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On the Circumscribing Ellipse of Three Concentric Ellipses

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Abstract. Consider three coplanar non-degenerate line segments OA, OB, and OC, where only two of them can be collinear. Three concentric ellipses are then formed, say c_1', c_2', and c_3', where (OA, OB), (OB, OC) and (OC, OA) are being respectively the corresponding three pairs of their defining conjugate semi-diameters. Then, there exist another concentric ellipse c^* which circumscribes (i.e. being tangent to) all the ellipses c_i', $i = 1, 2, 3$. Moreover, the common tangent line on each common (contact) point between each c_i' and their tangent ellipse c^*, is parallel to the line segment (from the bundle of OA, OB, and OC) which does not belong to the pair of conjugate semi-diameters which forms each time the specific c_i'. The above result is derived through synthetic methods of the Projective Plane Geometry. Moreover, certain geometric properties (concerning, among others, the orthoptic circle of c^* or the existence of an involution between two bundles of rays of c^*), as well as the study of some special cases, are also discussed. A series of figures clarify the performed geometric constructions.

1. Introduction

Consider the following problem:

Problem 1. *Let A, B, C and O be four non-collinear points on a plane e, forming three line segments OA, OB and OC, where two of them can coincide. If the pairs (OA, OB), (OB, OC) and (OC, OA) are considered as the pairs of conjugate semi-diameters of three ellipses c_1, c_2 and c_3 respectively, then determine a new concentric (to c_i) ellipse c^* which circumscribes all c_i, $i = 1, 2, 3$.*

The above problem, depicted in Figure 1, was proved in [5] were the authors utilized synthetic methods of Projective Geometry, through their so-called "Four Ellipses Theorem". In particular, the proof of the Four Ellipses Theorem were delivered in [5] through the help of two lemmas, Lemma 1 which proved first, and its generalization Lemma 2. Note here that, the existence of total two circumscribing ellipses of c_i, $i = 1, 2, 3$, was proven in [14] utilizing an analytic-geometric methodology.

In the present work, the proof of the "Four Ellipses Theorem", i.e. the determination of the concentric and circumscribing ellipse of all c_i, $i = 1, 2, 3$, as described in Problem 1, is attained as a direct application of Lemma 1, i.e. without the use

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of the generalized Lemma 2, using, again, exclusively methods of Synthetic Plane Projective Geometry. For the facilitation of the development of this new proof, the presentation of the subject is being done in two stages. In the first stage, the basic elements of Lemma 1 are presented, supplemented by some new and extra properties that refer either to the ellipses of our theorem or to other parts of the topic. In the second stage, the new proof of the main theorem is developed, adding a study of the special case which is mentioned in the theorem.

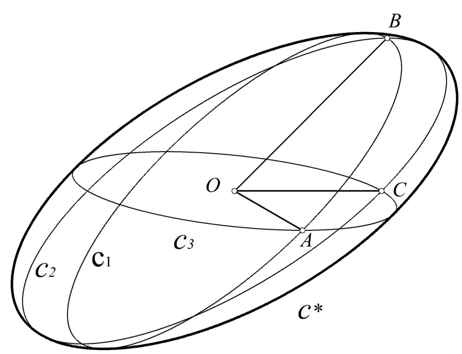

Figure 1. Illustration of Problem 1.

Note that, Problem 1 has appeared as an intermediate property (concerning these four ellipses) in a proof by G. A. Peschka [10] of the Karl Pohlke's Fundamental Theorem of Axonometry [6, pg. 250]. In Peschka's proof of the Pohlke's theorem, methods of the three-dimensional Euclidean Space were used, although Problem 1 is, by its description, a two-dimensional problem; see also [7, pg. 244] and [12]. In particular, a parallel projection was considered of an appropriate sphere onto the plane e where the concentric ellipses c_i, $i = 1, 2, 3$, lie. Figure 2 illustrates the above projection method.

Also, note that Evelyn et al. in [2] proved a more general form of "Four Ellipses Theorem" applying methods of the Analytic Plane Geometry with the use of suitable equations. Indeed, in their *"Double Contact Theorem"*, they stated that: *If three conics having a point (not on any of these conics) lying on a common (distinct) chord of each pair of the three conics, then there exist a conic which has a double contact with each of the three conics*. However, for the special case of Problem 1, the present paper provides a methodology of constructing the requested circumscribing ellipse as well as an investigation, where certain properties are derived. For the topic of concentric tangent ellipses see [3] among others.

In particular, in Section 2 we provide an outline of Lemma 1, proved in [5], while new properties concerning the construction are presented. Our main theorem (accompanying with the new corresponding geometric construction), is proven in Section 3, and certain special cases and properties are also presented in Section 4.

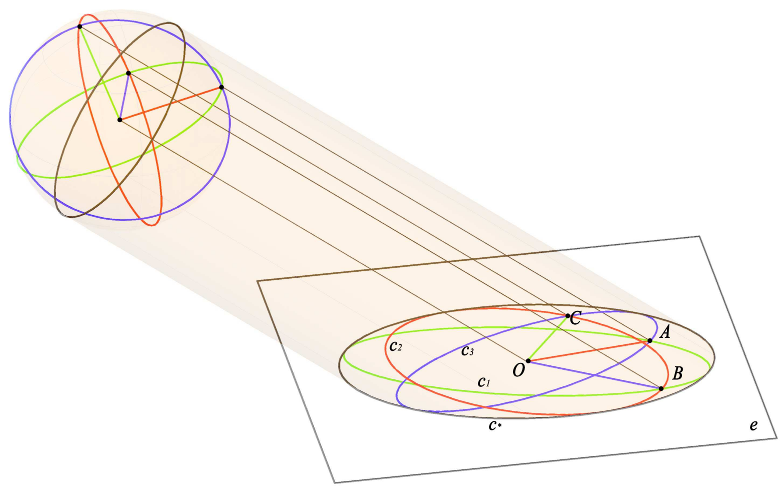

Figure 2. A parallel-projected sphere S onto the plane e where c_i, $i = 1, 2, 3$, lie.

2. A helpful construction

In this section a certain geometric construction is given, which shall be used for the proof of our main result in the next section, and is formulated through the following theorem. Since its proof is based on the properties of a projective transformation called homology, the plane —in which our problem is restricted— is considered to be the augmented Euclidean plane.

Proposition 1. *Let circle $c(O, \beta := |OB|)$ and an arbitrary point D' on the plane of c. We consider an arbitrary variable ellipse c'_1 having (OD', OH) as a pair of conjugate semi-diameters, where $H \in c$ an arbitrary point of circle c* (see Figure 3). *Then, it holds that:*

i. *The concentric and tangent to c ellipse c', which has point D' as one of its foci, is circumscribed (i.e. being tangent) to every variable ellipse c'_1 as the point $H \in c$ varies.*
ii. *If $0 < \beta' < \alpha'$ correspond to the semi-axes of the circumscribing ellipse c', then*
$$\alpha'^2 + \beta'^2 = |OD'|^2 + 2\beta^2. \tag{1}$$
iii. *If $0 < \beta'_1 < \alpha'_1$ correspond to the semi-axes of the varying ellipse c'_1, then*
$$\alpha'_1{}^2 + \beta'_1{}^2 = |OD'|^2 + |OH|^2 = |OD'|^2 + \beta^2 = \text{const.} \tag{2}$$

Proof. The proof of this proposition corresponds to the proof of Lemma 1 in [5] where the following procedure was adopted: Consider a homology f with the line spanned by BB' be f's homology axis, and the pair of points (A', A) be its homology pair of points. Since the requested ellipse c' is always tangent to the variable ellipse c'_1, then the corresponding homologue curves, under homology f, should

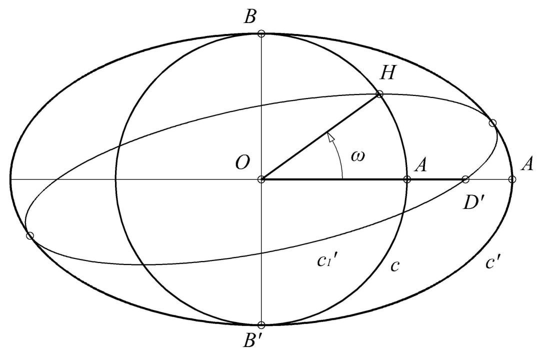

Figure 3. Illustration of Proposition 1.

be tangent with each other, i.e. circle c and ellipse c_1 have two (common) contact points, say A_1 and A_2; see Figure 4. Therefore, in order to complete the proof of this proposition, the variable ellipse c_1 should have $\alpha_1 = \beta$ as its major axis, as proved in [5, Lemma 1].

The following is essential for understanding of the procedure that finally yields $\alpha_1 = \beta$: The conjugate semi-diameters $(OD', OH = OL')$ of ellipse c_1', forming an angle $\omega \in (0, \pi/2)$, are homologue, under f, to the corresponding conjugate semi-diameters (OD, OL) of the ellipse c_1, which forms an angle $\varphi \in (0, \pi/2)$. For the ellipse c', consider the notations, $\alpha' := |OA'|$, $\beta' := \beta = |OB|$ and $\gamma' := |OD'|$. The following relations were proved; see [5, Lemma 1]:

$$|OD| = \frac{\beta\gamma'}{\alpha'}, \tag{3}$$

$$|OL| = \frac{\beta\sqrt{k}}{\alpha'}, \quad k := {\alpha'}^2 \sin^2\omega + \beta^2\cos^2\omega, \tag{4}$$

$$\sin\varphi = \frac{\alpha'\sin\omega}{\sqrt{k}}, \text{ and} \tag{5}$$

$$\alpha_1 = \beta, \tag{6}$$

where α_1 is the major semi-axis of ellipse c_1.

The following mappings, given in (7) and (8), clarify the notations, where $0 < \beta < \alpha$ are the principal semi-axes of an ellipse in general and f being the orthogonal homology:

$$\text{ellipse } c'(\alpha', \beta') \xmapsto{\text{orth. homology } f} \text{circle } c(O, |OB| = \beta), \tag{7}$$

$$\text{ellipse } c_1'(\alpha_1', \beta_1') \xmapsto{\text{orth. homology } f} \text{ellipse } c_1(\alpha_1 = \beta, \beta_1). \tag{8}$$

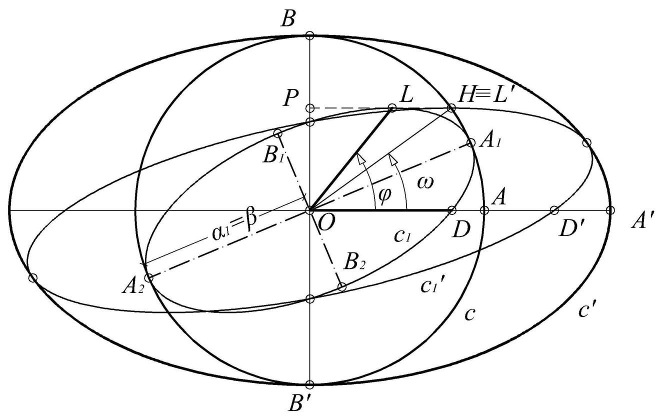

Figure 4. Illustration of the mappings (7) and (8).

We shall finally show that (1) and (2) hold. Indeed, for the ellipse c' with $0 < \beta' = \beta < \alpha'$ we derive that

$$|OA'|^2 + |OB|^2 = {\alpha'}^2 + \beta^2 = \left(\beta^2 + |OD'|^2\right) + \beta^2 = |OD'|^2 + 2\beta^2, \qquad (9)$$

while for the ellipse c_1' with $0 < \beta_1' < \alpha_1'$, Apollonius' theorem yields that

$$ {\alpha_1'}^2 + {\beta_1'}^2 = |OD'|^2 + |OH|^2 = |OD'|^2 + \beta^2 = \text{const.}, \qquad (10)$$

and theorem has been proved. □

The following remarks are new supplementary properties (not included in [5]) that derived from the proof of Proposition 1.

Remark 1. Alternatively, we can rewrite Proposition 1 as follows: *Let ellipse c' with principal semi-axes $0 < |OB'| = \beta' < |OA'| = \alpha'$ and a focus point D'. Let $\gamma' := |OD'|$. If H is an arbitrary point of the minor circle $c(O, \beta = |OB|)$ of the ellipse c', then the ellipse c_1' formed by its pair of conjugate semi-diameters (OD', OH) is tangent with the ellipse c'*; see Figures 3 and 5.

Remark 2. Notice that the angle φ between the conjugate semi-diameters OD and OL of c_1 can be written in a simpler form, i.e.

$$\tan\varphi = \frac{\alpha'}{\beta}\tan\omega, \quad \omega, \phi \in [0, \pi/2]. \qquad (11)$$

Indeed, (5) yields that

$$\tan^2\varphi = \frac{\sin^2\varphi}{1-\sin^2\varphi} = \frac{\left(\alpha'\frac{\sqrt{k}}{k}\sin\omega\right)^2}{1-\left(\alpha'\frac{\sqrt{k}}{k}\sin\omega\right)^2} = \frac{{\alpha'}^2\sin^2\omega}{k-{\alpha'}^2\sin^2\omega} = \frac{{\alpha'}^2\sin^2\omega}{\beta^2\cos^2\omega} = \frac{{\alpha'}^2}{\beta^2}\tan^2\omega,$$

and hence (11) holds.

Remark 3. An alternative method, other than the homology method, of determining point $L \in c_1$, homologue to $L' \in c_1'$, can be considered using (11). Figure 5 illustrates this method. In particular, if we consider circle $c(O, |OA|)$ as a trigonometric circle with $\overline{OA} = 1$, then $\tan\omega = \overline{AS}$. The line spanned by OS has a common point with the tangent line of ellipse c' at point, say J. From (11) or, alternatively, from $\tan\varphi / \tan\omega = \alpha'/\beta$ where $\alpha' = |OA'|$ and $\beta = |OA|$, and from the similarity of triangles (OAS) and $(OA'J)$, we obtain that $\tan\varphi = \overline{A'J}$. Therefore, if we set $A'J := AT$, the line spanned by OT forms angle φ, and hence the perpendicular line from L' to line spanned by OB intersects with OT at the requested point L.

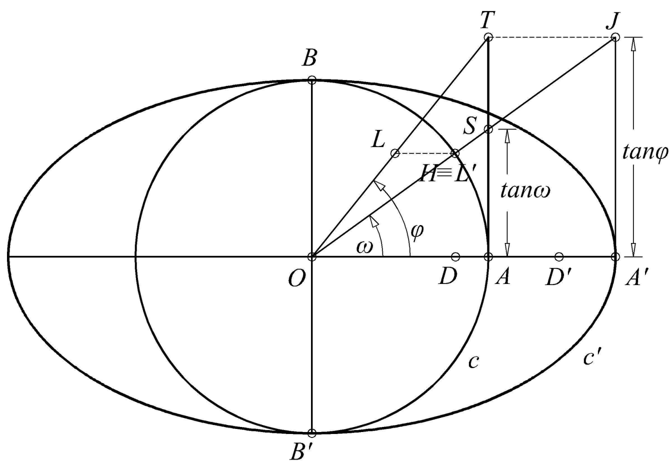

Figure 5. Alternative determination of point $L \in c_1$ with the help of the trigonometric circle (O,OA).

Remark 4. For the calculation of the minor semi-axis β_1 of c_1, Apollonius' theorem $\mu^2 + \nu^2 = \alpha_1^2 + \beta_1^2$ implies that $\beta_1 = \sqrt{\mu^2 + \nu^2 - \beta^2}$, where $\mu := |OD|$ and $\nu := |OL|$, and by substitution of

$$|OD| = \nu = \frac{\beta\gamma'}{\alpha'}, \tag{12}$$

(see [5, Lemma 1]) together with the fact that ${\alpha'}^2 = \beta^2 + {\gamma'}^2$, we finally obtain

$$\beta_1 = \frac{\beta}{\alpha'}\sqrt{\left(\beta^2 - \alpha'^2\right)\cos^2\omega + {\gamma'}^2} = \frac{\beta\gamma'}{\alpha'}\sin\omega, \tag{13}$$

Moreover, (13) and (12) yield the new property

$$\beta_1 = |OD|\sin\omega, \tag{14}$$

which means that the distance OD^* of point D from the corresponding variable $OH \equiv OL'$ is equal with the length of the minor semi-axis β_1 of the variable ellipse c_1 which is tangent to the circle c; see also Figure 6. Therefore, the length of β_1 can be determined from the beginning, i.e. when point $H \equiv L$ is considered.

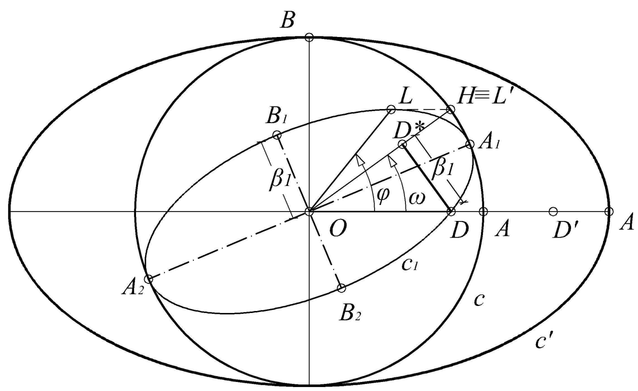

Figure 6. The c_1's minor semi-axis $\beta_1 = |DD^*|$ with $DD^* \perp OH$.

Remark 5. The following properties, which were not appeared in [5, Lemma 1], are hold; see Figure 7:

1. The orthogonal homology between ellipse c' and circle c has as homologue points the fixed pair (A', A). The absolute invariant of f is then $\lambda := \overline{OA}/\overline{OA'} = |OA|/|OA'| = \beta/\alpha'$. Therefore, the homologue ellipses c_1' and c_1 are intersect with each other onto the homology axis BB' of f.
2. Circle c and ellipse c_1 are orthogonally homologue to each other, under a well known homology f_1, with the corresponding homology axis spanned by the major axis A_1A_2 of ellipse c_1, which is also a diameter of circle c. The line spanned by the minor axis B_1B_2 of c_1 intersects circle c into two points, T and U. The orthogonal homology f_1 adopts (T, B_2) as its defining pair of points, with its absolute invariant being $\lambda_1 := \overline{OT}/\overline{OB}_2 = |OT|/\beta = \beta_1/\beta$, and via (13), $\lambda_1 = (\gamma'/\alpha)\sin\omega = \big(|OD'|/|OA'|\big)\sin\omega$.
3. If c_1' denotes a fixed ellipse, ellipses c_1' and c' are then homologue to each other under some homology f_1'.

The first two properties are trivial, and thus we shall now focus in the proof of the above third property: Let a point $S' \in c'$. The homologue point of S', under orthogonal homology f, is point $S \in c$, where the homology axis of f is spanned by BB', and (A', A) is the f's defining pair of homology points. The homologue point of S, under an orthogonal homology f_1, is point $S_1 \in c_1$, where f_1 is defined with homology axis spanned by A_1A_2 and (T, B_2) is considered as the homology pair of points. Moreover, the homologue point of S_1, under the inverse f^{-1} of the orthogonal homology f (defined as above), is point $S_1' \in c_1'$. Since the pairs (S', S) and (S_1', S_1) are homologue to each other, under homology f, the corresponding lines (spanned by) $S'S_1'$ and SS_1 are having a common point, say Q, onto the homology axis of f (spanned by BB'). As SS_1 is parallel to the minor axis B_1B_2 of ellipse c_1, the homologue under f line (spanned by) $S'S_1'$ is then parallel to the corresponding homologue fixed line (spanned by) B_1B_2, i.e. to $OB_2'T'$, where

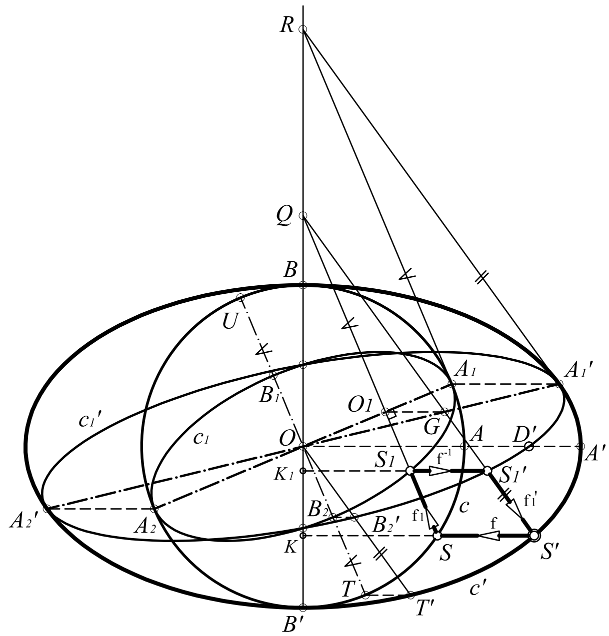

Figure 7. The consecutive homology mappings $c' \overset{f}{\mapsto} c \overset{f_1}{\mapsto} c_1 \overset{f^{-1}}{\mapsto} c_1' \overset{f_1'}{\mapsto} c'$.

the direction of $OB_2'T'$ remains invariant as S' moves along ellipse c'. Moreover, since the common tangent line (spanned by) A_1R at point A_1, between circle c and ellipse c_1, is perpendicular to the diameter A_1A_2 (i.e. parallel to B_1B_2), the homologue $S'S_1'$ of SS_1, under f, is then parallel to the common tangent line RA_1' at point A_1' of both homologue to c_1 and c (under f) ellipses c' and c_1'. As the pairs (A_1', A_1) and (A_2', A_2) are belonging to homology f, then the lines (spanned by) $A_1'A_2'$ and A_1A_2 are homologue. Therefore, if the line (spanned by) SS_1Q intersects with the line (spanned by) A_1A_2 at point, say O_1, then the corresponding homologue under f line (spanned by) $S'S_1'Q$ intersects with the line of $A_1'A_2'$ at, the homologue to O_1, point G. As a result, $GO_1 \parallel S_1'S_1 \parallel S'S$, since the homologue points are correspond to each other under f (see Figure 7). Therefore,

$$\frac{|GS_1'|}{|GS'|} = \frac{|O_1S_1|}{|O_1S|} = \frac{|OB_2|}{|OT|} = \lambda_1 = \text{const.},$$

for each angle ω and thus for specific (each time) ellipses c_1 and c_1'. From the above discussion it is concluded that ellipse c' and each of the variable ellipses c_1', tangent to c', are homologue with each other under homology, say f_1', which is defined by the homologue axis (spanned by) $A_1'A_2'$ and having absolute invariant ratio $\lambda_1' = |GS'|/|GS_1'| = |OT|/|OB_2| = \lambda_1^{-1} = \big(|OA'|/|OD'|\big)\csc\omega = \text{const.}$ The direction of this homology is parallel to the common tangent line (spanned by) RA_1' between the two ellipses c' and c_1' at their contact point A_1'. Conclusively, the two homologies f_1 and f_1' have:

- Inverse absolute invariant ratios λ_1 and λ'_1, i.e. $\lambda'_1 = 1/\lambda_1$.
- The axes A_1A_2 and $A'_1A'_2$ of homologies f_1 and f'_1 respectively are homologue lines under f.
- Homologue directions, determined by line (spanned by) OB_2T and its homologue OB'_2T', also under f.
- The three axes BB', A_1A_2 and $A'_1A'_2$ of homologies f, f_1 and f'_1 respectively are intersecting at point O.
- It holds that $\lambda \times \lambda_1 \times \lambda^{-1} \times \lambda'_1 = 1$.

The homology mappings in (7) and (8) can now be completed in Figure 8 with the cyclic representation of homologies according to Figure 7.

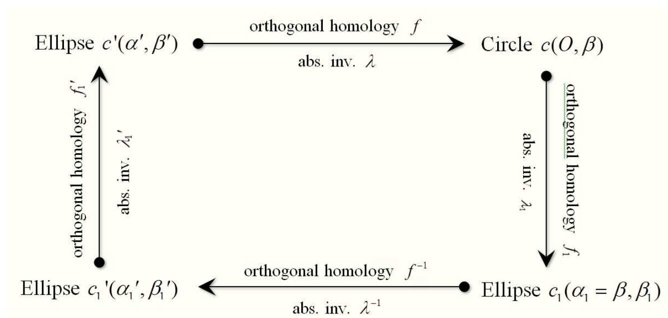

Figure 8. The cyclic action of homologies $f \to f_1 \to f^{-1} \to f'_1$ with their corresponding absolute invariants λ, λ_1, λ^{-1} and λ'_1.

3. Circumscribing ellipse of three concentric ellipses

We present now the plane-geometric proof of Problem 1, which is proved through the construction in Proposition 1.

Theorem 2. *Consider three given arbitrary coplanar and non-degenerated line segments OA, OB and OC, where only two of them can lie on the same line. Let c'_1, c'_2, and c'_3, denote the ellipses defined by the pairs of conjugate semi-diameters (OA, OB), (OB, OC), and (OC, OA), respectively. Then, there always exists an ellipse c^*, concentric to c'_i, $i = 1, 2, 3$, which circumscribes all c'_i ellipses, i.e. being tangent at two points with each one of them. Moreover, the common tangent line on each contact point between each c'_i and their tangent ellipse c^*, is parallel to the line segment which does not belong to the pair of conjugate semi-diameters that form each time the specific c'_i* (see Figure 9).

Proof. Consider a bundle of three arbitrary coplanar line segments OA, OB and OC, where $O \notin \{A, B, C\}$, i.e. OA, OB and OC are non-degenerated line segments. Two cases are then distinguished regarding the collinearity of those segments:

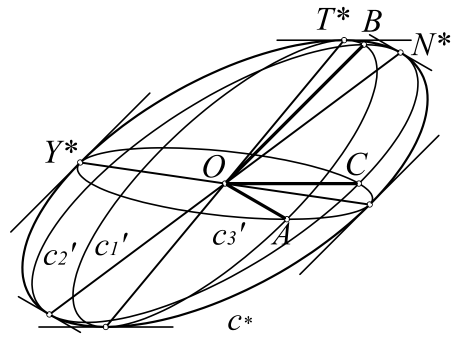

Figure 9. Contact points and the corresponding tangent lines between c^* and c_i', $i = 1, 2, 3$.

• *General case.* Assume that none of the OA, OB and OC, are collinear; see Figure 10. Let c_1' be an ellipse defined by its conjugate semi-diameters (OA, OB) via through Rytz's construction which is presented below. Figure 11 depicts the requested principal axes G_1D_1 and G_2D_2 of c_1'. For the Rytz's construction see also [8, pg. 69], [4, pg. 183] and [13] among others. In particular, we consider the following steps:

i. First we form the perpendicular line segment OP to OD, such that $|OP| = |OA|$.
ii. We then form the line segment PB, and from the middle point L of PB we draw a circle of radius LO. This circle intersects with PB at the points, say T and S.The principal axes of the requested ellipse c_1 are thus spanned by the line segment OT and OS.
iii. We then place $|BT| = \beta_1$ onto the line spanned by OS, and the apexes G_1 and D_2 of the c_1's minor axis are determined as $|OG_2| = |OD_2| = |BT| = \beta_1$.
iv. Similarly, the apexes G_1 and D_1 of the c_1's major axis are determined by placing $|OS| = |OG_1| = |OD_1| = \alpha_1$ onto the line spanned by OT.

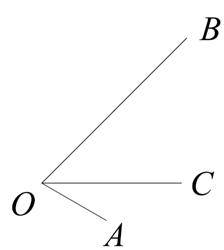

Figure 10. The given bundle of the three line segments OA, OB and OC.

Consider the major circle c of ellipse c_1' (i.e. the concentric circle which corresponds to ellipse's major radius) with diameter G_1D_1, as in Figure 12, which

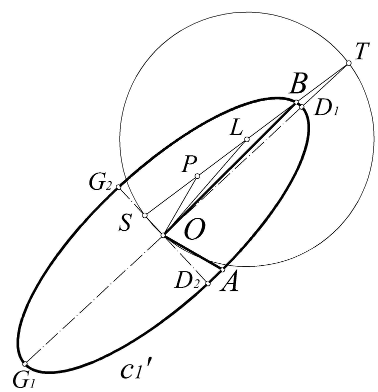

Figure 11. The principal semi-axes G_1D_1 and G_2D_2 of c_1' as obtained through Rytz's construction.

is the homologue of c_1', according to the orthogonal homology f with homology pair of points (G_2, G_2') and homology axis G_1D_1. In this homology, the points A and B of ellipse c_1' correspond to the points A' and B' respectively of circle c. Similarly, the third point C is having C' as its homologue point. Therefore, the given triplet (OA, OB, OC) is having (OA', OB', OC') as its corresponding homologue triplet.

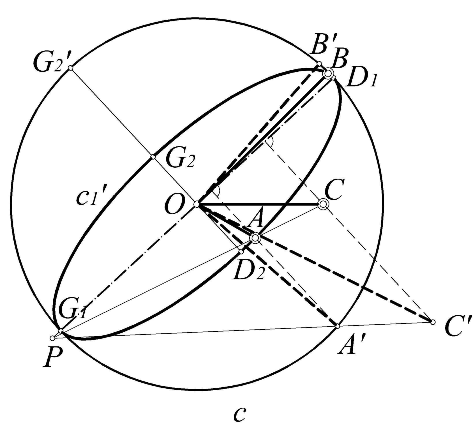

Figure 12. Construction of the homologue triplet (OA', OB', OC') of (OA, OB, OC).

Since the pair of line segments (OA, OB) is the pair of conjugate semi-diameters of ellipse c_1', then, according from a known property of homology, the homologue pair (OA', OB') of (OA, OB) is then a pair of perpendicular radii of circle c. Figure 13 presents a detail of Figure 12 in which included only the final triplet (OA', OB', OC') as well as the circle c.

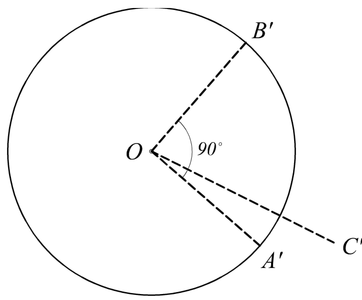

Figure 13. Triplet (OA', OB', OC'); detail from Figure 12.

It is then sufficient to construct an ellipse c' which it would be tangent to circle c, while it would be also tangent with two ellipses having (OB', OC') and (OC', OA') as pairs of their conjugate semi-diameters. With the construction of ellipse c' (presented in Step I below), the homologue of c' (under homology f) ellipse c^* (presented in Step II below), it would be simultaneously tangent with the three given ellipses c_1', c_2', and c_3'. Note that the following proof is based only to Proposition 1, which corresponds to [5, Lemma 1].
Step I. (Construction of the ellipse c'). Consider the perpendicular diameter of circle c to OC' which is intersecting c in two points T' and R'. We construct an ellipse with minor semi-axis OT' and focus point C'. Trivially, $T'C'$ equals with the major semi-axis OS' of the ellipse; see Figure 14. We shall show that the above constructed ellipse is the requested c'. Indeed, since C' is considered to be the focus point of c', and points A' and B' are belonging to circle c (Figure 14), then from Proposition 1 it is clear that the two ellipses defined by the pairs of conjugate semi-diameters (OB', OC') and (OC', OA'), depicted in Figure 15, are both tangent to ellipse c'; see also Figure 3.

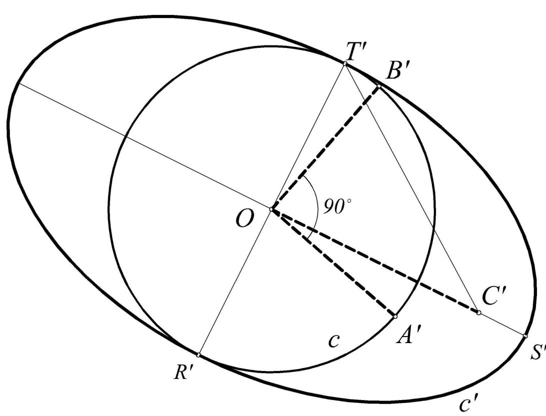

Figure 14. Construction of ellipse c'.

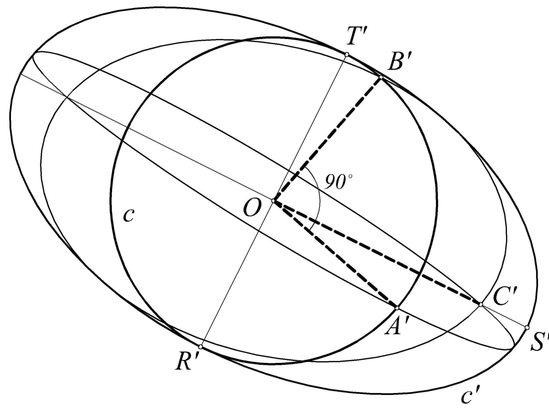

Figure 15. The ellipses defined by (OB', OC') and (OC', OA') are both tangent to ellipse c'.

Step II. (Construction of the ellipse c^*, homologue to c'). Figure 16 depicts the construction of the requested ellipse c^*, which is homologue (under orthogonal homology f) to ellipse c'. The corresponding homology axis is considered to be the line spanned by G_1D_1 while the pair of homology points is (C, C'); see Figure 12. Indeed, the pair of principal semi-axes (OS', OT') of ellipse c' adopts (OS^*, OT^*) as its homologue pair of conjugate semi-diameters of the homologue ellipse c^*. The points S^* and T^* are obtained through the known construction of homologue points:

- The perpendicular line to the homology axis G_1D_1 that passes through S', intersects line segment OC at the requested point S^*.
- For the point T^* we consider the line spanned by $S'T'$ which intersects with the homology axis at a point, say Q. Thus, the perpendicular line to homology axis that passes through T', intersects with S^*Q at the requested point T^*.
- Similarly, the point R^* of ellipse c^*, which is homologue to point R', is determined.

In order to complete the proof, the contact points between the circumscribing ellipse c^* and the given ellipse c'_1 have to be determined, as well as the corresponding tangent lines on them. Figure 17 depicts the contact point T^* between the ellipses c'_1 and c^* as well as their corresponding common tangent line t^* at T^*. Note that the second contact point R^* is diametrical to T^*. For the derivation of T^* we work as follows (recall Figure 16): Since ellipse c' and the circle c are in contact at point T', then the corresponding homologue curves c^* and c'_1 are in contact at the already known point T^*, which is homologue of T'. Moreover, since the tangent line t' at T' is parallel to OC' (Figure 16), the homologue of this tangent line is then also parallel to OC which is the homologue line of OC'. The tangent line r^* of c^* at

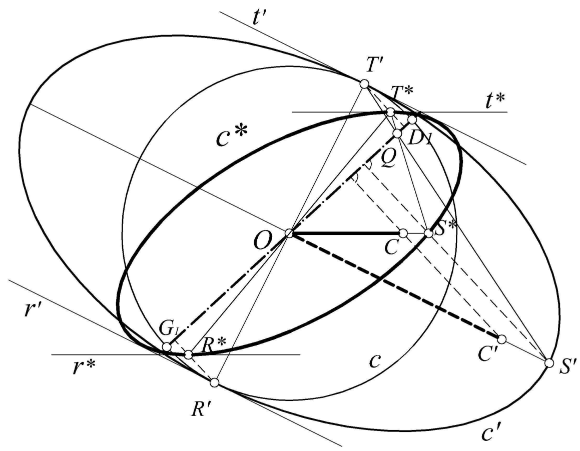

Figure 16. construction of the homologue to c' ellipse c^*.

point R^* can be determined in the same way. Figure 17 is a detail of Figure 16 where ellipse c_1', homologue to circle c, has been included.

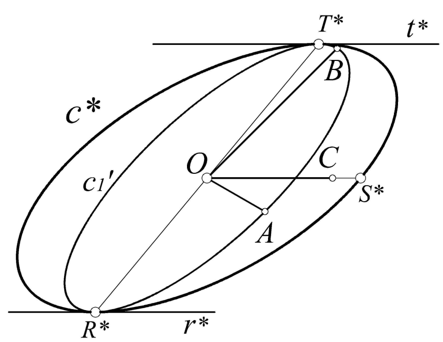

Figure 17. Contact points T^* and R^*, between c_1' and c^*, and their corresponding tangent lines t^* and r^*.

Similarly to the above described procedure, the contact points and the corresponding tangent lines between the requested circumscribing ellipse c^* and the other two ellipses c_1' and c_2' can be determined. In the final Figure 9 depicts all the ellipses c_i', $i = 1, 2, 3$, their common tangent ellipse c^*, as well as all the six contact points between them together with their corresponding tangent lines on them.

• *Degenerate case.* Assume that two line segments, say AO and OB, out of given three (OA, OB, OC) are being collinear with $A \neq B$; see Figure 18.

Then, two ellipses can be defined, say c_2' and c_3', when the pairs (OB, OC) and (OC, OA) are considered, respectively, to be their conjugate semi-diameters. These two ellipse are then tangent with each other at their common diametrical points T^* and C, and their corresponding common tangent lines on these points

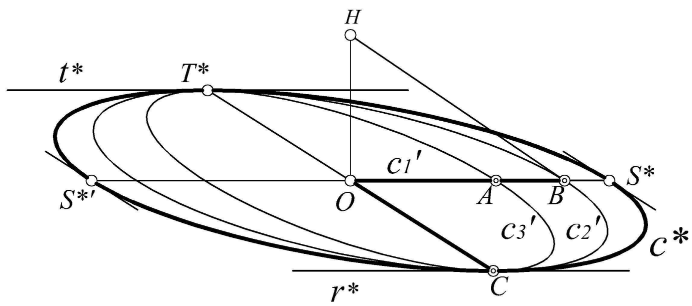

Figure 18. Degenerate circumscribing ellipse c^* for the degenerate case of ellipse c_1'.

are $t^* \parallel r^* \parallel OA$. In such a case, there exist an infinite number of real ellipses c^* circumscribing both c_2' and c_3' having two contact points on each of them. These contact points can only be T^* and C; see Figure 18. However, we can define one unique ellipse c^* as a solution for this special case of collinearity by assuming the collinear segments (OA, OB) as a degenerate pair of conjugate semi-diameters of a degenerate third ellipse, say c_1'. Indeed, in every ellipse, the Apollonius theorem can be applied, i.e.

$$|OA|^2 + |OB|^2 \;=\; \alpha^2 + \beta^2 \;\text{ and} \tag{15}$$

$$|OA||OB| \sin\varphi \;=\; \alpha\beta, \tag{16}$$

where $0 < \beta < \alpha$ denote the principal semi-axes of the ellipse defined by its pair of conjugate semi-diameters (OA, OB), and $\varphi := \measuredangle(OA, OB) \in [0,\, \pi/2]$. For the special degenerate case of ellipse c_1', the second relation (16) implies that

$$\alpha\beta = 0, \tag{17}$$

since $\varphi = 0$ which is due to the collinearity of OA and OB. However, as —in principle— $\beta < |OA| < \alpha$ and $\beta < |OB| < \alpha$ are both hold in the case of the non-degenerate ellipse where $|OA| > 0$ and $|OB| > 0$, we derive clearly that $\alpha > 0$, due to the fact that (17) should hold with $\alpha > \beta$. Therefore, (17) yields $\beta = 0$, and (15) is then written as

$$|OA|^2 + |OB|^2 = \alpha^2, \tag{18}$$

which means that the major semi-axis α of the degenerate ellipse c_1' equals with the hypotenuse HB of the orthogonal triangle (HOB), where $OA = OH$; see Figure 18. Consider the line segments $OS^* = OS^{*\prime} = HB$ onto OA, and admit as the solution ellipse c^* only one circumscribing ellipse around c_2' and c_3', with contact points T^* and C, which also includes the points S^* and $S^{*\prime}$ of the degenerate third ellipse c_1'. It is then clear that ellipse c^* has (OC, OS^*) as a pair of conjugate semi-diameters, since $OS^* \parallel r^*$. Therefore, at the points S^* and $S^{*\prime}$ the corresponding tangent lines of c^* are being parallel to OC, preserving the corresponding result of Theorem 2. Similarly, the tangent lines t^* and r^* at the contact points T^* and C are parallel to OA and OB. □

Recall the notion of the involution transform between two bundles of rays both sharing their common centers; see [1, 11] among others.

Corollary 3. *Consider the contact points T^*, N^* and Y^* of each ellipse c_i, $i = 1, 2, 3$* (as in Theorem 2) *with their common circumscribing ellipse c^** (see Figure 9). *The six rays spanned by the pairs (OT^*, OC), (ON^*, OA) and (OY^*, OB) are then correspond to two bundles of rays, i.e. (OT^*, ON^*, OY^*) and (OC, OB, OA), being in involution having center point O.*

Proof. The tangent line on each contact point T^*, N^* and Y^* of the ellipse c^* with one the concentric ellipses c_1, c_2 and c_3, is parallel each time to one of the three given line segments OC, OA and OB respectively (recall Theorem 2). Therefore, the rays spanned by (OT^*, ON^*, OY^*) correspond to rays being in involution with the rays spanned by (OC, OA, OB) respectively, due to the known fact the conjugate diameters of an ellipse (c^* in our case) are being in involution with each other; see also [4, pg. 175]. Figure 19 clarifies the above discussion, where the corresponding Fregier point of the formed involution (related to a certain circle passing through O) is also depicted, confirming the existence of the involution. For the Fregier point see also [9] among others. □

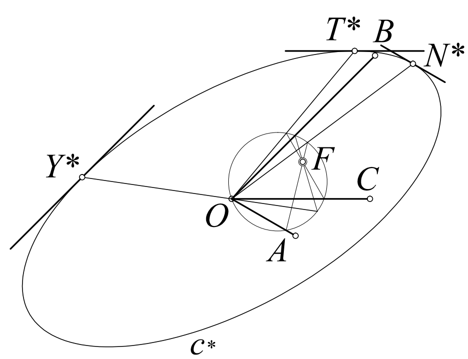

Figure 19. The rays spanned by (OT^*, ON^*, OY^*) are being in involution with the rays spanned by the given bundle (OC, OA, OB).

4. Properties and special cases

In the following we present some interesting properties concerning the orthoptic circle of ellipse c^*. Recall the notion of the *orthoptic circle* c_{M} (also known as *directive circle*, or *Fermat-Apollonius circle*, or *Monge circle*) of a given ellipse c, with principal semi-axes $0 < \beta < \alpha$. It is well known that is the concentric circle to c, having radius $\sqrt{\alpha^2 + \beta^2}$; see [1, pg. 261] among others. The characteristic property of c_{M} is that it is formed by the intersecting points between all the pairs of orthogonal (to each other) tangent lines of the ellipse c or, vice-versa, is the concentric to c circle c_{M} where from every point of c_{M} a pair of tangent lines to c is always orthogonal.

The following results are based on the relation

$$\alpha^{*2} + \beta^{*2} = |OA|^2 + |OB|^2 + |OC|^2, \tag{19}$$

where $0 < \beta^* < \alpha^*$ denote the principle semi-axes of c^*, which proved in [5, Theorem 3.1]. Note that (19) still holds for the special collinearity case of the above Theorem 2. An alternative interpretation of relation (19), is given with the following:

Corollary 4. *Consider three concentric and coplanar ellipses, say* c_i, $i = 1, 2, 3$ (recall Problem 1 or Theorem 2). *Then, the sum of the areas of the three orthoptic circles* $c_{\mathrm{M},i}$ *of* c_i, $i = 1, 2, 3$, *is two times the area of the orthoptic circle* c^*_{M} *of an ellipse* c^* *that circumscribes* (according to Theorem 2) *all* c_i, $i = 1, 2, 3$, *i.e.* $2A(c^*_{\mathrm{M}}) = \sum_{i=1}^{3} A(c_{\mathrm{M},i})$. *For the special case where one of the ellipses is a circle, say* c_1, *the sum of the areas of the two orthoptic circles of the other two non-circular ellipses is two times the area of the major circle* c^*_{maj} *of the circumscribed ellipse* c^*, *i.e.* $2A\big(c^*_{\mathrm{maj}}\big) = A(c_{\mathrm{M},2}) + A(c_{\mathrm{M},3})$.

Proof. Denote with OA, OB and OC the three common conjugate semi-diameters of c_i, $i = 1, 2, 3$, meaning that each two of them form each one of three given ellipses c_i, $i = 1, 2, 3$. Therefore, according to [5, Theorem 3.1], property (19) holds true. Utilizing Apollonius' theorem, i.e. $\alpha^2 + \beta^2 = \mu^2 + \nu^2$, where $\alpha, \beta > 0$ being the principal semi-diameters of an ellipse while $\mu, \nu > 0$ being a pair conjugate semi-diameters of the ellipse, relation (19) can be written as

$$\begin{aligned} 2\big(\alpha^{*2} + \beta^{*2}\big) &= \big(|OA|^2 + |OB|^2\big) + \big(|OB|^2 + |OC|^2\big) + \big(|OC|^2 + |OA|^2\big) \\ &= \sum_{i=1}^{3} \big(\alpha_i^2 + \beta_i^2\big), \end{aligned} \tag{20}$$

where $0 < \beta_i < \alpha_i$, $i = 1, 2, 3$, denote the principal semi-axes of c_i, $i = 1, 2, 3$, respectively. Therefore, multiplying (20) with π, and recalling the notion of the orthoptic circle given earlier, it holds that $2A(c^*_{\mathrm{M}}) = A(c_{\mathrm{M},1}) + A(c_{\mathrm{M},2}) + A(c_{\mathrm{M},3})$.

For the special case where, for instance, c_1 is being a circle, the minor semi-axes β^* of c^* must coincide with β_1, since circle c_1 must be the minor circle of its circumscribing ellipse c^*, Therefore, (20) yields $2\alpha^{*2} + 2\beta_1^2 = 2\beta_1^2 + \alpha_2^2 + \beta_2^2 + \alpha_3^2 + \beta_3^2$, i.e. $2\alpha^{*2} = \alpha_2^2 + \beta_2^2 + \alpha_3^2 + \beta_3^2$, and hence the special case $2A\big(c^*_{\mathrm{maj}}\big) = A(c_{\mathrm{M},2}) + A(c_{\mathrm{M},3})$ has been proved. □

A special case of relation (19) is given in the following:

Corollary 5. *Consider three line segments* OA, OB *and* OC, *as in Theorem 2, that can be freely rotated around their common point* O. *Then, all the circumscribed ellipses* c^* (derived via Theorem 2, as OA, OB and OC rotated independently around their common point O) *have the same orthoptic circle or, equivalently, the diagonals of the corresponding bounding rectangle of all* c^* (as well as on every rectangle that circumscribes c^*) *are of fixed length, which are the diameters of the orthoptic circle* c^*_{M}.

Proof. The orthoptic circle c^*_{M} of the variable common tangent ellipse c^* should have radius $\sqrt{\alpha^{*2}+\beta^{*2}}$, where $0<\beta^*<\alpha^*$ denote the c^*'s principal semi-axes. However, according to (19) it holds that each orthoptic circle c^*_{M} has the same radius since line segments OA, OB and OC preserve their length by assumption as they are rotated around O. Figure 20 depicts the common orthoptic circle c^*_{M} of c^* for two cases of OA, OB and OC. □

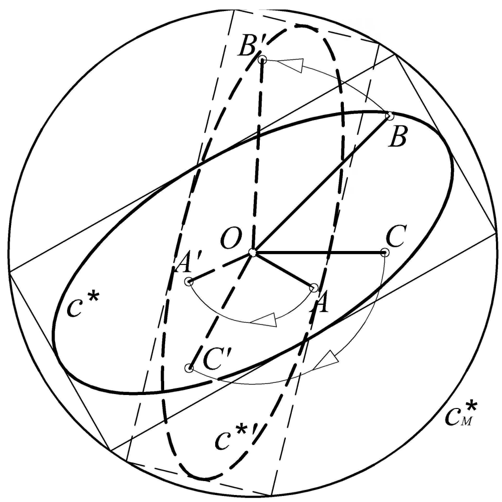

Figure 20. Common orthoptic circle c^*_{M} of two circumscribing ellipses c^* and $c^{*\prime}$.

Remark 6. It is clear that every rectangle that circumscribes each one c^*, an in Corollary 5, has a diagonal of fixed length, which is the diameter of the orthoptic circle c^*_{M}.

Remark 7. Note that, if we consider a fixed point, say Q, on the orthoptic circle c^*_{M}, as in Corollary 5, it is clear that all the rectangles that have point Q as one of their vertices and circumscribing each one c^*, would also have another fixed common vertex, which is diametrical to Q with respect to the orthoptic circle c^*_{M}.

Corollary 5 can be alternatively stated in the following form:

Corollary 6. *Consider a bundle of three line segments, that can be freely rotated around their* (pivoting) *common point and, therefore, forming three circles, say* κ_1, κ_2 *and* κ_3. *The area* $A\big(c^*_{\mathrm{M}}\big)$ *of the orthoptic circle* c^*_{M} *of the circumscribed ellipse* c^* (according to Theorem 2) *is always constant as the bundle rotates around its pivot, and equal to the sum of the areas of* κ_1, κ_2 *and* κ_3, *i.e.* $A\big(c^*_{\mathrm{M}}\big) = A\big(c^*_{\mathrm{maj}}\big) + A\big(c^*_{\mathrm{min}}\big) = \sum_{i=1}^{3} A(\kappa_i)$, *where* c^*_{maj} *and* c^*_{min} *denote the principle circles* (major and minor) *of the circumscribing ellipse* c^* *respectively. For the special case when two of the given three line segments remain always orthogonal and of the same length, the area of the ring between the major and minor circle of the* *circumscribing ellipse c^* is always equal to the area of the circle κ_n formed by the third, say $n \in \{1,2,3\}$, rotating line segment, i.e. $A\big(c^*_{\mathrm{maj}}\big) - A\big(c^*_{\mathrm{min}}\big) = A(\kappa_n)$, $n \in \{1,2,3\}$.*[1]

Proof. The non-special case is derived straightforward via (19). For the special case, assuming that $\beta := |OA| = |OB|$ and $OA \perp OB$, (19) yields $\alpha^{*2} + \beta^2 = 2\beta^2 + |OC|^2$, i.e. $\alpha^{*2} - \beta^2 = |OC|^2$, due to the fact that $\beta^* = \beta$ since OA and OB form a circle of radius β centered at point O, which must coincide with the minor circle of its circumscribing ellipse c^*. This completes the proof of the special case. □

Corollary 7. *Consider three line segments OA, OB and OC, as in Theorem 2, that can be rotated around their common point O. Assume now that two of the given three line segments remain always orthogonal and of the same length. It then holds that all the circumscribing ellipses c^** (recall Theorem 2) *are the same* (up to rotation)*, i.e. they all have the same length of principal axes* (see Figure 21).

Proof. Without loss of generality, consider that OA and OB are being always orthogonal and of equal length, as they rotate around O, i.e. $\beta := |OA| = |OB| =$ const. and $\measuredangle(OA, OB) := \pi/2 =$ const. It is then concluded that each common tangent ellipse c^* (derived via Theorem 2 at every position of OA, OB and OC) should have a minor semi-axis of the same length as the two orthogonal line segments, i.e. $\beta^* = \beta$, since (OA, OB) always form a circle of radius β centered at O when it considered as a pair of conjugate semi-diameters. This circle trivially corresponds to the minor circle of the circumscribing ellipse c^*, i.e. the circle of the minor radius of c^*. Relation (19) now yields $\alpha^{*2} = \beta^2 + |OC|^2 =$ const., and hence the principal semi-axes α^* and β^* of c^* are preserving their length or, alternatively, c^* is always preserving its shape as OA, OB and OC are rotated around O. Figure 21 depicts two cases of circumscribing ellipses, say c^* and $c^{*\prime}$, where $OA \perp OB$ and OC are rotated around O to $OA' \perp OB'$ and OC'. □

Discussion

Consider the problem of determining a concentric ellipse c^* which circumscribes three coplanar and concentric ellipses c_i, $i = 1, 2, 3$. Each one of the above ellipses is defined by two conjugate semi-diameters, taken from a bundle of three given coplanar line segments assumed that only two of them may coincide.

In the present paper the authors prove the existence of a circumscribing ellipse c^*, providing also a construction methodology of c^* in terms of plane Projective Geometry. This was achieved utilizing only a form of Lemma 1 in [5], since a different approach was adopted. Note that the construction of c^* provided in [5] were performed through a generalization of [5, Lemma 1]. Moreover, certain properties arising from the new process were also revealed and discussed. In addition, the degenerate case where two of the given three line segments can coincide were also studied. Finally, some geometric properties concerning the orthoptic circle of c^*

[1] For example, if $OA \perp OB$ and $|OA| = |OB|$, then κ_3 is the circle (O, OC).

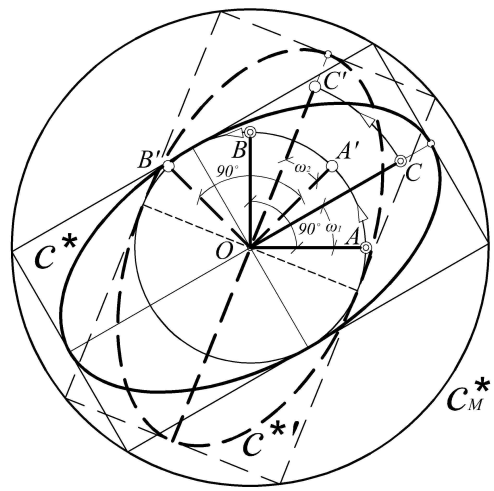

Figure 21. Two circumscribing ellipses c^* and $c^{*\prime}$ with the same orthoptic circle c^*_{M}.

were given. The provided figures illustrate the corresponding geometric constructions of the proofing process.

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