

On the bicycle transformation and the filament equation: results and conjectures

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1 Bicycle model, bicycle monodromy, and bicycle correspondence

This paper concerns a simple model of bicycle kinematics and continues the work done in [8, 14, 15, 19, 20]. In this model, a bicycle is represented by an oriented segment of constant length ℓ that can move in such a way that the velocity of its rear end is aligned with the segment (the rear wheel is fixed on the bicycle frame). The motion takes place in the Euclidean plane but, equally well, albeit less physically, we consider the motion in \mathbb{R}^n (see [10] for bicycle motion in the hyperbolic and elliptic planes).

The configuration space of oriented segments of fixed length in \mathbb{R}^n is the spherization of the tangent bundle $ST\mathbb{R}^n$, and the non-holonomic constraint that we have imposed on the motion defines an n -dimensional distribution \mathcal{D} therein. The motion of the bicycle is a smooth curve in $ST\mathbb{R}^n$ tangent to the distribution \mathcal{D} .

The two projections $ST\mathbb{R}^n \rightarrow \mathbb{R}^n$, to the front and to the rear ends of the segment, yield the front and rear bicycle tracks. The former projection is transverse to \mathcal{D} , and the front track is a smooth curve, whereas the direction of the latter projection is contained in \mathcal{D} , and the rear track may have singularities (generically, semi-cubical cusps). This happens when the front wheel of the bicycle turns 90° . We denote the rear track by γ and the front track by Γ .

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In this section, we review the relevant results from [14, 19, 20] that will be used later.

The bicycle segment gives the rear track γ coorientation (the orientation of the normal bundle), well defined even at cusps. Conversely, given a cooriented curve γ , possibly, with cusps, one uniquely reconstructs the respective curve Γ as the locus of endpoints of the tangent segments of length ℓ . That is, the rear track and the direction of the motion determine the front track.

On the other hand, the front track Γ determines the motion of the bicycle, and hence the rear track γ , only after its initial position is chosen (this is obvious geometrically, and it follows from the fact that the motion of the bicycle with a given front track is described by a first order differential equation, see [14] and equation (6) below). Thus the *bicycle monodromy* map $M_{\Gamma,\ell}$ arises that takes the initial position of the bicycle to the terminal one.

Assume that Γ is a closed curve. Then $M_{\Gamma,\ell}$ is a self-map of the sphere S^{n-1} , well defined up to conjugation (due to the freedom of choice of the initial point on Γ). Consider S^{n-1} as the sphere at infinity of the hyperbolic space \mathbb{H}^n . Then the Möbius group $O(n, 1)$ of isometries of \mathbb{H}^n acts on S^{n-1} . The next result was proved in [14].

Theorem 1 *The bicycle monodromy $M_{\Gamma,\ell}$ is a Möbius transformation.*

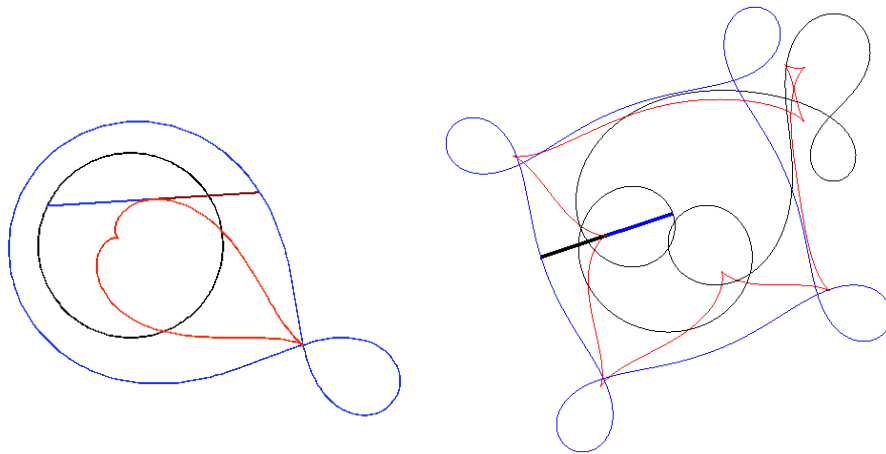


Figure 1: Bicycle correspondence (figures courtesy of R. Perline).

Let γ be a closed rear track. It has two coorientations, and they correspond to two front tracks, Γ_1 and Γ_2 (one may think of a forward-facing and

a backward-facing bicycle joined together). We say that Γ_1 and Γ_2 are in the *bicycle correspondence* and write $\mathcal{B}_{2\ell}(\Gamma_1, \Gamma_2)$. Equivalently, two points, x_1 and x_2 , traverse the curves Γ_1 and Γ_2 in such a way that the distance x_1x_2 is equal to 2ℓ , and the velocity of the midpoint of the segment x_1x_2 is aligned with x_1x_2 (it follows that the speeds of x_1 and x_2 are equal). See Figure 1.

The bicycle monodromy and the bicycle correspondence have discrete versions [9, 17, 20]. Let $P = (P_1, P_2, \dots, P_n)$ be an n -gon in \mathbb{R}^n , and let P_1Q_1 be a segment of length 2ℓ . Let Q_2 be the point such that $P_1Q_1P_2Q_2$ is an isosceles trapezoid. In other words, one parallel translated the segment P_1Q_1 to $P_2Q'_1$ and then reflects Q'_1 in the line Q_1P_2 , see Figure 2. Continuing this, one constructs a polygon Q_1, Q_2, \dots, Q_n .

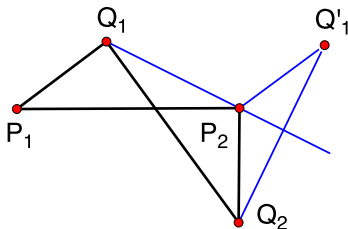


Figure 2: Discrete bicycle transformation.

In the limit $n \rightarrow \infty$, the two polygons become curves in the bicycle correspondence. The point Q_{n+1} does not necessarily coincide with Q_1 , and we obtain the discrete bicycle monodromy $Q_1 \mapsto Q_{n+1}$. Theorem 1 holds in this case as well, see [20]. If the polygon Q closes up, we say that the polygons P and Q are in the discrete bicycle correspondence, see Figure 3.

The next two theorems hold both for closed curves and for polygons, see [20] for proofs in the discrete setting. We formulate the continuous versions.

Theorem 2 *If closed curves Γ_1 and Γ_2 are in the bicycle correspondence, $\mathcal{B}_\ell(\Gamma_1, \Gamma_2)$, then the bicycle monodromies $M_{\Gamma_1, \lambda}$ and $M_{\Gamma_2, \lambda}$ are conjugated for every value of λ .*

Theorem 3 (Bianchi permutability) *Let Γ_1, Γ_2 and Γ_3 be three closed curves such that $\mathcal{B}_\ell(\Gamma_1, \Gamma_2)$ and $\mathcal{B}_\lambda(\Gamma_1, \Gamma_3)$ hold. Then there exists a closed curve Γ_4 such that $\mathcal{B}_\lambda(\Gamma_2, \Gamma_4)$ and $\mathcal{B}_\ell(\Gamma_3, \Gamma_4)$ hold.*

The bicycle correspondence is not a mapping: for example, the bicycle monodromy of a curve Γ_1 may be the identity, and then there is a whole

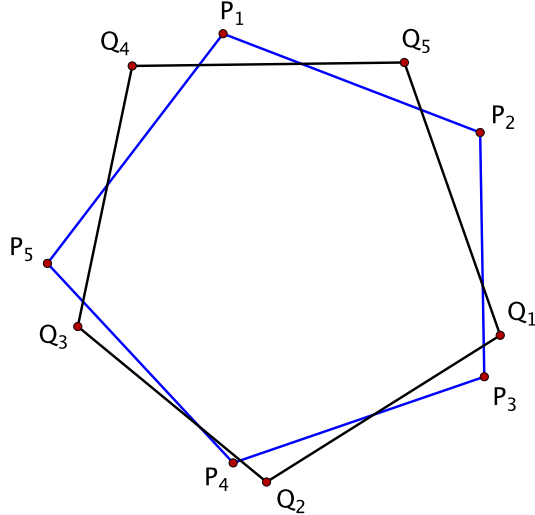


Figure 3: Two polygons in the discrete bicycle correspondence.

sphere worth of corresponding curves Γ_2 . For instance, this happens when Γ_1 is a circle, see Figure 1 on the left.

According to Theorem 1, in dimension three, the bicycle monodromy is a complex fractional-linear transformation and, for a generic curve and generic value of the length, it has two fixed points. Therefore the bicycle correspondence is a 2-2 map and, up to a binary choice, one can consider it as a mapping of space curves.

In the plane, the bicycle monodromy is a real fractional-linear transformation, and one has the usual trichotomy: it can be elliptic, parabolic, or hyperbolic. Let Γ_1 be a plane curve such that $M_{\Gamma_1, \ell}$ is hyperbolic, that is, it has two fixed points. Then one can choose one of the two respective curves, Γ_2 such that $\mathcal{B}_{2\ell}(\Gamma_1, \Gamma_2)$. By Theorem 2, $M_{\Gamma_2, \ell}$ is again hyperbolic, and there is a unique curve Γ_3 , different from Γ_1 , with $\mathcal{B}_{2\ell}(\Gamma_2, \Gamma_3)$, and so on. In this way the bicycle correspondence becomes a mapping of the space of plane curves.

Our ultimate goal is the following conjecture which we formulate in somewhat loose terms.

Conjecture 1 *The bicycle correspondence in \mathbb{R}^n is Liouville integrable.*

This means that the correspondence has an invariant Poisson bracket

and an infinite collection of independent integrals in involution that is, in an appropriate sense, complete.

Theorem 2 is a source of integrals. Let

$$\det(E - tM_{\Gamma,\lambda}) = \sum_{i=1}^{n+1} \sum_{j=1}^{\infty} I_{ij}(\Gamma) t^i \lambda^j$$

be the Taylor expansion of the characteristic polynomial of the monodromy, considered as an element of the Möbius group $O(n, 1)$. Then the functions $I_{ij}(\Gamma)$ on the space of curves in \mathbb{R}^n are conserved by the bicycle correspondence. We call them the *monodromy integrals*.

2 Two differential 2-forms

Consider two differential 2-forms on the space of smooth embedded parameterized curves, the first defined in \mathbb{R}^n for all n , and the second in \mathbb{R}^3 :

$$\omega(u, v) = \int u'(t) \cdot v(t) dt, \quad \Omega(u, v) = \int \det(\Gamma'(t), u(t), v(t)) dt.$$

Here $\Gamma(t)$ is a curve, and $u(t), v(t)$ are vector fields along Γ .

Note that the integrals do not depend on the parameterization. Note also that Ω vanishes on the submanifold of plane curves. The form ω can be defined on any Riemannian manifold, and the form Ω depends only on the volume form (and not on the metric).

Proposition 2.1 *The 2-forms ω and Ω are closed. The forms are degenerate: the kernel of ω consists of the parallel vector fields along the curve, and the kernel of Ω consists of the tangent vector fields.*

Proof. Consider the following 1-forms:

$$\alpha(u) = \frac{1}{2} \int \Gamma'(t) \cdot u(t) dt, \quad \beta(u) = -\frac{1}{3} \int \det(\Gamma(t), \Gamma'(t), u(t)) dt.$$

We claim that

$$\omega = d\alpha, \quad \Omega = d\beta.$$

Indeed, let u and v be commuting vector fields in a neighborhood of Γ . One has

$$d\alpha(u, v) = L_u(\alpha(v)) - L_v(\alpha(u)) - \alpha([u, v]).$$

The first two terms on the right hand side are easy to compute:

$$L_u(\alpha(v)) = \int u'(t) \cdot v(t) dt, \quad L_v(\alpha(u)) = \int v'(t) \cdot u(t) dt = - \int u'(t) \cdot v(t) dt,$$

and the result for ω follows. A similar computation works for Ω .

If u is in the kernel of ω , then $\int u'(t) \cdot v(t) dt = 0$ for all vector fields v along Γ . It follows that $u' = 0$, that is, u is parallel along Γ . Likewise, if u is in the kernel of Ω , then $\int \det(\Gamma'(t), u(t), v(t)) dt = 0$ for all vector fields v along Γ . This implies that u is collinear with Γ' . \square

Corollary 2.2 *The form ω descends on the quotient space of curves modulo parallel translations and is symplectic therein, and the form Ω descends on the quotient space of curves mod $\text{Diff}_+(S^1)$, that is, on the space of non-parameterized oriented curves, and is a symplectic structure therein.*

The latter is the celebrated Marsden-Weinstein symplectic structure, see [1, 5, 16]. One can identify isotopic equivalence classes of smooth knots with coadjoint orbits of the group of volume-preserving diffeomorphisms of \mathbb{R}^3 (the pairing of a curve with a divergence-free vector field is given by the flux across a film that spans the curve). Under this identification, the Marsden-Weinstein symplectic structure coincides with the Kirillov-Kostant-Souriau symplectic structure on coadjoint orbits of a Lie group.

Our main observation is as follows.

Theorem 4 *The bicycle correspondence preserves the forms ω and Ω , the former in any dimension, and the latter in dimension three.*

Proof. Consider a smooth arc of the rear track γ between two consecutive cusps and give it arc length parameterization with parameter $t \in [a, b]$. Let $\Gamma_{\pm} = \gamma \pm \ell\gamma'$ be the respective arcs of the two front tracks which are in the bicycle correspondence.

Let $u(t), v(t)$ be two vector fields along $\gamma(t)$. The differentials of the maps $\gamma \mapsto \Gamma_{\pm}$ take u and v to vector fields U and V along Γ_{\pm} . We claim that

$$U = u + \ell(u' - (\gamma' \cdot u')\gamma'), \tag{1}$$

and likewise for V ; here ℓ may be positive or negative, depending on whether one considers Γ_+ or Γ_- .

To prove (1), let $\gamma_1 = \gamma + \varepsilon u$ be a variation of γ . Then $|\gamma'_1| = 1 + \varepsilon \gamma' \cdot u'$, and one has:

$$\Gamma_1 = \gamma_1 + \ell \frac{\gamma'_1}{|\gamma'_1|} = \gamma + \ell \gamma' + \varepsilon(u + \ell(u' - (\gamma' \cdot u')\gamma')),$$

as claimed (all computations are mod ε^2 , and ℓ is signed).

It follows from (1) that

$$U' = u' + \ell(u'' - (\gamma'' \cdot u')\gamma' - (\gamma' \cdot u'')\gamma' - (\gamma' \cdot u')\gamma'').$$

We want to show that the part of the integral $\int U' \cdot V dt$ which is odd in ℓ vanishes. The only odd term is linear in ℓ ; it is equal to

$$\int_a^b [u' \cdot v' + u'' \cdot v] - [(\gamma' \cdot v')(\gamma' \cdot u') + (\gamma' \cdot v)(\gamma'' \cdot u') + (\gamma' \cdot v)(\gamma' \cdot u'') + (\gamma'' \cdot v)(\gamma' \cdot u')] dt.$$

The integral of the first bracket is equal to $\ell(u' \cdot v)|_a^b$, and of the second to $\ell(\gamma' \cdot v)(\gamma' \cdot u')|_a^b$ (this would finish the proof if γ had no cusps).

Now consider the next smooth arc of γ , arc length parameterized by $t \in [b, c]$. A similar computation applies to this arc, with ℓ changing the sign. The contribution of the cusp, corresponding to the value of parameter b , is $\ell(u' \cdot v)(b)$, computed for the first arc, and $\ell(u' \cdot v)(b)$, computed for the second one. They cancel each other: the sign of u' also changes to the opposite because the direction of motion reverses in the cusp. Likewise, one has cancelation of the terms $\ell(\gamma' \cdot v)(\gamma' \cdot u')$, and this implies that $\bar{T}_\ell^*(\omega) = \omega$.

Similarly, the computation for Ω involves $\int \det(\Gamma', U, V) dt$. There are two terms odd in ℓ , the linear and the cubic ones. The linear term is $\int \det(\gamma', u, v)' dt$, again, the integral of a derivative. And again, the total integral of a full derivative vanishes, taking the sign changes at the cusps into account.

Finally, consider the cubic in ℓ term,

$$\int \det(\gamma'', u' - (\gamma' \cdot u')\gamma', v' - (\gamma' \cdot v')\gamma') dt.$$

The determinant is identically zero since all three vectors are orthogonal to γ' , and hence linearly dependent. \square

The monodromy integrals descend to both quotient spaces, curves modulo parallel translations and non-parameterized curves.

Conjecture 2 *The monodromy integrals Poisson commute on these quotient spaces with respect to the Poisson bracket induced, respectively, by the forms ω (in every dimension) and Ω (in dimension three).*

3 The bicycle transformation in dimension three and the filament equation

The *filament (binormal, localized induction, smoke ring) equation* is an evolution on curves in \mathbb{R}^3

$$\dot{\Gamma} = \Gamma' \times \Gamma''$$

where $\Gamma(x)$ is arc length parameterized curve, prime is d/dx , and dot is the time derivative.

This system (equivalent to the non-linear Schrödinger equation via the Hasimoto transformation) is completely integrable in the following sense, see [11, 13]. It is a Hamiltonian system with respect to the symplectic structure induced by Ω on the space of arc length parameterized curves, the Hamiltonian function being the perimeter length of the curve. It has a hierarchy of Poisson commuting integrals F_1, F_2, \dots that starts with

$$\int 1 \, dx, \int \tau \, dx, \int \kappa^2 \, dx, \int \kappa^2 \tau \, dx, \int \left((\kappa')^2 + \kappa^2 \tau^2 - \frac{1}{4} \kappa^4 \right) \, dx, \dots \quad (2)$$

where τ is the torsion, and κ is the curvature. One also has a hierarchy of commuting Hamiltonian vector fields X_0, X_1, X_2, \dots that starts with

$$-T, \kappa B, \frac{\kappa^2}{2} T + \kappa' N + \kappa \tau B, \kappa^2 \tau T + (2\kappa' \tau + \kappa \tau') N + \left(\kappa \tau^2 - \kappa'' - \frac{\kappa^3}{2} \right) B, \dots \quad (3)$$

where T, N, B is the Frenet frame along Γ , and where X_i is the Hamiltonian vector field of F_i . The vector fields X_i satisfy the recurrence relation

$$T \times X_n = X'_{n-1}. \quad (4)$$

In dimension three, the bicycle transformation is completely integrable, in the same sense as the filament equation.

Theorem 5 *The functions (2) are integrals of the bicycle transformation (independently of the length parameter ℓ). The bicycle transformation commutes with the vector fields (3).*

Proof. We claim that

$$\omega(X_{n-1}, \cdot) = \Omega(X_n, \cdot) = dF_n. \quad (5)$$

Indeed, the first equality follows from (4) and the definitions of the 2-forms, and the second from the fact that X_n is the Hamiltonian vector field of the function F_n .

Assume that the bicycle transformation preserves the field X_{n-1} . By Theorem 4, it preserves ω , and hence by (5), it preserves dF_n . Assume now that the bicycle transformation preserves the differential dF_n . By Theorem 4, it preserves Ω , and hence the Hamiltonian vector field X_n .

Since the bicycle transformation preserves dF_n , it changes F_n by a constant: $F_n \mapsto F_n + c$. To see that this constant is zero, it suffices to consider a circle of sufficiently large radius. Such a circle is fixed by the bicycle transformation, hence $c = 0$.

Thus one may argue inductively, starting with the function F_1 , the perimeter length, clearly preserved by the bicycle transformation. \square

Theorem 5 is hardly new: the bicycle transformation is the Bäcklund transformation for the filament equation, see, e.g., [6, 18]. One of the reasons to use the term ‘bicycle’ (rather than Bäcklund, or Darboux) transformation is to decouple it from the filament equation which, unlike the bicycle transformation, is specifically 3-dimensional.

Conjecture 3 *In dimensions three and two, the monodromy integrals and the integrals (2) are functionally dependent of each other.*

Specifically, we conjecture that the integrals (2) are the consecutive terms in the Taylor expansion at $\lambda = 0$ of the conjugacy invariant

$$\frac{\text{Tr}^2(M_{\Gamma, \lambda})}{\det(M_{\Gamma, \lambda})}$$

where the monodromy is considered as a 2×2 complex (in dimension 3) or real (in dimension 2) matrix.

Remark 3.1 In dimension two, the motion of the bicycle is described by a differential equation

$$\frac{d\alpha(x)}{dx} + \frac{\sin \alpha(x)}{\ell} = \kappa(x), \quad (6)$$

where α is the steering angle, κ is the curvature of the front track Γ , x is the arc length parameter on the front track Γ , and ℓ the length of the bicycle segment, see [14]. It is proved in [14] that the derivatives $\sigma_{1,2}$ of the monodromy at its fixed points are equal to $\exp(\pm \int \cos \alpha dx)$, where $\alpha(x)$ is a periodic solution of (6). One has

$$\frac{1}{\sigma_1} + \frac{1}{\sigma_2} = \frac{\text{Tr}^2(M_{\Gamma,\lambda})}{\det(M_{\Gamma,\lambda})} - 2,$$

hence the quantity $\int \cos \alpha dx$ is an integral of the bicycle transformation.

This suggests a method of obtaining integrals: consider a periodic solution $\alpha(x)$ of equation (6) as a function of ℓ and expand $\cos \alpha$ as a Taylor series in ℓ . Then the integral of each coefficient of this series is a conserved quantity of the bicycle transformation. The first three integrals obtained this way are

$$\int 1 dx, \int \kappa^2 dx, \int \left((\kappa')^2 - \frac{1}{4} \kappa^4 \right) dx,$$

as predicted by Conjecture 3.

Now let M be an oriented 3-dimensional Riemannian manifold. One can define the differential 2-forms ω and Ω and considers the filament equation on M . One can also define an analog of the bicycle correspondence: the bicycle frame is a geodesic segment of fixed length. It is natural to ask when one has an analog of the recursion scheme (5) for the filament equation and an analog of Theorem 5.

Conjecture 4 *This holds if and only if M has constant curvature.*

4 In the plane. Zindler curves

A plane analog of the filament equation is the *planar filament equation*

$$\dot{\Gamma} = \frac{\kappa^2}{2} T + \kappa' N$$

where (T, N) is the planar Frenet frame. This system is equivalent to the modified Korteweg-deVries equation, the same way as the filament equation to the non-linear Schrödinger equation.

It is shown in [12] that the planar filament equation has an infinite hierarchy of integrals, namely, the odd-numbered ones in the sequence (2) (the torsion τ vanishes for plane curves). Likewise, the odd-numbered vector fields in the list (3) commute and preserve the planarity of curves. With this modification, Theorem 5 holds for the bicycle transformation in dimension two.

An interesting ‘bicycle’ problem is to describe the situations when, given closed rear and front tracks, γ and Γ , one cannot tell which way the bicycle went [7]. A trivial example is a pair of concentric circles; for numerous non-trivial examples, see [21]–[26] and Figure 4.

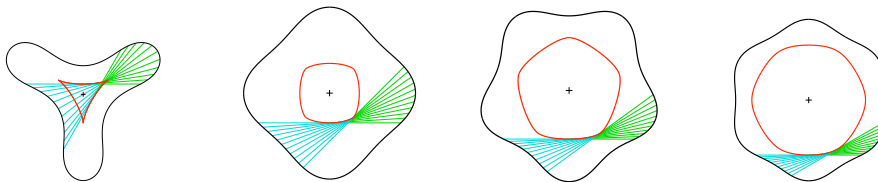


Figure 4: Examples of Zindler curves from [26].

Such an ‘ambiguous’ front track Γ can be characterized by the property that a chord of fixed length ℓ can traverse the curve in such a way that the velocity of its mid-point is aligned with the chord. That is, Γ is in the bicycle correspondence with itself: $\mathcal{B}_\ell(\Gamma, \Gamma)$. Following [3, 4], we call such curves *Zindler curves* (see [27]).¹ Of course, Zindler curves can be defined in other dimensions as well.

The 2-dimensional version of Theorem 5 has the following corollary.

Corollary 4.1 *Let Γ be a Zindler curve, and let Γ_1 be the result of evolving Γ along the planar filament flow. Then Γ_1 is also a Zindler curve with the same length parameter ℓ . The same is true if the planar filament field is replaced by any of the odd-numbered vector fields in (3) or their linear combination.*

Proof. As we mentioned earlier, the end points of the segment of fixed length move along a Zindler curve with the same speed. Let $\Gamma(x)$ be an arc length parameterization. Then $\mathcal{B}_\ell(\Gamma(x), \Gamma(x+c))$ for some constant c . Since

¹Incidentally, (the interiors of) Zindler curves are solutions to the 2-dimensional case of Ulam’s problem: which bodies float in equilibrium in all positions? See [2].

the bicycle transformation commutes with the planar filament flow, and the latter preserves the arc length, we have $\mathcal{B}_\ell(\Gamma_1(x), \Gamma_1(x+c))$, as needed. \square

Recall that a soliton of the filament equation is a curve that is a critical point for a linear combination of integrals of motion.

Conjecture 5 *The Zindler curves are solitons of the planar filament equation.*

In particular, according to preliminary computations of R. Perline, the Zindler curves constructed by Wegner [25] are pressurized elastica, that is, extrema of a linear combination of area, length, and total squared curvature.

5 Other integrals

The bicycle correspondence in \mathbb{R}^n has additional integrals.

Theorem 6 *The bivector and the vector,*

$$A(\Gamma) = \int_{\Gamma} \Gamma(t) \wedge \Gamma'(t) dt \quad \text{and} \quad J(\Gamma) = \int_{\Gamma} (\Gamma(t) \cdot \Gamma'(t)) \Gamma(t) dt,$$

are preserved by the bicycle correspondence.

Proof. As before, $\Gamma = \gamma + \ell\gamma'$ with t being an arc length parameter along γ . The odd in ℓ part of $A(\Gamma)$ is $\ell \int \gamma \wedge \gamma'' dt$. Note that $\gamma \wedge \gamma'' = (\gamma \wedge \gamma')'$. This would prove that the integral vanishes for a smooth rear track γ . If γ has cusps, we note that γ' and ℓ change sign in each cusp, and we conclude similarly to the proof of Theorem 4.

Likewise, for $J(\Gamma)$, the odd in ℓ parts are

$$\ell \int ((1 + \gamma \cdot \gamma'') \gamma + (\gamma \cdot \gamma') \gamma') dt \quad \text{and} \quad \ell^3 \int (\gamma' \cdot \gamma'') \gamma' dt.$$

Concerning the first integral, the integrand is $((\gamma \cdot \gamma') \gamma)'$, and that $(\gamma \cdot \gamma') \gamma$ changes sign at cusps, along with ℓ . The second integral vanishes because $\gamma(t)$ is an arc length parameterization, and $\gamma' \cdot \gamma'' = 0$. \square

If the ambient dimension is n then the area bivector $A(\Gamma)$ provides $\binom{n}{2}$ integrals, the areas of the projections on all the coordinate 2-planes, and the

“centroid” vector $J(\Gamma)$ provides n integrals. In dimension two, the meaning of these integrals is as follows: $A(\Gamma)$ is twice the area bounded by the curve Γ , and the vector $J(\Gamma)$, rotated 90° and divided by the area, is the center of mass of the domain bounded by Γ .

We finish with a remark about the area vector as an integral of the filament equation. Let a be a constant vector field in \mathbb{R}^3 (that is in the kernel of the form ω), and let

$$F_a = \frac{1}{2}A(\Gamma) \cdot a = \frac{1}{2} \int_{\Gamma} (\Gamma(t) \times \Gamma'(t)) \cdot a \, dt,$$

the projection of the area vector along a . Likewise, define

$$J_a = -J(\Gamma) \cdot a = - \int_{\Gamma} (\Gamma(t) \cdot \Gamma'(t)) (\Gamma(t) \cdot a) \, dt.$$

Lemma 5.1 *One has:*

$$i_a \Omega = dF_a = i_{\Gamma \times a} \omega, \quad i_{\Gamma \times a} \Omega = dJ_a.$$

Proof. Let u be a test vector field along Γ . Then

$$dF_a(u) = \frac{1}{2} \int_{\Gamma} [(u \times \Gamma') + (\Gamma \times u')] \cdot a \, dt = \int_{\Gamma} \det(\Gamma', a, u) \, dt = i_a \Omega(u),$$

proving the first equality. The second equality follows from the identity $(a \times b) \cdot c = \det(a, b, c)$ for any triple of vectors.

For the last equality, let v be a test vector field along Γ . Then

$$i_{\Gamma \times a} \Omega(v) = \int_{\Gamma} \det(\Gamma', \Gamma \times a, v) \, dt = \int_{\Gamma} [(\Gamma' \cdot a)(\Gamma \cdot v) - (\Gamma' \cdot \Gamma)(a \cdot v)] \, dt,$$

where we use the identity

$$\det(a, b \times c, d) = (a \cdot c)(b \cdot d) - (a \cdot b)(c \cdot d)$$

that holds for every quadruple of vectors. On the other hand,

$$\begin{aligned} dJ_a(v) &= - \int_{\Gamma} [(v \cdot \Gamma')(\Gamma \cdot a) + (\Gamma \cdot v')(\Gamma \cdot a) + (\Gamma \cdot \Gamma')(v \cdot a)] \, dt \\ &= \int_{\Gamma} [(\Gamma' \cdot a)(\Gamma \cdot v) - (\Gamma' \cdot \Gamma)(a \cdot v)] \, dt, \end{aligned}$$

where we used integration by parts. This completes the proof. \square

Therefore, a is a Hamiltonian vector field of the function F_a with respect to Ω . The constant vector field a commutes with the filament flow and all higher flows X_n , and hence F_a , and therefore $A(\Gamma)$, is an integral of the filament equation. For a discrete version of the filament equation, the area vector is a monodromy integral, see [17].

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