On Numbers and Games

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Abstract— In this research we study construction of the Surreal Numbers, showing it is a class that forms the totally ordered Field, and then explore some of new numbers, we present the reader some algebraic operations related to combinatorial games and gives a detailed outlook of the Surreal Numbers. A fresh outlook to some combinatorial mathematical algebraic operations, through the evaluation of a deduced several algebraic concepts.

Index Terms— Combinatorial games, Surreal Numbers, Groups, Fields, Algebraic Studies.

1 Introduction

In 1902, combinatorial game theory was born by Bouton at Harvard [1]. In the 1930s, Sprague and Grundy extended this theory to cover all impartial games. In the 1970s, the theory was extended to partisan games, a large collection which includes the ancient Hawaiian game called Konane [2], many variations of Hackenbush, cutcakes, ski-jumps, Domineering, Toads-and-Frogs, etc. [3]. One reviewer remarked that although there were over a hundred such games in Winning Ways, most of them had been invented by the authors. John Conway axiomatized this important branch of the subject [4].

In this paper we construct objects called games, which are ordered pairs of independent sets containing already constructed games, one of which we call the left set and the other the right set. These fractal constructions, first discovered and studied by John H. Conway, can describe both the games of Game Theory after which they are named, as well the real and ordinal numbers, and more besides. This paper introduces games, and specifically to Conway's Surreal Numbers, as introduced in his 1976 "book On Numbers and Games", a subclass of Games that contains and extends the real and ordinal numbers to a field [5]. We study the Surreal Numbers, showing they form a totally ordered field, comparing the surreal constructions of the reals and ordinals to their usual constructions and exploring the new numbers that emerge from this method of construction. After that we introduce the concept of combinatorial games and how they relate to the mathematical objects of games, some of the mechanisms and arguments by which we try to analyze these games, and then work through and prove the complete theory for a few of these games.

Most of the proofs in the first half of the paper follow those of Conway in On Numbers and Games, though they are here expanded and explicitly reasoned, and the paper attempts to be comprehensive in the material that it covers, omitting no proofs except those which are obvious repetitions.

In short if we want to talk about the surreal numbers, the Surreal Numbers are a totally ordered class that form a Field (a field of a class, rather than a set) that extends the real and ordinal numbers, and is the largest totally ordered Field described yet in history , it has been shown to be the largest possible ordered field [6]. In the next few sections we shall show this rigorously, but in this section, we less formally explore the (literally) simplest surreal numbers and arguments to help develop the reader's intuition around this unique construction, as well as defining many key terms.

2 Numbers Constructions

Surreal numbers were invented (some prefer to say "discovered") by John Horton Conway of Cambridge University and described in his book On Numbers and Games. Conway used surreal numbers to describe various aspects of game theory, the term "surreal number" was invented by Donald Knuth.

What can surreal numbers be used for? Not very much at present, except for some use in game theory. But it is still a new field, and the future may show uses that we haven't thought of. Nevertheless, surreal numbers are worth studying for two reasons. First, as a study in pure math they are a fascinating – even beautiful – subject. Secondly, they provide a good introduction to and exercise in abstract algebra, and as such they serve a didactic purpose.

So, what are surreal numbers? Before we start looking at the definition, you must forget everything you know about numbers. You don't know what "less than" means. You don't know what "equal to" means. You don't know what "one" or "two" or "three" means. You don't know what addition and multiplication are. Okay?

Now we can begin with two definitions that must be considered in tandem:

2.1 Definition

For any two sets of numbers L and R,

$$\exists \text{ the number } \{L|R\} \Leftrightarrow \nexists x \epsilon L \ : x \geq y, \forall y \in R.$$

That is, there exists a new number $\{L|R\}$ if and only if no member of L is greater-than-or-equal to any member of R.

We denote the left set of a number a as A^L and the right set as A^R, so

$a = \{A^L|A^R\}$.

It is important to distinguish between numbers and sets of numbers, so we use lower case letters to denote numbers, and

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upper-case letters to denote sets of numbers. We also denote an arbitrary member of A^Las a^l, and write $a^l = \{A^{LL}|A^{LR}\}$, and similarly write $a^R = \{A^{RL}|A^{RR}\}$for a member of A^R.

2.2 Definition

For any two numbers $x = \{X^L \middle| X^R\}$ and$y = \{Y^L|Y^R\}$,

$$x \le y \Leftrightarrow \nexists x^L \epsilon X^L: x^L \ge y \wedge \nexists y^R \epsilon Y^R: y^R \le x$$

That is $x \le y$, if and only if no member of X^L (the left set of x) is greater-than-or-equal to y and no member of Y^R(the right set of y) is less-than-or-equal to x. We will also often use the inverse of this:

$$x \nleq y \Leftrightarrow \exists x^L \epsilon X^L: x^L \ge y \vee \exists y^R \epsilon Y^R: y^R \le x$$

We also define equality here as: $x = y \Leftrightarrow x \le y \wedge x \ge y$, define less-than as $x < y \Leftrightarrow x \le y \wedge y \nleq x$
, and define greater-than as $x > y \Leftrightarrow x \ge y \wedge y \ngeq x$
We from previous can see directly then that equality is a symmetric relation between two numbers (i.e. $x = y \Leftrightarrow y = x$), also both less-than and greater-than are asymmetric relations (i.e. $x < y \Leftrightarrow y \nless x$), and that both less-than-or-equal and greater-than-or-equal are antisymmetric (i.e. $x \le y \wedge y \le x \Rightarrow x = y$).

From these two definitions we can begin to construct and order numbers. First, we know there exists a set containing no elements, called the empty set, written $\emptyset = \{\,\}$ and consider the empty set. Then by 2.1 definition, letting L=R= $\emptyset$, there exists a number$\{\emptyset|\emptyset\}$, since the empty set has no elements and so 2.1 definition's requirement is automatically fulfilled. Let us call this number $0 := \{\emptyset|\emptyset\}$ (and we will later show that it is the additive identity for the surreal numbers as we expect from zero), giving us two sets of numbers, $\emptyset$and $\{0\}$. Then we can construct three more possible numbers:

$a := \{\{0\} \middle| \emptyset\}$ $\quad b := \{\emptyset \middle| \{0\}\}$ $\quad c := \{\{0\} \middle| \{0\}\}$

(For ease, since in all our constructions there is only one left set, and only one right set, we usually omit the outmost brackets of these sets, and if one of these sets is the empty set, we leave that side empty. So, we rewrite:

$a = \{0|\,\}, b = \{\,|0\}, c = \{0|0\}, 0 = \{\,|\,\}$).

For a, the only member of A^L is 0, and there are no members of A^R, so from 2.1 definition, a is a number. Similarly, as there are no members of B^L,b is a number, Generally, any construction $x = \{X^L \middle| X^R\}$ with either $X^L = \emptyset$ or $X^R = \emptyset$ will be a number, as 2.1 definition will hold vacuously. For c, we have $0 \epsilon C^L$ and $0 \epsilon C^R$. But from 2.2 definition we know that 0≥0, so by 2.1 definition we see that c is not a number [7].

Now we have three numbers,0, a, and b. We can use 2.2 definition to order them. First, we consider 0 and a:

Is $0 \le a$? There are no members of 0^L, so $\nexists x \epsilon 0^L: x \ge a$.Similarly, there are no members of A^R,

so $\nexists a^R \epsilon A^R: a^R \le 0$.
So, by 2.2 definition, $0 \le a$.
Is $a \le 0$? $0 \epsilon A^L$and $0 \le 0$, so by 2.2 definition, $0 \nleq a$.
So, we have $0 \le a$ and $a \nleq 0$, which implies. $0 < a$.
Next, we consider 0 and b:
Since $0 \epsilon B^R$ and $0 \le 0$, we have $0 \nleq b$. And since $B^L = 0^R = \emptyset$, the conditions for $b \le 0$ hold vacuously, so $0 \nleq b \wedge b \le 0 \Rightarrow b < 0$.
We can now order our three numbers: $b < 0 < a$, and it is easy to check directly by the same method that $b < a$.
We write $1 := a = \{0|\,\}$ and □ $-1 := b = \{\,|0\}$,
these names later. For now, we consider the eight sets of numbers we can now form:

$$\emptyset \quad \{-1\} \quad \{0\} \quad \{1\}$$
$$\{-1,0\} \quad \{-1,1\} \quad \{0,1\} \quad \{-1,0,1\}$$

Pairing these up into left and right sets, we get 64 candidates for numbers. However, since we have ordered -1,0 and 1, we can quickly show using 2.1 definition that most of them are not numbers, leaving us with the following:

$\{-1|\,\}$ $\quad \{-1,0|\,\}$ $\quad \{-1,1|\,\}$ $\quad \{-1,0,1|\,\}$ $\quad \{0,1|\,\}$ $\quad \{1|\,\}$
$\{\,|-1\}$ $\quad \{\,|-1,0\}$ $\quad \{\,|-1,1\}$ $\quad \{\,|-1,0,1\}$ $\quad \{\,|0,1\}$ $\quad \{\,|1\}$
□$\{-1|0\}$ $\quad \{-1|0,1\}$ $\quad \{-1|1\}$ $\quad \{-1,0|1\}$ $\quad \{0|1\}$

However, by using the Truncation Theorem which tells us we can remove all but the greatest element in the left set, and all but the least in the right set, and this new construction will be equal to the original one, and noting that, for example , $0 \le \{-1|1\,\}$and $\{-1|1\,\} \le 0$(meaning $\{-1|1\,\} = 0$), we can show that many of these candidates are equal to each other (we formalize this in the next section after showing = is an equivalence relation), leaving us finally with only these four new numbers created (and ordered using 2.2 definition):

$$\{\,|-1\} < -1 < \{-1|0\} < 0 < \{0|1\} < 1 < \{1|\,\}$$

Indeed, we will show later that we are justified in calling these numbers
$-2 := \{\,|-1\}, -1/2 := \{-1|0\,\}$, $1/2 := \{0|1\,\}$ and $2 := \{1|\,\}$, that is that they have the properties we expect, such as $1/2 + 1/2 = 1$.

We began with one number, 0, from which we constructed two more numbers, -1and 1, from which in turn we constructed four more numbers. Notice that it would be impossible to construct 2 before constructing 1, or 1 before constructing 0. This is because 0 was constructed in a previous step, or day. We call one number simpler than another if it was constructed on an earlier day and call the day a number is constructed on is its birthday. Thus 1 is simpler than 2, but 2 is as simple as -2 or 1/2, since they have the same birthday. We say then that 0 was created on day 0, 1 on day 1, &c. Similarly, if we have a set of numbers, we call the sum of their birthdays their day sum, and if one set of numbers has a lower day sum than a different set of numbers, we say that set is the simpler set.

The definition of the birthday of a number is the day on which it is first constructed, and we say that a number x is simpler than another number y if x has an earlier birthday than y.

2.3 Surreal Numbers

The existence of a surreal number x implies at the very least that the new surreal number $\{x|\ \}$ exists. Thus, the Surreal Numbers are not a set, but a proper class [8].

In the next paragraph we will explain the appropriate mathematical methods that we will use to prove some important theorems.

2.4 Methods of Proof

We will call the elements of X^L the left options of x, and the elements of X^R the right options of x. Often, we use induction on the options of a number x to show that a property P holds for that x. In terms of birthdays, we can express this method of proof as follows: suppose that, for some property P, x is the simplest number for which P does not hold. Then if we can show that this implies that P does not hold for one of the options of x ,we have a contradiction, since we can always express x in a form where it has only options simpler than x, since this is the only way to express x on its birthday. This means that either P holds for no numbers, or if we show that P holds for the simplest number,0, that P holds for all numbers. In the first proof below we will repeat this argument explicitly, but after that we will use it implicitly, and use phrases like 'we eventually only have to check the case of 0' or 'we inductively reduce the question down to 0' to refer to this reasoning [9].

We will in general be proving theorems on all numbers, including infinite ordinals. Usually in transfinite induction we must show that P holds for the base case (P(0)holds),the successor case (for any successor ordinal α+1,P($\alpha+1$) holds if P(α) holds),and the limit case (for any limit ordinal β, P(β) holds if P($\acute{\beta}$) holds for all $\acute{\beta} < \beta$) [8]. However, since here we do induction on the birthday of a number, we treat each day after 0 as a limit case, and only need to show P(0) holds, and then that for any number x, including ordinals, P(x)follows from P($\acute{x}$) for some $\acute{x}$ simpler than x,since we assume that P($\acute{x}$) holds for all $\acute{x}$ created on an earlier day than x.

2.5 Ordering the Surreal Numbers

In this section we build on our two definitions and show that the Surreal Numbers are totally ordered, and then prove some more general properties of the surreals, which allow us to manipulate them up to equality, which will be very useful for later proofs.

2.5.1 Lemma

For any number x , $x \le x$ *and* $x = x$.

Proof.

(a) $x \le x$: From 2.2 definition, we must show

$$\nexists x^L \epsilon X^L : x^L \ge x, \text{ or equivalently, } x \nleq x^L, \forall x^L \epsilon X^L \quad (1)$$

$$\nexists x^R \epsilon X^R : x^R \le x, \text{ or equivalently, } x \ngeq x^R, \forall x^R \epsilon X^R \quad (2)$$

but $x \nleq x^L$ if there exists an $\tilde{x}^L \epsilon X^L$ such that $\tilde{x}^L \ge x^L$ for any x^L, and $x \ngeq x^R$ if there exists an $\tilde{x}^R \epsilon X^R$ such that $\tilde{x}^R \le x^R$ for any x^R. But we can choose $\tilde{x}^L = x^L$ and $\tilde{x}^R = x^R$, and in this way we have reduced the question on x to questions on the options of x, that is we now know that

$x \le x$ if both $x^L \le x^L$ and $x^R \le x^R$.

Now suppose that the theorem does not hold for x, and that furthermore x is the simplest number for which the theorem does not hold. Then we must also have either that the theorem holds for no numbers, because the theorem holding for simpler numbers would imply it holding for numbers with those simpler numbers as options, or that the theorem holds for all numbers, if the theorem holds for the simplest number, 0. So we only have to consider only whether $0 \le 0$. But this follows from the fact that 0 has no options, so by induction, $x \le x$ all x. This method of argument will we use often, and from now on, implicitly [10].

(b) $x = x$: This follows directly from our definition of = and (a).

2.5.2 Theorem

For any number x, $x^L < x$ for all $x^L \epsilon X^L$.

Proof.

We first show that $x^L \le x$ for all $x^L \epsilon X^L$.. For this to be true we must have from 2.2 definition that both

$$\nexists x^{LL} \epsilon X^{LL} : x^{LL} \ge x \quad (3)$$

$$\text{And } \nexists x^R \epsilon X^R : x^R \le x^L, \forall x^L \epsilon X^L . \quad (4)$$

Note that (4) is just restating of 2.1 definition, so we just need to show (3), which can be equivalently written as

$$\forall x^{LL} \epsilon X^{LL}, x \nleq x^{LL}. \quad (5)$$

Then by the inverse form of 2.2 definition, (5) is true if

$$\exists\, x^L \epsilon X^L : x^L \ge x^{LL}, \forall x^{LL} \epsilon X^{LL} \quad (6)$$

but this is the same condition that we wanted to show at the start of (a), replacing x with x^L and x^L with x^{LL}, that is to say, we have $x^L \le x$ only if $x^{LL} \le x^L$, for all $x^{LL} \epsilon X^{LL}$. By repeating this process, we will eventually only have to consider sets whose only left option is 0, so (3) will hold vacuously. Thus, by induction $x^L \le x$ for all $x^L \epsilon X^L$.

Now we show that $x \nleq x^L$ for all $x^L \epsilon X^L$. By the inverse form of 2.2 definition, to prove this it is enough to show that

$$\exists\, \tilde{x}^L \epsilon X^L : \tilde{x}^L \ge x^L \quad (7)$$

and we can choose $\tilde{x}^L = x^L$, and then from Lemma 2.4.1 Thus $x^L < x$ for all $x^L \epsilon X^L$.

2.5.3 Theorem

[11] For any number x, $x < x^R$ for all $x^R \epsilon X^R$.

For any number x, $x^L < x$ for all $x^L \epsilon X^L$.

Proof.

By a symmetric argument the proof for this is the same as the proof from Theorem 2.5.2 but considering the right options of

x .

We say that any two numbers $x = \{X^L | X^R\}$ and $y = \{Y^L|Y^R\}$are identical if and only if $X^L = Y^L$ and $X^R = Y^R$,and write $x \equiv y$ to express identity. We also simplify the notation here by writing 'for all/there existsx^L' instead of the longer 'for all/there exists $x^L \epsilon X^L$' , since there is no ambiguity, as every number has only one left set and one right set.

2.5.4 Theorem

If two numbers x and y are identical, then they are also equal.

Proof.

Since x and y yare identical, $X^L = Y^L$ and $X^R = Y^R$. Then $x \le y$if there does not exist any$y^L \ge y$or any$x^R \le x$, and$y \le x$ if there does not exist any$x^L \ge x$ or any$y^R \le y$. But from Theorems 2.5.2/2.5.3 none of these exist, so $x = y$.

2.5.5 Theorem

For all numbers x , y, and $\mathcal{Z}$, if $x \le y$and $y \le \mathcal{Z}$, then $x \le \mathcal{Z}$.

Proof.

We will do a proof by contradiction, so assume that the proposition$\pi(x , y, \mathcal{Z})$: $(x \le y \wedge y \le \mathcal{Z} \wedge x \nleq \mathcal{Z}$holds. Then the following all hold by 2.2 definition:

$$\nexists x^L : x^L \ge y \tag{8}$$

$$\nexists \mathcal{Z}^R : \mathcal{Z}^R \le y \tag{9}$$

$$\exists x^L : x^L \ge \mathcal{Z} \vee \exists \mathcal{Z}^R : \mathcal{Z}^R \le x \tag{10}$$

If we suppose the first proposition of (10) holds, then by considering also (8) the following holds:

$$\exists x^L : y \le \mathcal{Z} \wedge \mathcal{Z} \le x^L \wedge y \nleq x^L \tag{11}$$

That is, $\pi(y , \mathcal{Z}, x^L)$. If we suppose the second proposition of (10) holds, then by considering also (9) the following holds:

$$\exists \mathcal{Z}^R : \mathcal{Z}^R \le x \wedge x \le y \wedge \mathcal{Z}^R \nleq y \tag{12}$$

That is, $\pi(\mathcal{Z}^R , x , y)$. So, in either case the truth of $\pi(x , y, \mathcal{Z})$depends on the truth of π with one of x or $\mathcal{Z}$ replaced by one of their options. Then, since (10) will not hold for any$X^L = Z^R = \emptyset$, by induction $\pi(x , y, \mathcal{Z})$not hold. So, we must have $x \le y \wedge y \le \mathcal{Z} \Rightarrow x \le \mathcal{Z}$, that is that numbers are transitive under$\le$.

It follows directly from Theorem 2.5.5 that = is transitive, so we have now shown that= is an equivalence relation on numbers, as it is reflexive, symmetric, and transitive.

Then equality partitions the surreals into equivalence classes, and in general when we talk of constructing a new number, we mean a number that is not equal to an already constructed number, that is it does not belong to any already existing equivalence class. We call a construction that is not identical to an already constructed number ,but that is equal to one, a new form of that number, so that on day 2 we construct the new number 2= $\{0,1|\,\}$, but the new construction$\{-1,0|\,\}$ is just a new form of the number 1. In section 5 we define the natural form of a number, which is the simplest representation of any equivalence class that can be constructed in a finite number of days [12].

2.5.6 Theorem

For any numbers x and y, either $x \le y$ or $y \le x$.

Proof.

By contradiction, suppose neither $x \le y$ nor $y \le x$. Then by the inverse of definition2:

$$\exists x^L : x^L \ge y \vee \exists y^R : y^R \le x \tag{13}$$

$$\exists y^L : y^L \ge x \vee \exists x^R : x^R \le y \tag{14}$$

Then there are four combinations of statements that we must show are contradictory [13].

$\exists x^L : x^L \ge y$and$\exists y^L : y^L \ge x$. From Theorem 2.4.2 we know$x^L \le x$ and$y^L \le y$. It follows from Theorem 2.4.5 that$y \le x^L \le x$ and $x \le y^L \le y$. But then $x = y$, which contradicts our supposition.

$\exists x^L : x^L \ge y$ and $\exists x^R : x^R \le y$. It follows that $\exists x^L, x^R : x^R \le x^L$, but this contradicts 2.1 definition.

$\exists y^R : y^R \le x$ and $\exists y^L : y^L \ge x$. The argument is as in (b).

$\exists y^R : y^R \le x$ and $\exists x^R : x^R \le y$. Similarly, to (a), from Theorem 2.4.3 it follows that $y \le y^R \le x$ and$x \le x^R \le y$, so $x = y$.

So, numbers are total under$\le$.

We have now shown that$\le$ is a non-strict total order on numbers, as it is reflexive (from Lemma 2.5.1), antisymmetric (from the definition of equality), transitive (from Theorem 2.5.5), and total (from Theorem 2.5.6).

2.5.7 Theorem

For any number $\mathcal{Z} = \{Z^L | Z^R\}$, if Z^L has a greatest member a , we can write $\mathcal{Z} = \{a|Z^R\}$, and if Z^R has a least member b, we can write $\mathcal{Z} = \{Z^L|b\}$ [14].

Proof.

If a is the greatest element of Z^L, then for all $Z^L \ne a, Z^L < a$, so we can apply the Truncation Theorem on Z^L and rewrite$\mathcal{Z} = \{a|Z^R\}$. We can similarly do this for Z^R, and write $\mathcal{Z} = \{a|b\}$

A word more on notation: every number has only one left set and one right set. But in specifying that a number's left or right set contains the elements from the union of more than one set, we wish to omit to union sign for ease and esthetics [15]. Thus, we read

$a = \{A^{L_1}, A^{L_2}|A^{R_1}, A^{R_2}\}$ as $a = \{A^{L_1} \cup A^{L_2}|A^{R_1} \cup A^{R_2}\}$.

3. Algebraic properties on the Surreal Numbers

Our aim in this section is to show that we can define an arithmetic on the Surreal Numbers such that its equivalence classes have a field structure. That is, for any three surreal numbers x,y and z, we can define the two operations + (addition) and $\cdot$ (multiplication), such that all the following field definitions hold [16]:

(a) Closure under addition and multiplication: $x + y$ is a number and $x \cdot y$ is a number.

(b) Commutativity under addition and multiplication: $x + y = y + x$ and $x \cdot y = y \cdot x$.
(c) Associativity under addition and multiplication: $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
(d) Existence of an addition identity 0 such that $x + 0 = x$.
(e) Existence of a multiplicative identity 1 such that $x \cdot 1 = x$.
(f) Existence of an additive inverse $-x$ such that $x + -x = 0$.
(g) Existence of a multiplicative inverse x^{-1} such that $x \cdot x^{-1} = 1$, for all $x \neq 0$.
(h) Distributivity of multiplication over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$

Note that we will often write $x \cdot y$ as simply xy. We begin now with addition.

3.1 Addition

Conway says in On Numbers and Games: "The spirit of definitions is to ask what we already know about the object being defined, and to make the answers part of our definition." [5] In defining addition then, what we want (if we are to construct an arithmetic that matches our intuitive understanding of the world) is that, for example, $x + y > x^L + y$, for all x and y. Let us then define addition on numbers then as:

$$x + y := \{X^L + y, x + Y^L | X^R + y, x + Y^R\}$$

3.1.1 Example

Calculate $1 + 1/2$. To do this, we must first remember that $1 := \{0| \ \}$ and $1/2 := \{0|1\}$. So we have

$1 + 1/2 := \{0 + 1/2, 1 + 0 | \emptyset + 1/2, 1 + 1\}$
$\quad := \{0 + 1/2, 1 + 0 | 1 + 1\}$

This expresses $1 + 1/2$ in terms of other sums. So, let's have a go at $0 + 1/2$. Remember that

$0 := \{\ |\ \}$.
$0 + 1/2 := \{\emptyset + 1/2, 0 + 0 | \emptyset + 1/2, 0 + 1\}$
$\quad := \{0 + 0 | 0 + 1\}$

Let's find $0 + 0$

$0 + 0 := \{\emptyset + 0, 0 + \emptyset | \emptyset + 0, 0 + \emptyset\}$
$\quad := \{\ |\ \} := 0$

So, what about $0 + 1$?

$0 + 1 := \{\emptyset + 1, 0 + 0 | \emptyset + 1, 0 + \emptyset\}$
$\quad := \{0 |\ \} := 1$

Similarly, we find that $1 + 0 := 1$

Now

$0 + 1/2 := \{0 + 0 | 0 + 1\}$
$\quad := \{0|1\}$
$\quad := 1/2$

Now, let's have a go at $1 + 1$

$1 + 1 := \{0 + 1, 1 + 0 | \emptyset + 1, 1 + \emptyset\}$
$\quad := \{1|\ \}$
$\quad := 2$

Finally, we can go back to find $1 + 1/2$

$1 + 1/2 := \{0 + 1/2, 1 + 0 | 1 + 1\}$
$\quad := \{1/2, 1 | 2\}$
$\quad := \{1|2\}$
$\quad := 1\ 1/2$

3.1.2 Example

As another example, let us calculate $1/2 + 1/2$:

$1/2 + 1/2 := \{0 + 1/2, 1/2 + 0 | 1 + 1/2, 1/2 + 1\}$
$\quad := \{1/2 | 1\ 1/2\} \ := 1$

3.1.1 Theorem

For any numbers x, y and z, we have:
(a) $0 \equiv \{\ |\ \}$ as the identity element: $x + 0 \equiv x$
(b) Commutativity: $x + y \equiv y + x$
(c) Associativity: $(x + y) + z \equiv x + (y + z)$

Proof.

(a) $x + 0 = \{X^L + 0, x + \emptyset | X^R + 0, x + \emptyset\} = \{{x_1}^L + 0, \ldots | {x_1}^R + 0, \ldots\}$

Then by induction $x + 0 \equiv x$ if $0 + 0 \equiv 0$, but this follows from $0 + 0 = 0$ as shown above.

(b) We have

$x + y \equiv \{X^L + y, x + Y^L | X^R + y, x + Y^R\} \equiv \{{x_1}^L + y, \ldots, x + {y_1}^L, \ldots | {x_1}^R + y, \ldots, x + {y_1}^R, \ldots\}$

and

$y + x \equiv \{Y^L + x, y + X^L | Y^R + x, y + X^R\} \equiv \{{y_1}^L + x, \ldots, y + {x_1}^L, \ldots | {y_1}^R + x, \ldots, y + {x_1}^R, \ldots\}$

The commutativity of x and y then depends on the commutativity of the pairs formed of one of x or y and an option of the other. So inductively we need to check only that $x + 0 \equiv 0 + x$. But $0 + x \equiv \{0 + {x_1}^L, \ldots | 0 + {x_1}^R, \ldots\}$, so inductively $x + 0 \equiv x \equiv 0 + x$, since
$0 + 0 \equiv 0 + 0$.

(c) We have

$\{({x_1}^L + y) + z, \ldots, (x + {y_1}^L) + z, \ldots, (x + y) + {z_1}^L, \ldots\}$

for the left set of $(x + y) + z$, and

$\{{x_1}^L + (y + z), \ldots, x + ({y_1}^L + z), \ldots, x + (y + {z_1}^L), \ldots\}$

for the left set of $x + (y + z)$. So again, we inductively reduce the question down to associativity on 0, but clearly
$(x + y) + z \equiv x + (y + z)$ when one of x, y, or z is equal to 0 from (a). The same argument shows that

$$((x + y) + z)^R \equiv (x + (y + z))^R$$

, finishing the proof

3.1.2 Theorem

If x and y are numbers, then $x + y$ is a number.

Proof.

From 2.1 definition we must show that no element in $(x + y)^L$ is greater-than-or-equal to any element in $(x + y)^R$. But if all of $x^L + y, x + y^L, x^R + y, x + y^R$ are numbers, this follows from Theorems. For example, since $x^L < x$ and $y < y^R$, we know $x^L + y < x + y < x + y^R$. So inductively we reduce the question on $x + y$ down to questions on the sums of one of x or y and an option of the other. Eventually then we only need to show that for a number z that z +0 and $0+z$ are numbers, which follows from 0 being the additive identity, so the theorem holds for any numbers x and y.

3.2 Negation

We define the negation of a number x as:

$$-x = \{-\mathrm{X}^L | -\mathrm{X}^R\}$$

where for any set of numbers A

$$-A = \{-a_1, -a_2, -a_3, \dots\} \; for\ all\ a_i \in A$$

3.2.1 Theorem

If any x is a number, then so is $-x$

Proof.

We have $x = \{\mathrm{X}^L | \mathrm{X}^R\}$ and $-x = \{-\mathrm{X}^R | -\mathrm{X}^L\}$. We need to show by definition1 that no $-x^R$ is greater-than-or-equal to any $-x^L$. But since x is a number we have $x^L < x^R$, for all x^L, x^R, and if $-x^L$ and $-x^R$ are also numbers, then we must have that $-x^R < -x^L$, that is that x is a number. So inductively we reduce the question on x to questions of the options of x , and eventually we only have to consider the theorem for 0, but since $0 \equiv 0$, the theorem holds for 0 and therefore inductively for any number x. [17]

3.3 Multiplication

In defining multiplication, we can use the fact that we know, for example, $x - x^L > 0\ and\ y - y^L > 0$, and that we want the property that $(x - x^L)(y - y^L) > 0$, and also the property that we can expand this to get $xy - x^L y - x y^L + x^L y^L > 0$.So we want $xy > x^L y + x y^L - x^L y^L$, and similarly we can formulate inequalities for all the other combinations of $x - x^L > 0$, $y - y^L > 0$, $x - x^R < 0$, $y - y^R < 0$, to give us a tentative definition of multiplication as: $xy =$

$$\{X^L y + xY^L - X^L Y^L, \mathrm{X}^R y + x\mathrm{Y}^R - \mathrm{X}^R\mathrm{Y}^R | X^L y + x\mathrm{Y}^R - \mathrm{X}^L\mathrm{Y}^R, \mathrm{X}^R y + xY^L - \mathrm{X}^R\mathrm{Y}^L\}.$$

which we will now check has all the properties we expect of it. $Ax = \{ax : a \in A\}$,Similarly, to addition and negation, we use the notation $AB = \{a \in b : aA, b \in B\}$to express multiplication on sets.

3.3.1 Theorem

For any numbers$x$, $y\ and\ z$:

(a) $\pi(x, y, z)$: $(x + y)z = xz + yz$

(b) $\Omega(x, y, z)$: $(xy)z = x(yz)$

Proof.

(a)We have

$$xz + yz \equiv \{(XZ)^L + yz, \dots | \dots\} \equiv \{X^L z + xZ^L - X^L Z^L + yz, \dots | \dots\}$$

and

$$(x + y)z \equiv \{(X^L + y)z + (x + y)Z^L - (X^L + y)Z^L, \dots | \dots\}$$

Now suppose$\pi(X^L, y, z), \pi(x, y, Z^L), \pi(X^L, y, Z^L)$ all hold. Then $(x + y)z$ becomes, using the equality $x + -x = 0$:

$$(x + y)z \equiv \{X^L z + yz + xZ^L + yZ^L - X^L Z^L - yZ^L, \dots | \dots\}$$

$$= \{X^L z + yz + xZ^L - X^L Z^L, \dots | \dots\} \equiv xz + yz$$

So $\pi(x, y, z)$ depends on π holding for the left options of x and z, but if $x^L = 0$ or $z^L = 0$, then clearly π holds, so inductively$(x + y) = xz + yz$.

(b)We have $x(yz) \equiv \{X^L(yz) + x(YZ)^L - X^L(YZ)^L, \dots | \dots\} \equiv \{X^L(yz) + x(Y^L z + yZ^L - Y^L Z^L) - X^L(Y^L z + yZ^L - Y^L Z^L, \dots | \dots\}$

then using (a), $-(-x) \equiv x$ and

$yx \equiv xy$,

$$= \left\{ \begin{matrix} X^L(yz) + x(Y^L z) + x(yZ^L) - x(Y^L Z^L) - X^L(Y^L z) - X^L(yZ^L) \\ +X^L(Y^L Z^L), .. \end{matrix} \middle| \dots \right\}$$

And

$$(xy)z \equiv \{(XY)^L z + (xy)Z^L - (xy)^L Z^L, \dots | \dots\} \equiv$$
$$\{(X^L y + xY^L - X^L Y^L)z + (xy)Z^L - (X^L y + xY^L - X^L Y^L)Z^L, \dots | \dots\}$$

then using (a) and $(- x) \equiv x$,

$$= \left\{ \begin{matrix} (X^L y)z + (xY^L)z - (X^L Y^L)z + (xy)Z^L - (X^L y)Z^L - \\ (xY^L)Z^L + (X^L Y^L)Z^L, \dots \end{matrix} \middle| \dots \right\}$$

which, if $\Omega(X^L, y, z), \Omega(x, Y^L, z)$, ... all hold,

$$\equiv \left\{ \begin{matrix} X^L(yz) + x(y^L z) - X^L(Y^L z) + x(yZ^L) - \\ X^L(yZ^L) - x(Y^L Z^L) + X^L(Y^L Z^L), \dots \end{matrix} \middle| \dots \right\} \equiv x(yz)$$

So again we reduce $\Omega(x, y, z)$ down to the same proposition on its options, but when one of x, y or z is equal to 0, clearly Ω holds, so by induction, $x(yz) = (xy)z$.

Note that since we evoke the equality $x + -x = 0$ in the proof, the theorem only holds up to equality, not identity.

3.4 Division

The last thing we must show to have a field is how to find the multiplicative inverse of a number, that is, for any $x \neq 0$, if there exists a number y, such that $xy = t$, then we need to show how to find this y. But note that if, for a positive x, we could find a ý such that xý $= 1$, then we would also know that $t(x\,ý) = \mathrm{t} \Rightarrow \mathrm{x}(\mathrm{t}\,y)^{\prime} = t$, that is that $y = t$ý . If x were instead negative, we would just need to multiply the equation through by -1. So, we only need show how to find y for some $xy = 1$, where x is positive.

3.4.1 Lemma

For every positive $x = \{\mathrm{X}^L | \mathrm{X}^R\}$, we can write x in a form with$\mathrm{X}^L = \{0, \mathrm{x}^L\}$, where all $\mathrm{x}^L$are positive, and this new form is equal to the original one.

Proof.

Since $0 < x$ is positive, by the Extension Theorem we can append 0 to X^L. Then from the Truncation Theorem we can remove any element of X^Lless than 0 .

For the rest of this section when we write x^Lwe are referring only to the non-zero terms, and since x here is positive, we must have all $\mathrm{x}^R > 0$. Now we define y recursively. That is, every element of Y^Lgenerates a new element inY^L, and similarly forY^R . We write

$$y = \left\{ \begin{matrix} 0, \dfrac{1 + (x^R - x)\,y^L}{x^R}, \\ \dfrac{1 + (x^L - x)\,y^R}{x^L} \end{matrix} \middle| \begin{matrix} \dfrac{1 + (x^L - x)\,y^L}{x^L}, \\ \dfrac{1 + (x^R - x)\,y^R}{x^R} \end{matrix} \right\}$$

which has y^L and y^R in the definition of y ! What we mean by this is that we build up these left and right sets by using

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elements already in them, so that if $y_1^L$is in Y^L, then ,for example, $(1 + (x^R - x)y_1^L)/ x^R$ is also in Y^L. Conway gives the following elucidation [5]:

Let $x = \{0,2|\}$. Then the only (non-zero) x^L is 2, giving us $1/x^L = 1/2$ and$(x^L - x) = -1$, and there is no x^L,

so, we have

$y = \left\{0, \frac{1}{2}(1 - y^R) \middle| \frac{1}{2}(1 - y^L)\right\}$.

Putting in $y^L = 0$ into the right option updates y to

$y = \left\{0, \frac{1}{2}(1 - y^R) \middle| \frac{1}{2}, \frac{1}{2}(1 - y^L)\right\}$, and we can now put this new right option into the left set,

giving us

$y = \left\{0, \frac{1}{4}, \frac{1}{2}(1 - y^R) \middle| \frac{1}{2}, \frac{1}{2}(1 - y^L)\right\}$. We can then repeat this process endlessly.

In the next section we will explain the relationship between Surreal numbers and games and how players choose their movements based on Surreal numbers

4. Surreal Numbers and Game

We can think of any number $g = \{a, b, c, \ldots | d, e, f, \ldots\}$[18] as a game where the elements of the left set represent moves that one player can make, and the elements in the right set represent the moves that another player can make. For example, if g was a game between players Left and Right, then Left could move from some starting point, g, to any of a, b, c, ..., and Right could move from g to any of d, e, f, ... If Left starts the game and moves to a, then the representation of the game is changed to$a = \{A, B, C, \ldots | D, E, F, \ldots\}$. Thus Right can now move to any of D, E, F, ... If she moves to $E = \{\alpha, \beta, \gamma, \ldots | \epsilon, \delta, \zeta, \ldots\}$, then Left can then move to any of $\epsilon, \delta, \zeta, \ldots$, and so on. The last person to make a move wins the game.

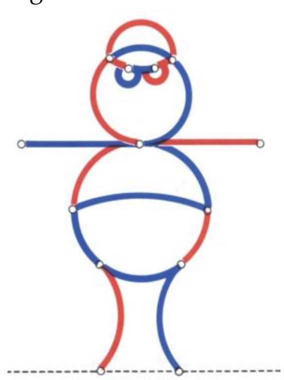


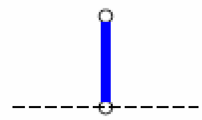
Figure 1:
A Hackenbush game [19]

One specific game that we can consider is Hackenbush (Figure 1 provides a fancy example). Hackenbush is a two-player game played with a picture of nodes joined by edges that are colored with two different colors (we will use red and blue). The picture must be constructed so that you can reach the ground (which is the dotted line in Figure 2) from any node by travelling along a series of adjacent edges. The two players, Left and Right, take turns alternately. Left can delete only blue edges and Right can delete only red edges. After one edge is deleted, any edges no longer connected to the ground are also deleted. The last player to delete an edge wins.

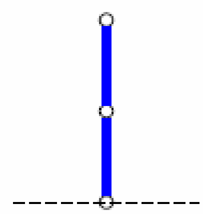
4.1 Basic Games

We will now analyze some simple games.

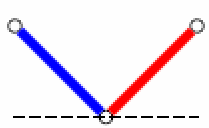
If there are no red or blue edges then neither player has any moves, meaning that the game would be $\{\emptyset | \emptyset\} = 0$. We call this state endgame [18]. Note that the first person to move automatically loses.



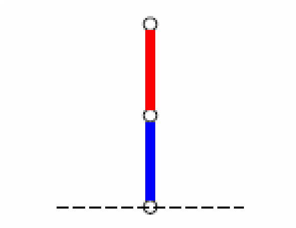
If there is just one blue edge, then Left can move to 0 while Right has no moves; thus the game would be $\{0|\} = 1$. In this case, Left automatically wins since there are no legal moves for Right. If there were just one red edge, then Left would have no moves while Right could move to 0. Thus the game would be $\{|0\} = -1$, and Right would win.



If there are two blue edges stacked on top of one another, then Left can pull the bottom edge to form game 0, or the top edge to form 1. Thus the game would be $\{0,1|\} \equiv \{1|\} = 2$. Again, Left automatically wins. If we had two red edges stacked on top of each other then we would have the game $\{|0,-1\} \equiv \{|-1\} = -2$. In general, if we have a chain n blue edges then the game will have a value of n, and if we have a chain of n red edges then the game will have a value of $-n$, where n is a positive integer.



If there is one red edge and one blue edge coming from one node, then the game would be $\{-1|1\}$.Since the second player to move will cause an endgame, we see that the first player to move will lose. Thus this game is equivalent to 0, meaning the game 0 has multiple forms.



If there is one red edge on top of one blue edge, then Left can delete the bottom edge to form game 0, while Right can delete the top edge to form game 1. Thus, the game would be $\{0|1\} = 1/2$. We note that Left will win regardless of who goes first. If we had one blue edge stacked on top of one red edge, then Left could delete the top edge to form -1 while Right could delete the bottom to form 0. Thus, we would have $\{-1|0\} = 1/2$, and Right would win regardless of who goes first.

In general, we use the following notations from On Numbers and Games [18]:

$G > 0$(G is positive) if there is a winning strategy for Left

$G < 0$ (G is negative) if there is a winning strategy for Right

$G \equiv 0$ (G is like 0) if there is a winning strategy for the second person to move

$G \| 0$ (G is fuzzy) if there is a winning strategy for the first person to move

$G \geq 0$ if $G > 0$ or $G \equiv 0$, which means that if Right starts there is a winning strategy for Left, since Left would then be the second to move.

$G \leq 0$ if $G < 0$ or $G \equiv 0$, which means that if Left starts there is a winning strategy for Right.

$G \, \|> 0$ if $G > 0$ or $G \| 0$, which means if Left starts then there is a winning strategy for Left, since they would be the first to move.

$G \, <\| \, 0$ if $G < 0$ or $G \| 0$, which means that if Right starts then there is a winning strategy for Right.

5. Conclusion

we have shown surreal number and how we construct them and the all operations on them and that we have a multiplicative inverse for any non-zero number. We also showed before that we have an additive inverse, additive and multiplicative identities, that both addition and multiplication are associative, commutative, and closed, and finally that multiplication is distributive under addition. That is, we have defined addition and multiplication on the equivalence classes of the Surreal Numbers in a way that satisfies all the field definitions, as mentioned previously. We also showed in previous section that numbers are totally ordered. In sum then, we have shown that the surreals form a totally ordered Field. Finally, we mentioned the relationship between the Surreal numbers and the games and showed examples of the basic games.

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