Any cover of curves over any field K has only finitely many equivalence classes of decompositions. Moreover, the number of such equivalence classes of decompositions is bounded above by a constant which depends only on the degree of the cover.*

Proof. By Lemma 2.8, the number of equivalence classes of decompositions of a degree-n cover is at most the number of decreasing chains of subgroups of S_n. □

Remark 2.11. In the case of covers $\mathbb{P}^1 \to \mathbb{P}^1$, Corollary 2.10 asserts that a rational function in $K(X)$ with nonzero derivative has only finitely many equivalence classes of decompositions. In fact the proof of Lemma 2.8 implies the same conclusion for rational functions with zero derivative, since

the number of equivalence classes of decompositions equals the number of decreasing chains of fields between $K(x)$ and $K(f(x))$ (where x is transcendental over K), and there are only finitely many fields between $K(x)$ and $K(f(x))$ since $K(x)/K(f(x))$ is a simple extension [22, Thm. V.4.6]. However, there exist inseparable finite morphisms between curves which admit infinitely many equivalence classes of decompositions [22, Exerc. V.24]. This behavior is typical for questions about decompositions of inseparable morphisms: inseparable morphisms between curves can have completely different properties than do separable morphisms, but inseparable rational functions behave in exactly the same way as do separable rational functions (and likewise for polynomials).

Remark 2.12. In the case of covers $\mathbb{P}^1 \to \mathbb{P}^1$, Lemma 2.8 may be compared with the assertion in [17, p. 666] that in this setting "there is no bijection between groups and intermediate fields".

3. Functional decomposition of polynomials

Functional decomposition is often studied for polynomials $f(X) \in K[X]$, where one is interested in the expressions of $f(X)$ as the composition of polynomials in $K[X]$ of degree at least 2. Here we say that two decompositions $f = f_n \circ f_{n-1} \circ \cdots \circ f_1$ and $f = g_m \circ g_{m-1} \circ \cdots \circ g_1$ are equivalent if $m = n$ and there are degree-one $\mu_i \in K[X]$ (for $0 \le i \le n$) such that $\mu_0 = \mu_n = X$ and $g_i \circ \mu_{i-1} = \mu_i \circ f_i$ for $1 \le i \le n$. Note that we have already defined a different notion of decompositions of a polynomial, since a polynomial may be viewed as a rational function. In this section we show that these two notions are compatible, and we also show that if $\mathrm{char}(K) \nmid \deg(f)$ then the monodromy group of $f(X)$ has a transitive cyclic subgroup.

Lemma 3.1. *Let K be a field and pick any $f(X) \in K[X]$. Then every equivalence class of decompositions of $f(X)$ in the sense of rational functions contains exactly one equivalence class of decompositions of $f(X)$ in the sense of polynomials.*

Proof. Rational functions (or polynomials) of degree less than 2 have no decompositions according to our definitions, so we may assume that $\deg(f) \ge 2$. Write $f = f_n \circ f_{n-1} \circ \cdots \circ f_1$ where $f_i \in K(X)$. Since f is a polynomial, we have $f^{-1}(\infty) = \{\infty\}$, so that ∞ has a unique preimage under $f_n \circ \cdots \circ f_i$ whenever $1 \le i \le n$. Define $\mu_n = \mu_0 = X$ and, for each $i = n-1, n-2, \ldots, 1$ in succession, let $\mu_i \in K(X)$ be a degree-one rational

function for which $\hat{f}_{i+1} := \mu_{i+1}^{-1} \circ f_{i+1} \circ \mu_i$ fixes ∞, and hence is a polynomial. Then also $\hat{f}_1 := \mu_1^{-1} \circ f_1 \circ \mu_0$ fixes ∞, and $f = \hat{f}_n \circ \hat{f}_{n-1} \circ \cdots \circ \hat{f}_1$ is a decomposition of f which is equivalent to our original decomposition. Finally, our procedure for defining the μ_i's shows that any two choices yield decompositions which are equivalent in the sense of polynomials. □

In light of this result, Remark 2.11 implies the following:

Corollary 3.2. *Any $f(X) \in K[X]$ has only finitely many equivalence classes of decompositions in the sense of polynomials.*

Remark 3.3. The special case of this result for $K = \mathbb{C}$ was first proved in 1922 [36, §2] via essentially the same method as above. Corollary 3.2 also follows at once from [24, §2] and [11, Cor. 2.3], where a method is used which only applies to polynomials. After appearing in dozens of papers and books over the next several decades, the case $K = \mathbb{C}$ of Corollary 3.2 arose again in 2000 as one of the main "new" results of [2], where it was proved by a more complicated version of the proof in [24, §2]. The authors of [2] motivated the $K = \mathbb{C}$ case of Corollary 3.2 by making the curious assertion that no previous authors had noticed the special role of degree-one polynomials in the theory of functional decomposition; however, this special role is addressed in nearly every treatment of this topic, for instance [3, 8, 10, 11, 23, 24, 36, 38]. Indeed, this is an instance of the special role units play in the theory of factorization in any monoid.

Next we show that, if $f(X) \in K[X]$ has degree not divisible by $\mathrm{char}(K)$, then the monodromy group of $f(X)$ contains a transitive cyclic subgroup. Here the monodromy group is just the Galois group of $f(X) - t$ over $K(t)$, where t is transcendental over K. One such transitive cyclic subgroup is the inertia group at any place of the splitting field of $f(X) - t$ which lies over the infinite place of $K(t)$, as has been well-known for over a hundred years. For the benefit of authors unfamiliar with inertia groups, we include here a self-contained proof of the existence of a transitive cyclic subgroup (based on Newton's ideas as arranged in [41, Lemma 3.3]).

Lemma 3.4. *If $f(X) \in K[X]$ is a degree-n polynomial over a field K for which $\mathrm{char}(K) \nmid n$, then the monodromy group of $f(X)$ contains a transitive cyclic subgroup.*

Proof. Let t be transcendental over K, and let $\overline{K}$ be an algebraic closure of K. Let $L := \overline{K}((1/t))$ be the field of formal Laurent series over $\overline{K}$, namely

the set of expressions $\sum_{i=-\infty}^{\infty} a_i t^i$ where $a_i \in \overline{K}$ and where in addition there exists an integer N for which $a_i = 0$ whenever $i > N$. We first show that the Galois group of $f(X) - t$ over L has a transitive cyclic subgroup. Let s be any root of $X^n - t$ in an extension of L, and note that $L(s) = \overline{K}((1/s))$. Let b be the leading coefficient of $f(X)$. For any $c \in \overline{K}$ such that $c^n = b$, if we write $x_c := cs + \sum_{i=-\infty}^{0} a_i s^i$ then there is a unique choice of elements $a_i \in \overline{K}$ for which $f(x_c) = s^n$, since for each $i = 0, 1, 2, 3, \ldots$ in succession we can uniquely determine a_i from the condition that the coefficient of s^{n-1-i} in $f(x_c) - s^n$ equals zero. Now let $\theta \in \overline{K}$ be a primitive n-th root of unity, and let σ be the automorphism of $L(s)$ which maps $\sum_i a_i s^i$ to $\sum_i \sigma(a_i) s^i$. Since σ fixes every element of L, it must permute the roots of $f(X) - t$, namely the n elements x_c with $c^n = b$. By considering the action of σ on the coefficient of s in the various elements x_c, we see that σ induces a transitive permutation on the x_c's. Therefore $\langle \sigma \rangle$ is a transitive cyclic subgroup of the Galois group of $f(X) - t$ over L, so the restriction of $\langle \sigma \rangle$ to the splitting field of $f(X) - t$ over $K(t)$ is a transitive cyclic subgroup of the monodromy group of $f(X)$. □

4. Ritt's first theorem

In this section we show that if $\phi\colon C \to D$ is a cover of curves whose monodromy group has a transitive quasi-Hamiltonian subgroup, then any two complete decompositions of ϕ have the same length and the same multiset of degrees of the involved indecomposable subcovers. Moreover, we prove that we can pass from any complete decomposition of ϕ to any other via finitely many steps of a special form; this will play a crucial role in subsequent sections.

Theorem 4.1. *Let $\phi\colon C \to D$ be a cover of curves over a field K, and suppose that the monodromy group of ϕ has a transitive quasi-Hamiltonian subgroup. Then any complete decomposition $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$ can be obtained from any other complete decomposition $\phi = \psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_1$ through finitely many steps, where in each step we replace two adjacent indecomposable covers θ_2, θ_1 in a complete decomposition by two other indecomposable covers $\hat{\theta}_2, \hat{\theta}_1$ such that $\theta_2 \circ \theta_1 = \hat{\theta}_2 \circ \hat{\theta}_1$ and $\{\deg \theta_1, \deg \theta_2\} = \{\deg \hat{\theta}_1, \deg \hat{\theta}_2\}$. In particular, $m = n$ and the sequence $(\deg \phi_i)_{1 \le i \le n}$ is a permutation of the sequence $(\deg \psi_i)_{1 \le i \le m}$.*

By Lemma 2.8, Theorem 4.1 is a consequence of the following group-theoretic assertion.

Lemma 4.2. *Let A and B be finite groups with $B \leq A$. Let S be a set of groups between A and B such that $A, B \in S$ and S contains both IJ and $I \cap J$ whenever it contains groups I and J. Let $A = V_n > V_{n-1} > \cdots > V_0 = B$ and $A = W_m > W_{m-1} > \cdots > W_0 = B$ be two maximal decreasing chains of groups in S which lie between A and B. Then one can pass from the first chain to the second chain through finitely many steps, where in each step a chain $A = C_k > C_{k-1} > \cdots > C_0 = B$ is replaced by a chain $A = D_k > D_{k-1} > \cdots > D_0 = B$ of groups in S such that $D_i = C_i$ for all but one i with $0 < i < k$. Moreover, if $C_i \neq D_i$ then $[C_i : C_{i-1}] = [C_{i+1} : D_i]$.*

Proof. We prove the result by induction on $|A|$, noting that the result is vacuously true when $|A| = 0$. Let A be a finite group, and assume that the assertion holds for all groups of order less than $|A|$. Pick any B and S as in the lemma, and let $A = V_n > V_{n-1} > \cdots > V_0 = B$ and $A = W_m > W_{m-1} > \cdots > W_0 = B$ be two maximal increasing chains of groups in S. If $V_{n-1} = W_{m-1}$ then the inductive hypothesis implies that the chains $V_{n-1} > V_{n-2} > \cdots > V_0$ and $W_{m-1} > W_{m-2} > \cdots > W_0$ satisfy the desired conclusion, so the desired conclusion also holds for the two chains obtained by appending A to both of these chains. Henceforth we assume that $V_{n-1} \neq W_{m-1}$. Then $V_{n-1}W_{m-1}$ is a group in S which is strictly larger than at least one of V_{n-1} or W_{m-1}, so the maximality of the chains implies that $V_{n-1}W_{m-1} = A$, whence $[A : W_{m-1}] = [V_{n-1} : V_{n-1} \cap W_{m-1}]$. Let U be a group in S such that $V_{n-1} \geq U > V_{n-1} \cap W_{m-1}$. Then S contains UW_{m-1}, and $[UW_{m-1} : W_{m-1}] = [U : U \cap W_{m-1}] = [U : V_{n-1} \cap W_{m-1}] > 1$. Maximality of the chains implies that $UW_{m-1} = A$, so that $[A : W_{m-1}] = [U : V_{n-1} \cap W_{m-1}]$, whence $U = V_{n-1}$. Now let $V_{n-1} \cap W_{m-1} = Y_r > \cdots > Y_0 = B$ be any maximal chain of groups in S which lie between $V_{n-1} \cap W_{m-1}$ and B. Appending V_{n-1} yields a maximal chain of groups in S which lie between V_{n-1} and B, so by inductive hypothesis we can pass from this chain to the chain $V_{n-1} > \cdots > V_0 = B$ by steps of the required type. Therefore if we augment both chains by appending A, we can still pass between these augmented chains via steps of the required type. The same argument shows that steps of the required type enable us to pass from $A > W_{m-1} > \cdots > W_0$ to $A > W_{m-1} > Y_r > \cdots > Y_0$, which implies the

desired conclusion since the replacement of $A > W_{m-1} > Y_r > \cdots > Y_0$ by $A > V_{n-1} > Y_r > \cdots > Y_0$ is a step of the required type. □

Remark 4.3. Our proof of Theorem 4.1 actually shows something slightly stronger, since we do not need the monodromy group G of ϕ to contain a transitive quasi-Hamiltonian subgroup. What we actually need is that, if H is a one-point stabilizer of G, then G contains a transitive subgroup A with the property that $IJ = JI$ for all groups I, J such that $H \cap A \leq I, J \leq A$ and $HI = IH$ and $HJ = JH$. We do not know whether there are any natural situations in which Theorem 4.1 does not apply but this stronger version does.

Remark 4.4. In case ϕ is given by a polynomial $f(X) \in K[X]$ of degree not divisible by $\mathrm{char}(K)$, the monodromy group of ϕ contains a transitive cyclic subgroup by Lemma 3.4, so the conclusion of Theorem 4.1 holds. In this case Theorem 4.1 is known as Ritt's First Theorem, and it was first proved by Ritt for $K = \mathbb{C}$ [36]. A different proof was given by Engstrom [9, Thm. 4.1] in case K is an arbitrary field of characteristic zero, and Engstrom's proof extends at once to polynomials over any field with $\mathrm{char}(K) \nmid \deg(f)$ (cf. [23, Thm. 4.1.34], [39, Thm. 7], [3, Thm. 5.11], [43, Thm. VII.5]). Ritt's proof may be viewed as a special case of the proof given above, although it is presented in a different language. Alternate versions of Ritt's proof are given in [10, Thm. 3.1] and [8, Thm. 2] for polynomials of degree less than $\mathrm{char}(K)$ (see also [44, Thm. 2.1 and Cor. 2.12]). Yet another version of Ritt's proof for polynomials in characteristic zero is given in [31, Thm. R.1], where it is noted that the transitive cyclic subgroup used in Ritt's proof can be replaced by a transitive abelian subgroup. A slightly weaker version of Theorem 4.1 is stated in [21, Cor. 1.5].

5. The monodromy invariant

In the previous section we showed that, if ϕ is a cover of curves whose monodromy group $\mathrm{Mon}(\phi)$ has a transitive quasi-Hamiltonian subgroup, then any two complete decompositions of ϕ have the same length and the same collection of degrees of the involved indecomposable subcovers. In this section we show that a stronger conclusion holds under a slightly more restrictive hypothesis: specifically, if $\mathrm{Mon}(\phi)$ has a transitive Dedekind subgroup then any two complete decompositions of ϕ have the same collection of monodromy groups of the involved indecomposable subcovers. Here, as

usual, the monodromy groups are viewed as permutation groups, so that the degree of the monodromy group equals the degree of the corresponding cover, and hence covers with isomorphic monodromy groups have the same degrees as one another.

The main result of this section is as follows.

Theorem 5.1. *Let $\phi\colon C \to D$ be a cover of curves over a field K, and suppose that the monodromy group $\mathrm{Mon}(\phi)$ has a transitive Dedekind subgroup. If $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$ and $\phi = \psi_n \circ \psi_{n-1} \circ \cdots \circ \psi_1$ are complete decompositions of ϕ, then there is a permutation π of $\{1, 2, \ldots, n\}$ such that, for each $i = 1, 2, \ldots, n$, the groups $\mathrm{Mon}(\phi_i)$ and $\mathrm{Mon}(\psi_{\pi(i)})$ are isomorphic as permutation groups.*

Remark 5.2. As noted above, covers with isomorphic monodromy groups must have the same degree. We will show in Lemma 6.3 that two such covers must also have isomorphic automorphism groups, so that Theorem 1.1 follows from Theorem 5.1 if $\mathrm{Mon}(\phi)$ has a transitive Dedekind subgroup.

In light of Theorem 4.1, it suffices to prove Theorem 5.1 when $n = 2$. In fact we will prove the following refinement of the case $n = 2$ of Theorem 5.1:

Proposition 5.3. *Let $\phi\colon C \to D$ be a cover of curves over a field K, and suppose that the monodromy group $\mathrm{Mon}(\phi)$ has a transitive Dedekind subgroup. If $\phi = \phi_2 \circ \phi_1$ and $\phi = \psi_2 \circ \psi_1$ are inequivalent complete decompositions of ϕ, then $\mathrm{Mon}(\phi_2) \cong \mathrm{Mon}(\psi_1)$ (as permutation groups) and likewise $\mathrm{Mon}(\phi_1) \cong \mathrm{Mon}(\psi_2)$.*

In order to prove Proposition 5.3, we first translate it into a group-theoretic statement. This requires the following terminology.

Definition 5.4. If W is a subgroup of a group G, then the *core* of W in G is the largest normal subgroup of G which is contained in W, and is denoted $\mathrm{core}_G(W)$.

Remark 5.5. Two basic properties of cores are as follows: first, $\mathrm{core}_G(W) = \cap_{g \in G} W^g$, where $W^g := g^{-1} W g$. Second, $\mathrm{core}_G(W)$ is the kernel of the homomorphism $G \to \mathrm{Sym}(G/W)$ induced by the action of G by left multiplication on the set G/W of left cosets of W in G.

Next we use cores to describe the monodromy groups of the subcovers occurring in a decomposition of a cover.

Lemma 5.6. *Let $\phi\colon C \to D$ be a cover of curves over a field K, and let $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$ be a decomposition of ϕ, where $\phi_i\colon C_{i-1} \to C_i$. Write $G_i := \mathrm{Gal}(\Omega/K(C_i))$, where Ω is the Galois closure of $K(C)/K(D)$. Then $\mathrm{Mon}(\phi_i)$ is isomorphic as a permutation group to the group $G_i/\operatorname{core}_{G_i}(G_{i-1})$ in its action on the set of left cosets of G_{i-1} in G_i.*

Proof. By definition, $\mathrm{Mon}(\phi_i)$ is the Galois group of the Galois closure of $K(C_{i-1})/K(C_i)$, and hence as an abstract group $\mathrm{Mon}(\phi_i) \cong G_i/\operatorname{core}_{G_i}(G_{i-1})$. We view $\mathrm{Mon}(\phi_i)$ as a group of permutations of the set Λ of homomorphisms $K(C_{i-1}) \to \overline{K(C_i)}$ which restrict to the identity on $K(C_i)$. The image of any such homomorphism is contained in Ω (since $\Omega/K(C_i)$ is normal), so we can identify Λ with G_i/G_{i-1} without changing the action of $\mathrm{Mon}(\phi_i)$. □

In combination with Lemma 2.8, this lemma shows that Proposition 5.3 is a consequence of the following result.

Proposition 5.7. *Let G be a permutation group with a transitive Dedekind subgroup A, and let H be a one-point stabilizer of G. If $G > U > H$ and $G > V > H$ are distinct maximal decreasing chains of groups between G and H, then $G/\operatorname{core}_G(U)$ (in its action as a permutation group on G/U) is isomorphic to $V/\operatorname{core}_V(H)$ (in its action as a permutation group on V/H).*

The following lemma exhibits the portion of Proposition 5.7 which we can prove under the weaker hypothesis that G has a transitive quasi-Hamiltonian subgroup.

Lemma 5.8. *Let G be a permutation group which has a transitive quasi-Hamiltonian subgroup A, and let H be a one-point stabilizer of G. If $G > U > H$ and $G > V > H$ are distinct maximal decreasing chains of groups between G and H, then $N := \operatorname{core}_G(U)$ and $C := \operatorname{core}_V(H)$ satisfy either $G/N \cong V/C$ or $N = C = \operatorname{core}_G(H)$.*

Proof. By Lemma 2.9 we have $U = HI = IH$ and $V = HJ = JH$ where $I := U \cap A$ and $J := V \cap A$, and also $U \cap V = H(I \cap J)$ and $\langle U, V\rangle = H\langle I, J\rangle$. Maximality (and distinctness) of the chains implies that $U \cap V = H$ and $\langle U, V\rangle = G$, so that $I \cap J = H \cap A$ and $\langle I, J\rangle = A$. Since A is quasi-Hamiltonian we have $IJ = \langle I, J\rangle = A$, and since $HI = IH$ and $HJ = JH$ we find that $UV = HIHJ = HIJ = HA = G$. The facts that $U \cap V = H$

and $UV = G$ imply that

$$C = \bigcap_{g \in V} H^g = \bigcap_{g \in V} (U \cap V)^g = \big(\bigcap_{g \in V} U^g\big) \cap V$$
$$= \big(\bigcap_{g \in G} U^g\big) \cap V = N \cap V.$$

Since N is normal in G, we know that NV is a subgroup of G, so maximality of the chain $G > V > H$ implies that either $NV = V$ or $NV = G$. If $NV = V$ then $V \geq N$ so $C = N \cap V = N$, whence $C = N = \mathrm{core}_G(H)$. Finally, suppose that $NV = G$. Since $N \cap V = C$ is a normal subgroup of V and $NH = U$ (by maximality), the natural map $V/(N \cap V) \to NV/N$ is an isomorphism of permutation groups $V/C \cong G/N$. □

We now prove Proposition 5.7, which as we have seen implies Proposition 5.3 and Theorem 5.1. In the notation of Lemma 5.8, all that must be shown is that if A is Dedekind then $N \neq C$.

Proof of Proposition 5.7. We may assume that $\mathrm{core}_G(U) = \mathrm{core}_V(H) = \mathrm{core}_G(H)$, since otherwise Lemma 5.8 implies the desired conclusion. Since $G = HA$ and $U > H$ we have $G = UA$, so that

$$\mathrm{core}_G(U) = \bigcap_{g \in G} U^g = \bigcap_{g \in A} U^g \geq \bigcap_{g \in A} (U \cap A)^g = \mathrm{core}_A(U \cap A).$$

But $\mathrm{core}_A(U \cap A) = U \cap A$ (since A is Dedekind), so $U \cap A \leq \mathrm{core}_G(U) = \mathrm{core}_G(H)$, which yields the contradiction $U = H(U \cap A) \leq H$. □

Remark 5.9. We do not know whether Theorem 5.1 and Proposition 5.3 would remain true if we assumed only that $\mathrm{Mon}(\phi)$ has a transitive quasi-Hamiltonian subgroup, rather than a transitive Dedekind subgroup. Any counterexample to this generalization of Proposition 5.3 would have $N = C = 1$ in the notation of Lemma 5.8 (since $G := \mathrm{Mon}(\phi)$ is faithful so that $\mathrm{core}_G(H) = 1$), but we do not know whether this can happen. We note that the proof of Proposition 5.7 shows that this cannot happen if every minimal nontrivial subgroup of A is normal.

Remark 5.10. Proposition 5.3 was first proved for complex polynomials as a step in the proof of [31, Thm. R.2]; in this case [44, Thm. 2.13] shows that the conclusion holds if we replace the hypothesis that the decompositions are inequivalent and complete by the hypothesis that $\gcd(\deg \phi_2, \deg \psi_2) = 1 = \gcd(\deg \phi_1, \deg \psi_1)$. Theorem 5.1 was first proved for complex polynomials in [44, Thm. 1.3].

6. Automorphism groups of covers

In this section we examine the automorphism group of a cover $\phi\colon C \to D$, and show that if the monodromy group of ϕ has a transitive quasi-Hamiltonian subgroup then the collection of automorphism groups of the indecomposable covers in a complete decomposition of ϕ is uniquely determined by ϕ.

Definition 6.1. If $\phi\colon C \to D$ is a cover of curves over a field K, then an *automorphism* of ϕ is an automorphism σ of C which is defined over K and which satisfies $\phi \circ \sigma = \phi$.

We write $\mathrm{Aut}(\phi)$ to denote the set of all automorphisms of ϕ, and note that $\mathrm{Aut}(\phi)$ is a group under the operation of composition. We will prove the following result.

Theorem 6.2. *Let $\phi\colon C \to D$ be a cover of curves over a field K, and suppose that the monodromy group $\mathrm{Mon}(\phi)$ has a transitive quasi-Hamiltonian subgroup. If $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$ and $\phi = \psi_n \circ \psi_{n-1} \circ \cdots \circ \psi_1$ are complete decompositions of ϕ, then there is a permutation π of $\{1, 2, \ldots, n\}$ such that, for each i with $1 \le i \le n$, we have $\deg \phi_i = \deg \psi_{\pi(i)}$ and $\mathrm{Aut}(\phi_i) \cong \mathrm{Aut}(\psi_{\pi(i)})$.*

We begin with the following simple result.

Lemma 6.3. *If $\phi\colon C \to D$ is a cover of curves over a field K, then we can write $\phi = \phi_2 \circ \phi_1$ where $\phi_2\colon C_1 \to D$ and $\phi_1\colon C \to C_1$ are covers of curves over K such that $K(C)/K(C_1)$ is Galois with Galois group $\mathrm{Aut}(\phi)$. For any expression $\phi_2 = \psi_2 \circ \psi_1$ as the composition of two covers, we have $\mathrm{Aut}(\phi) = \mathrm{Aut}(\psi_1 \circ \phi_1)$. Finally, $\mathrm{Aut}(\phi) \cong N_G(H)/H$, where G is the monodromy group of ϕ and H is a one-point stabilizer of G.*

Proof. Via the standard equivalence of categories between curves and function fields, we see that $\mathrm{Aut}(\phi)$ is isomorphic to the group of automorphisms of the function field $K(C)$ which act as the identity on $K(D)$. In other words, $\mathrm{Aut}(\phi)$ is the Galois group of the largest Galois extension $K(C)/L$ where L is a field between $K(D)$ and $K(C)$. Now let Ω be the Galois closure of $K(C)/K(D)$, and write $G := \mathrm{Gal}(\Omega/K(D))$ and $\tilde{H} := \mathrm{Gal}(\Omega/K(C))$. Then $\mathrm{Aut}(\phi) \cong \mathrm{Gal}(K(C)/L) \cong N_G(\tilde{H})/\tilde{H}$. Since G is transitive, any one-point stabilizer H of G is conjugate to $\tilde{H}$ in G, so that $N_G(H)/H \cong N_G(\tilde{H})/\tilde{H}$. Finally, let $\phi_2\colon C_1 \to D$ and $\phi_1\colon C \to C_1$ be covers of curves over K which

correspond to the field extensions $L/K(D)$ and $K(C)/L$; then $\phi = \phi_2 \circ \phi_1$ and $K(C)/L$ is Galois with Galois group $\mathrm{Aut}(\phi) = \mathrm{Aut}(\phi_1) = \mathrm{Aut}(\psi_1 \circ \phi_1)$, as required. □

We now record an immediate geometric reformulation of the condition that $K(C)/K(C_1)$ is Galois in the above result.

Lemma 6.4. *Let $\phi\colon C \to D$ be a cover of curves over a field K. Then the function field extension $K(C)/K(D)$ is Galois if and only if the irreducible components of the fibered product $C \times_D C$ are precisely the graphs of the functions $\nu\colon C \to C$ for $\nu \in \mathrm{Aut}(\phi)$.*

Remark 6.5. In the geomeric setting, one says that ϕ is Galois when $K(C)/K(D)$ is Galois. In the algebraic setting, where ϕ is a polynomial $f(X)$, the above result says (for x transcendental over K) that $K(x)/K(f(x))$ is Galois if and only if $f(X) - f(Y)$ is a constant times the product of all $X - \nu(Y)$ where $\nu \in K[X]$ satisfies $f \circ \nu = f$. Polynomials with this property are called "factorable" [5], and if ϕ is a polynomial then the polynomial playing the role of ϕ_1 in Lemma 6.3 is called the "factorable core" of ϕ.

Next we show that indecomposable covers with nontrivial automorphism groups are highly restricted.

Corollary 6.6. *If $\phi\colon C \to D$ is an indecomposable cover of curves over a field K, then the following are equivalent:*

(1) $\deg \phi$ *is prime and both* $\mathrm{Aut}(\phi)$ *and* $\mathrm{Mon}(\phi)$ *are cyclic of order* $\deg \phi$
(2) $\mathrm{Mon}(\phi)$ *is abelian*
(3) $\mathrm{Mon}(\phi)$ *is regular*
(4) $|\mathrm{Aut}(\phi)| > 1$.

Proof. We show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1). The first implication is immediate. Now let Ω be the Galois closure of $K(C)/K(D)$, and put $H := \mathrm{Gal}(\Omega/K(C))$. If (2) holds then H is a normal subgroup of $\mathrm{Mon}(\phi)$, so that $K(C)/K(D)$ is Galois and thus $\Omega = K(C)$, whence $H = 1$ so (3) holds. If (3) holds then Lemma 6.3 implies that $\mathrm{Aut}(\phi) \cong \mathrm{Mon}(\phi)$ is nontrivial. Finally, suppose that (4) holds. Write $\phi = \phi_2 \circ \phi_1$ as in Lemma 6.3. Since ϕ is indecomposable and $\deg \phi_1 = |\mathrm{Aut}(\phi)| > 1$, we must have $\deg \phi_1 = \deg \phi$. Therefore $K(C)/K(D)$ is Galois with Galois group $\mathrm{Aut}(\phi)$, so that $\mathrm{Mon}(\phi) \cong \mathrm{Gal}(K(C)/K(D)) \cong \mathrm{Aut}(\phi)$. Since ϕ is indecomposable, Lemma 2.8 implies that $\mathrm{Mon}(\phi)$ has no nontrivial proper subgroups, so that $\mathrm{Mon}(\phi)$ must have prime order. □

Proof of Theorem 6.2. By Theorem 4.1 it suffices to prove the result when $n = 2$. Assuming $n = 2$, let H be a one-point stabilizer in $G := \mathrm{Mon}(\phi)$, and let A be a transitive quasi-Hamiltonian subgroup of G. Let U and V be groups between H and G which correspond to ϕ_1 and ψ_1 via Lemma 2.8. We assume $U \neq V$, since otherwise the conclusion is immediate. By symmetry, it suffices to show that $\mathrm{Aut}(\phi_2) \cong \mathrm{Aut}(\psi_1)$ and $\deg \phi_2 = \deg \psi_1$. By Lemma 6.3, this holds if $\mathrm{Mon}(\phi_2) \cong \mathrm{Mon}(\psi_1)$, which is the case unless $\mathrm{core}_G(U) = \mathrm{core}_V(H) = \mathrm{core}_G(H)$ (by Lemmas 5.6 and 5.8). Since G is the Galois group of a field extension, in particular it is a faithful permutation group, so that $\mathrm{core}_G(H) = 1$. Henceforth assume that $\mathrm{core}_G(U) = \mathrm{core}_V(H) = 1$. Therefore U is not normal in G, so Corollary 6.6 implies that $\mathrm{Aut}(\phi_2) = 1$. If $\mathrm{Aut}(\psi_1) \neq 1$ then Corollary 6.6 implies that $H = 1$, so $G = HA = A$ and thus U is a maximal proper subgroup of A. This contradicts Ore's result [34, Thm. 5] that every maximal proper subgroup of a quasi-Hamiltonian group is normal, so in fact $\mathrm{Aut}(\psi_1) = 1$. Finally, maximality and distinctness of the chains forces $UV = G$ and $U \cap V = H$, so that $[G : U] = [V : H]$ and thus $\deg \phi_2 = \deg \psi_1$. □

Remark 6.7. In the above proof we used Ore's result that every maximal proper subgroup of a quasi-Hamiltonian group is normal. In other words, we used a portion of the defining property of Dedekind groups which remains true for the larger class of quasi-Hamiltonian groups. It would be interesting to generalize Theorem 6.2 by replacing quasi-Hamiltonian groups by an even larger class of groups.

Remark 6.8. Since Ore's proof is quick, we include it here for the reader's convenience. If U is a non-normal maximal subgroup of a quasi-Hamiltonian group G then U has a conjugate $V \neq U$, and plainly V cannot be a subgroup of U. Therefore UV is a group which strictly contains U, so maximality implies that $UV = G$, whence $V = v^{-1}Uv$ for some $v \in V$, yielding the contradiction $V \neq U = vVv^{-1} = V$.

In the remainder of this section we describe some ways in which the concept of the automorphism group of a cover has arisen (in the special case of polynomials or rational functions) in the literature on complex dynamics and value sets of polynomials over finite fields. We note that previous authors have had to work much harder in order to prove some of the above results in the case of polynomials, and their proofs do not extend to treat more general covers. For $f(X) \in K(X)$, the group $\mathrm{Aut}(f)$ consists of the degree-one

rational functions $\mu(X) \in K(X)$ for which $f \circ \mu = f$; if $f(X) \in K[X] \setminus K$ then every element of $\mathrm{Aut}(f)$ must permute the set $f^{-1}(\infty) = \{\infty\}$, and hence must be a degree-one polynomial.

In case $f(X) \in \mathbb{C}[X]$ has degree $n \geq 2$, we write $\Gamma(f)$ for the group of degree-one $\mu \in \mathbb{C}[X]$ for which there exists a degree-one $\nu \in \mathbb{C}[X]$ satisfying $\nu \circ f \circ \mu = f$. One can easily check that $\Gamma(f)$ is infinite if and only if $f(X)$ is conjugate to X^n, and that if $\Gamma(f)$ is finite then it is cyclic of order m; here, for any degree-one $\theta \in \mathbb{C}[X]$ such that $\hat{f} := \theta^{-1} \circ f \circ \theta$ has no term of degree $n-1$, the number m is the greatest common divisor of the collection of all differences between degrees of pairs of terms of $\hat{f}$. In case $\Gamma(f)$ is finite, it coincides with the group of symmetries of the Julia set of $f(X)$ [1]. The paper [2] studies the subgroup $\mathrm{Aut}(f)$ of $\Gamma(f)$, and gives a lengthy proof of Theorem 6.2 in the special case of complex polynomials by making use of the much more difficult "Ritt's Second Theorem" (which should not be confused with Ritt's First Theorem as discussed in Remark 4.4). In a subsequent paper we will discuss the analogue of $\Gamma(f)$ for arbitrary covers of curves (over an arbitrary field), and in the case of rational functions we will discuss the union of the groups $\Gamma(f^{\circ n})$ where $f^{\circ n}$ denotes the n-th iterate of $f(X)$. Some results in this direction appear in [25].

Now consider $f(X) \in K[X] \setminus K[X^p]$, where K is a field of characteristic $p \geq 0$. As noted in Remark 6.5, we can write $f = g \circ h$ where $g, h \in K[X]$ and $K(x)/K(h(x))$ is Galois with Galois group $\mathrm{Aut}(f)$. Now $\mathrm{Aut}(f)$ consists of all degree-one $\mu \in K[X]$ for which $X - \mu(Y)$ divides $f(X) - f(Y)$, so in particular $h(X) - h(Y)$ is the product of degree-one polynomials in $K[X, Y]$. Together with the known list of possibilities for $h(X)$ [40, Thm. 1], this implies the main results of [5]. The elements of $\mathrm{Aut}(f)$ play a prominent role in the study of the image set $f(K)$ where K is a finite field. This classical topic has a rich tradition, with important contributions by Betti, Mathieu, Hermite, Dickson, Schur, Davenport, Carlitz, Birch, Swinnerton-Dyer, Mordell, Bombieri, and many others. The group $\mathrm{Aut}(f)$ plays an especially fundamental role in case the ratio $|f(K)|/|K|$ is either large [19, 26] or small [13, 29], and also arises in other well-behaved cases [4, 42]. See [30] for a recent survey on this topic.

7. A divisibility property of automorphism groups of subcovers

In this section we prove the following divisibility result.

Theorem 7.1. *Let ϕ be a cover of curves over a field K, and assume that the monodromy group $\mathrm{Mon}(\phi)$ has a transitive Dedekind subgroup. Let $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$ be a decomposition of ϕ, and for each $i = 1, 2, \ldots, n$ write $\psi_i := \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1$. Then for $1 \le i < n$ the group $\mathrm{Aut}(\psi_i)$ is a normal subgroup of $\mathrm{Aut}(\psi_{i+1})$, and the quotient group is isomorphic to a subgroup of $\mathrm{Aut}(\phi_{i+1})$. In particular,*

$$|\mathrm{Aut}(\phi)| \text{ divides } \prod_{i=1}^{n} |\mathrm{Aut}(\phi_i)|.$$

The crucial case in the proof of Theorem 7.1 is contained in the following result which is of independent interest.

Proposition 7.2. *Let $\theta\colon C_1 \to D$ and $\rho\colon C \to C_1$ be covers of curves over K for which the monodromy group of $\psi := \theta \circ \rho$ has a transitive Dedekind subgroup A. Then, for each $\mu \in \mathrm{Aut}(\psi)$, there is a unique $\nu \in \mathrm{Aut}(\theta)$ for which $\rho \circ \mu = \nu \circ \rho$. Moreover, the map $\mu \mapsto \nu$ defines a homomorphism $\mathrm{Aut}(\psi) \to \mathrm{Aut}(\theta)$ with kernel $\mathrm{Aut}(\rho)$.*

Proof. Let $G > U > H$ be the chain of groups corresponding to the decomposition $\psi = \theta \circ \rho$, where H is a one-point stabilizer of $G := \mathrm{Mon}(\psi)$. Then $J := N_G(H) \cap A$ and $J_1 := U \cap A$ satisfy $N_G(H) = HJ$ and $U = HJ_1$. Since $J \le N_G(H)$ and J normalizes J_1 (because A is Dedekind), it follows that $J \le N_G(U)$, so also $N_G(H) = HJ \le N_G(U)$ (since $H \le U$). Thus $N_G(H)U$ is a group which normalizes U. Let $\theta = \theta_2 \circ \theta_1$ be the chain of covers which corresponds to the chain of groups $G \ge N_G(H)U \ge U$. Writing θ_1 as $\theta_1\colon C_1 \to E$, we see that $K(C_1)/K(E)$ is Galois, so that $\mathrm{Aut}(\theta_1) \cong \mathrm{Mon}(\theta_1) \cong N_G(H)U/U$ has order equal to $\deg \theta_1$. By Lemma 6.4, the irreducible components of the fibered product $Z := C_1 \times_E C_1$ are the graphs of the functions $\nu\colon C_1 \to C_1$ for $\nu \in \mathrm{Aut}(\theta_1)$. Lemma 6.3 implies that $\mathrm{Aut}(\psi) = \mathrm{Aut}(\theta_1 \circ \rho)$, so for $\mu \in \mathrm{Aut}(\psi)$ we have $\theta_1 \circ \rho \circ \mu = \theta_1 \circ \rho$. Therefore the map $P \mapsto (\rho \circ \mu(P), \rho(P))$ is a morphism $C_0 \to Z$, so its image is a component of Z, whence there is some $\nu \in \mathrm{Aut}(\theta_1)$ for which $\rho \circ \mu = \nu \circ \rho$. Now it is clear that the map $\mu \mapsto \nu$ is a homomorphism $\mathrm{Aut}(\psi) \to \mathrm{Aut}(\theta_1)$ with kernel $\mathrm{Aut}(\rho)$, which proves the result since $\mathrm{Aut}(\theta_1) \le \mathrm{Aut}(\theta)$. □

Proof of Theorem 7.1. Since $\mathrm{Mon}(\phi)$ has a transitive Dedekind subgroup, so too does $\mathrm{Mon}(\psi_{i+1})$ for any $i < n$. Apply Proposition 7.2 with $(\psi, \theta, \rho) = (\psi_{i+1}, \phi_{i+1}, \psi_i)$ to conclude that $\mathrm{Aut}(\psi_i) \trianglelefteq \mathrm{Aut}(\psi_{i+1})$ and $\mathrm{Aut}(\psi_{i+1})/\mathrm{Aut}(\psi_i)$

is isomorphic to a subgroup of $\mathrm{Aut}(\phi_{i+1})$. It follows that

$$|\mathrm{Aut}(\phi)| = |\mathrm{Aut}(\psi_1)| \cdot \prod_{i=1}^{n-1} \frac{|\mathrm{Aut}(\psi_{i+1})|}{|\mathrm{Aut}(\psi_i)|}$$

divides $|\mathrm{Aut}(\phi_1)| \cdot \prod_{i=1}^{n-1} |\mathrm{Aut}(\phi_{i+1})|$, which concludes the proof. □

Remark 7.3. Theorem 7.1 would not remain true if we weakened the hypothesis to require only that $\mathrm{Mon}(\phi)$ contains a transitive quasi-Hamiltonian subgroup. For, let G be any finite quasi-Hamiltonian group which is not a Dedekind group, and let U be a non-normal subgroup of G. Let $L/\mathbb{C}(x)$ be a Galois extension with group G (this exists for any choice of G, essentially by Riemann's existence theorem). Then the chain of fields $\mathbb{C}(x) \subseteq L^U \subseteq L$ corresponds to a chain of covers $\phi = \phi_2 \circ \phi_1$ where $\mathrm{Aut}(\phi_1) \cong U$ and $\mathrm{Aut}(\phi) \cong G$ but $\mathrm{Aut}(\phi_2) \cong N_G(U)/U$ has order strictly less than $[G : U]$.

Remark 7.4. The final assertion in Theorem 7.1 has been considered previously when ϕ is given by a polynomial $f(X)$. This assertion was proved in [2, Thm. 1.2] for $f(X) \in \mathbb{C}[X]$, and in [16, Thm. 8] for $f(X) \in K[X]$ where K is a field and $\mathrm{char}(K) \nmid \deg f$. The proofs in those papers apply only to polynomials, and make no mention of monodromy groups. This difference in perspective perhaps accounts for the conjecture by Gutierrez and Sevilla [16, Conj. 1] that the final assertion in Theorem 7.1 is true whenever ϕ is a rational function $f(X) \in K(X)$ over a field K for which $\mathrm{char}(K) \nmid \deg f$. One counterexample is $f(X) := X^3 + X^{-3}$ over the field $K = \mathbb{C}$: for $f_2(X) := X^3 - 3X$ and $f_1(X) := X + X^{-1}$ we have $f = f_2 \circ f_1$ but $|\mathrm{Aut}(f)| = 6$ does not divide $|\mathrm{Aut}(f_2)| \cdot |\mathrm{Aut}(f_1)| = 1 \cdot 2$. There are many further counterexamples, and indeed the perspective of the present paper explains why one should not expect the conjecture to be true.

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