ON THE CONSTRUCTION OF NEW TOPOLOGICAL SPACES FROM EXISTING ONES

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Abstract. In this note, we introduce a guiding principle to define topologies for a wide variety of spaces built from existing topological spaces. The topologies so-constructed will have a *universal property* taking one of two forms. If the topology is the *coarsest* so that a certain condition holds, we will give an elementary characterization of all continuous functions taking values in this new space. Alternatively, if the topology is the *finest* so that a certain condition holds, we will characterize all continuous functions whose domain is the new space.

Consider a function $f\colon X \to Y$ between a pair of sets. If Y is a topological space, we could define a topology on X by asking that it is the coarsest topology so that f is continuous. (The finest topology making f continuous is the discrete topology.) Explicitly, a subbasis of open sets of X is given by the preimages of open sets of Y. With this definition, a function $W \to X$, where W is some other space, is continuous if and only if the composite function $W \to Y$ is continuous.

On the other hand, if X is assumed to be a topological space, we could define a topology on Y by asking that it is the finest topology so that f is continuous. (The coarsest topology making f continuous is the indiscrete topology.) Explicitly, a subset of Y is open if and only if its preimage in X is open. With this definition, a function $Y \to Z$, where Z is some other space, is continuous if and only if the composite function $X \to Z$ is continuous.

In this what follows, we use a mild generalization of this principle (where single maps are replaced by families of maps) to define topologies on new spaces constructed from old. We first describe the construction of the underlying sets of these spaces.

1. On the construction of new sets

There are many ways to build new sets from existing sets:

Disjoint unions. Given sets A and B, their *disjoint union* is the set $A \coprod B$ whose elements are elements of exactly one of A or B.

For example, the integers $\mathbb{Z}$ are the disjoint union of the odd integers and the even integers. Or, iterating the disjoint union construction, the set of n elements is the disjoint union of n copies of the set $*$ with a single element.

Products. Given sets A and B their (*cartesian*) *product* is the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Date: March 23, 2015.

Elements of $A \times B$ are called *ordered pairs*.[1] Infinite products may be formed similarly: the product of sets $A_1, A_2, \ldots$ is the set

$$\prod_{n \in \mathbb{N}} A_n = \{(a_1, a_2, \ldots) \mid a_1 \in A_1, a_2 \in A_2, \ldots\}$$

More generally, a product can be indexed by any (possibly uncountable) set. Given a set A_α for each $\alpha \in I$ — this I referred to as the *index set* — an element of $\prod_{\alpha \in I} A_\alpha$ is a collection of elements $(a_\alpha)_{\alpha \in I}$ with a *coordinate* $a_\alpha \in A_\alpha$ for each $\alpha \in I$.

For example, the Euclidean plane $\mathbb{R}^2$ is the product $\mathbb{R} \times \mathbb{R}$. The lattice of integer points is the product $\mathbb{Z} \times \mathbb{Z}$.

Subsets. The *subsets* of a set X are again sets in their own right.

Quotients. A *quotient* of a set X is a set whose elements are thought of as "points of X subject to certain identifications." For example, there is a quotient of $\mathbb{R}$ which we might call the set "$\mathbb{R}$ mod $\mathbb{Z}$". Elements are real numbers plus some arbitrary unspecified integer. There is a bijection between the set $\mathbb{R}$ mod $\mathbb{Z}$ and the set $[0, 1)$.

If X is equipped with an equivalence relation $\sim$, then the set $X/_\sim$ of equivalence classes is a quotient of the set X. More generally, any binary relation $\sim$ generates an equivalence relation: by definition, the equivalence relation generated by $\sim$ is the smallest equivalence relation on X so that x and x' are in the same equivalence class if $x \sim x'$. In this case, we again write $X/_\sim$ for the set of equivalence classes of X in the equivalence relation generated by $\sim$ and refer to this set as the quotient of X by $\sim$.

Gluings. Suppose X and Y are two sets that are not disjoint but share a common subset $A = X \cap Y$. We can form a new set by *gluing* X to Y along A. We denote this set by $X \cup_A Y$. Formally, $X \cup_A Y$ is the quotient of $X \coprod Y$ by the relation that identifies those points in X and in Y that lie in the intersection. If we label the inclusions, as in the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & Y \arrow[d] \\ X \arrow[r] & X \cup_A Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{1.1}$$

we say that points in $X \cup_A Y$ are points in $X \coprod Y$ subject to the identifications $i(a) \sim j(a)$ for each $a \in A$.[2] We arrange inclusions into a square (1.1) to represent that the inclusions *commute*: a point $a \in A$ has the same image in $X \cup_A Y$ regardless of whether it is mapped first into X and then into $X \cup_A Y$ or mapped first into Y and then into $X \cup_A Y$.

Gluings are also called *pushouts*. The symbol "$\ulcorner$" is included to remind the reader that the set in the lower right-hand corner is constructed as a gluing.

For example, when two 2-dimensional closed disks are glued together along their boundary circles, the result is a 2-dimensional sphere. We visualize one disk and the "northern hemisphere", the other as the "southern hemisphere", and their common boundary as the "equator".

[1] Note that the elements (a, a') and (a', a) in the product $A \times A$ are distinct.

[2] Technically we should say "subject to the equivalence relation generated by $i(a) \sim j(a)$ for each $a \in A$." Note, that the maps i and j need not be injective.

Pullbacks. We can now "dualize"[3] the previous picture. A *pullback* is a subset of a product space, subject to certain conditions. More precisely, suppose X and Y are sets equipped with functions $f\colon X \to A$ and $g\colon Y \to A$. The *pullback* is the set

$$X \times_A Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

For example, the pullback of the "pairity" map $\mathbb{Z} \to \mathbb{Z}/2$ along the "inclusion of 0" map $* \to \mathbb{Z}/2$ is the set of even integers. In general the pullback of an arbitrary function $f\colon X \to A$ along a function $a\colon * \to A$ that picks out a single element $a \in A$ is called the *fiber*: it consists of the set of elements of X whose image under f is a.

Here is another example. Let $f\colon X \to Y$ be any function. The pullback of f along the identity is a set we might call the *graph* of f. It is the set of points $(x, f(x)) \in X \times Y$. To understand our choice of terminology, it might be helpful to draw a picture in the case of $f\colon \mathbb{R} \to \mathbb{R}$.

2. Functions that remember constructions

Now suppose these existing sets were topological spaces. How can we topologize the newly constructed sets in a sensible way? It turns out there is a uniform procedure for doing this that encompasses each of the examples introduced above, producing the "correct" answer in each context. What "correct" means is that the result is mathematically interesting, something that one must convince oneself of gradually through accumulated experience.

A recognition problem. But before we address this issue, we must find a way for these sets to "remember" how they were constructed. To explain what we mean by this, let us describe what it means to "forget" the mechanism of construction. For instance, the sets $\mathbb{Z}$, $\mathbb{Z} \sqcup \mathbb{Z}$, and $\mathbb{Z} \times \mathbb{Z}$ are isomorphic, all having the same cardinality. Without extra data it is impossible to tell that the second and third are respectively a disjoint union and a product of two copies of the first. Or given finite sets A and X with $|A| < |X|$ there are many possible ways to regard A as a subset of X. A priori there is no way to prefer one to the $\binom{|X|}{|A|} \cdot |A|!$ others. Similarly, without any extra information, it would be impossible to recognize which circle on the surface of a 2-sphere is the equator along which the two disks were glued.

The key insight is to use certain canonical functions between the sets we've just constructed and the sets these sets were built from to remember the constructions. The reader is encouraged, as an exercise, to revisit the above examples and guess what these functions might be.

Done? Here are the answers.

Disjoint unions. A disjoint union can be recognized by the injections

$$A \hookrightarrow A \coprod B \hookleftarrow B$$

More generally, given sets A_α, there is a canonical injection $A_\alpha \hookrightarrow \coprod_\alpha A_\alpha$ for each α in the index set. These maps are jointly surjective: each element in $\coprod_\alpha A_\alpha$ is in the image of (exactly) one of the canonical injections.

[3]Informally, *dualize* means "turn around all the arrows." This doesn't have anything to do with inverse functions. Instead, in any place where the previous discussion refers to a function $X \to Y$, consider instead a (a priori unrelated) function $Y \to X$.

Products. Products can be recognized by the projections

$$A \twoheadleftarrow A \times B \twoheadrightarrow B$$

onto each coordinate. More generally, given sets A_α, there is a canonical projection function

$$\begin{aligned} \prod_\alpha A_\alpha &\to A_\alpha \\ (\ldots, a_\alpha, \ldots) &\mapsto a_\alpha \end{aligned}$$

for each α.

Subsets. Subsets are of course recognized from the inclusion map $A \hookrightarrow X$.

Quotients. When a quotient of X is formed from an equivalence relation $\sim$ there is a canonical surjective function $X \to X/_\sim$ that takes an element to its equivalence class. More generally, we can think of **any** surjective function $X \twoheadrightarrow Y$ as defining an equivalence relation on X. Equivalence classes correspond bijectively to elements of Y. Explicitly, $x \sim x'$ if x and x' lie in the same fiber, i.e., if x and x' have the same image in Y.

With this perspective, a quotient of X is any set Y for which there is a surjective function $X \to Y$, and such a function witnesses the fact that Y is a quotient of X.

Gluings. The components of a gluing $X \cup_A Y$ are identified via the maps in the diagram (1.1). We refer to the maps $X \to X \cup_A Y \leftarrow Y$ as the canonical inclusions, though this terminology is a bit of an abuse: if the maps i and j in (1.1) are not injective, then these maps will likely not be injections.

Pullbacks. Dually, a pullback $X \times_A Y$ is recognized from the maps

$$\begin{tikzcd} X \times_A Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & A \end{tikzcd}$$

The arrows $X \leftarrow X \times_A Y \to Y$ are also called projections, mapping an element (x, y) to x and y respectively. We use the symbol "$\lrcorner$" to indicate that the set in the upper left-hand corner is constructed as a pullback.

The interesting uses of the pullback construction in topology are of a somewhat less elementary nature, so pullbacks will not be discussed in detail in this paper.

3. The universal definition and its characterizing theorem

Now we are ready to introduce the appropriate topology on the spaces formed by each of our constructions. To do so, first note that our six examples come in two flavors. For disjoint unions, quotients, and gluings, the functions that recognize each construction map **to** the set so-constructed. By contrast, for products, subsets, and pullbacks, these functions map **from** the set so-constructed. The topologies we assign to the spaces in these two classes of examples will take dual forms and the resulting spaces will satisfy two different (dual) sorts of *universal properties.*

The general definitions.

Definition 3.1. Given spaces X_α, the *disjoint union topology* on $\coprod_\alpha X_\alpha$ is the finest topology so that the canonical injections $X_\alpha \hookrightarrow \coprod_\alpha X_\alpha$ are continuous.

Definition 3.2. Given a space X and a quotient Y, the *quotient topology* on Y is the finest topology so that the canonical projection $X \twoheadrightarrow Y$ is continuous.

Definition 3.3. Given spaces and continuous functions $X \xleftarrow{i} A \xrightarrow{j} Y$, the *gluing topology* on $X \cup_A Y$ is the finest topology so that the maps $X \to X \cup_A Y \leftarrow Y$ are continuous.

Definition 3.4. Given spaces X_α, the *product topology* on $\prod_\alpha X_\alpha$ is the coarsest topology so that the canonical projections $\prod_\alpha X_\alpha \to X_\alpha$ are continuous.

Definition 3.5. Given a space X and a subset A, the *subspace topology* on A is the coarsest topology so that the canonical inclusion $A \hookrightarrow X$ is continuous.

Definition 3.6. Given spaces and continuous functions $X \xrightarrow{f} A \xleftarrow{g} Y$, the *pullback topology* on $X \times_A Y$ is the coarsest so that the projection maps $X \leftarrow X \times_A Y \to Y$ are continuous.

Note that the indiscrete topology makes the functions of definitions 3.1, 3.2, and 3.3 continuous — but the indiscrete topology is not very interesting. This is why we asked for the **finest** topology and not the coarsest one. Similarly, the discrete topology makes the functions of definitions 3.4, 3.5, and 3.6 continuous; this is why we asked for the **coarsest** topology and not the finest one. The upshot is that we do not get to choose whether the coarsest or finest possible topologies are used in our definitions: in each example only one of these will produce an interesting result.

The fact that the topologies specified by these definitions exist follows from the following general result whose proof is left as an exercise.

Lemma. *Let $\{\mathcal{T}_\alpha\}$ be any collection of topologies on a space X. Then there is a unique topology $\mathcal{T}_F$ which is finer than each topology $\mathcal{T}_\alpha$ and is the coarsest topology with this property, and there is a unique topology $\mathcal{T}_C$ which is coarser than each topology $\mathcal{T}_\alpha$ and is the finest topology with this property.*

The universal properties. We have not yet given an explicit description of these topologies. Nonetheless, we can prove the following theorems.

Theorem 3.7 (theorem-schema F[4])**.** *Consider a family of spaces X_α and a set X given the finest topology so that certain maps $X_\alpha \to X$ are continuous. Then if Z is any space, a function $X \to Z$ is continuous if and only if the composite maps $X_\alpha \to X \to Z$ are continuous.*

Theorem 3.8 (theorem-schema C[5])**.** *Consider a family of spaces X_α and a set X given the coarsest topology so that certain maps $X \to X_\alpha$ are continuous. Then if Z is any space, a function $Z \to X$ is continuous if and only if the composite maps $Z \to X \to X_\alpha$ are continuous.*

We realize that the "certain maps" in these theorem statements is somewhat vague. The point is in each example, there will be a theorem of this form that is

[4]For "finest".
[5]For "coarsest".

provable without any further information about the topologies we have just defined! Even more remarkably, though we won't prove this here, these *universal properties*,[6] not only give rise to an explicit description of each of these topologies but also determine the underlying sets of these spaces uniquely up to isomorphism.

Explicit descriptions. It turns out that it is not difficult to give a direction characterization of the topologies introduced in definitions 3.1-3.6. For products, subspaces, pullbacks, or indeed for any topology defined to be the coarsest structure on X so that a given collection of functions with domain X are continuous, there is a concrete description of a subbasis for this topology.

Theorem 3.9. *Let X be any set equipped with functions $f_\alpha \colon X \to X_\alpha$ taking values in topological spaces X_α, and suppose the topology on X is defined to be the coarsest topology so that these maps are continuous. Then the collection*

$$\bigcup_\alpha \{f_\alpha^{-1}(U) \mid U \subset X_\alpha \text{ is open}\}$$

defines a subbasis for the topology on X.

Proof. For f_α to be continuous, $f_\alpha^{-1}(U)$ must be open in X for each open $U \subset X_\alpha$. These sets define a subbasis for some topology: the open sets in this topology are precisely the unions of finite intersections of sets of the form $f_\alpha^{-1}(U)$. This is clearly the coarsest topology so that the f_α are continuous. □

Remark. Indeed, the conclusion of Theorem 3.9 remains true if we only consider preimages of basis elements or even subbasis elements for each space X_α. A general open set $U \subset X_\alpha$ is expressible as a union of finite intersections of subbasis elements; its preimage will then be a union of the (finite) intersections of the preimages of these subbasis elements.

Similarly, for disjoint unions, quotients, gluings, or indeed for any topology defined to be the finest structure so that a certain collection of maps to a given set are continuous, there is an explicit characterization of the open sets in this topology.

Theorem 3.10. *Let X be any set equipped with functions $f_\alpha \colon X_\alpha \to X$ whose domains are topological spaces X_α, and suppose the topology on X is defined to be the finest topology so that these maps are continuous. Then $U \subset X$ is open if and only if $f_\alpha^{-1}(U) \subset X_\alpha$ is open for each α.*

Proof. Let $\mathcal{T}$ be the finest topology on X so that the maps f_α are continuous. If $U \in \mathcal{T}$, then each $f_\alpha^{-1}(U) \subset X_\alpha$ must be open because each f_α is supposed to be continuous. Conversely, suppose $A \subset X$ is any set for which $f_\alpha^{-1}(A) \subset X_\alpha$ is open. Define a topology on X consisting of all finite intersections and arbitrary unions of A and elements of $\mathcal{T}$. The preimage under f_α of an open set in this topology is expressible as a union of finite intersections of $f_\alpha^{-1}(A)$ and preimages of elements of $\mathcal{T}$. By hypothesis, these sets are open in X_α. Hence, f_α is continuous, and the assumption that $\mathcal{T}$ is the finest topology with this property implies that $A \in \mathcal{T}$. □

Now let's see how this plays out in our examples.

[6] As a rough approximation, a *universal property* refers to the existence of certain maps satisfying some specified conditions.

4. Disjoint unions

Let's prove this theorem for our first example: disjoint unions. The essential details in the proofs for arbitrary disjoint unions and for finite, or indeed binary, disjoint unions are identical, so to simplify our notation, we'll just consider the latter.

Theorem (universal property of the disjoint union topology)**.** *Suppose A and B are spaces and give $A \coprod B$ the disjoint union topology. Then any set-function $f\colon A \coprod B \to Z$ taking values in a topological space Z is continuous if and only if the restrictions $f|_A$ and $f|_B$ are continuous.*

Note $f|_A$ is precisely the composite function $A \hookrightarrow A \coprod B \xrightarrow{f} Z$. Hence this theorem is a special case of our theorem-schema F. Note also that we have yet to give a concrete description of the disjoint union topology. Nonetheless, we can give a proof!

Proof. If f is continuous then so are the composite functions $f|_A$ and $f|_B$, so one direction is clear. For the converse implication, consider an open subset $U \subset Z$. By hypothesis $f|_A^{-1}(U)$ is open in A and $f|_B^{-1}(U)$ is open in B. Clearly $f^{-1}(U) = f|_A^{-1}(U) \coprod f|_B^{-1}(U)$, so to prove that this is open it suffices to show that $f|_A^{-1}(U)$ and $f|_B^{-1}(U)$ are open in $A \coprod B$.

The proof of this fact uses a general argument. Recall $A \coprod B$ was given the finest topology so that the inclusions $A \hookrightarrow A \coprod B \hookleftarrow B$ are continuous. So to show that any subset $V \subset A \coprod B$ is open, we claim that it suffices to show that $V \cap A$ and $V \cap B$, the preimages along these inclusions, are open. Provided these two conditions hold, then V could be added to the topology on $A \coprod B$ without violating the continuity of these maps. But because $A \coprod B$ has the finest topology with this property, it must be the case that V is already open in $A \coprod B$.

Note that $(f|_A^{-1}(U)) \cap A = f|_A^{-1}(U)$ and $(f|_A^{-1}(U)) \cap B = \emptyset$, both of which are open. Hence $f|_A^{-1}(U)$ is open in $A \coprod B$. Similarly $f|_B^{-1}(U)$ is open. This completes the proof. □

Indeed, the image of any open subset in A is open in $A \coprod B$. In general a continuous map $f\colon X \to Y$ is called *open* if f maps open sets in X to open sets in Y and *closed* if f maps closed sets to closed sets.

Lemma. *The canonical inclusions $A \hookrightarrow A \coprod B \hookleftarrow B$ are both open and closed.*

Proof. Let $V \subset A$ be open. Note $V \cap B = \emptyset$ is open in B. So V could be added to the topology of $A \coprod B$ without disrupting the continuity of the maps $A \hookrightarrow A \coprod B \hookleftarrow B$. Because $A \coprod B$ has the finest such topology, V must therefore be open in A. Replacing "open" by "closed", this argument also proves that the maps $A \hookrightarrow A \coprod B \hookleftarrow B$ are closed. □

The idea that underpinned the proof of these two results allows us to immediately characterize all open sets in the disjoint union topology. The following definition is equivalent to definition 3.1.

Definition (the disjoint union topology)**.** Given spaces X_α, a subset is open in the disjoint union topology on $\coprod_\alpha X_\alpha$ if and only if it is a union $\coprod_\alpha U_\alpha$ of open sets $U_\alpha \subset X_\alpha$.

We should argue that the topology defined here is the finest topology that makes the injections $X_\alpha \hookrightarrow \coprod_\alpha X_\alpha$ continuous. To prove this, it suffices to show that any non-open subset of the coproduct has some non-open preimage in one of the X_α. Suppose $A \subset \coprod_\alpha X_\alpha$ is not open in this topology. Note we can write

$$A = \coprod_\alpha (A \cap X_\alpha).$$

Hence one of the $A \cap X_\beta$ is not open. But this means that if A were open, then the inclusion $X_\beta \hookrightarrow \coprod_\alpha X_\alpha$ would not be continuous. This is a special case of the proof of Theorem 3.10.

Exercise. Let $*$ denote the one-point space given the only possible topology. Let S be a set. What is the topology on the set $\coprod_S *$ defined to be the S-indexed coproduct of $*$ with itself?

Exercise. The set $\mathbb{R}$ is the disjoint union of the rationals $\mathbb{Q}$ and the irrationals. Suppose $\mathbb{R}$ is given the standard topology of 1-dimensional Euclidean space. Is it possible to topologize the rationals and the irrationals so that the standard topology on $\mathbb{R}$ agrees with the disjoint union topology?

5. Products

Theorem 3.9 provides a concrete description of the product topology:

Definition (the product topology)**.** Given spaces X_α the product topology on $\prod_\alpha X_\alpha$ has open sets: $\prod_\alpha U_\alpha$ where $U_\alpha \subset X_\alpha$ is open and $U_\alpha = X_\alpha$ for all but finitely many indices α.

The reader is encouraged to give a direct proof that this definition is equivalent to definition 3.4.

Example. Recall a basis for the standard topology on $\mathbb{R}$ is given by open intervals (a, b). Thus, a subbasis for the product topology on $\mathbb{R}^2$ is given by "open strips" of the form $(a_1, b_1) \times \mathbb{R}$ or $\mathbb{R} \times (a_2, b_2)$. Taking intersections, we see that a basis for the product topology is given by open rectangles $(a_1, b_1) \times (a_2, b_2)$. This basis defines the standard topology on the Euclidean plane.

Analogously, the product topology on $\mathbb{R}^n$ has a basis given by open rectangular prisms whose edges are parallel to the coordinate axes. This basis defines the *standard topology* on n-dimensional Euclidean space.

Example. As the notation suggests, the set $2^{\mathbb{N}}$ of "coin flips" is isomorphic to the $\mathbb{N}$-indexed product of the set 2 of two elements. Give each two-element set the discrete topology. An element in the subbasis for the product topology on $2^{\mathbb{N}}$ is an "observation", i.e., a set consisting of all sequences of coin flips so that the k-th term takes a specific value, for some $k \in \mathbb{N}$. This subbasis defines the *observable topology* on coin flips.

We should prove the analog of Theorem 3.8 for the product topology.

Theorem (universal property of the product topology)**.** *Let Z be a topological space, let X_α be a collection of spaces, and give $\prod_\alpha X_\alpha$ the product topology. Then a function $f\colon Z \to \prod_\alpha X_\alpha$ is continuous if and only if each coordinate function $f_\alpha\colon Z \to X_\alpha$ is continuous.*

Note the coordinate function f_α is precisely the composite of f with the projection $\prod_\alpha X_\alpha \to X_\alpha$.

Proof. If f is continuous, then the coordinate functions f_α are composites of continuous functions and hence continuous. Conversely, suppose each coordinate function is continuous. To see that $f\colon Z \to \prod_\alpha X_\alpha$ is continuous, it suffices to check that the preimages of subbasis elements are open in Z. By Theorem 3.9 a subbasis is given by the collection of sets $f_\alpha^{-1}(U)$ for some open $U \subset X_\alpha$. But these are open by hypothesis, completing the proof. $\square$

The box topology. We conclude this section with a quick aside describing another topology on an infinite product of spaces. Given spaces X_α, the *box topology* on the set $\prod_\alpha X_\alpha$ has as open sets products $\prod_\alpha U_\alpha$ of open sets $U_\alpha \subset X_\alpha$ subject to no additional restrictions.

At first glance this seems like a reasonable definition. But the following example illustrates how this topology can be poorly behaved with respect to continuous functions.

Example. Giving $\mathbb{R}^\omega$ the box topology, the function $t \mapsto (t, t, t, \ldots)\colon \mathbb{R} \to \mathbb{R}^\omega$ is not continuous even though its coordinate functions clearly are: the preimage of the open set

$$(-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots$$

is the set consisting of a single point, the origin, which is not open.

6. Subspaces

Theorem 3.9 gives an explicit description of the subspace topology on a subset A of a topological space X.

Definition (the subspace topology)**.** The subspace topology on A has a subbasis given by the sets $A \cap U$ with $U \subset X$ open.

Note the collection of sets $\{A \cap U \mid U \text{ is open in } X\}$ is closed under finite intersection and arbitrary union. Thus **all** open sets in the subspace topology have this form.

Exercise. What is the subspace topology assigned to $\mathbb{Z} \hookrightarrow \mathbb{R}$?

We have the following analog of Theorem 3.8 for the subspace topology.

Theorem (universal property of the subspace topology)**.** *Let X and Z be spaces and let $A \subset X$ be given the subspace topology. A function $f\colon Z \to A$ is continuous if and only if the composite $Z \to A \hookrightarrow X$ is continuous.*

Proof. Open subsets in the subspace topology have the form $A \cap U$ where $U \subset X$ is open. The preimage in Z of $A \cap U$ under f is the same as the preimage in Z of U under the composite function $Z \to X$. It is therefore open provided this composite is continuous. $\square$

In particular, any continuous function $Z \to X$ whose image is contained in A restricts to a continuous function $Z \to A$.

Exercise. Suppose $A \subset X$ is given the subspace topology. Is the inclusion $A \hookrightarrow X$ open?

More generally, we refer to any continuous injective function $A \hookrightarrow X$ as a subspace inclusion if the topology on A coincides with the subspace topology on X.

Example. The space $2^{\mathbb{N}}$ of coin flips is a subspace of the Euclidean unit interval $[0,1]$ via the function

$$(e_1, e_2, e_3, \ldots) \mapsto \sum_{n>0} \frac{2e_n}{3^n} \colon 2^{\mathbb{N}} \to [0,1]$$

Put another way, this function defines an imbedding from the observable topology to the Euclidean topology. Its image is a subset of the real line called the *Cantor set*.

7. Gluings

The gluing construction introduced above extends to arbitrary (non-injective) functions. Given sets and functions $X \xleftarrow{f} A \xrightarrow{g} Y$ define the *gluing* or *pushout*

(7.1)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\tilde{g}"] \\ Y \arrow[r, "\tilde{f}"'] & X \cup_A Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

to be the quotient of the disjoint union $X \coprod Y$ by the equivalence relation generated by the relation $f(a) \sim g(a)$ for each $a \in A$. Intuitively, this means that points in the image of f are identified with corresponding points in the image of g. But this quotienting process can also be used to identify points in X together. For example, if Y is the singleton set $*$, the underlying set[7] of the pushout $X \cup_A *$ will be the set $X \backslash f(A) \coprod *$. Every point in the image of f will be collapsed to a single point $*$.

Now suppose the maps f and g are continuous. By definition 3.3, the space $X \cup_A Y$ is assigned the finest topology so that the maps $\tilde{f}$ and $\tilde{g}$ are continuous. Theorem 3.10 leads to a concrete description of the gluing topology.

Definition (the gluing topology)**.** A set $U \subset X \cup_A Y$ is open in the gluing topology if and only if $\tilde{f}^{-1}(U) \subset Y$ and $\tilde{g}^{-1}(U) \subset X$ are open.

In the case where f and g are inclusions, this definition may be restated in the following form: $U \subset X \cup_A Y$ is open if and only if $U \cap X$ and $U \cap Y$ are open.

Example. The *real line with two origins* is a space which we will denote $\mathbb{R} \diamond \mathbb{R}$ obtained as a gluing

$$\begin{tikzcd} \mathbb{R}\backslash\{0\} \arrow[r, hook] \arrow[d, hook] & \mathbb{R} \arrow[d, hook] \\ \mathbb{R} \arrow[r, hook] & \mathbb{R} \diamond \mathbb{R} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

It has the property that any neighborhood of one origin must intersect any neighborhood of the other.

[7]We say "underlying *set*" because the space $X \cup_A *$ is **not** topologized as a disjoint union.

Example. Let $p \in S^n$ be any fixed point. The *wedge of two n-spheres* is obtained by gluing S^n to S^n at this point.

$$\begin{tikzcd} * \arrow[r, "p"] \arrow[d, "p"'] & S^n \arrow[d] \\ S^n \arrow[r] & S^n \vee S^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

For instance the space $S^1 \vee S^1$ is the figure eight. Iterating this construction produces a *bouquet of n-spheres* glued together along a single point.

Warning (gluings vs unions). Let X be a space with subspaces A and B so that $X = A \cup B$. Consider the commutative square of inclusions

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

If A and B are both closed subspaces, then this square is a gluing square. This square is also a gluing square if both A and B are open. But in general, the topology on X might not agree with the gluing topology even though X is the set-theoretic union of A and B. For instance, if B is the complement of A, $A \cap B$ is empty, and the gluing topology assigned to $A \cup B$ is such that the subsets A and B are both open and closed. In general, the gluing topology might be finer than the pre-existing topology on X.

A particular type of gluing has a special name.

Definition (attaching an n-cell)**.** Let X be a topological space and let $f\colon S^{n-1} \to X$ be any continuous function. We may *attach an n-cell* to X by gluing D^n to X along the *attaching map* f by means of the following pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \\ D^n \arrow[r] & X \cup_{S^{n-1}} D^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

In this terminology, (discrete) points are also called 0-cells. Their boundary consists of the empty space $S^{-1} = \emptyset$. "Attaching" a point just means taking a disjoint union. A vast library of topological spaces can be constructed by repeatedly attaching cells in various dimensions.

Definition 7.2. A *cell complex* is a space constructed by repeatedly attaching cells. A *CW-complex* is a cell complex constructed from a set of zero cells by first attaching 1-cells to the 0-cells, then attaching 2-cells to the 0-cells and 1-cells, then attaching 3-cells to the 0-, 1-, and 2-cells, and so on.

In this context, Theorem 3.7 takes the following form.

Theorem (the universal property of the gluing topology)**.** *Consider a space $X \cup_A Y$ constructed as a pushout* (7.1)*. A function $j\colon X \cup_A Y \to Z$ is continuous if and only if the composite functions $X \to X \cup_A Y \to Z$ and $Y \to X \cup_A Y \to Z$ are continuous.*

Proof. Suppose the composite functions are continuous and consider an open set $U \subset Z$. By hypothesis, $\tilde{f}^{-1}j^{-1}(U)$ and $\tilde{g}^{-1}j^{-1}(U)$ are open. Now the explicit characterization of the gluing topology implies that $j^{-1}(U)$ is open, as desired. □

In fact, the universal property of gluings is even stronger than asserted.

Theorem (the universal property of the gluing topology II)**.** *Let $h\colon X \to Z$ and $k\colon Y \to Z$ be any continuous functions so that $hf = kg$. Then there exists a unique continuous function $j\colon X \cup_A Y \to Z$ so that its restriction to X is the function h and its restriction to Y is the function k, i.e., so that the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\tilde{g}"] \arrow[ddr, bend left, "h"] & \\ Y \arrow[r, "\tilde{f}"] \arrow[drr, bend right, "k"'] & X \cup_A Y \arrow[dr, dashed, "\exists!"', "j"] \arrow[ul, phantom, "\ulcorner", very near start] & \\ & & Z \end{tikzcd}$$

commutes.

In words, a function from a space constructed by gluing X to Y along A is uniquely determined by a pair of functions defined on X and Y, provided that these functions agree on A. Note that nothing in this result requires that the functions f or g are injective. In this context, the restriction of j to X means the composite of j with $\tilde{g}$; similar remarks apply to the restriction to Y.

Proof. The functions $\tilde{f}$ and $\tilde{g}$ are jointly surjective onto $X \cup_A Y$, meaning that every point is in the image of either $\tilde{f}$ or $\tilde{g}$. We define the value of j on a point $\tilde{f}(y)$ to be $k(y)$ and define j on a point $\tilde{g}(x)$ to be $h(x)$. We must show that this is well defined.

As a set, the gluing $X \cup_A Y$ is constructed by taking the disjoint union $X \coprod Y$ and then quotienting by the equivalence relation generated by the relation $f(a) \sim g(a)$ for all $a \in A$. But note, given a point $f(a) = g(a)$ in $X \cup_A Y$, the first definition for the image under j is $kg(a)$ while the second is $hf(a)$; because $hf = kg$, these two candidate definitions agree. It follows that j is well defined. Continuity is given by the previous theorem. □

The following example illustrates how this result might be used in a very simple case.

Example. Let X be a topological space. A *loop* in X is a continuous function $S^1 \to X$. It is common to choose a "base point" in S^1 at say that the loop $S^1 \to X$ is a *loop based at the point* x in the image of this chosen point.

Equivalently, recall S^1 is formed by the following gluing square

$$\begin{tikzcd} * \coprod * \arrow[r] \arrow[d, "i_0 \coprod i_1"'] & * \arrow[d] \\ I \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & S^1 \end{tikzcd} \tag{7.3}$$

By the previous theorem, a loop in X is given by a function $* \to X$, whose image determines a point $x \in X$, together with a continuous map $I \to X$, called a *path*, subject to the condition that the starting and ending points of the path are the point $x \in X$. But this is precisely what it means to define a loop at x in X!

8. Quotients

Let $p\colon X \twoheadrightarrow Y$ be a surjective function between sets X and Y. Recall that we can use this function to think of points in Y as equivalence classes of points in X. Here, the equivalence classes are the *fibers* of the map p, i.e., the sets $p^{-1}(y)$ for some $y \in Y$.

Now suppose X is a space. Recall definition 3.2: the quotient topology on Y is the finest topology so that p is continuous. Once more, Theorem 3.10 provides an explicit description of the open sets in this topology.

Definition (the quotient topology)**.** Suppose given a surjective continuous function $p\colon X \twoheadrightarrow Y$. A set $U \subset Y$ is open in the quotient topology if and only if $p^{-1}(U)$ is open.

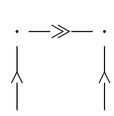
To review the ideas introduced above, let us give a direct proof that this definition agrees with the topology defined in 3.2. Because X has a topology and the preimage function p^{-1} respects intersections and unions, this definition defines a topology on Y. To show that it agrees with the topology of definition 3.2, we must show that every open set in this topology is open in the other topology and visa versa. To show that the topology of definition 3.2 is finer than this topology, it suffices to note that p is continuous with respect to this topology and 3.2 was defined to be the finest such topology. To show that this topology is finer than the topology of 3.2, we note that if U is open in 3.2, then $p^{-1}(U)$ is open by continuity, and hence U is open in the topology defined here.

When Y is given the quotient topology, the map $p\colon X \twoheadrightarrow Y$ satisfies a particular property, which is stronger than continuity. Sometimes maps with this property are found "in nature" in which case it is convenient to have a name by which to recognize them.

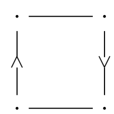
Definition (quotient maps)**.** A surjective map $p\colon X \twoheadrightarrow Y$ is a *quotient map* if $U \subset Y$ is open if and only if $p^{-1}(U) \subset X$ is open.

If p is a quotient map, then the topology on Y is necessarily the quotient topology with respect to p. In particular, the universal properties described below apply equally well to quotient maps found "in nature."

Example. Form a set T as a quotient of the square $I \times I$ by identifying points $(x, 0) \sim (x, 1)$ and points $(0, y) \sim (1, y)$. The resulting space is homeomorphic to the torus and the map $I \times I \twoheadrightarrow T$ is a quotient map with respect to the usual (subspace) topologies on these spaces.



Example. The *Möbius strip* M is defined to be the quotient of the space $I \times I$ by the relation $(0, y) \sim (1, 1-y)$ for all $y \in I$.



Example. Attaching an n-disk to a point defines a surjective map $D^n \twoheadrightarrow S^n$. Points in the interior of the disk D^n map injectively into S^n whereas points on the boundary $S^{n-1} \subset D^n$ map to the "south pole" of S^n. This map is a quotient map when both spaces are given their standard (subspace) topologies.

This example can be generalized. We leave the proof as an exercise:

Lemma. *Let $i \colon A \hookrightarrow X$ be a subspace inclusion and let $p \colon A \twoheadrightarrow Y$ be a quotient map. Then the pushout $\tilde{p} \colon X \twoheadrightarrow X \cup_A Y$ is a quotient map.*

Example. Let $\mathbb{R}P^n$ be the set of lines through the origin in $\mathbb{R}^{n+1}$. There is a surjective function $S^n \twoheadrightarrow \mathbb{R}P^n$ that takes a point on the unit sphere to the line it inhabits. The topology on the space $\mathbb{R}P^n$ is defined to be the quotient topology with respect to this map.

Example. The (*unreduced*) *suspension* of a space X is the quotient of the space $X \times I$, given the product topology, by the relation $(x, 0) \sim (x, 1)$ for all $x \in X$. For example, the suspension of S^1 is the quotient of the cylinder $S^1 \times I$ by the relation that collapses the top circle to a point and collapses the bottom circle to a point. This space is homeomorphic to S^2. Similarly, the suspension of S^n is homeomorphic to S^{n+1}.

Theorem 3.7 encodes the universal property of the quotient topology.

Theorem (the universal property of the quotient topology)**.** *Let $p \colon X \twoheadrightarrow Y$ be a quotient map and let Z be a topological space. A set-function $g \colon Y \to Z$ is continuous if and only if the composite $gp \colon X \to Z$ is continuous.*

Proof. The hard part is to show that if gp is continuous then g is. Consider an open set $U \subset Z$. Because gp is continuous, $(gp)^{-1}(U) = p^{-1}(g^{-1}(U))$ is open. But this implies that $g^{-1}(U)$ is open by the definition of the quotient topology, proving continuity of g. □

Once more, the universal property of the quotient topology can be strengthened.

Theorem (the universal property of the quotient topology II)**.** *Let $p \colon X \twoheadrightarrow Y$ be a quotient map and let Z be a topological space. Given any continuous function $h \colon X \to Z$ that is constant on fibers, i.e., so that $h(x) = h(x')$ whenever $p(x) = p(x')$, then there is a unique continuous function $g \colon Y \to Z$ so that $gp = h$.*

$$\begin{tikzcd} X \arrow[d, two heads, "p"'] \arrow[dr, "h"] & \\ Y \arrow[r, dashed, "\exists!"', "g"] & Z \end{tikzcd}$$

This result is often summarized by the following slogan: to define a continuous function whose domain is a quotient space, it suffices to define the function "upstairs", provided that the "upstairs" function is constant on fibers. For example, to define a function whose domain is $\mathbb{R}P^n$, it suffices to define a function whose domain is S^n and that is constant on antipodes.

Proof. Define $g(y)$ to be $h(x)$ for any point $x \in p^{-1}(y)$. Because h is constant on fibers, this is well-defined. The function g is continuous by the previous theorem. Uniqueness is obvious. $\square$

We close our discussion of quotient spaces with a final general example. Suppose given continuous functions $X \xleftarrow{f} A \xrightarrow{g} Y$. As a set $X \cup_A Y$ is a quotient of $X \coprod Y$ by the equivalent relation generated by $f(a) \sim g(a)$ for all $a \in A$. There is a canonical function $X \coprod Y \to X \cup_A Y$ which takes points to their equivalence classes.

Theorem. *The canonical map $X \coprod Y \to X \cup_A Y$ is a quotient map from disjoint union topology to gluing topology.*

Proof. $U \subset X \cup_A Y$ is open if and only if $U \cap X$ and $U \cap Y$ are open, i.e., if and only if its preimage in $X \coprod Y$ is open. $\square$

By the universal property of this quotient map, continuous functions $X \cup_A Y \to Z$ correspond to continuous functions $X \to Z$ and $Y \to Z$ that agree upon restriction to A. This is precisely the universal property of the gluing topology.

9. An application of the universal properties

In practice, particularly as the examples get more complicated, universal properties are a very useful way to define continuous functions. We give a few quick examples.

Example. Suppose we wanted to define an "interesting" function $S^n \vee S^n \to S^n \times S^n$. In the case $n = 1$, we hope to define a map from the figure eight to the torus. To define a map taking values in a product space $S^n \times S^n$, it suffices, by the universal property of the product topology, to define a pair of continuous coordinate functions $S^n \vee S^n \to S^n$. Because $S^n \vee S^n$ is constructed by gluing, for each map $S^n \vee S^n \to S^n$ it suffices, by the universal property of the gluing topology, to define the two components $S^n \to S^n$, provided these functions agree at the point $p \in S^n$ at which the two spheres are attached.

The map we have in mind might be called

$$S^n \vee S^n \xrightarrow{(1 \vee p, p \vee 1)} S^n \times S^n.$$

The first coordinate function is the identity on the first sphere and constant at the point p on the second. The second coordinate function is the identity on the second sphere and constant at the point p on the first. In the case $n = 1$, the image of the figure eight consists of the union of a loop around the rim of the torus and a loop around the handle.

Example. The inclusion $S^1 \hookrightarrow S^2$ whose image is the "equator" defines a map $\mathbb{R}P^1 \to \mathbb{R}P^2$. To see this, note that the composite $S^1 \hookrightarrow S^2 \twoheadrightarrow \mathbb{R}P^2$ is constant on antipodes and continuous. Hence, the universal property of the quotient implies that this map defines a unique continuous function $\mathbb{R}P^1 \to \mathbb{R}P^2$ so that the diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, two heads] & S^2 \arrow[d, two heads] \\ \mathbb{R}P^1 \arrow[r] & \mathbb{R}P^2 \end{tikzcd}$$

commutes. This map $\mathbb{R}P^1 \to \mathbb{R}P^2$ includes lines through the origin in $\mathbb{R}^2$ as lines through the origin that lie in the xy-plane in $\mathbb{R}^3$. This construction can be generalized to maps of arbitrary increasing dimension $S^n \to S^{n+k}$ and their quotients $\mathbb{R}P^n \to \mathbb{R}P^{n+k}$.

Our final example, also involving real projective spaces, is more exotic. Let us first try and get our bearings. The space $\mathbb{R}P^1$ turns out to be homeomorphic to S^1. Note however that the quotient map $S^1 \to \mathbb{R}P^1$ is not a homeomorphism: it's not bijective![8]

The space $\mathbb{R}P^2$ can be built as a CW-complex with one 0-cell, one 1-cell, and one 2-cell. First, the 1-cell is attached the 0-cell as displayed in (7.3) to form the space S^1. Then, the 2-cell is attached to the space S^1 along the "squaring" map $S^1 \to S^1$; if $S^1 \subset \mathbb{C}$ is the subspace of complex numbers of norm 1, this map squares each complex number. The pushout

$$\begin{tikzcd} S^1 \arrow[r, "(-)^2"] \arrow[d, hook] & S^1 \arrow[d] \\ D^2 \arrow[r, "q"'] & \mathbb{R}P^2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

defines a space homeomorphic to $\mathbb{R}P^2$. To see this, note that D^2 is homeomorphic to the northern hemisphere of S^2 together with the equator. The squaring map $S^1 \to S^1$ is a quotient map; hence its pushout $D^2 \to \mathbb{R}P^2$ is also a quotient map. It is easy to see that this quotient map identifies antipodal points along the equator of the northern hemisphere. But if we superimpose lines through the origin in $\mathbb{R}^3$ onto our picture we can easily define a bijection between $\mathbb{R}P^2$ and the resulting quotient space.

Unlike the case in dimension 1, it turns out that $\mathbb{R}P^2$ and S^2 are not homeomorphic. We don't have the tools to prove this result yet, but we can at least provide some fairly convincing intuition. An important property of the surface $\mathbb{R}P^2$ is that it is *non-orientable*. A surface is non-orientable if and only if it contains a subspace that is homeomorphic to the Möbius strip M. The 2-sphere S^2, by contrast, is orientable and does not have this property.

How might we define the embedding $M \hookrightarrow \mathbb{R}P^2$? Why by appealing to the universal properties of the quotient space of course! Recall D^2 is homeomorphic to $I \times I$. First define a map $f\colon I \times I \to I \times I$ by

$$(x, y) \mapsto (x, \frac{1}{2} + \frac{y - \frac{1}{2}}{2}).$$

The image is the strip of radius $\frac{1}{2}$ around the horizontal bisector of $I \times I$. Next consider the composite

$$I \times I \xrightarrow{f} I \times I \xrightarrow{\cong} D^2 \xrightarrow{q} \mathbb{R}P^2.$$

Here the middle map is the homeomorphism defined by placing the closed unit disk D^2 on top of the closed square in such a way that their centers of mass coincide and then "stretching" the points of the square radially so that they map bijectively to the disk. By design this homeomorphism sends "antipodal points" on the boundary of $I \times I$ to antipodal points on $S^1 \subset D^2$.

[8]As an exercise, write down a homeomorphism between these spaces.

Recall we have a quotient map $p\colon I \times I \twoheadrightarrow M$. We can check that the composite map above is constant on fibers of p: the points $(y, 0)$ and $(1, 1-y)$ map to antipodal points on $S^1 \subset D^2$ which are identified by the quotient map q. Hence this continuous function determines a unique continuous map $M \to \mathbb{R}P^2$. This is our desired embedding of the Möbius band as the "prime meridian" of $\mathbb{R}P^2$.

10. Discrete and indiscrete topologies

We end with one final pair of examples. We could imagine a version of definitions 3.1-**??** in which there were no maps (and hence no conditions) for the topology on the set X except that it were the finest or coarsest possible. It turns out these topologies are quite familiar.

Definition (the discrete topology)**.** The finest possible topology on a set X is called the discrete topology.

Definition (the indiscrete topology)**.** The coarsest possible topology on a set X is called the indiscrete topology.

It is easy to argue that every subset of X must be open in the discrete topology, and that only X and the empty set are open in the indiscrete topology. The reason for mentioning these examples is to note the following consequences of Theorems 3.7 and 3.8. We leave the proof to the reader.

Theorem 10.1 (universal properties of the discrete and indiscrete topologies)**.** *Let Z be a topological space. Any set-function $Z \to X$ is continuous if X is given the indiscrete topology. Any set-function $X \to Z$ is continuous if X is given the discrete topology.*

11. A concluding thought

The insights presented here all derived from a branch of mathematics called *category theory*. If you liked this story, you might want to check it out. There's a lot more where this came from!

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