

IX. *On the Mathematical Foundations of Theoretical Statistics.*

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## DEFINITIONS.

*Centre of Location.*—That abscissa of a frequency curve for which the sampling errors of optimum location are uncorrelated with those of optimum scaling. (9.)

*Consistency.*—A statistic satisfies the criterion of consistency, if, when it is calculated from the whole population, it is equal to the required parameter. (4.)

*Distribution.*—Problems of distribution are those in which it is required to calculate the distribution of one, or the simultaneous distribution of a number, of functions of quantities distributed in a known manner. (3.)

*Efficiency.*—The efficiency of a statistic is the ratio (usually expressed as a percentage) which its intrinsic accuracy bears to that of the most efficient statistic possible. It

expresses the proportion of the total available relevant information of which that statistic makes use. (4 and 10.)

*Efficiency (Criterion).*—The criterion of efficiency is satisfied by those statistics which, when derived from large samples, tend to a normal distribution with the least possible standard deviation. (4.)

*Estimation.*—Problems of estimation are those in which it is required to estimate the value of one or more of the population parameters from a random sample of the population. (3.)

*Intrinsic Accuracy.*—The intrinsic accuracy of an error curve is the weight in large samples, divided by the number in the sample, of that statistic of location which satisfies the criterion of sufficiency. (9.)

*Isostatistical Regions.*—If each sample be represented in a generalized space of which the observations are the co-ordinates, then any region throughout which any set of statistics have identical values is termed an isostatistical region.

*Likelihood.*—The likelihood that any parameter (or set of parameters) should have any assigned value (or set of values) is proportional to the probability that if this were so, the totality of observations should be that observed.

*Location.*—The location of a frequency distribution of known form and scale is the process of estimation of its position with respect to each of the several variates. (8.)

*Optimum.*—The optimum value of any parameter (or set of parameters) is that value (or set of values) of which the likelihood is greatest. (6.)

*Scaling.*—The scaling of a frequency distribution of known form is the process of estimation of the magnitudes of the deviations of each of the several variates. (8.)

*Specification.*—Problems of specification are those in which it is required to specify the mathematical form of the distribution of the hypothetical population from which a sample is to be regarded as drawn. (3.)

*Sufficiency.*—A statistic satisfies the criterion of sufficiency when no other statistic which can be calculated from the same sample provides any additional information as to the value of the parameter to be estimated. (4.)

*Validity.*—The region of validity of a statistic is the region comprised within its contour of zero efficiency. (10.)

## 1. THE NEGLECT OF THEORETICAL STATISTICS.

SEVERAL reasons have contributed to the prolonged neglect into which the study of statistics, in its theoretical aspects, has fallen. In spite of the immense amount of fruitful labour which has been expended in its practical applications, the basic principles of this organ of science are still in a state of obscurity, and it cannot be denied that, during the recent rapid development of practical methods, fundamental problems have been ignored and fundamental paradoxes left unresolved. This anomalous state of statistical science is strikingly exemplified by a recent paper (1) entitled “The Funda-

mental Problem of Practical Statistics," in which one of the most eminent of modern statisticians presents what purports to be a general proof of BAYES' postulate, a proof which, in the opinion of a second statistician of equal eminence, "seems to rest upon a very peculiar—not to say hardly supposable—relation." (2.)

Leaving aside the specific question here cited, to which we shall recur, the obscurity which envelops the theoretical bases of statistical methods may perhaps be ascribed to two considerations. In the first place, it appears to be widely thought, or rather felt, that in a subject in which all results are liable to greater or smaller errors, precise definition of ideas or concepts is, if not impossible, at least not a practical necessity. In the second place, it has happened that in statistics a purely verbal confusion has hindered the distinct formulation of statistical problems; for it is customary to apply the same name, *mean*, *standard deviation*, *correlation coefficient*, etc., both to the true value which we should like to know, but can only estimate, and to the particular value at which we happen to arrive by our methods of estimation; so also in applying the term probable error, writers sometimes would appear to suggest that the former quantity, and not merely the latter, is subject to error.

It is this last confusion, in the writer's opinion, more than any other, which has led to the survival to the present day of the fundamental paradox of inverse probability, which like an impenetrable jungle arrests progress towards precision of statistical concepts. The criticisms of BOOLE, VENN, and CHRYSTAL have done something towards banishing the method, at least from the elementary text-books of Algebra; but though we may agree wholly with CHRYSTAL that inverse probability is a mistake (perhaps the only mistake to which the mathematical world has so deeply committed itself), there yet remains the feeling that such a mistake would not have captivated the minds of LAPLACE and POISSON if there had been nothing in it but error.

## 2. THE PURPOSE OF STATISTICAL METHODS.

In order to arrive at a distinct formulation of statistical problems, it is necessary to define the task which the statistician sets himself: briefly, and in its most concrete form, the object of statistical methods is the reduction of data. A quantity of data, which usually by its mere bulk is incapable of entering the mind, is to be replaced by relatively few quantities which shall adequately represent the whole, or which, in other words, shall contain as much as possible, ideally the whole, of the relevant information contained in the original data.

This object is accomplished by constructing a hypothetical infinite population, of which the actual data are regarded as constituting a random sample. The law of distribution of this hypothetical population is specified by relatively few parameters, which are sufficient to describe it exhaustively in respect of all qualities under discussion. Any information given by the sample, which is of use in estimating the values of these parameters, is relevant information. Since the number of independent facts supplied in

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the data is usually far greater than the number of facts sought, much of the information supplied by any actual sample is irrelevant. It is the object of the statistical processes employed in the reduction of data to exclude this irrelevant information, and to isolate the whole of the relevant information contained in the data.

When we speak of the *probability* of a certain object fulfilling a certain condition, we imagine all such objects to be divided into two classes, according as they do or do not fulfil the condition. This is the only characteristic in them of which we take cognisance. For this reason probability is the most elementary of statistical concepts. It is a parameter which specifies a simple dichotomy in an infinite hypothetical population, and it represents neither more nor less than the frequency ratio which we imagine such a population to exhibit. For example, when we say that the probability of throwing a five with a die is one-sixth, we must not be taken to mean that of any six throws with that die one and one only will necessarily be a five; or that of any six million throws, exactly one million will be fives; but that of a hypothetical population of an infinite number of throws, with the die in its original condition, exactly one-sixth will be fives. Our statement will not then contain any false assumption about the actual die, as that it will not wear out with continued use, or any notion of approximation, as in estimating the probability from a finite sample, although this notion may be logically developed once the meaning of probability is apprehended.

The concept of a *discontinuous frequency distribution* is merely an extension of that of a simple dichotomy, for though the number of classes into which the population is divided may be infinite, yet the frequency in each class bears a finite ratio to that of the whole population. In *frequency curves*, however, a second infinity is introduced. No finite sample has a frequency curve: a finite sample may be represented by a histogram, or by a frequency polygon, which to the eye more and more resembles a curve, as the size of the sample is increased. To reach a true curve, not only would an infinite number of individuals have to be placed in each class, but the number of classes (arrays) into which the population is divided must be made infinite. Consequently, it should be clear that the concept of a frequency curve includes that of a hypothetical infinite population, distributed according to a mathematical law, represented by the curve. This law is specified by assigning to each element of the abscissa the corresponding element of probability. Thus, in the case of the normal distribution, the probability of an observation falling in the range  $dx$ , is

$$\frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx,$$

in which expression  $x$  is the value of the variate, while  $m$ , the mean, and  $\sigma$ , the standard deviation, are the two parameters by which the hypothetical population is specified. If a sample of  $n$  be taken from such a population, the data comprise  $n$  independent facts. The statistical process of the reduction of these data is designed to extract from them all relevant information respecting the values of  $m$  and  $\sigma$ , and to reject all other information as irrelevant.

It should be noted that there is no falsehood in interpreting any set of independent measurements as a random sample from an infinite population ; for any such set of numbers are a random sample from the totality of numbers produced by the same matrix of causal conditions : the hypothetical population which we are studying is an aspect of the totality of the effects of these conditions, of whatever nature they may be. The postulate of randomness thus resolves itself into the question, " Of what population is this a random sample ? " which must frequently be asked by every practical statistician.

It will be seen from the above examples that the process of the reduction of data is, even in the simplest cases, performed by interpreting the available observations as a sample from a hypothetical infinite population ; this is *a fortiori* the case when we have more than one variate, as when we are seeking the values of coefficients of correlation. There is one point, however, which may be briefly mentioned here in advance, as it has been the cause of some confusion. In the example of the frequency curve mentioned above, we took it for granted that the values of both the mean and the standard deviation of the population were relevant to the inquiry. This is often the case, but it sometimes happens that only one of these quantities, for example the standard deviation, is required for discussion. In the same way an infinite normal population of two correlated variates will usually require five parameters for its specification, the two means, the two standard deviations, and the correlation ; of these often only the correlation is required, or if not alone of interest, it is discussed without reference to the other four quantities. In such cases an alteration has been made in what is, and what is not, relevant, and it is not surprising that certain small corrections should appear, or not, according as the other parameters of the hypothetical surface are or are not deemed relevant. Even more clearly is this discrepancy shown when, as in the treatment of such fourfold tables as exhibit the recovery from smallpox of vaccinated and unvaccinated patients, the method of one school of statisticians treats the proportion of vaccinated as relevant, while others dismiss it as irrelevant to the inquiry. (3.)

### 3. THE PROBLEMS OF STATISTICS.

The problems which arise in reduction of data may be conveniently divided into three types :—

- (1) Problems of Specification. These arise in the choice of the mathematical form of the population.
- (2) Problems of Estimation. These involve the choice of methods of calculating from a sample statistical derivatives, or as we shall call them statistics, which are designed to estimate the values of the parameters of the hypothetical population.
- (3) Problems of Distribution. These include discussions of the distribution of statistics derived from samples, or in general any functions of quantities whose distribution is known.

It will be clear that when we know (1) what parameters are required to specify the

population from which the sample is drawn, (2) how best to calculate from the sample estimates of these parameters, and (3) the exact form of the distribution, in different samples, of our derived statistics, then the theoretical aspect of the treatment of any particular body of data has been completely elucidated.

As regards problems of specification, these are entirely a matter for the practical statistician, for those cases where the qualitative nature of the hypothetical population is known do not involve any problems of this type. In other cases we may know by experience what forms are likely to be suitable, and the adequacy of our choice may be tested *a posteriori*. We must confine ourselves to those forms which we know how to handle, or for which any tables which may be necessary have been constructed. More or less elaborate forms will be suitable according to the volume of the data. Evidently these are considerations the nature of which may change greatly during the work of a single generation. We may instance the development by PEARSON of a very extensive system of skew curves, the elaboration of a method of calculating their parameters, and the preparation of the necessary tables, a body of work which has enormously extended the power of modern statistical practice, and which has been, by pertinacity and inspiration alike, practically the work of a single man. Nor is the introduction of the Pearsonian system of frequency curves the only contribution which their author has made to the solution of problems of specification: of even greater importance is the introduction of an objective criterion of goodness of fit. For empirical as the specification of the hypothetical population may be, this empiricism is cleared of its dangers if we can apply a rigorous and objective test of the adequacy with which the proposed population represents the whole of the available facts. Once a statistic, suitable for applying such a test, has been chosen, the exact form of its distribution in random samples must be investigated, in order that we may evaluate the probability that a worse fit should be obtained from a random sample of a population of the type considered. The possibility of developing complete and self-contained tests of goodness of fit deserves very careful consideration, since therein lies our justification for the free use which is made of empirical frequency formulæ. Problems of distribution of great mathematical difficulty have to be faced in this direction.

Although problems of estimation and of distribution may be studied separately, they are intimately related in the development of statistical methods. Logically problems of distribution should have prior consideration, for the study of the random distribution of different suggested statistics, derived from samples of a given size, must guide us in the choice of which statistic it is most profitable to calculate. The fact is, however, that very little progress has been made in the study of the distribution of statistics derived from samples. In 1900 PEARSON (15) gave the exact form of the distribution of  $\chi^2$ , the Pearsonian test of goodness of fit, and in 1915 the same author published (18) a similar result of more general scope, valid when the observations are regarded as subject to linear constraints. By an easy adaptation (17) the tables of probability derived from this formula may be made available for the more numerous cases in which linear con-



straints are imposed upon the hypothetical population by the means which we employ in its reconstruction. The distribution of the mean of samples of  $n$  from a normal population has long been known, but in 1908 "Student" (4) broke new ground by calculating the distribution of the ratio which the deviation of the mean from its population value bears to the standard deviation calculated from the sample. At the same time he gave the exact form of the distribution in samples of the standard deviation. In 1915 FISHER (5) published the curve of distribution of the correlation coefficient for the standard method of calculation, and in 1921 (6) he published the corresponding series of curves for intraclass correlations. The brevity of this list is emphasised by the absence of investigation of other important statistics, such as the regression coefficients, multiple correlations, and the correlation ratio. A formula for the probable error of any statistic is, of course, a practical necessity, if that statistic is to be of service: and in the majority of cases such formulæ have been found, chiefly by the labours of PEARSON and his school, by a first approximation, which describes the distribution with sufficient accuracy if the sample is sufficiently large. Problems of distribution, other than the distribution of statistics, used to be not uncommon as examination problems in probability, and the physical importance of problems of this type may be exemplified by the chemical laws of mass action, by the statistical mechanics of GIBBS, developed by JEANS in its application to the theory of gases, by the electron theory of LORENTZ, and by PLANCK'S development of the theory of quanta, although in all these applications the methods employed have been, from the statistical point of view, relatively simple.

The discussions of theoretical statistics may be regarded as alternating between problems of estimation and problems of distribution. In the first place a method of calculating one of the population parameters is devised from common-sense considerations: we next require to know its probable error, and therefore an approximate solution of the distribution, in samples, of the statistic calculated. It may then become apparent that other statistics may be used as estimates of the same parameter. When the probable errors of these statistics are compared, it is usually found that, in large samples, one particular method of calculation gives a result less subject to random errors than those given by other methods of calculation. Attacking the problem more thoroughly, and calculating the surface of distribution of any two statistics, we may find that the whole of the relevant information contained in one is contained in the other: or, in other words, that when once we know the other, knowledge of the first gives us no further information as to the value of the parameter. Finally it may be possible to prove, as in the case of the Mean Square Error, derived from a sample of normal population (7), that a particular statistic summarises the whole of the information relevant to the corresponding parameter, which the sample contains. In such a case the problem of estimation is completely solved.

## 4. CRITERIA OF ESTIMATION.

The common-sense criterion employed in problems of estimation may be stated thus :— That when applied to the whole population the derived statistic should be equal to the parameter. This may be called the *Criterion of Consistency*. It is often the only test applied : thus, in estimating the standard deviation of a normally distributed population, from an ungrouped sample, either of the two statistics—

$$\sigma_1 = \frac{1}{n} \sqrt{\frac{\pi}{2}} S(|x - \bar{x}|) \quad (\text{Mean error})$$

and

$$\sigma_2 = \sqrt{\frac{1}{n} S(x - \bar{x})^2} \quad (\text{Mean square error})$$

will lead to the correct value,  $\sigma$ , when calculated from the whole population. They both thus satisfy the criterion of consistency, and this has led many computers to use the first formula, although the result of the second has 14 per cent. greater weight (7), and the labour of increasing the number of observations by 14 per cent. can seldom be less than that of applying the more accurate formula.

Consideration of the above example will suggest a second criterion, namely :—That in large samples, when the distributions of the statistics tend to normality, that statistic is to be chosen which has the least probable error.

This may be called the *Criterion of Efficiency*. It is evident that if for large samples one statistic has a probable error double that of a second, while both are proportional to  $n^{-\frac{1}{2}}$ , then the first method applied to a sample of  $4n$  values will be no more accurate than the second applied to a sample of any  $n$  values. If the second method makes use of the whole of the information available, the first makes use of only one-quarter of it, and its efficiency may therefore be said to be 25 per cent. To calculate the efficiency of any given method, we must therefore know the probable error of the statistic calculated by that method, and that of the most efficient statistic which could be used. The square of the ratio of these two quantities then measures the efficiency.

The criterion of efficiency is still to some extent incomplete, for different methods of calculation may tend to agreement for large samples, and yet differ for all finite samples. The complete criterion suggested by our work on the mean square error (7) is :—

That the statistic chosen should summarise the whole of the relevant information supplied by the sample.

This may be called the *Criterion of Sufficiency*.

In mathematical language we may interpret this statement by saying that if  $\theta$  be the parameter to be estimated,  $\theta_1$  a statistic which contains the whole of the information as to the value of  $\theta$ , which the sample supplies, and  $\theta_2$  any other statistic, then the



surface of distribution of pairs of values of  $\theta_1$  and  $\theta_2$ , for a given value of  $\theta$ , is such that for a given value of  $\theta_1$ , the distribution of  $\theta_2$  does not involve  $\theta$ . In other words, when  $\theta_1$  is known, knowledge of the value of  $\theta_2$  throws no further light upon the value of  $\theta$ .

It may be shown that a statistic which fulfils the criterion of sufficiency will also fulfil the criterion of efficiency, when the latter is applicable. For, if this be so, the distribution of the statistics will in large samples be normal, the standard deviations being proportional to  $n^{-\frac{1}{2}}$ . Let this distribution be

$$df = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} e^{-\frac{1}{1-r^2}\left\{\frac{\theta_1-\theta}{2\sigma_1^2} - \frac{2r\theta_1-\theta}{2\sigma_1\sigma_2} + \frac{\theta_2-\theta}{2\sigma_2^2}\right\}^2} d\theta_1 d\theta_2,$$

then the distribution of  $\theta_1$  is

$$df = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{\theta_1-\theta}{2\sigma_1^2}} d\theta_1,$$

so that for a given value of  $\theta_1$  the distribution of  $\theta_2$  is

$$df = \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left\{\frac{r\theta_1-\theta}{\sigma_1} - \frac{\theta_2-\theta}{\sigma_2}\right\}^2} d\theta_2;$$

and if this does not involve  $\theta$ , we must have

$$r\sigma_2 = \sigma_1;$$

showing that  $\sigma_1$  is necessarily less than  $\sigma_2$ , and that the efficiency of  $\theta_2$  is measured by  $r^2$ , when  $r$  is its correlation in large samples with  $\theta_1$ .

Besides this case we shall see that the criterion of sufficiency is also applicable to finite samples, and to those cases when the weight of a statistic is not proportional to the number of the sample from which it is calculated.

## 5. EXAMPLES OF THE USE OF THE CRITERION OF CONSISTENCY.

In certain cases the criterion of consistency is sufficient for the solution of problems of estimation. An example of this occurs when a fourfold table is interpreted as representing the double dichotomy of a normal surface. In this case the dichotomic ratios of the two variates, together with the correlation, completely specify the four fractions into which the population is divided. If these are equated to the four fractions into which the sample is divided, the correlation is determined uniquely.

In other cases where a small correction has to be made, the amount of the correction is not of sufficient importance to justify any great refinement in estimation, and it is sufficient to calculate the discrepancy which appears when the uncorrected method is applied to the whole population. Of this nature is SHEPPARD'S correction for grouping,

and it will illustrate this use of the criterion of consistency if we derive formulæ for this correction without approximation.

Let  $\xi$  be the value of the variate at the mid point of any group,  $a$  the interval of grouping, and  $x$  the true value of the variate at any point, then the  $k^{\text{th}}$  moment of an infinite grouped sample is

$$\sum_{p=-\infty}^{p=\infty} \int_{\xi-\frac{1}{2}a}^{\xi+\frac{1}{2}a} \xi^k f(x) dx,$$

in which  $f(x) dx$  is the frequency, in any element  $dx$ , of the ungrouped population, and

$$\xi = \left(p + \frac{\theta}{2\pi}\right)a,$$

$p$  being any integer.

Evidently the  $k^{\text{th}}$  moment is periodic in  $\theta$ , we will therefore equate it to

$$A_0 + A_1 \sin \theta + A_2 \sin 2\theta \dots \\ + B_1 \cos \theta + B_2 \cos 2\theta \dots$$

Then

$$A_0 = \frac{1}{2\pi} \sum_{p=-\infty}^{p=\infty} \int_0^{2\pi} d\theta \int_{\xi-\frac{1}{2}a}^{\xi+\frac{1}{2}a} \xi^k f(x) dx$$

$$A_s = \frac{1}{\pi} \sum_{p=-\infty}^{p=\infty} \int_0^{2\pi} \sin s\theta d\theta \int_{\xi-\frac{1}{2}a}^{\xi+\frac{1}{2}a} \xi^k f(x) dx,$$

$$B_s = \frac{1}{\pi} \sum_{p=-\infty}^{p=\infty} \int_0^{2\pi} \cos s\theta d\theta \int_{\xi-\frac{1}{2}a}^{\xi+\frac{1}{2}a} \xi^k f(x) dx.$$

But

$$\theta = \frac{2\pi}{a} \xi - 2\pi p,$$

therefore

$$d\theta = \frac{2\pi}{a} d\xi,$$

$$\sin s\theta = \sin \frac{2\pi}{a} s\xi,$$

$$\cos s\theta = \cos \frac{2\pi}{a} s\xi,$$

hence

$$A_0 = \frac{1}{a} \int_{-\infty}^{\infty} d\xi \int_{\xi-\frac{1}{2}a}^{\xi+\frac{1}{2}a} \xi^k f(x) dx = \frac{1}{a} \int_{-\infty}^{\infty} f(x) dx \int_{x-\frac{1}{2}a}^{x+\frac{1}{2}a} \xi^k d\xi.$$

Inserting the values 1, 2, 3 and 4 for  $k$ , we obtain for the aperiodic terms of the four moments of the grouped population

$$\begin{aligned} {}_1A_0 &= \int_{-\infty}^{\infty} x f(x) dx, \\ {}_2A_0 &= \int_{-\infty}^{\infty} \left(x^2 + \frac{\alpha^2}{12}\right) f(x) dx, \\ {}_3A_0 &= \int_{-\infty}^{\infty} \left(x^3 + \frac{\alpha^2 x}{4}\right) f(x) dx, \\ {}_4A_0 &= \int_{-\infty}^{\infty} \left(x^4 + \frac{\alpha^2 x^2}{2} + \frac{\alpha^4}{80}\right) f(x) dx. \end{aligned}$$

If we ignore the periodic terms, these equations lead to the ordinary SHEPPARD corrections for the second and fourth moment. The nature of the approximation involved is brought out by the periodic terms. In the absence of high contact at the ends of the curve, the contribution of these will, of course, include the terms given in a recent paper by PEARSON (8); but even with high contact it is of interest to see for what degree of coarseness of grouping the periodic terms become sensible.

Now

$$\begin{aligned} A_s &= \frac{1}{\pi} \sum_{p=-\infty}^{p=\infty} \int_0^{2\pi} \sin s\theta d\theta \int_{x-\frac{1}{2}a}^{x+\frac{1}{2}a} \xi^k f(x) dx, \\ &= \frac{2}{a} \int_{-\infty}^{\infty} \sin \frac{2\pi s\xi}{a} d\xi \int_{x-\frac{1}{2}a}^{x+\frac{1}{2}a} \xi^k f(x) dx, \\ &= \frac{2}{a} \int_{-\infty}^{\infty} f(x) dx \int_{x-\frac{1}{2}a}^{x+\frac{1}{2}a} \xi^k \sin \frac{2\pi s\xi}{a} d\xi. \end{aligned}$$

But

$$\frac{2}{a} \int_{x-\frac{1}{2}a}^{x+\frac{1}{2}a} \xi \sin \frac{2\pi s\xi}{a} d\xi = -\frac{a}{\pi s} \cos \frac{2\pi s x}{a} \cos \pi s,$$

therefore

$${}_1A_s = (-)^{s+1} \frac{a}{\pi s} \int_{-\infty}^{\infty} \cos \frac{2\pi s x}{a} f(x) dx;$$

similarly the other terms of the different moments may be calculated.

For a normal curve referred to the true mean

$${}_1A_s = (-)^{s+1} \frac{2\epsilon}{s} e^{-\frac{s^2\sigma^2}{2\epsilon^2}},$$

$${}_1B_s = 0,$$

in which

$$a = 2\pi\epsilon.$$

The error of the mean is therefore

$$-2\epsilon \left( e^{-\frac{\sigma^2}{2\epsilon^2}} \sin \theta - \frac{1}{2} e^{-\frac{4\sigma^2}{2\epsilon^2}} \sin 2\theta + \frac{1}{3} e^{-\frac{9\sigma^2}{2\epsilon^2}} \sin 3\theta - \dots \right).$$

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To illustrate a coarse grouping, take the group interval equal to the standard deviation : then

$$\epsilon = \frac{\sigma}{2\pi},$$

and the error is

$$-\frac{\sigma}{\pi} e^{-2\pi^2} \sin \theta$$

with sufficient accuracy. The standard error of the mean being  $\frac{\sigma}{\sqrt{n}}$ , we may calculate the size of the sample for which the error due to the periodic terms becomes equal to one-tenth of the standard error, by putting

$$\frac{\sigma}{10\sqrt{n}} = \frac{\sigma}{\pi} e^{-2\pi^2},$$

whence

$$n = \frac{\pi^2}{100} e^{4\pi^2} = 13,790 \text{ billion.}$$

For the second moment

$$B_s = (-)^s 4 \left( \sigma^2 + \frac{\epsilon^2}{s^2} \right) e^{-\frac{s^2 \sigma^2}{2\epsilon^2}},$$

and, if we put

$$\frac{\sqrt{2\sigma^2}}{10\sqrt{n}} = 4\sigma^2 e^{-2\pi^2},$$

there results

$$n = \frac{1}{800} e^{4\pi^2} = 175 \text{ billion.}$$

The error, while still very minute, is thus more important for the second than for the first moment.

For the third moment

$$A_s = (-)^s \frac{6\sigma^4 s}{\epsilon} \left\{ 1 + \frac{\epsilon^2}{s^2 \sigma^2} - \frac{\epsilon^4}{3s^4 \sigma^4} (\pi^2 s^2 - 6) \right\} e^{-\frac{s^2 \sigma^2}{2\epsilon^2}};$$

putting

$$\frac{\sqrt{15}\sigma^3}{10\sqrt{n}} = 12\pi\sigma^3 e^{-2\pi^2},$$

$$n = \frac{1}{960\pi^2} e^{4\pi^2} = 14.7 \text{ billion.}$$

While for the fourth moment

$$B_s = (-)^{s+1} \frac{8\sigma^6 s^2}{\epsilon^2} \left\{ 1 - (\pi^2 s^2 - 3) \frac{\epsilon^4}{s^4 \sigma^4} - (\pi^2 s^2 - 6) \frac{\epsilon^6}{s^6 \sigma^6} \right\} e^{-\frac{s^2 \sigma^2}{2\epsilon^2}},$$

so that, if we put,

$$\frac{\sqrt{96}\sigma^4}{10\sqrt{n}} = 32\pi^2 \sigma^4 e^{-2\pi^2},$$

$$n = \frac{3}{3200\pi^4} e^{4\pi^2} = 1.34 \text{ billion.}$$

In a similar manner the exact form of SHEPPARD'S correction may be found for other curves ; for the normal curve we may say that the periodic terms are exceedingly minute so long as  $a$  is less than  $\sigma$ , though they increase very rapidly if  $a$  is increased beyond this point. They are of increasing importance as higher moments are used, not only absolutely, but relatively to the increasing probable errors of the higher moments. The principle upon which the correction is based is merely to find the error when the moments are calculated from an infinite grouped sample ; the corrected moment therefore fulfils the criterion of consistency, and so long as the correction is small no greater refinement is required.

Perhaps the most extended use of the criterion of consistency has been developed by PEARSON in the "Method of Moments." In this method, which is without question of great practical utility, different forms of frequency curves are fitted by calculating as many moments of the sample as there are parameters to be evaluated. The parameters chosen are those of an infinite population of the specified type having the same moments as those calculated from the sample.

The system of curves developed by PEARSON has four variable parameters, and may be fitted by means of the first four moments. For this purpose it is necessary to confine attention to curves of which the first four moments are finite ; further, if the accuracy of the fourth moment should increase with the size of the sample, that is, if its probable error should not be infinitely great, the first eight moments must be finite. This restriction requires that the class of distribution in which this condition is not fulfilled should be set aside as "heterotypic," and that the fourth moment should become practically valueless as this class is approached. It should be made clear, however, that there is nothing anomalous about these so-called "heterotypic" distributions except the fact that the method of moments cannot be applied to them. Moreover, for that class of distribution to which the method can be applied, it has not been shown, except in the case of the normal curve, that the best values will be obtained by the method of moments. The method will, in these cases, certainly be serviceable in yielding an approximation, but to discover whether this approximation is a good or a bad one, and to improve it, if necessary, a more adequate criterion is required.

A single example will be sufficient to illustrate the practical difficulty alluded to above. If a point  $P$  lie at known (unit) distance from a straight line  $AB$ , and lines be drawn at random through  $P$ , then the distribution of the points of intersection with  $AB$  will be distributed so that the frequency in any range  $dx$  is

$$df = \frac{1}{\pi} \cdot \frac{dx}{1 + (x-m)^2},$$

in which  $x$  is the distance of the infinitesimal range  $dx$  from a fixed point  $0$  on the line, and  $m$  is the distance, from this point, of the foot of the perpendicular  $PM$ . The distri-





is the best statistic for locating the curve, is the normal or gaussian curve of errors. If the curve is not of this form the mean is not necessarily, as we have seen, of any value whatever. The determination of the true curves of variation for different types of work is therefore of great practical importance, and this can only be done by different workers recording their data in full without rejections, however they may please to treat the data so recorded. Assuredly an observer need be exposed to no criticism, if after recording data which are not probably normal in distribution, he prefers to adopt some value other than the arithmetic mean.

## 6. FORMAL SOLUTION OF PROBLEMS OF ESTIMATION.

The form in which the criterion of sufficiency has been presented is not of direct assistance in the solution of problems of estimation. For it is necessary first to know the statistic concerned and its surface of distribution, with an infinite number of other statistics, before its sufficiency can be tested. For the solution of problems of estimation we require a method which for each particular problem will lead us automatically to the statistic by which the criterion of sufficiency is satisfied. Such a method is, I believe, provided by the Method of Maximum Likelihood, although I am not satisfied as to the mathematical rigour of any proof which I can put forward to that effect. Readers of the ensuing pages are invited to form their own opinion as to the possibility of the method of the maximum likelihood leading in any case to an insufficient statistic. For my own part I should gladly have withheld publication until a rigorously complete proof could have been formulated; but the number and variety of the new results which the method discloses press for publication, and at the same time I am not insensible of the advantage which accrues to Applied Mathematics from the co-operation of the Pure Mathematician, and this co-operation is not infrequently called forth by the very imperfections of writers on Applied Mathematics.

If in any distribution involving unknown parameters  $\theta_1, \theta_2, \theta_3, \dots$ , the chance of an observation falling in the range  $dx$  be represented by

$$f(x, \theta_1, \theta_2, \dots) dx,$$

then the chance that in a sample of  $n$ ,  $n_1$  fall in the range  $dx_1$ ,  $n_2$  in the range  $dx_2$ , and so on, will be

$$\frac{n!}{\Pi (n_p!)} \Pi \{f(x_p, \theta_1, \theta_2, \dots) dx_p\}^{n_p}.$$

The method of maximum likelihood consists simply in choosing that set of values for the parameters which makes this quantity a maximum, and since in this expression the parameters are only involved in the function  $f$ , we have to make

$$S(\log f)$$

a maximum for variations of  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$ , &c. In this form the method is applicable to the fitting of populations involving any number of variates, and equally to discontinuous as to continuous distributions.

In order to make clear the distinction between this method and that of BAYES, we will apply it to the same type of problem as that which BAYES discussed, in the hope of making clear exactly of what kind is the information which a sample is capable of supplying. This question naturally first arose, not with respect to populations distributed in frequency curves and surfaces, but with respect to a population regarded as divided into two classes only, in fact in problems of *probability*. A certain proportion,  $p$ , of an infinite population is supposed to be of a certain kind, *e.g.*, "successes," the remainder are then "failures." A sample of  $n$  is taken and found to contain  $x$  successes and  $y$  failures. The chance of obtaining such a sample is evidently

$$\frac{n!}{x!y!} p^x (1-p)^y.$$

Applying the method of maximum likelihood, we have

$$S(\log f) = x \log \hat{p} + y \log (1-\hat{p})$$

whence, differentiating with respect to  $p$ , in order to make this quantity a maximum,

$$\frac{x}{p} = \frac{y}{1-\hat{p}}, \quad \text{or} \quad \hat{p} = \frac{x}{n}.$$

The question then arises as to the accuracy of this determination. This question was first discussed by BAYES (10), in a form which we may state thus. After observing this sample, when we know  $\hat{p}$ , what is the *probability* that  $p$  lies in any range  $dp$ ? In other words, what is the frequency distribution of the values of  $p$  in populations which are selected by the restriction that a sample of  $n$  taken from each of them yields  $x$  successes. Without further data, as BAYES perceived, this problem is insoluble. To render it capable of mathematical treatment, BAYES introduced the *datum*, that among the populations upon which the experiment was tried, those in which  $p$  lay in the range  $dp$  were equally frequent for all equal ranges  $dp$ . The probability that the value of  $p$  lay in any range  $dp$  was therefore assumed to be simply  $dp$ , before the sample was taken. After the selection effected by observing the sample, the probability is clearly proportional to

$$p^x (1-p)^y dp.$$

After giving this solution, based upon the particular datum stated, BAYES adds a *scholium* the purport of which would seem to be that in the absence of all knowledge save that supplied by the sample, it is reasonable to assume this particular *a priori* distribution of  $p$ . The *result*, the *datum*, and the *postulate* implied by the *scholium*, have all been somewhat loosely spoken of as BAYES' Theorem.

The postulate would, if true, be of great importance in bringing an immense variety of questions within the domain of probability. It is, however, evidently extremely arbitrary. Apart from evolving a vitally important piece of knowledge, that of the exact form of the distribution of values of  $p$ , out of an assumption of complete ignorance, it is not even a unique solution. For we might never have happened to direct our attention to the particular quantity  $p$ : we might equally have measured probability upon an entirely different scale. If, for instance,

$$\sin \theta = 2p - 1,$$

the quantity,  $\theta$ , measures the degree of probability, just as well as  $p$ , and is even, for some purposes, the more suitable variable. The chance of obtaining a sample of  $x$  successes and  $y$  failures is now

$$\frac{n!}{2^n x! y!} (1 + \sin \theta)^x (1 - \sin \theta)^y;$$

applying the method of maximum likelihood,

$$S(\log f) = x \log (1 + \sin \theta) + y \log (1 - \sin \theta) - n \log 2,$$

and differentiating with respect to  $\theta$ ,

$$\frac{x \cos \theta}{1 + \sin \theta} = \frac{y \cos \theta}{1 - \sin \theta}, \quad \text{whence} \quad \sin \theta = \frac{x - y}{2n},$$

an exactly equivalent solution to that obtained using the variable  $p$ . But what *a priori* assumption are we to make as to the distribution of  $\theta$ ? Are we to assume that  $\theta$  is equally likely to lie in all equal ranges  $d\theta$ ? In this case the *a priori* probability will be  $d\theta/\pi$ , and that after making the observations will be proportional to

$$(1 + \sin \theta)^x (1 - \sin \theta)^y d\theta.$$

But if we interpret this in terms of  $p$ , we obtain

$$p^x (1 - p)^y \frac{dp}{\sqrt{p(1 - p)}} = p^{x-\frac{1}{2}} (1 - p)^{y-\frac{1}{2}} dp,$$

a result inconsistent with that obtained previously. In fact, the distribution previously assumed for  $p$  was equivalent to assuming the special distribution for  $\theta$ ,

$$df = \frac{\cos \theta}{2} d\theta,$$

the arbitrariness of which is fully apparent when we use any variable other than  $p$ .

In a less obtrusive form the same species of arbitrary assumption underlies the method

known as that of inverse probability. Thus, if the same observed result  $A$  might be the consequence of one or other of two hypothetical conditions  $X$  and  $Y$ , it is assumed that the probabilities of  $X$  and  $Y$  are in the same ratio as the probabilities of  $A$  occurring on the two assumptions,  $X$  is true,  $Y$  is true. This amounts to assuming that before  $A$  was observed, it was known that our universe had been selected at random for an infinite population in which  $X$  was true in one half, and  $Y$  true in the other half. Clearly such an assumption is entirely arbitrary, nor has any method been put forward by which such assumptions can be made even with consistent uniqueness. There is nothing to prevent an irrelevant distinction being drawn among the hypothetical conditions represented by  $X$ , so that we have to consider two hypothetical possibilities  $X_1$  and  $X_2$ , on both of which  $A$  will occur with equal frequency. Such a distinction should make no difference whatever to our conclusions; but on the principle of inverse probability it does so, for if previously the relative probabilities were reckoned to be in the ratio  $x$  to  $y$ , they must now be reckoned  $2x$  to  $y$ . Nor has any criterion been suggested by which it is possible to separate such irrelevant distinctions from those which are relevant.

There would be no need to emphasise the baseless character of the assumptions made under the titles of inverse probability and BAYES' Theorem in view of the decisive criticism to which they have been exposed at the hands of BOOLE, VENN, and CHRYSTAL, were it not for the fact that the older writers, such as LAPLACE and POISSON, who accepted these assumptions, also laid the foundations of the modern theory of statistics, and have introduced into their discussions of this subject ideas of a similar character. I must indeed plead guilty in my original statement of the Method of the Maximum Likelihood (9) to having based my argument upon the principle of inverse probability; in the same paper, it is true, I emphasised the fact that such inverse probabilities were relative only. That is to say, that while we might speak of one value of  $p$  as having an inverse probability three times that of another value of  $p$ , we might on no account introduce the differential element  $dp$ , so as to be able to say that it was three times as probable that  $p$  should lie in one rather than the other of two equal elements. Upon consideration, therefore, I perceive that the word probability is wrongly used in such a connection: probability is a ratio of frequencies, and about the frequencies of such values we can know nothing whatever. We must return to the actual fact that one value of  $p$ , of the frequency of which we know nothing, would yield the observed result three times as frequently as would another value of  $p$ . If we need a word to characterise this relative property of different values of  $p$ , I suggest that we may speak without confusion of the *likelihood* of one value of  $p$  being thrice the likelihood of another, bearing always in mind that likelihood is not here used loosely as a synonym of probability, but simply to express the relative frequencies with which such values of the hypothetical quantity  $p$  would in fact yield the observed sample.

The solution of the problems of calculating from a sample the parameters of the hypothetical population, which we have put forward in the method of maximum likeli-

hood, consists, then, simply of choosing such values of these parameters as have the maximum likelihood. Formally, therefore, it resembles the calculation of the mode of an inverse frequency distribution. This resemblance is quite superficial: if the scale of measurement of the hypothetical quantity be altered, the mode must change its position, and can be brought to have any value, by an appropriate change of scale; but the optimum, as the position of maximum likelihood may be called, is entirely unchanged by any such transformation. Likelihood also differs from probability\* in that it is not a differential element, and is incapable of being integrated: it is assigned to a particular point of the range of variation, not to a particular element of it. There is therefore an absolute measure of probability in that the unit is chosen so as to make all the elementary probabilities add up to unity. There is no such absolute measure of likelihood. It may be convenient to assign the value unity to the maximum value, and to measure other likelihoods by comparison, but there will then be an infinite number of values whose likelihood is greater than one-half. The sum of the likelihoods of admissible values will always be infinite.

Our interpretation of BAYES' problem, then, is that the likelihood of any value of  $p$  is proportional to

$$p^x (1-p)^y,$$

and is therefore a maximum when

$$p = \frac{x}{n},$$

which is the best value obtainable from the sample; we shall term this the *optimum* value of  $p$ . Other values of  $p$  for which the likelihood is not much less cannot, however, be deemed unlikely values for the true value of  $p$ . We do not, and cannot, know, from the information supplied by a sample, anything about the probability that  $p$  should lie between any named values.

The reliance to be placed on such a result must depend upon the frequency distribution of  $x$ , in different samples from the same population. This is a perfectly objective statistical problem, of the kind we have called problems of distribution; it is, however, capable of an approximate solution, directly from the mathematical form of the likelihood.

When for large samples the distribution of any statistic,  $\theta_1$ , tends to normality, we

\* It should be remarked that likelihood, as above defined, is not only fundamentally distinct from mathematical probability, but also from the logical "probability" by which Mr. KEYNES (21) has recently attempted to develop a method of treatment of uncertain inference, applicable to those cases where we lack the statistical information necessary for the application of mathematical probability. Although, in an important class of cases, the likelihood may be held to measure the degree of our rational belief in a conclusion, in the same sense as Mr. KEYNES' "probability," yet since the latter quantity is constrained, somewhat arbitrarily, to obey the addition theorem of mathematical probability, the likelihood is a quantity which falls definitely outside its scope.

may write down the chance for a given value of the parameter  $\theta$ , that  $\theta_1$  should lie in the range  $d\theta_1$  in the form

$$\Phi = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\theta_1 - \theta)^2}{2\sigma^2}} d\theta_1.$$

The mean value of  $\theta_1$  will be the true value  $\theta$ , and the standard deviation is  $\sigma$ , the sample being assumed sufficiently large for us to disregard the dependence of  $\sigma$  upon  $\theta$ .

The likelihood of any value,  $\theta$ , is proportional to

$$e^{-\frac{(\theta_1 - \theta)^2}{2\sigma^2}},$$

this quantity having its maximum value, unity, when

$$\theta = \theta_1;$$

for

$$\frac{\partial}{\partial \theta} \log \Phi = \frac{\theta_1 - \theta}{\sigma^2}.$$

Differentiating now a second time

$$\frac{\partial^2}{\partial \theta^2} \log \Phi = -\frac{1}{\sigma^2}.$$

Now  $\Phi$  stands for the total frequency of all samples for which the chosen statistic has the value  $\theta_1$ , consequently  $\Phi = S'(\phi)$ , the summation being taken over all such examples, where  $\phi$  stands for the probability of occurrence of a certain specified sample. For which we know that

$$\log \phi = C + S(\log f),$$

the summation being taken over the individual members of the sample.

If now we expand  $\log f$  in the form

$$\log f(\theta) = \log f(\theta_1) + \overline{\theta - \theta_1} \frac{\partial}{\partial \theta} \log f(\theta_1) + \frac{\overline{\theta - \theta_1}^2}{2} \frac{\partial^2}{\partial \theta^2} \log f(\theta_1) + \dots,$$

or

$$\log f = \log f_1 + a \overline{\theta - \theta_1} + \frac{b}{2} \overline{\theta - \theta_1}^2 + \dots,$$

we have

$$\log \phi = C + \overline{\theta - \theta_1} S(a) + \frac{1}{2} \overline{\theta - \theta_1}^2 S(b) + \dots;$$

now for optimum statistics

$$S(a) = 0,$$

and for sufficiently large samples  $S(b)$  differs from  $n\bar{b}$  only by a quantity of order  $\sqrt{n}\sigma_b$ ; moreover,  $\theta - \theta_1$  being of order  $n^{-\frac{1}{2}}$ , the only terms in  $\log \phi$  which are not reduced without limit, as  $n$  is increased, are

$$\log \phi = C + \frac{1}{2} n \bar{b} \overline{\theta - \theta_1}^2;$$



hence

$$\phi \propto e^{\frac{1}{2}n \bar{b} \overline{\theta - \theta_1}^2}.$$

Now this factor is constant for all samples which have the same value of  $\theta_1$ , hence the variation of  $\Phi$  with respect to  $\theta$  is represented by the same factor, and consequently

$$\log \Phi = C' + \frac{1}{2}n \bar{b} \overline{\theta - \theta_1}^2;$$

whence

$$-\frac{1}{\sigma_{\theta_1}^2} = \frac{\partial^2}{\partial \theta^2} \log \Phi = n \bar{b},$$

where

$$b = \frac{\partial^2}{\partial \theta^2} \log f(\theta_1),$$

$\theta_1$  being the optimum value of  $\theta$ .

The formula

$$-\frac{1}{\sigma_{\theta}^2} = x \frac{\partial^2}{\partial \theta^2} \log f$$

supplies the most direct way known to me of finding the probable errors of statistics. It may be seen that the above proof applies only to statistics obtained by the method of maximum likelihood.\*

For example, to find the standard deviation of

$$\hat{p} = \frac{x}{n}$$

\* A similar method of obtaining the standard deviations and correlations of statistics derived from large samples was developed by PEARSON and FILON in 1898 (16). It is unfortunate that in this memoir no sufficient distinction is drawn between the *population* and the *sample*, in consequence of which the formulæ obtained indicate that the likelihood is always a maximum (for continuous distributions) when the *mean* of each variate in the sample is equated to the corresponding mean in the population (16, p. 232, "A<sub>r</sub> = 0"). If this were so the mean would always be a sufficient statistic for location; but as we have already seen, and will see later in more detail, this is far from being the case. The same argument, indeed, is applied to all statistics, as to which nothing but their *consistency* can be truly affirmed.

The probable errors obtained in this way are those appropriate to the method of maximum likelihood, but not in other cases to statistics obtained by the method of moments, by which method the examples given were fitted. In the 'Tables for Statisticians and Biometricians' (1914), the probable errors of the constants of the Pearsonian curves are those proper to the method of moments; no mention is there made of this change of practice, nor is the publication of 1898 referred to.

It would appear that shortly before 1898 the process which leads to the correct value, of the probable errors of *optimum* statistics, was hit upon and found to agree with the probable errors of statistics found by the method of moments for *normal* curves and surfaces; without further enquiry it would appear to have been assumed that this process was valid in all cases, its directness and simplicity being peculiarly attractive. The mistake was at that time, perhaps, a natural one; but that it should have been discovered and corrected without revealing the inefficiency of the method of moments is a very remarkable circumstance.

In 1903 the correct formulæ for the probable errors of statistics found by the method of moments are given in 'Biometrika' (19); references are there given to SHEPPARD (20), whose method is employed, as well as to PEARSON and FILON (16), although both the method and the results differ from those of the latter.

in samples from an infinite population of which the true value is  $p$ ,

$$\log f = \log p + y \log (1-p),$$

$$\frac{\partial}{\partial p} \log f = \frac{x}{p} - \frac{y}{1-p},$$

$$\frac{\partial^2}{\partial p^2} \log f = -\frac{x}{p^2} - \frac{y}{(1-p)^2}.$$

Now the mean value of  $x$  in  $pn$ , and of  $y$  is  $(1-p)n$ , hence the mean value of  $\frac{\partial^2}{\partial p^2} \log f$  is

$$-\left(\frac{1}{p} + \frac{1}{1-p}\right)n;$$

therefore

$$\sigma_p^2 = \frac{p(1-p)}{n},$$

the well-known formula for the standard error of  $p$ .

## 7. SATISFACTION OF THE CRITERION OF SUFFICIENCY.

That the criterion of sufficiency is generally satisfied by the solution obtained by the method of maximum likelihood appears from the following considerations.

If the individual values of any sample of data are regarded as co-ordinates in hyperspace, then any sample may be represented by a single point, and the frequency distribution of an infinite number of random samples is represented by a density distribution in hyperspace. If any set of statistics be chosen to be calculated from the samples, certain regions will provide identical sets of statistics; these may be called *isostatistical* regions. For any particular space element, corresponding to an actual sample, there will be a particular set of parameters for which the frequency in that element is a maximum; this will be the optimum set of parameters for that element. If now the set of statistics chosen are those which give the optimum values of the parameters, then all the elements of any part of the same isostatistical region will contain the greatest possible frequency for the same set of values of the parameters, and therefore any region which lies wholly within an isostatistical region will contain its maximum frequency for that set of values.

Now let  $\theta$  be the value of any parameter,  $\hat{\theta}$  the statistic calculated by the method of maximum likelihood, and  $\theta_1$  any other statistic designed to estimate the value of  $\theta$ , then for a sample of given size, we may take

$$f(\theta, \hat{\theta}, \theta_1) d\hat{\theta} d\theta_1$$

to represent the frequency with which  $\hat{\theta}$  and  $\theta_1$  lie in the assigned ranges  $d\hat{\theta}$  and  $d\theta_1$ .

The region  $d\hat{\theta} d\theta_1$  evidently lies wholly in the isostatistical region  $d\hat{\theta}$ . Hence the equation

$$\frac{\partial}{\partial \theta} \log f(\theta, \hat{\theta}, \theta_1) = 0$$

is satisfied, irrespective of  $\theta_1$ , by the value  $\theta = \hat{\theta}$ . This condition is satisfied if

$$f(\theta, \hat{\theta}, \theta_1) = \phi(\theta, \hat{\theta}) \cdot \phi'(\hat{\theta}, \theta_1);$$

for then

$$\frac{\partial}{\partial \theta} \log f = \frac{\partial}{\partial \theta} \log \phi,$$

and the equation for the optimum degenerates into

$$\frac{\partial}{\partial \theta} \log \phi(\theta, \hat{\theta}) = 0,$$

which does not involve  $\theta_1$ .

But the factorisation of  $f$  into factors involving  $(\theta, \hat{\theta})$  and  $(\hat{\theta}, \theta_1)$  respectively is merely a mathematical expression of the condition of sufficiency; and it appears that any statistic which fulfils the condition of sufficiency must be a solution obtained by the method of the optimum.

It may be expected, therefore, that we shall be led to a sufficient solution of problems of estimation in general by the following procedure. Write down the formula for the probability of an observation falling in the range  $dx$  in the form

$$f(\theta, x) dx,$$

where  $\theta$  is an unknown parameter. Then if

$$L = S(\log f),$$

the summation being extended over the observed sample,  $L$  differs by a constant only from the logarithm of the likelihood of any value of  $\theta$ . The most likely value,  $\hat{\theta}$ , is found by the equation

$$\frac{\partial L}{\partial \theta} = 0,$$

and the standard deviation of  $\hat{\theta}$ , by a second differentiation, from the formula

$$\frac{\partial^2 L}{\partial \theta^2} = -\frac{1}{\sigma_{\hat{\theta}}^2};$$

this latter formula being applicable only where  $\hat{\theta}$  is normally distributed, as is often the case with considerable accuracy in large samples. The value  $\sigma_{\hat{\theta}}$  so found is in these cases the least possible value for the standard deviation of a statistic designed to

estimate the same parameter ; it may therefore be applied to calculate the efficiency of any other such statistic.

When several parameters are determined simultaneously, we must equate the second differentials of  $L$ , with respect to the parameters, to the coefficients of the quadratic terms in the index of the normal expression which represents the distribution of the corresponding statistics. Thus with two parameters,

$$\frac{\partial^2 L}{\partial \theta_1^2} = -\frac{1}{1-r_{\theta_1\theta_2}^2} \cdot \frac{1}{\sigma_{\theta_1}^2}, \quad \frac{\partial^2 L}{\partial \theta_2^2} = -\frac{1}{1-r_{\theta_1\theta_2}^2} \cdot \frac{1}{\sigma_{\theta_2}^2},$$

$$\frac{\partial^2 L}{\partial \theta_1 \partial \theta_2} = +\frac{1}{1-r_{\theta_1\theta_2}^2} \cdot \frac{r}{\sigma_{\theta_1} \sigma_{\theta_2}},$$

or, in effect,  $\sigma_{\theta}^2$  is found by dividing the Hessian determinant of  $L$ , with respect to the parameters, into the corresponding minor.

The application of these methods to such a series of parameters as occur in the specification of frequency curves may best be made clear by an example.

#### 8. THE EFFICIENCY OF THE METHOD OF MOMENTS IN FITTING CURVES OF THE PEARSONIAN TYPE III.

Curves of PEARSON'S Type III. offer a good example for the calculation of the efficiency of the Method of Moments. The chance of an observation falling in the range  $dx$  is

$$df = \frac{1}{a \cdot p!} \cdot \left(\frac{x-m}{a}\right)^p e^{-\frac{x-m}{a}} dx.*$$

By the method of moments the curve is located by means of the statistic  $\mu_1$ , its dimensions are ascertained from the second moment  $\mu_2$ , and the remaining parameter  $p$  is determined from  $\beta_1$ . Considering first the problem of location, if  $a$  and  $p$  were known and we had only to determine  $m$ , we should take, according to the method of moments,

$$\mu_1 = m_\mu + a(p+1),$$

where  $m_\mu$  represents the estimate of the parameter  $m$ , obtained by using the method of moments. The variance of  $m_\mu$  is, therefore,

$$\sigma_{m_\mu}^2 = \sigma_{\mu_1}^2 = \frac{\mu_2}{n} = \frac{a^2(p+1)}{n}.$$

If, on the other hand, we aim at greater accuracy, and make the likelihood of the sample a maximum for variations of  $m$ , we have

$$L = -n \log a - n \log(p!) + pS\left(\log \frac{x-m}{a}\right) - S\left(\frac{x-m}{a}\right),$$

\* The expression,  $x!$ , is used here and throughout as equivalent to the Gaussian  $\Pi(x)$ , or to  $\Gamma(x+1)$ , whether  $x$  is an integer or not.

and the equation to determine  $m$  is

$$\frac{\partial L}{\partial m} = -pS\left(\frac{1}{x-\hat{m}}\right) + \frac{n}{a} = 0; \quad (1)$$

the accuracy of the value so obtained is found from the second differential,

$$\frac{\partial^2 L}{\partial m^2} = -pS\left(\frac{1}{x-m}\right),$$

of which the mean value is

$$-\frac{n}{a^2(p+1)},$$

whence

$$\sigma_m^2 = \frac{a^2(p+1)}{n}.$$

We now see that the efficiency of location by the method of moments is

$$\frac{p-1}{p+1} = 1 - \frac{2}{p+1}.$$

Efficiencies of over 80 per cent. for location are therefore obtained if  $p$  exceeds 9; for  $p = 1$  the efficiency of location vanishes, as in other cases where the curve makes an angle with the axis at the end of its range.

Turning now to the problem of scaling, we have, by the method of moments,

$$\mu_2 = a^2(p+1),$$

whence, knowing  $p$ ,  $a$  is obtained. Since

$$\sigma_{\mu_2}^2 = \frac{\beta_2 - 1}{n} \mu_2^2,$$

we must have

$$\sigma_{a\mu}^2 = \frac{\beta_2 - 1}{4n} a^2 = \frac{4 + 3\beta_1}{8n} a^2 = \frac{p+4}{2(p+1)n} a^2;$$

on the other hand, from the value of  $L$ , we find the equation

$$\frac{\partial L}{\partial a} = -\frac{n}{a}(p+1) + \frac{1}{a^2}S(x-m) = 0, \quad (2)$$

to be solved for  $m$  and  $a$  as a simultaneous equation with (1); whence

$$\frac{\partial^2 L}{\partial m \partial a} = -\frac{n}{a^2},$$

and

$$\frac{\partial^2 L}{\partial a^2} = \frac{n}{a^2}(p+1) - \frac{2}{a^3}S(x-m),$$

of which the mean value is

$$-\frac{n(p+1)}{a^2}.$$

The variance of  $a$ , determined from this pair of simultaneous equations, is found by dividing

$$\frac{\partial^2 L}{\partial m^2} = -\frac{n}{a^2(p-1)}$$

by the determinant

$$\begin{vmatrix} -\frac{n}{a^2(p-1)} & -\frac{n}{a^2} \\ -\frac{n}{a^2} & -\frac{n(p+1)}{a^2} \end{vmatrix}$$

which reduces to

$$-\frac{2}{p-1} \cdot \frac{n^2}{a^4},$$

whence

$$\sigma_a^2 = \frac{a^2}{2n},$$

and the efficiency of scaling by the method of moments is

$$\frac{p+1}{p+4} = 1 - \frac{3}{p+4}.$$

Efficiency of over 80 per cent. for scaling are, therefore, obtained when  $p$  exceeds 11. The efficiency of scaling does not, however, vanish for any possible value of  $p$ , though it tends to zero, as  $p$  approaches its limiting value,  $-1$ .

Lastly,  $p$  is found by the method of moments by putting

$$\frac{4}{p+1} = \beta_1.$$

Now

$$\sigma_{\beta_1}^2 = \frac{\beta_1}{n} (4\beta_4 - 24\beta_2 + 36 + 9\beta_1\beta_2 - 12\beta_3 + 35\beta_1),$$

and for curves of Type III,

$$\beta_2 = 3 + \frac{3}{2}\beta_1,$$

$$\beta_3 = 2\beta_1\beta_2 + 4\beta_1 = \beta_1(3\beta_1 + 10),$$

$$\beta_4 = \frac{5}{2}(\beta_3 + 2\beta_2) = \frac{5}{2}(3\beta_1^2 + 13\beta_1 + 6),$$

hence

$$\begin{aligned} \sigma_{\beta_1}^2 &= \frac{3\beta_1}{n} (5\beta_1 + 4)(\beta_1 + 4), \\ &= \frac{\beta_1^2}{n} \cdot \frac{6(p+2)(p+6)}{p+1}, \end{aligned}$$



whence it follows, since  $n$  is large, that

$$\sigma_{p^2}^2 = \frac{\overline{p+1}^2}{n} \cdot \frac{6(p+2)(p+6)}{p+1} = \frac{6}{n}(p+1)(p+2)(p+6).$$

From the value of  $L$ ,

$$\frac{\partial L}{\partial p} = -n \frac{d}{dp} \log(p!) + S \left( \log \frac{x-m}{a} \right),$$

which equation solved for  $m$ ,  $a$  and  $p$  as a simultaneous equation with (1) and (2), will yield the set of values for the parameters which has the maximum likelihood. To find the variance of the value of  $p$ , so obtained, observe that

$$\frac{\partial^2 L}{\partial m \partial p} = -S \left( \frac{1}{x-m} \right),$$

of which the mean value is

$$-\frac{n}{ap},$$

$$\frac{\partial^2 L}{\partial a \partial p} = -\frac{n}{a^2},$$

and

$$\frac{\partial^2 L}{\partial p^2} = -n \frac{d^2}{dp^2} \log(p!).$$

The variance of  $p$ , derived from this set of simultaneous equations, is therefore found by dividing the minor of  $\frac{\partial^2 L}{\partial p^2}$ , namely

$$\frac{2}{p-1} \cdot \frac{n^2}{a^4},$$

by the determinant

$$\frac{n^3}{a^4} \begin{vmatrix} \frac{1}{p-1} & 1 & \frac{1}{p} \\ 1 & p+1 & 1 \\ \frac{1}{p} & 1 & \frac{d^2}{dp^2} \log(p!) \end{vmatrix} = \frac{n^3}{a^4} \cdot \frac{1}{p-1} \left\{ 2 \frac{d^2}{dp^2} \log(p!) - \frac{2}{p} + \frac{1}{p^2} \right\};$$

hence

$$\sigma_p^2 = \frac{2}{n \left\{ 2 \frac{d^2}{dp^2} \log(p!) - \frac{2}{p} + \frac{1}{p^2} \right\}}.$$

When  $p$  is large,

$$2 \frac{d^2}{dp^2} \log(p!) - \frac{2}{p} + \frac{1}{p^2} = \frac{1}{3} \left( \frac{1}{p^3} - \frac{1}{5p^5} + \frac{1}{7p^7} \dots \right),$$

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so that, approximately,

$$\sigma_p^2 = \frac{6}{n} (p^3 + \frac{1}{5}p);$$

for large values of  $p$ , the efficiency of the method of moments is, therefore, approximately

$$\frac{p^3 + \frac{1}{5}p}{p+1 \cdot p+2 \cdot p+6}.$$

Efficiencies of over 80 per cent. occur when  $p$  exceeds 38.1 ( $\beta_1 = 0.102$ ); evidently the method of moments is effective for determining the form of the curve only when it is relatively close to the normal form. For small values of  $p$ , the above approximation for the efficiency is not adequate. The true values can easily be obtained from the recently published tables of the Trigamma\* function (11). The following values are obtained for the integral values of  $p$  from 0 to 5.

$p$	0	1	2	3	4	5
Efficiency . . .	0	0.0274	0.0871	0.1532	0.2159	0.2727

An interesting point which may be resolved at this stage of the enquiry is to find the variance of  $m$ , when  $a$  and  $p$  are not known, derived from the above set of simultaneous equations; that is to say, to calculate the accuracy with which the limiting point of the curve is determined; such determinations are often stated as the result of fitting curves of limited range, but their probable errors are seldom, if ever, evaluated. To obtain the greatest possible accuracy with which such a point can be determined we must divide the minor of  $\frac{\partial^2 L}{\partial m^2}$ , namely,

$$\frac{n^2}{a^2} \left\{ \overline{p+1} \frac{d^2}{dp^2} \log (p!) - 1 \right\}.$$

by

$$\frac{n^3}{a^4} \cdot \frac{1}{p-1} \left\{ 2 \frac{d^2}{dp^2} \log (p!) - \frac{2}{p} + \frac{1}{p^2} \right\},$$

whence

$$\sigma_m^2 = \frac{a^2}{n} \frac{\overline{p-1} \left\{ \overline{p+1} \frac{d^2}{dp^2} \log (p!) - 1 \right\}}{2 \frac{d^2}{dp^2} \log (p!) - \frac{2}{p} + \frac{1}{p^2}}.$$

The position of the limiting point will, when  $p$  is at all large, evidently be determined with much less accuracy than is the position, as a whole, of a curve of known form and size. Let  $n'$  be a multiplier such that the position of the extremity of a curve calculated

\* It is sometimes convenient to write  $F(x)$  for  $\frac{d^2}{dx^2} \log (x!)$ .

from  $nn'$  observations will be determined with the same accuracy as the position, as a whole, of a curve of known form and size, can be determined from a sample of  $n$  observations when  $n$  is large. Then

$$n' = \frac{\overline{p+1} \frac{d^2}{dp^2} \log(p!) - 1}{2 \frac{d^2}{dp^2} \log(p!) - \frac{2}{p} + \frac{1}{p^2}};$$

but, when  $p$  is large,

$$\overline{p+1} \frac{d^2}{dp^2} \log(p!) - 1 = \frac{1}{2p} \left( 1 - \frac{2}{3p} + \frac{1}{3p^2} \dots \right)$$

and

$$2 \frac{d^2}{dp^2} \log(p!) - \frac{2}{p} + \frac{1}{p^2} = \frac{1}{3p^3} \left( 1 - \frac{1}{5p^2} \dots \right);$$

therefore

$$\begin{aligned} n' &= \frac{3}{2} p^2 \left( 1 - \frac{2}{3p} p + \frac{8}{15p^2} \dots \right) \\ &= \frac{3}{2} p^2 - p + \frac{4}{5}. \end{aligned}$$

For large values of  $p$  the probable error of the determination of the end-point may be found approximately by multiplying the probable error of location by

$$\left(p - \frac{1}{3}\right) \sqrt{\frac{3}{2}}.$$

As  $p$  grows smaller,  $n'$  diminishes until it reaches unity, when  $p = 1$ . For values of  $p$  less than 1 it would appear that the end-point had a smaller probable error than the probable error of location, but, as a matter of fact, for these values location is determined by the end-point, and as we see from the vanishing of  $\sigma_{\hat{m}}$ , whether or not  $p$  and  $a$  are known, when  $p = 1$ , the weight of the determination from this point onwards increases more rapidly than  $n$ , as the sample increases. (See Section 10.)

The above method illustrates how it is possible to calculate the variance of any function of the population parameters as estimated from large samples; by comparing this variance with that of the same function estimated by the method of moments, we may find the efficiency of that method for any proposed function. The above examination, in which the determinations of the locus, the scale, and the form of the curve are treated separately, will serve as a general criterion of the application of the method of moments to curves of Type III. Special combinations of the parameters will, however, be of interest in special cases. It may be noted here that by virtue of equation (2) the function of  $m + a(p + 1)$  is the same, whether determined by moments or by the method of the optimum:

$$m_{\mu} + a_{\mu}(p_{\mu} + 1) = \hat{m} + \hat{a}(\hat{p} + 1).$$

The efficiency of the method of moments in determining this function is therefore 100 per cent. That this function is the abscissa of the mean does not imply 100 per cent. efficiency of location, for the centre of location of these curves is not the mean (see p. 340).

## 9. LOCATION AND SCALING OF FREQUENCY CURVES IN GENERAL.

The general problem of the location and scaling of curves may now be treated more generally. This is the problem which presents itself with respect to *error curves* of assumed form, when to find the best value of the quantity measured we must *locate* the curve as accurately as possible, and to find the probable error of the result of this process we must, as accurately as possible, estimate its *scale*.

The *form* of the curve may be specified by a function  $\phi$ , such that

$$df \propto e^{\phi(\xi)} d\xi, \text{ when } \xi = \frac{x-m}{a}.$$

In this expression  $\phi$  specifies the form of the curve, which is unaltered by variations of  $a$  and  $m$ .

When a sample of  $n$  observations has been taken, the likelihood of any combination of values of  $a$  and  $m$  is

$$L = (1-n) \log a + S(\phi),$$

whence

$$\frac{\partial L}{\partial m} = S\left(\frac{d\phi}{d\xi} \cdot \frac{d\xi}{dm}\right) = -\frac{1}{a} S(\phi'),$$

since

$$\frac{\partial \xi}{\partial m} = -\frac{1}{a};$$

also

$$\frac{\partial L}{\partial a} = -\frac{1}{a} S(\xi\phi') - \frac{n}{a},$$

since

$$\frac{\partial \xi}{\partial a} = -\frac{\xi}{a}.$$

Differentiating a second time,

$$\frac{\partial^2 L}{\partial m^2} = \frac{1}{a^2} S(\phi'');$$

therefore

$$\sigma_m^2 = -\frac{a^2}{n\phi''}.$$

This expression enables us to compare the accuracy of error curves of different form, when the location is performed in each case by the method which yields the minimum error.

*Example* :—The curve

$$df = \frac{1}{\pi} \frac{d\xi}{1+\xi^2}$$

referred to in Section 5 has an infinite standard deviation, but it is not on that account an error curve of zero accuracy, for

$$\phi = -\log(1+\xi^2), \quad \phi' = -\frac{2\xi}{1+\xi^2}, \quad \phi'' = -\frac{2(1-\xi^2)}{(1+\xi^2)^2}.$$

Now

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-\xi^2}{(1+\xi^2)^3} d\xi = \frac{1}{4},$$

hence

$$\overline{\phi''} = -\frac{1}{2} \quad \text{and} \quad \sigma_m^2 = \frac{2a^2}{n}.$$

The quantity,

$$-\frac{\overline{\phi''}}{a^2} = \frac{1}{2a^2},$$

which is the factor by which  $n$  is multiplied in calculating the weight of the estimate made from  $n$  measurements, may be called the *intrinsic accuracy* of an *error curve*. In the above example we see that errors distributed so that

$$df = \frac{a}{\pi} \frac{dx}{a^2 + x^2}$$

have the same intrinsic accuracy as errors distributed according to the normal curve

$$df = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx,$$

provided

$$\sigma^2 = 2a^2.$$

Fig. 1 illustrates two such curves of equal intrinsic accuracy.

Returning now to the general problem in which

$$L = C - n \log a + S(\phi),$$

we have

$$\frac{\partial^2 L}{\partial m \partial a} = \frac{1}{a^2} S(\phi' + \xi \phi'') = \frac{1}{a^2} S(\xi \phi'')$$

and

$$\frac{\partial^2 L}{\partial a^2} = \frac{1}{a^2} S(2\xi \phi' + \xi^2 \phi'') + \frac{n}{a^2} = \frac{1}{a^2} S(\xi^2 \phi'' - 1).$$

The latter expression will directly give the accuracy with which  $a$  is determined only if

$$\frac{\partial^2 L}{\partial m \partial a} = 0,$$

and we can always arrange that this shall be so by subtracting from  $\xi$  the quantity

$$\frac{\overline{\xi \phi''}}{\phi''}.$$

Thus in a Type III. curve where, referred to the end of the range,

$$\overline{\xi \phi''} = -1, \quad \overline{\phi''} = -\frac{1}{p-1},$$

instead of

$$\phi = p \log \xi - \bar{\xi}$$

we must write

$$\phi = p \log \overline{\xi + p - 1} - \overline{\xi + p - 1};$$

then

$$\phi' = \frac{p}{\overline{\xi + p - 1}} - 1, \quad \phi'' = -\frac{p}{\overline{\xi + p - 1}^2},$$

hence

$$\begin{aligned} \frac{\partial^2 L}{\partial a^2} &= \frac{1}{a^2} S(\xi^2 \phi'' - 1) \\ &= \frac{1}{a^2} S\left(-p + \frac{2p \overline{p - 1}}{\overline{\xi + p - 1}} - \frac{p \overline{p - 1}^2}{\overline{\xi + p - 1}^2} - 1\right), \end{aligned}$$

of which the mean value is

$$\frac{n}{a^2}(-p + 2\overline{p - 1} - \overline{p - 1} - 1) = -\frac{2n}{a^2},$$

hence

$$\sigma_a^2 = \frac{a^2}{2n}.$$

For one particular point of origin, therefore, the variations of the abscissa are uncorrelated with those of  $a$ ; this point may be termed the *centre of location*.

*Example* :—To determine the centre of location of the curve of Type IV.,

$$df \propto e^{-\nu \tan^{-1} \xi} (1 + \xi^2)^{-\frac{r+2}{2}}.$$

Here

$$\phi = -\nu \tan^{-1} \xi - \frac{r+2}{2} \log \overline{1 + \xi^2},$$

$$\phi' = -(\nu + \overline{r+2} \xi) \overline{1 + \xi^2}^{-1},$$

$$\phi'' = \overline{r+2} \overline{1 + \xi^2}^{-1} + 2(\nu \xi - \overline{r+2}) \overline{1 + \xi^2}^{-2};$$

from these we find

$$\overline{\phi''} = -\frac{\overline{r+1} \overline{r+2} \overline{r+4}}{\overline{r+4}^2 + \nu^2},$$

$$\overline{\xi \phi''} = \frac{\overline{r+1} \overline{r+2} \nu}{\overline{r+4}^2 + \nu^2},$$

so that

$$\frac{\overline{\xi \phi''}}{\overline{\phi''}} = -\frac{\nu}{\overline{r+4}}.$$

The centre of location, therefore, at the distance from the mode,

$$= -\frac{\nu a}{\overline{r+4}}.$$



*Example* :—Determine the intrinsic accuracy of an error curve of Type IV. and the efficiency of the method of moments in location and scaling.

Since

$$\overline{\phi''} = -\frac{\overline{r+1} \overline{r+2} \overline{r+4}}{\overline{r+4}^2 + \nu^2},$$

$$\sigma_{\mu}^2 = \frac{\alpha^2}{n} \cdot \frac{\overline{r+4}^2 + \nu^2}{\overline{r+1} \overline{r+2} \overline{r+4}};$$

and the intrinsic accuracy of the curve is

$$\frac{1}{\alpha^2} \frac{\overline{r+1} \overline{r+2} \overline{r+4}}{\overline{r+4}^2 + \nu^2};$$

but

$$\sigma_{m_{\mu}}^2 = \frac{\alpha^2}{n} \cdot \frac{r^2 + \nu^2}{r^2 r - 1},$$

therefore the efficiency of the method of moments in location is

$$\frac{r^2 r - 1 (\overline{r+4}^2 + \nu^2)}{\overline{r+1} \overline{r+2} \overline{r+4} (r^2 + \nu^2)}. \quad \dots \dots \dots (3)$$

When  $\nu = 0$ , we have for curves of Type VII. an efficiency of location

$$1 - \frac{6}{\overline{r+1} \overline{r+2}}.$$

The efficiency of location of these curves vanishes at  $r = 1$ , at which value the standard deviation becomes infinite. Although values down to  $-1$  give admissible frequency curves, the conventional limit at which curves are reckoned as heterotypic is at  $r = 7$ . For this value the efficiency is

$$\frac{49}{132} \cdot \frac{121 + \nu^2}{49 + \nu^2},$$

which varies from 91.67 per cent. for the symmetrical Type VII. curve, to 37.12 per cent. when  $\nu \rightarrow \infty$  and the curve to Type V.

Turning to the question of scaling, we find

$$\overline{\xi^2 \phi''} - 1 = -\frac{\overline{r+1} (2\overline{r+4} + \nu^2)}{\overline{r+4}^2 + \nu^2},$$

whence

$$\overline{\xi^2 \phi''} - 1 - \frac{\overline{\xi^2 \phi''^2}}{\overline{\phi''^2}} = -\frac{2\overline{r+1}}{\overline{r+4}},$$

and

$$\sigma_a^2 = \frac{\alpha^2}{n} \cdot \frac{r+4}{2\overline{r+1}};$$

the intrinsic accuracy of scaling is therefore independent of  $\nu$ . Now for these curves

$$\beta_2 = \frac{3 \overline{r-1}}{\overline{r-2} \overline{r-3}} \left( r + 6 - \frac{8r^2}{r^2 + \nu^2} \right),$$

so that

$$\frac{\beta_2 - 1}{4} = \frac{r^3 \overline{r-2} + \nu^2 (r^2 + 10r - 12)}{2 \overline{r-2} \overline{r-3} (r^2 + \nu^2)},$$

and

$$\sigma_{a_\mu}^2 = \frac{a^2}{n} \cdot \frac{r^3 \overline{r-2} + \nu^2 (r^2 + 10r - 12)}{2 \overline{r-2} \overline{r-3} (r^2 + \nu^2)}.$$

The efficiency of the method of moments for scaling is thus

$$\frac{\overline{r-2} \overline{r-3} \overline{r+4} (r^2 + \nu^2)}{r+1 \{ r^3 \overline{r-2} + \nu^2 (r^2 + 10r - 12) \}}; \quad \dots \quad (4)$$

when  $\nu = 0$ , we have for curves of Type VII. an efficiency of scaling

$$1 - \frac{12}{r \overline{r+1}}.$$

The efficiency of the method of moments in scaling these curves vanishes at  $r = 3$ , where  $\beta_2$  becomes infinite; for  $r = 7$ , the efficiency of scaling is

$$\frac{55}{2} \cdot \frac{49 + \nu^2}{1715 + 107\nu^2},$$

varying in value from 78.57 per cent. for the symmetrical Type VII. curve, to 25.70 per cent. when  $\nu \rightarrow \infty$  and the curve to Type V.

#### 10. THE EFFICIENCY OF THE METHOD OF MOMENTS IN FITTING THE PEARSONIAN CURVES.

The Pearsonian group of skew curves are obtained as solutions of the equation

$$\frac{1}{y} \frac{dy}{dx} = \frac{-(x-m)}{a+bx+cx^2}; \quad \dots \quad (5)$$

algebraically these fall into two main classes,

$$df \propto \left( 1 + \frac{x}{a_1} \right)^{m_1} \left( 1 - \frac{x}{a_2} \right)^{m_2} dx$$

and

$$df \propto \left( 1 + \frac{x^2}{a^2} \right)^{-\frac{r+2}{2}} e^{-\nu \tan^{-1} \frac{x}{a}} dx,$$

according as the roots of the quadratic expression in (5) are real or imaginary.

The first of these forms may be rewritten

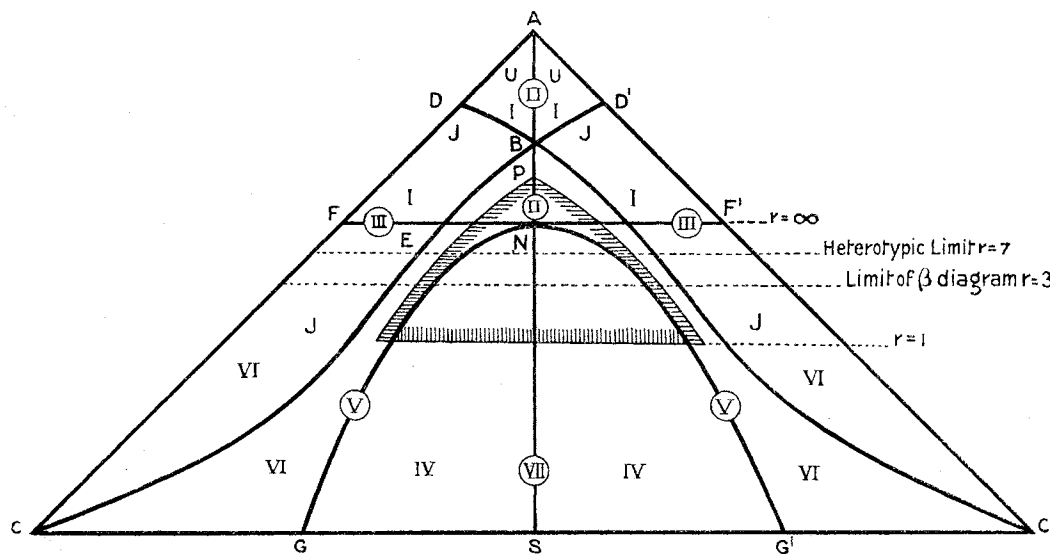
$$df \propto \left(1 - \frac{x^2}{a^2}\right)^{-\frac{r+2}{2}} e^{-\nu \tanh^{-1} \frac{x}{a}} dx,$$

$r$  being negative, showing its affinity with the second class.

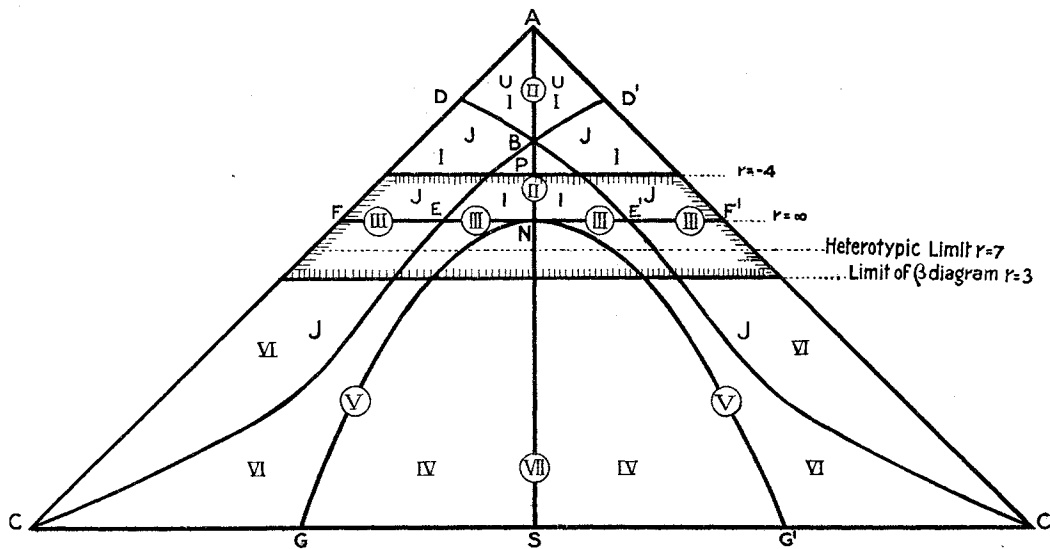
In order that these expressions may represent frequency curves, it is necessary that the integral over the whole range of the curve should be finite; this restriction acts in two ways:—

- (1) When the curve terminates at a finite value of  $x$ , say  $x = a_2$ , the power to which  $a_2 - x$  is raised must be greater than  $-1$ .
- (2) When the curve extends to infinity, the ordinate, when  $x$  is large, must diminish more rapidly than  $\frac{1}{x}$ ;

In Fig. 2 is shown a conspectus of all possible frequency curves of the Pearsonian type;



A. Showing region of validity of first moment.



B. Showing region of validity of second moment.

Fig. 2. Conspectus of Pearsonian system of frequency curves.

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the lines AC and AC' represent the limits along which the area between the curve and a vertical ordinate tends to infinity, and on which  $m_1$ , or  $m_2$ , takes the value  $-1$ ; the line CC' represents the limit at which unbounded curves enclose an infinite area with the horizontal axis; at this limit  $r = -1$ .

The symmetrical curves of Type II.

$$df \propto \left(1 - \frac{x^2}{a^2}\right)^{-\frac{r+2}{2}}$$

extend from the point N, representing the normal curve, at which  $r$  is infinite, through the point P at which  $r = -4$ , and the curve is a *parabola*, to the point B ( $r = -2$ ), where the curve takes the form of a *rectangle*; from this point the curves are U-shaped, and at A, when the arms of U are hyperbolic, we have the limiting curve of this type, which is the discontinuous distribution of *equal or unequal dichotomy* ( $r = 0$ ).

The unsymmetrical curves of Type I. are divided by PEARSON into three classes according as the terminal ordinate is infinite at neither end, at one end (J curves), or at both ends (U curves); the dividing lines are C'BD and CBD', along which one of the terminal ordinates are finite ( $m_1$ , or  $m_2 = 0$ ); at the point B, as we have seen, both terminal ordinates are finite.

The same line of division divides the curves of Type III.,

$$df \propto x^p e^{-x} dx,$$

at the point E ( $p = 0$ ), representing a simple exponential curve; the J curves of Type III. extend to F ( $p = -1$ ), at which point the integral ceases to converge. In curves of Type III.,  $r$  is infinite;  $\nu$  is also infinite, but one of the quantities  $m_1$  and  $m_2$  is finite, or zero ( $= p$ ); as  $p$  tends to infinity we approach the normal curve

$$df \propto e^{-\frac{1}{2}x^2} dx.$$

Type VI., like Type III., consists of curves bounded only at one end; here  $r$  is positive, and both  $m_1$  and  $m_2$  are finite or zero. For the J curves of Type VI. both  $m_1$  and  $m_2$  are negative, but for the remainder of these curves they are of opposite sign, the negative index being the greater by at least unity in order that the representative point may fall above CC' ( $r = -1$ ).

Type V. is here represented by a parabola separating the regions of Types IV. and VI.; the typical equation of this type of curve is

$$df \propto x^{-\frac{r+3}{2}} e^{-\frac{1}{2}x} dx.$$

As  $r$  tends to infinity the curve tends to the normal form; the integral does not become divergent until  $\frac{r+3}{2} = 1$ , or  $r = -1$ . On curves of Type V., then,  $r$  is finite or zero, but  $\nu$  is infinite.

In Type IV.

$$df \propto \left(1 + \frac{x^2}{\alpha^2}\right)^{-\frac{r+2}{2}} e^{-\nu \tan^{-1} \frac{x}{\alpha}};$$

we have written  $\nu$ , not as previously for the difference between  $m_1$  and  $m_2$ , for these quantities are now complex, and their difference is a pure imaginary, but for the difference divided by  $\sqrt{-1}$ ;  $\nu$  is then real and finite throughout Type IV., and it vanishes along the line NS, representing the symmetrical curves of Type VII.

$$df \propto \left(1 + \frac{x^2}{\alpha^2}\right)^{-\frac{r+2}{2}}$$

from  $r = \infty$  to  $r = -1$ .

The Pearsonian system of frequency curves has hitherto been represented by the diagram (13, p. 66), in which the co-ordinates are  $\beta_1$  and  $\beta_2$ . This is an unsymmetrical diagram which, since  $\beta_1$  is necessarily positive, places the symmetrical curves on a boundary, whereas they are the central types from which the unsymmetrical curves diverge on either hand; further, neither of the limiting conditions of these curves can be shown on the  $\beta$  diagram; the limit of the U curves is left obscure,\* and the other limits are either projected to infinity, or, what is still more troublesome, the line at infinity cuts across the diagram, as occurs along the line  $r = 3$ , for there  $\beta_2$  becomes infinite. This diagram thus excludes all curves of Types VII., IV., V., and VI., for which  $r < 3$ .

In the  $\beta$  diagram the condition  $r = \text{constant}$  yields a system of concurrent straight lines. The basis of the representation in fig. 2 lies in making these lines parallel and horizontal, so that the ordinate is a function of  $r$  only. We have chosen  $r = y - \frac{1}{y}$ , and have represented the limiting types by the simplest geometrical forms, straight lines and parabolas, by taking

$$4\epsilon = r^2 + \nu^2 = \frac{(\overline{1+y^2+x^2})(\overline{1-y^2-x^2})}{y(x^2+y)}.$$

It might have been thought that use could have been made of the criterion,

$$\kappa_2 = \frac{\beta_1(\beta_2+3)^2}{4(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)} = 1 - \frac{r^2}{4\epsilon},$$

by which PEARSON distinguishes these curves; but this criterion is only valid in the region treated by PEARSON. For when  $r = 0$ ,  $\kappa_2 = 1$ , and we should have to place a variety of curves of Types VII., IV., V., and VI., all in Type V. in order to adhere to the criterion.

This diagram gives, I believe, the simplest possible *conspectus* of the whole of the Pearsonian system of curves; the inclusion of the curves beyond  $r = 3$  becomes neces-

\* The true limit is the line  $\beta_2 = \beta_1 + 1$ , along which the curves degenerate into simple dichotomies.



In fig. 3 are shown the contours along which the efficiency is 20, 40, 60, and 80 per cent. For high efficiencies these contours tend to the system of ellipses,

$$8x^2 + 6y^2 = 1 - E.$$

In a similar manner, we have obtained in expression (4) the efficiency of the second moment in fitting Pearsonian curves. The region of validity in this case is shown in fig. 4; this region is bounded by the lines  $r = 3$ ,  $r = -4$ , and by the limits

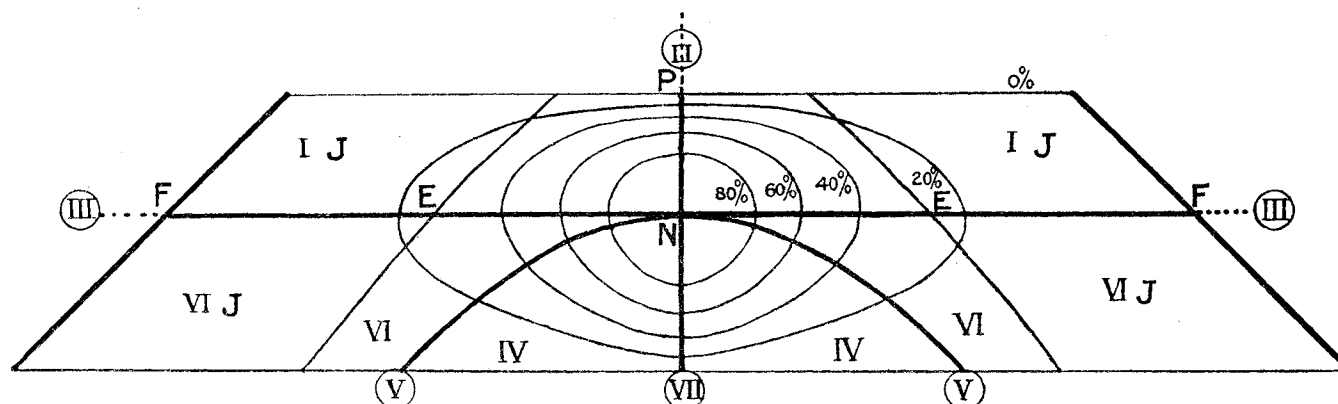


Fig. 4. Region of validity of the second moment (standard deviation) applied in scaling of Pearsonian curves, showing contours of efficiency.

( $m_1$ , or  $m_2$ ,  $= -1$ ) on which  $r^2 + v^2$  vanishes. This statistic is therefore valid for certain J curves, though the maximum efficiency among the J curves is about 30 per cent. As before, the contours are centred about the normal curve (N) and for high efficiencies tend to the system of concentric circles,

$$12x^2 + 12y^2 = 1 - E,$$

showing that the region of high efficiency is somewhat more restricted for the second moment, as compared to the first.

The lower boundary to the efficiencies of these statistics is due merely to their probable errors becoming infinite, a weakness of the method of moments which has been partially recognised by the exclusion of the so-called heterotypic curves ( $r < 7$ ). The stringency of the upper boundary is much more unexpected; the probable errors of the moments do not here become infinite; only the ratio of the probable errors of the moments to the probable error of the corresponding optimum statistics is great and tends to infinity as the size of the sample is increased.

That this failure as regards location occurs when the curve makes a finite angle with the axis may be seen by considering the occurrence of observations near the terminus of the curve.

Let

$$df = kx^a dx$$



in the neighbourhood of the terminus, then the chance of an observation falling within a distance  $x$  of the terminus is

$$f = \frac{k}{\alpha + 1} x^{\alpha+1} = k' x^{\alpha+1},$$

and the chance of  $n$  observations all failing to fall in this region is

$$(1-f)^n$$

or, when  $n$  is great, and  $f$  correspondingly small,

$$e^{-fn}.$$

Equating this to any finite probability,  $e^{-a}$ , we have

$$k' x^{\alpha+1} = \frac{a}{n},$$

or, in other words, if we use the extreme observation as a means of locating the terminus, the error,  $x$ , is proportional to

$$n^{-\frac{1}{\alpha+1}};$$

when  $\alpha < 1$ , this quantity diminishes more rapidly than  $n^{-\frac{1}{2}}$ , and consequently for large samples it is much more accurate to locate the curve by the extreme observation than by the mean.

Since it might be doubted whether such a simple method could really be more accurate than the process of finding the actual mean, we will take as example the location of the curve (B) in the form of a rectangle,

$$df = \frac{dx}{a}, \quad m - \frac{a}{2} < x < m + \frac{a}{2},$$

and

$$df = 0,$$

outside these limits.

This is one of the simplest types of distribution, and we may readily obtain examples of it from mathematical tables. The mean of the distribution is  $m$ , and the standard deviation  $\frac{a}{\sqrt{12}}$ , the error  $m_\mu - m$ , of the mean obtained from  $n$  observations, when  $n$  is reasonably large, is therefore distributed according to the formula

$$\frac{1}{a} \sqrt{\frac{6n}{\pi}} e^{-\frac{6nx^2}{a^2}} dx.$$

The difference of the extreme observation from the end of the range is distributed according to the formula

$$\frac{n}{a} e^{-\frac{n\xi}{a}} d\xi;$$

if  $\xi$  is the difference at one end of the range and  $\eta$  the difference at the other end, the joint distribution (since, when  $n$  is considerable, these two quantities may be regarded as independent) is

$$\frac{n^2}{a^2} e^{-\frac{n}{a}(\xi+\eta)} d\xi d\eta.$$

Now if we take the mean of the extreme observations of the sample, our error is

$$\frac{1}{2} \overline{\xi - \eta},$$

for which we write  $x$ ; writing also  $y$  for  $\xi + \eta$ , we have the joint distribution of  $x$  and  $y$ ,

$$\frac{n^2}{a^2} e^{-\frac{n}{a}y} dx dy.$$

For a given value of  $x$  the values of  $y$  range from  $2|x|$  to  $\infty$ , whence, integrating with respect to  $y$ , we find the distribution of  $x$  to be

$$df = \frac{n}{a} e^{-\frac{2n}{a}|x|} dx,$$

the double exponential curve shown in fig. 5.

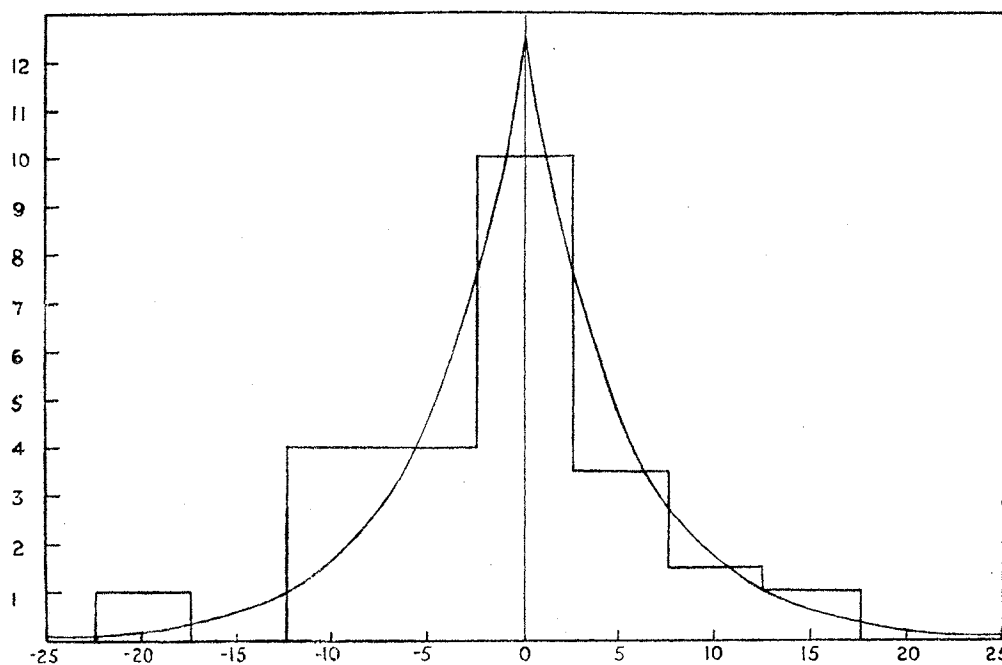


Fig. 5. Double exponential frequency curve, showing distribution of 25 deviations.

The two error curves are thus of a radically different form, and strictly no value for the efficiency can be calculated; if, however, we consider the ratio of the two standard deviations, then

$$\frac{\sigma_m^2}{\sigma_{m_\mu}^2} = \frac{a^2}{2n^2} \div \frac{1}{12n} = \frac{6}{n}$$

when  $n$  is large, a quantity which diminishes indefinitely as the sample is increased.

For example, we have taken from VEGA (14) sets of digits from the table of Natural Logarithms to 48 places of decimals. The last block of four digits was taken from the logarithms of 100 consecutive numbers from 101 to 200, giving a sample of 100 numbers distributed evenly over a limited range. It is sufficient to take the three first digits to the nearest integer; then each number has an equal chance of all values between 0 and 1000. The true mean of the population is 500, and the standard deviation 289. The standard error of the mean of a sample of 100 is therefore 28.9.

Twenty-five such samples were taken, using the last five blocks of digits, for the logarithms of numbers from 101 to 600, and the mean determined merely from the highest and lowest number occurring, the following values were obtained :—

Digits.	1st hundred.			2nd hundred.			3rd hundred.			4th hundred.			5th hundred.		
	Lowest.	Highest.	$\hat{m}-m$ .	Lowest.	Highest.	$\hat{m}-m$ .	Lowest.	Highest.	$\hat{m}-m$ .	Lowest.	Highest.	$\hat{m}-m$ .	Lowest.	Highest.	$\hat{m}-m$ .
45-48	24	978	+ 1.0	39	980	+ 9.5	1	999	0	16	983	- 0.5	18	994	+6.0
41-44	35.5	993	+14.0	3	960	-18.5	6	997	+1.5	1	978	-10.5	4	979	-8.5
37-40	9	988	- 1.5	11	999	+ 5.0	31	984	+7.5	4	978	- 9.0	2	986	-6.0
33-36	7	995	+ 1.0	13	997	+ 5.0	4	998	+1.0	0	994	- 3.0	3	981	-8.0
29-32	1	988	- 5.5	3	988	- 4.5	4	992	-2.0	1	996	- 1.5	21	977	-1.0

It will be seen that these errors rarely exceed one-half of the standard error of the mean of the sample. The actual mean square error of these 25 values is 6.86, while the calculated value,  $\sqrt{50}$ , is 7.07. It will therefore be seen that, with samples of only 100, there is no exaggeration in placing the efficiency of the method of moments as low as 6 per cent. in comparison with the more accurate method, which in this case happens to be far less laborious.

Such a value for the efficiency of the mean in this case is, however, purely conventional, since the curve of distribution is outside the region of its valid application, and the two curves of sampling do not tend to assume the same form. It is, however, convenient to have an estimate of the effectiveness of statistics for small samples, and in such cases we should prefer to treat the curve of distribution of the statistic as an error curve, and to judge the effectiveness of the statistic by the *intrinsic accuracy* of the curve as defined in Section 9. Thus the intrinsic accuracy of the curve of distribution of the mean of all the observations is

$$\frac{12n}{\sigma^2},$$

while that of the mean of the extreme values is

$$\frac{4n^2}{\alpha^2},$$

so yielding a ratio  $3/n$ . It is probable that this quantity may prove a suitable substitute for the efficiency of a statistic for curves beyond its region of validity.

To determine the efficiency of the moment coefficients  $\beta_1$  and  $\beta_2$  in determining the form of a Pearsonian curve, we must in general apply the method of Section 8 to the calculation of the simultaneous distribution of the four parameters of those curves when estimated by the method of maximum likelihood. Expressing the curve by the formula appropriate to Type IV., we are led to the determinant

$$\begin{vmatrix} \frac{r+1}{\alpha^2} \frac{r+2}{(r+4^2+\nu^2)} & -\frac{r+1}{\alpha^2} \frac{r+2}{(r+4^2+\nu^2)} \nu & \frac{r+1}{\alpha} \frac{r+2}{(r+2^2+\nu^2)} & -\frac{r+1}{\alpha} \frac{\nu}{(r+2^2+\nu^2)} \\ -\frac{r+1}{\alpha^2} \frac{r+2}{(r+4^2+\nu^2)} \nu & \frac{r+1}{\alpha^2} \frac{(2r+4+\nu^2)}{(r+4^2+\nu^2)} & -\frac{r+1}{\alpha} \frac{\nu}{(r+2^2+\nu^2)} & \frac{r+2+\nu^2}{\alpha (r+2^2+\nu^2)} \\ \frac{r+1}{\alpha} \frac{r+2}{(r+2^2+\nu^2)} & -\frac{r+1}{\alpha} \frac{\nu}{(r+2^2+\nu^2)} & \frac{\partial^2}{\partial \nu^2} \log F & \frac{\partial^2}{\partial \nu \partial r} \log F \\ -\frac{r+1}{\alpha} \frac{\nu}{(r+2^2+\nu^2)} & \frac{r+2+\nu^2}{\alpha (r+2^2+\nu^2)} & \frac{\partial^2}{\partial \nu \partial r} \log F & \frac{\partial^2}{\partial r^2} \log F \end{vmatrix}$$

as the Hessian of  $-L$ , when

$$F = e^{-\frac{1}{2}\nu\pi} \int_0^\pi e^{\nu\theta} \sin^r \theta d\theta.$$

The ratios of the minors of this determinant to the value of the determinant give the standard deviations and correlations of the optimum values of the four parameters obtained from a number of large samples.

In discussing the efficiency of the method of moments in respect of the *form* of the curve, it is doubtful if it be possible to isolate in a unique and natural manner, as we have done in respect of *location* and *scaling*, a series of parameters which shall successively represent different aspects of the process of curve fitting. Thus we might find the efficiencies with which  $r$  and  $\nu$  are determined by the method of moments, or those of the parametric functions corresponding to  $\beta_1$  and  $\beta_2$ , or we might use  $m_1$  and  $m_2$  as independent parameters of form; but in all these cases we should be employing an arbitrary pair of measures to indicate the relative magnitude of corresponding contour ellipses of the two frequency surfaces.

For the symmetrical series of curves, the Types II. and VII., the two systems of

ellipses are coaxial, the deviations of  $r$  and  $\nu$  being uncorrelated ; in the case of Type VII. we put  $\nu = 0$ , in the determinant given above, which then becomes

$$\begin{vmatrix} \frac{\overline{r+1} \overline{r+2}}{r+4} & 0 & \frac{r+1}{r+2} & 0 \\ 0 & \frac{2\overline{r+1}}{r+4} & 0 & \frac{1}{r+2} \\ \frac{r+1}{r+2} & 0 & \frac{1}{2} \mathbb{F}\left(\frac{r}{2}\right) & 0 \\ 0 & \frac{1}{r+2} & 0 & \frac{1}{4} \left\{ \mathbb{F}\left(\frac{r-1}{2}\right) - \mathbb{F}\left(\frac{r}{2}\right) \right\} \end{vmatrix}$$

and falls in the two factors

$$\left[ \frac{r+1}{2 \overline{r+4}} \left\{ \mathbb{F}\left(\frac{r-1}{2}\right) - \mathbb{F}\left(\frac{r}{2}\right) \right\} - \frac{1}{\overline{r-2^2}} \right] \left[ \frac{\overline{r+1} \overline{r+2}}{2 \overline{r+4}} \mathbb{F}\left(\frac{r}{2}\right) - \frac{\overline{r+1^2}}{\overline{r+2^2}} \right],$$

so that

$$n\sigma_{\epsilon}^2 = \frac{2 \overline{r+2^3}}{\overline{r+2^3} \mathbb{F}\left(\frac{r}{2}\right) - 2 \overline{r+1} \overline{r+4}},$$

and

$$n\sigma_{\nu}^2 = \frac{4 \overline{r+1} \overline{r+2^2}}{\overline{r+1} \overline{r+2^2} \left\{ \mathbb{F}\left(\frac{r-1}{2}\right) - \mathbb{F}\left(\frac{r}{2}\right) \right\} - 2 \overline{r+4}}.$$

The corresponding expressions for the method of moments are

$$n\sigma_{\nu_{\mu}}^2 = \frac{3}{8} \cdot \frac{r^2 \overline{r-2^2} (r^2 + r + 10)}{r-1 \overline{r-3} \overline{r-5}},$$

and

$$n\sigma_{r_{\mu}}^2 = \frac{2}{3} \cdot \frac{r \overline{r-1^2} \overline{r-3} (r^2 - r + 18)}{r-5 \overline{r-7}}.$$

Since for moderately large values of  $r$ , we have, approximately,

$$\overline{r+2^3} \mathbb{F}\left(\frac{r}{2}\right) - 2 \overline{r+1} \overline{r+4} = \frac{16}{3} \left( 1 - \frac{1}{5 \overline{r+2^2}} \right),$$

and

$$\overline{r+1} \overline{r+2^2} \left\{ \mathbb{F}\left(\frac{r-1}{2}\right) - \mathbb{F}\left(\frac{r}{2}\right) \right\} - 2 \overline{r+1} \overline{r+4} = 6 - \frac{4}{\overline{r+2^2}};$$

we have, approximately, for the efficiency of  $\nu_\mu$ ,

$$\frac{(\overline{r+2^3} + \frac{1}{5} \overline{r+2} \dots) \overline{r-1} \overline{r-3} \overline{r-5}}{(\overline{r^2+r+10}) \overline{r^2} \overline{r-2}^2},$$

or, when  $r$  is great,

$$1 - \frac{28.8}{r^2};$$

and for the efficiency of  $r_\mu$ ,

$$\frac{(\overline{r+2^2} + \frac{2}{3} \dots) \overline{r+1}^2 \overline{r-5} \overline{r-7}}{(\overline{r^2-r+18}) \overline{r} \overline{r-1}^2 \overline{r-3}};$$

or, when  $r$  is great,

$$1 - \frac{53.3}{r^2}.$$

The following table gives the values of the transcendental quantities required, and the efficiency of the method of moments in estimating the value of  $\nu$  and  $r$  from samples drawn from Type VII. distribution.

$r$ .	$\overline{r+2^3} \mathbb{F}\left(\frac{r}{2}\right) - 2\overline{r+1} \overline{r+4}.$	Efficiency of $\nu_\mu$ .	$\overline{r+1}^2 \overline{r+2}^2 \times \left\{ \mathbb{F}\left(\frac{r-1}{2}\right) - \mathbb{F}\left(\frac{r}{2}\right) \right\} - 2\overline{r+1} \overline{r+4}.$	Efficiency of $r_\mu$ .
5	5.31271	0		
6	5.31736	0.2572		
7	5.32060	0.4338		
8	5.32296	0.5569	5.9473	0
9	5.32472	0.6449	5.9574	0.1687
10	5.32607	0.7097	5.9649	0.3130
11	5.32607	0.7097	5.9706	0.4403
12	5.32713	0.7586	5.9706	0.5207
13	5.32797	0.7963	5.9750	0.5935
14	5.32866	0.8259	5.9787	0.6519
15	5.32919	0.8497	5.9810	0.6990
16			5.9839	0.7376
17			5.9853	0.7694
18			5.9870	0.7959
			5.9883	0.8182
			5.9895	

It will be seen that we do not attain to 80 per cent. efficiency in estimating the form of the curve until  $r$  is about 17.2, which corresponds to  $\beta_2 = 3.42$ . Even for symmetrical curves higher values of  $\beta_2$  imply that the method of moments makes use of less than four-fifths of the information supplied by the sample.

On the other side of the normal point, among the Type II. curves, very similar formulæ apply. The fundamental Hessian is

$$\begin{vmatrix} \frac{\overline{r-1} \overline{r-2}}{\overline{r-4}} & 0 & -\frac{\overline{r-1}}{\overline{r-2}} & 0 \\ 0 & \frac{2 \overline{r-1}}{\overline{r-4}} & 0 & -\frac{1}{\overline{r-2}} \\ -\frac{\overline{r-1}}{\overline{r-2}} & 0 & \frac{1}{2} \mathbb{F}\left(\frac{\overline{r-2}}{2}\right) & 0 \\ 0 & -\frac{1}{\overline{r-2}} & 0 & \frac{1}{4} \left\{ \mathbb{F}\left(\frac{\overline{r-2}}{2}\right) - \mathbb{F}\left(\frac{\overline{r-1}}{2}\right) \right\} \end{vmatrix}$$

where  $r$  is written for the positive quantity,  $-r$ , whence

$$n\sigma_{\theta}^2 = \frac{2 \overline{r-2}^3}{\overline{r-2}^3 \mathbb{F}\left(\frac{\overline{r-2}}{2}\right) - 2 \overline{r-1} \overline{r-4}}$$

and

$$n\sigma_{\varphi}^2 = \frac{4 \overline{r-1} \overline{r-2}^2}{\overline{r-1} \overline{r-2}^2 \left\{ \mathbb{F}\left(\frac{\overline{r-2}}{2}\right) - \mathbb{F}\left(\frac{\overline{r-1}}{2}\right) \right\} - 2 \overline{r-4}}$$

Now since

$$\mathbb{F}\left(\frac{\overline{r-2}}{2}\right) = \mathbb{F}\left(\frac{\overline{r-4}}{2}\right) - \frac{4}{\overline{r-2}^3},$$

it follows that

$$\overline{r-2}^3 \mathbb{F}\left(\frac{\overline{r-2}}{2}\right) - 2 \overline{r-1} \overline{r-4} = \overline{r-2}^3 \mathbb{F}\left(\frac{\overline{r-4}}{2}\right) - 2 \overline{r-3},$$

which is the same function of  $r-4$  as

$$\overline{r+2}^3 \mathbb{F}\left(\frac{r}{2}\right) - 2 \overline{r+1} \overline{r+4}$$

is of  $r$ .

In a similar manner

$$\begin{aligned} \overline{r+1}^2 \overline{r-2}^2 \left\{ \mathbb{F}\left(\frac{\overline{r-2}}{2}\right) - \mathbb{F}\left(\frac{\overline{r-1}}{2}\right) \right\} - 2 \overline{r-1} \overline{r-4} \\ = \overline{r-1}^2 \overline{r-2}^2 \left\{ \mathbb{F}\left(\frac{\overline{r-4}}{2}\right) - \mathbb{F}\left(\frac{\overline{r-3}}{2}\right) \right\} - 2 \overline{r-2} \overline{r+1}, \end{aligned}$$

which is the same function of  $r-3$  as

$$\overline{r+1}^2 \overline{r+2}^2 \left\{ \mathbb{F}\left(\frac{\overline{r-1}}{2}\right) - \mathbb{F}\left(\frac{r}{2}\right) \right\} - 2 \overline{r+1} \overline{r+4}$$

is of  $r$ .



In all these functions and those of the following table,  $r$  must be substituted as a positive quantity, although it must not be forgotten that  $r$  changes sign as we pass from Type VII. to Type II., and we have hitherto adhered to the convention that  $r$  is to be taken positive for Type VII. and negative for Type II.

$r.$	$\frac{r-2}{-2r-1} \mathbb{F}\left(\frac{r-2}{2}\right)$ $-2r-1$	Efficiency of $v_{\mu}$ .	$\frac{r-1}{-2r-1} \mathbb{F}\left(\frac{r-1}{2}\right)$ $-2r-1$	Efficiency of $r_{\mu}$ .
2	4	0	4	0
3	4.93480	0.0576	5.1595	0.0431
4	5.15947	0.2056	5.5648	0.1445
5	5.23966	0.3590	5.7410	0.2613
6	5.27578	0.4865	5.8305	0.3708
7	5.29472	0.5857	5.8813	0.4653
8	5.30576	0.6615	5.9126	0.5441
9	5.31271	0.7198	5.9331	0.6090
10	5.31736	0.7650	5.9473	0.6624
11	5.32060	0.8005	5.9574	0.7063
12	5.32296	0.8287	5.9649	0.7427
13	5.32472	0.8516	5.9706	0.7731
14	5.32607	0.8702	5.9750	0.7986
15			5.9787	0.8202

In both cases the region of validity is bounded by the rectangle, at the point B (fig. 2, p. 343). Efficiency of 80 per cent. is reached when  $r$  is about 14.1 ( $\beta_2 = 2.65$ ). Thus for symmetrical curves of the Pearsonian type we may say that the method of moments has an efficiency of 80 per cent. or more, when  $\beta_2$  lies between 2.65 and 3.42. The limits within which the values of the parameters obtained by moments cannot be greatly improved are thus much narrower than has been imagined.

#### 11. THE REASON FOR THE EFFICIENCY OF THE METHOD OF MOMENTS IN A SMALL REGION SURROUNDING THE NORMAL CURVE.

We have seen that the method of moments applied in fitting Pearsonian curves has an efficiency exceeding 80 per cent. only in the restricted region for which  $\beta_2$  lies between the limits 2.65 and 3.42, and as we have seen in Section 8, for which  $\beta_1$  does not exceed 0.1. The contours of equal efficiency are nearly circular or elliptical within these limits, if the curves are represented as in fig. 2, p. 343, and are ultimately centred round the normal point, at which point the efficiencies of all parameters tend to 100 per cent. It was, of course, to be expected that the first two moments would have 100 per cent. efficiencies at this point, for they happen to be the optimum statistics for fitting the normal curve. That the moment coefficients  $\beta_1$  and  $\beta_2$  also tend to 100 per cent. efficiency in this region suggests that in the immediate neighbourhood of the normal

curve the departures from normality specified by the Pearsonian formula agree with those of that system of curves for which the method of moments gives the solution of maximum likelihood.

The system of curves for which the method of moments is the best method of fitting may easily be deduced, for if the frequency in the range  $dx$  be

$$y(x, \theta_1, \theta_2, \theta_3, \theta_4) dx,$$

then

$$\frac{\partial}{\partial \theta} \log y$$

must involve  $x$  only as polynomials up to the fourth degree; consequently

$$y = e^{-a^2(x^4 + p_1x^3 + p_2x^2 + p_3x + p_4)},$$

the convergence of the probability integral requiring that the coefficient of  $x^4$  should be negative, and the five quantities  $a, p_1, p_2, p_3, p_4$  being connected by a single relation, representing the fact that the total probability is unity.

Typically these curves are bimodal, and except in the neighbourhood of the normal point are of a very different character from the Pearsonian curves. Near this point, however, they may be shown to agree with the Pearsonian type; for let

$$y = Ce^{-\frac{x^2}{2\sigma^2} + k_1\frac{x^3}{\sigma^3} + k_2\frac{x^4}{\sigma^4}}$$

represent a curve of the quartic exponent, sufficiently near to the normal curve for the squares of  $k_1$  and  $k_2$  to be neglected, then

$$\begin{aligned} \frac{d}{dx} \log y &= -\frac{x}{\sigma^2} \left( 1 - 3k_1 \frac{x}{\sigma} - 4k_2 \frac{x^2}{\sigma^2} \right) \\ &= -\frac{x}{\sigma^2 \left( 1 + 3k_1 \frac{x}{\sigma} + 4k_2 \frac{x^2}{\sigma^2} \right)}, \end{aligned}$$

neglecting powers of  $k_1$  and  $k_2$ . Since the only terms in the denominator constitute a quadratic in  $x$ , the curve satisfies the fundamental equation of the Pearsonian type of curves. In the neighbourhood of the normal point, therefore, the Pearsonian curves are equivalent to curves of the quartic exponent; it is to this that the efficiency of  $\mu_3$  and  $\mu_4$ , in the neighbourhood of the normal curve, is to be ascribed.

## 12. DISCONTINUOUS DISTRIBUTIONS.

The applications hitherto made of the optimum statistics have been problems in which the data are ungrouped, or at least in which the grouping intervals are so small as not to disturb the values of the derived statistics. By grouping, these continuous

distributions are reduced to discontinuous distributions, and in an exact discussion must be treated as such.

If  $p_s$  be the probability of an observation falling in the cell ( $s$ ),  $p_s$  being a function of the required parameters  $\theta_1, \theta_2 \dots$ ; and in a sample of  $N$ , if  $n_s$  are found to fall into that cell, then

$$S(\log f) = S(n_s \log p_s).$$

If now we write  $\bar{n}_s = p_s N$ , we may conveniently put

$$L = S\left(n_s \log \frac{n_s}{\bar{n}_s}\right),$$

where  $L$  differs by a constant only from the logarithm of the likelihood, with sign reversed, and therefore the method of the optimum will consist in finding the *minimum* value of  $L$ . The equations so found are of the form

$$\frac{\partial L}{\partial \theta} = -S\left(\frac{n_s}{\bar{n}_s} \frac{\partial \bar{n}_s}{\partial \theta}\right) = 0. \quad \dots \quad (6)$$

It is of interest to compare these formulæ with those obtained by making the Pearsonian  $\chi^2$  a minimum.

For

$$\chi^2 = S \frac{(n_s - \bar{n}_s)^2}{n_s},$$

and therefore

$$1 + \chi^2 = S\left(\frac{n_s^2}{\bar{n}_s}\right),$$

so that on differentiating by  $d\theta$ , the condition that  $\chi^2$  should be a minimum for variations of  $\theta$  is

$$-S\left(\frac{n_s^2}{\bar{n}_s^2} \frac{\partial \bar{n}_s}{\partial \theta}\right) = 0. \quad \dots \quad (7)$$

Equation (7) has actually been used (12) to “improve” the values obtained by the method of moments, even in cases of normal distribution, and the Poisson series, where the method of moments gives a strictly sufficient solution. The discrepancy between these two methods arises from the fact that  $\chi^2$  is itself an approximation, applicable only when  $\bar{n}_s$  and  $n_s$  are large, and the difference between them of a lower order of magnitude. In such cases

$$L = S\left(n_s \log \frac{n_s}{\bar{n}_s}\right) = S\left(\overline{m+x} \log \frac{m+x}{m}\right) = S\left\{x + \frac{x^2}{2m} - \frac{x^3}{6m^2} \dots\right\},$$

and since

$$S(x) = 0,$$

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we have, when  $x$  is in all cases small compared to  $m$ ,

$$L = \frac{1}{2} S \left( \frac{x^2}{m} \right) = \frac{1}{2} \chi^2$$

as a first approximation. In those cases, therefore, when  $\chi^2$  is a valid measure of the departure of the sample from expectation, it is equal to  $2L$ ; in other cases the approximation fails and  $L$  itself must be used.

The failure of equation (7) in the general problem of finding the best values for the parameters may also be seen by considering cases of fine grouping, in which the majority of observations are separated into units. For the formula in equation (6) is equivalent to

$$S \left( \frac{1}{\bar{n}_s} \frac{\partial \bar{n}_s}{\partial \theta} \right)$$

where the summation is taken over all the observations, while the formula of equation (7), since it involves  $n_s^2$ , changes its value discontinuously, when one observation is gradually increased, at the point where it happens to coincide with a second observation.

Logically it would seem to be a necessity that that population which is chosen in fitting a hypothetical population to data should also appear the best when tested for its goodness of fit. The method of the optimum secures this agreement, and at the same time provides an extension of the process of testing goodness of fit, to those cases for which the  $\chi^2$  test is invalid.

The practical value of  $\chi^2$  lies in the fact that when the conditions are satisfied in order that it shall closely approximate to  $2L$ , it is possible to give a general formula for its distribution, so that it is possible to calculate the probability,  $P$ , that in a random sample from the population considered, a worse fit should be obtained; in such cases  $\chi^2$  is distributed in a curve of the Pearsonian Type III.,

$$df \propto \left( \frac{\chi^2}{2} \right)^{\frac{n'-3}{2}} e^{-\frac{\chi^2}{2}} d \left( \frac{\chi^2}{2} \right)$$

or

$$df \propto L^{\frac{n'-3}{2}} e^{-L} dL,$$

where  $n'$  is one more than the number of degrees of freedom in which the sample may differ from expectation (17).

In other cases we are at present faced with the difficulty that the distribution  $L$  requires a special investigation. This distribution will in general be discontinuous (as is that of  $\chi^2$ ), but it is not impossible that mathematical research will reveal the existence of effective graduations for the most important groups of cases to which  $\chi^2$  cannot be applied.

We shall conclude with a few illustrations of important types of discontinuous distribution.

### 1. *The Poisson Series.*

$$e^{-m} \left( 1, m, \frac{m^2}{2!}, \dots, \frac{m^x}{x!}, \dots \right)$$

involves only the single parameter, and is of great importance in modern statistics. For the optimum value of  $m$ ,

$$S \left\{ \frac{\partial}{\partial m} (-m + x \log m) \right\} = 0,$$

whence

$$S \left( \frac{x}{\hat{m}} - 1 \right) = 0,$$

or

$$\hat{m} = \bar{x}.$$

The most likely value of  $m$  is therefore found by taking the first moment of the series.

Differentiating a second time,

$$-\frac{1}{\sigma_{\hat{m}}^2} = S \left( -\frac{x}{m^2} \right) = -\frac{n}{m},$$

so that

$$\sigma_{\hat{m}}^2 = \frac{m}{n},$$

as is well known.

### 2. *Grouped Normal Data.*

In the case of the normal curve of distribution it is evident that the second moment is a sufficient statistic for estimating the standard deviation; in investigating a sufficient solution for grouped normal data, we are therefore in reality finding the optimum correction for grouping; the SHEPPARD correction having been proved only to satisfy the criterion of consistency.

For grouped normal data we have

$$p_s = \frac{1}{\sigma \sqrt{2\pi}} \int_{x_s}^{x_{s+1}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx,$$

and the optimum values of  $m$  and  $\sigma$  are obtained from the equations,

$$\frac{\partial L}{\partial m} = S \left( \frac{n_s}{p_s} \frac{\partial p_s}{\partial m} \right) = 0,$$

$$\frac{\partial L}{\partial \sigma} = S \left( \frac{n_s}{p_s} \frac{\partial p_s}{\partial \sigma} \right) = 0;$$

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or, if we write,

$$z = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}},$$

we have the two conditions,

$$S\left(\frac{n_s}{p_s} \overline{z_s - z_{s+1}}\right) = 0$$

and

$$S\left\{\frac{n_s}{p_s} \left(\frac{x_s}{\sigma} z_s - \frac{x_{s+1}}{\sigma} z_{s+1}\right)\right\} = 0.$$

As a simple example we shall take the case chosen by K. SMITH in her investigation of the variation of  $\chi^2$  in the neighbourhood of the moment solution (12).

Three hundred errors in right ascension are grouped in nine classes, positive and negative errors being thrown together as shown in the following table :—

0''·1 are	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9
Frequency . .	114	84	53	24	14	6	3	1	1

The second moment, without correction, yields the value

$$\sigma_v = 2\cdot282542.$$

Using SHEPPARD'S correction, we have

$$\sigma_\mu = 2\cdot264214,$$

while the value obtained by making  $\chi^2$  a minimum is

$$\sigma_{\chi^2} = 2\cdot355860.$$

If the latter value were accepted we should have to conclude that SHEPPARD'S correction, even when it is small, and applied to normal data, might be altogether of the wrong magnitude, and even in the wrong direction. In order to obtain the optimum value of  $\sigma$ , we tabulate the values of  $\frac{\partial L}{\partial \sigma}$  in the region under consideration; this may be done without great labour if values of  $\sigma$  be chosen suitable for the direct application of the table of the probability integral (13, Table II.). We then have the following values :—

$\frac{1}{\sigma}$	0·43	0·44	0·45	0·46
$\frac{\partial L}{\partial \sigma}$	+15·135	+2·149	-11·098	-24·605
$\Delta^2 \frac{\partial L}{\partial \sigma}$		-0·261	-0·260	

By interpolation,

$$\frac{1}{\hat{\sigma}} = 0.441624$$

$$\hat{\sigma} = 2.26437.$$

We may therefore summarise these results as follows :—

Uncorrected estimate of $\sigma$ . . . . .	2.28254
SHEPPARD'S correction . . . . .	−0.01833
Correction for maximum likelihood . . . . .	−0.01817
“ Correction ” for minimum $\chi^2$ . . . . .	+0.07332

Far from shaking our faith, therefore, in the adequacy of SHEPPARD'S correction, when small, for normal data, this example provides a striking instance of its effectiveness, while the approximate nature of the  $\chi^2$  test renders it unsuitable for improving a method which is already very accurate.

It will be useful before leaving the subject of grouped normal data to calculate the actual loss of efficiency caused by grouping, and the additional loss due to the small discrepancy between moments with SHEPPARD'S correction and the optimum solution.

To calculate the loss of efficiency involved in the process of grouping normal data, let

$$v = \frac{1}{a} \int_{\xi - \frac{1}{2}a}^{\xi + \frac{1}{2}a} f(\xi) d\xi,$$

when  $a\sigma$  is the group interval, then

$$\begin{aligned} v &= f(\xi) + \frac{a^2}{24} f''(\xi) + \frac{a^4}{1920} f^{iv}(\xi) + \frac{a^6}{322,560} f^{vi}(\xi) + \dots \\ &= f(\xi) \left\{ 1 + \frac{a^2}{24} (\xi^2 - 1) + \frac{a^4}{1920} (\xi^4 - 6\xi^2 + 3) + \frac{a^6}{322,560} (\xi^6 - 15\xi^4 + 45\xi^2 - 15) + \dots \right\}, \end{aligned}$$

whence

$$\log v = \log f + \frac{a^2}{24} (\xi^2 - 1) - \frac{a^4}{2880} (\xi^4 + 4\xi^2 - 2) + \frac{a^6}{181,440} (\xi^6 + 6\xi^4 + 3\xi^2 - 1) - \dots,$$

and

$$\frac{\partial^2}{\partial m^2} \log v = -\frac{1}{\sigma^2} + \frac{1}{\sigma^2} \left\{ \frac{a^2}{12} - \frac{a^4}{720} (3\xi^2 + 2) + \frac{a^6}{30,240} (5\xi^4 + 12\xi^2 + 1) - \dots \right\},$$

of which the mean value is

$$-\frac{1}{\sigma^2} \left\{ 1 - \frac{a^2}{12} + \frac{a^4}{144} - \frac{a^6}{4320} \dots \right\},$$



neglecting the periodic terms ; and consequently

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} \left( 1 + \frac{\alpha^2}{12} - \frac{\alpha^6}{2880} \dots \right).$$

Now for the mean of ungrouped data

$$\sigma_{\bar{m}}^2 = \frac{\sigma^2}{n},$$

so that the loss of efficiency due to grouping is nearly  $\frac{\alpha^2}{12}$ .

The further loss caused by using the mean of the grouped data is very small, for

$$\sigma_{v_1}^2 = \frac{v_2}{n} = \frac{\sigma^2}{n} \left( 1 + \frac{\alpha^2}{12} \right),$$

neglecting the periodic terms ; the loss of efficiency by using  $v_1$  therefore is only

$$\frac{\alpha^6}{2880}.$$

Similarly for the efficiency for scaling,

$$\begin{aligned} \frac{\partial^2}{\partial \sigma^2} \log v &= \frac{1}{\sigma^2} - \frac{3\xi^2}{\sigma^2} + \frac{1}{\sigma^2} \left\{ \frac{\alpha^2}{12} (10\xi^2 - 3) - \frac{\alpha^4}{360} (9\xi^4 + 21\xi^2 - 5) \right. \\ &\quad \left. + \frac{\alpha^6}{30,240} (26\xi^6 + 110\xi^4 + 36\xi^2 - 7) - \frac{\alpha^8}{1,814,400} (51\xi^8 + 315\xi^6 + 351\xi^4 - 55\xi^2 + 9) + \dots \right\}, \end{aligned}$$

of which the mean value is

$$-\frac{2}{\sigma^2} \left\{ 1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{40} - \frac{\alpha^6}{270} + \frac{83\alpha^8}{129,600} \dots \right\},$$

neglecting the periodic terms ; and consequently

$$\sigma_{\bar{s}}^2 = \frac{\sigma^2}{2n} \left\{ 1 + \frac{\alpha^2}{6} + \frac{\alpha^4}{360} - \frac{\alpha^8}{10,800} \dots \right\}.$$

For ungrouped data

$$\sigma_{\bar{s}}^2 = \frac{\sigma^2}{2n},$$

so that the loss of efficiency in scaling due to grouping is nearly  $\frac{\alpha^2}{6}$ . This may be made as low as 1 per cent. by keeping  $\alpha$  less than  $\frac{1}{4}$ .

The further loss of efficiency produced by using the grouped second moment with SHEPPARD'S correction is again very small, for

$$\sigma_{v_2}^2 = \frac{v_4 - v_2^2}{n} = \frac{2\sigma^4}{n} \left( 1 + \frac{\alpha^2}{6} + \frac{\alpha^4}{360} \right)$$

neglecting the periodic terms.

Whence it appears that the further loss of efficiency is only

$$\frac{\alpha^8}{10,800}.$$

We may conclude, therefore, that the high agreement between the optimum value of  $\sigma$  and that obtained by SHEPPARD'S correction in the above example is characteristic of grouped normal data. The method of moments with SHEPPARD'S correction is highly efficient in treating such material, the gain in efficiency obtainable by increasing the likelihood to its maximum value is trifling, and far less than can usually be gained by using finer groups. The loss of efficiency involved in grouping may be kept below 1 per cent. by making the group interval less than one-quarter of the standard deviation.

Although for the normal curve the loss of efficiency due to moderate grouping is very small, such is not the case with curves making a finite angle with the axis, or having at an extreme a finite or infinitely great ordinate. In such cases even moderate grouping may result in throwing away the greater part of the information which the sample provides.

### 3. *Distribution of Observations in a Dilution Series.*

An important type of discontinuous distribution occurs in the application of the dilution method to the estimation of the number of micro-organisms in a sample of water or of soil. The method here presented was originally developed in connection with Mr. CUTLER'S extensive counts of soil protozoa carried out in the protozoological laboratory at Rothamsted, and although the method is of very wide application, this particular investigation affords an admirable example of the statistical principles involved.

In principle the method consists in making a series of dilutions of the soil sample, and determining the presence or absence of each type of protozoa in a cubic centimetre of the dilution, after incubation in a nutrient medium.

The series in use proceeds by powers of 2, so that the frequency of protozoa in each dilution is one-half that in the last.

The frequency at any stage of the process may then be represented by

$$m = \frac{n}{2^x},$$

when  $x$  indicates the number of dilutions.

Under conditions of random sampling, the chance of any plate receiving 0, 1, 2, 3 protozoa of a given species is given by the Poisson series

$$e^{-m} \left( 1, m, \frac{m^2}{2!}, \frac{m^3}{3!}, \dots \right),$$

and in consequence the proportion of sterile plates is

$$p = e^{-m},$$

and of fertile plates

$$q = 1 - e^{-m}.$$

In general we may consider a dilution series with dilution factor  $a$  so that

$$\log p = -\frac{n}{a^x},$$

and assume that  $s$  plates are poured from each dilution.

The object of the method being to estimate the number  $n$  from a record of the sterile and fertile plates, we have

$$L = S_1 (\log p) + S_2 (\log q)$$

when  $S_1$  stands for summation over the sterile plates, and  $S_2$  for summation over those which are fertile.

Now

$$\frac{\partial p}{\partial \log n} = -\frac{\partial q}{\partial \log n} = p \log p,$$

so that the optimum value of  $n$  is obtained from the equation,

$$\frac{\partial L}{\partial \log n} = S_1 (\log p) - S_2 \left( \frac{p}{q} \log p \right) = 0.$$

Differentiating a second time,

$$\frac{\partial^2 L}{\partial (\log n)^2} = S_1 (\log p) - S_2 \left\{ \frac{p \log p}{q} \left( \log p + 1 + \frac{p \log p}{q} \right) \right\};$$

now the mean number of sterile plates is  $ps$ , and of fertile plates  $qs$ , so that the mean value of  $\frac{\partial^2 L}{\partial (\log n)^2}$  is

$$-\frac{1}{\sigma_{\log n}^2} = sS \left\{ p \log p - p \log p \left( \log p + 1 + \frac{p}{q} \log p \right) \right\} = -sS \left\{ \frac{p}{q} (\log p)^2 \right\},$$

the summation,  $S$ , being extended over all the dilutions.

It thus appears that each plate observed adds to the weight of the determination of  $\log n$  a quantity

$$w = \frac{p}{q} (\log p)^2.$$

We give below a table of the values of  $p$ , and of  $w$ , for the dilution series  $\log p = 2^{-x}$  from  $x = -4$  to  $x = 11$ .

$x$ .	$p$ .	$w$ .	$S(w)$ (per cent.).
-4	0.00000014167	0.000036	0.002
-3	0.0003354626	0.021477	0.906
-2	0.01831564	0.298518	13.485
-1	0.1353353	0.626071	39.865
0	0.3678794	0.581977	64.388
1	0.6065306	0.385355	80.625
2	0.7788009	0.220051	89.897
3	0.8824969	0.117350	94.842
4	0.9394110	0.060565	97.394
5	0.9692333	0.030764	98.690
6	0.9844965	0.015503	99.343
7	0.9922179	0.007782	99.671
8	0.9961014	0.003899	99.836
9	0.9980488	0.001951	99.918
10	0.9990239	0.000976	99.959
11	0.9995119	0.000488	99.979
Remainder . . . . .		0.000488	
Total . . . . .		2.373251	

For the same dilution constant the total  $S(w)$  is nearly independent of the particular series chosen. Its average value being  $\frac{\pi^2}{6 \log \alpha}$ , or in this case 2.373138. The fourth column shows the total weight attained at any stage, expressed as a percentage of that obtained from an infinite series of dilutions. It will be seen that a set of eight dilutions comprise all but about 2 per cent. of the weight. With a loss of efficiency of only 2 to  $2\frac{1}{2}$  per cent., therefore, the number of dilutions which give information as to a particular species may be confined to eight. To this number must be added a number depending on the range which it is desired to explore. Thus to explore a range from 100 to 100,000 per gramme (about 10 octaves) we should require 10 more dilutions, making 18 in all, while to explore a range of a millionfold, or about 20 octaves, 28 dilutions would be needed.

In practice it would be exceedingly laborious to calculate the optimum value of  $n$  for each series observed (of which 38 are made daily). On the advice of the statistical department, therefore, Mr. CUTLER adopted the plan of counting the total number of sterile plates, and taking the value of  $n$  which on the average would give that number. When a sufficient number of dilutions are made,  $\log n$  is diminished by  $\frac{1}{s} \log \alpha$  for each additional sterile plate, and even near the ends of the series the appropriate values of  $n$  may easily be tabulated. Since this method of estimation is of wide application and appears at first sight to be a very rough one, it is important to calculate its efficiency.

For any dilution the variance in the number of sterile plates is

$$spq,$$

and as the several dilutions represent independent samples, the total variance is

$$sS(pq),$$

hence

$$\sigma^2_{\log n} = \frac{(\log \alpha)^2}{s} S(pq).$$

Now  $S(pq)$  has an average value  $\frac{\log 2}{\log \alpha}$ , therefore taking  $\alpha = 2$ ,

$$(\log \alpha)^2 = .480453,$$

and

$$S(pq) = 1$$

being very nearly constant and within a small fraction of unity ; whence the efficiency of the method of counting the sterile plates is

$$\frac{6}{\pi^2 \log 2} = 87.71 \text{ per cent.},$$

a remarkably high efficiency, considering the simplicity of the method, the efficiency being independent of the dilution ratio.

### 13. SUMMARY.

During the rapid development of practical statistics in the past few decades, the theoretical foundations of the subject have been involved in great obscurity. Adequate distinction has seldom been drawn between the sample recorded and the hypothetical population from which it is regarded as drawn. This obscurity is centred in the so-called "inverse" methods.

On the bases that the purpose of the statistical reduction of data is to obtain statistics which shall contain as much as possible, ideally the whole, of the relevant information contained in the sample, and that the function of Theoretical Statistics is to show how such adequate statistics may be calculated, and how much and of what kind is the information contained in them, an attempt is made to formulate distinctly the types of problems which arise in statistical practice.

Of these, problems of Specification are found to be dominated by considerations which may change rapidly during the progress of Statistical Science. In problems of Distribution relatively little progress has hitherto been made, these problems still affording a field for valuable enquiry by highly trained mathematicians. The principal purpose of this paper is to put forward a general solution of problems of Estimation.

Of the criteria used in problems of Estimation only the criterion of Consistency has hitherto been widely applied ; in Section 5 are given examples of the adequate and inadequate application of this criterion. The criterion of Efficiency is shown to be a special but important case of the criterion of Sufficiency, which latter requires that the whole of the relevant information supplied by a sample shall be contained in the statistics calculated.

In order to make clear the nature of the general method of satisfying the criterion of Sufficiency, which is here put forward, it has been thought necessary to reconsider BAYES' problem in the light of the more recent criticisms to which the idea of " inverse probability " has been exposed. The conclusion is drawn that two radically distinct concepts, both of importance in influencing our judgment, have been confused under the single name of *probability*. It is proposed to use the term *likelihood* to designate the state of our information with respect to the parameters of hypothetical populations, and it is shown that the quantitative measure of likelihood does not obey the mathematical laws of probability.

A proof is given in Section 7 that the criterion of Sufficiency is satisfied by that set of values for the parameters of which the likelihood is a maximum, and that the same function may be used to calculate the efficiency of any other statistics, or, in other words, the percentage of the total available information which is made use of by such statistics.

This quantitative treatment of the information supplied by a sample is illustrated by an investigation of the efficiency of the method of moments in fitting the Pearsonian curves of Type III.

Section 9 treats of the location and scaling of Error Curves in general, and contains definitions and illustrations of the *intrinsic accuracy*, and of the *centre of location* of such curves.

In Section 10 the efficiency of the method of moments in fitting the general Pearsonian curves is tested and discussed. High efficiency is only found in the neighbourhood of the normal point. The two causes of failure of the method of moments in locating these curves are discussed and illustrated. The special cause is discovered for the high efficiency of the third and fourth moments in the neighbourhood of the normal point.

It is to be understood that the low efficiency of the moments of a sample in estimating the form of these curves does not at all diminish the value of the notation of moments as a means of the comparative specification of the form of such curves as have finite moment coefficients.

Section 12 illustrates the application of the method of maximum likelihood to discontinuous distributions. The POISSON series is shown to be sufficiently fitted by the mean. In the case of grouped normal data, the SHEPPARD correction of the crude moments is shown to have a very high efficiency, as compared to recent attempts to improve such fits by making  $\chi^2$  a minimum ; the reason being that  $\chi^2$  is an expression only approximate to a true value derivable from likelihood. As a final illustration of

the scope of the new process, the theory of the estimation of micro-organisms by the dilution method is investigated.

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#### REFERENCES.

- (1) K. PEARSON (1920). "The Fundamental Problem of Practical Statistics," 'Biom.,' xiii., pp. 1-16.
- (2) F. Y. EDGEWORTH (1921). "Molecular Statistics," 'J.R.S.S.,' lxxxiv., p. 83.
- (3) G. U. YULE (1912). "On the Methods of Measuring Association between two Attributes." 'J.R.S.S.,' lxxv., p. 587.
- (4) STUDENT (1908). "The Probable Error of a Mean," 'Biom.,' vi., p. 1.
- (5) R. A. FISHER (1915). "Frequency Distribution of the Values of the Correlation Coefficient in Samples from an Indefinitely Large Population," 'Biom.,' x., 507.
- (6) R. A. FISHER (1921). "On the 'Probable Error' of a Coefficient of Correlation deduced from a Small Sample," 'Metron.,' i., pt. iv., p. 82.
- (7) R. A. FISHER (1920). "A Mathematical Examination of the Methods of Determining the Accuracy of an Observation by the Mean Error and by the Mean Square Error," 'Monthly Notices of R.A.S.,' lxxx., 758.
- (8) E. PAIRMAN and K. PEARSON (1919). "On Corrections for the Moment Coefficients of Limited Range Frequency Distributions when there are finite or infinite Ordinates and any Slopes at the Terminals of the Range," 'Biom.,' xii., p. 231.
- (9) R. A. FISHER (1912). "On an Absolute Criterion for Fitting Frequency Curves," 'Messenger of Mathematics,' xli., p. 155.
- (10) BAYES (1763). "An Essay towards Solving a Problem in the Doctrine of Chances," 'Phil. Trans.,' liii., p. 370.
- (11) K. PEARSON (1919). "Tracts for Computers. No. 1: Tables of the Digamma and Trigamma Functions," By E. PAIRMAN, Camb. Univ. Press.
- (12) K. SMITH (1916). "On the 'best' Values of the Constants in Frequency Distributions," 'Biom.,' xi., p. 262.
- (13) K. PEARSON (1914). "Tables for Statisticians and Biometricians," Camb. Univ. Press.
- (14) G. VEGA (1764). "Thesaurus Logarithmorum Completus," p. 643.
- (15) K. PEARSON (1900). "On the Criterion that a given System of Deviations from the Probable in the case of a Correlated System of Variables is such that it can be reasonably supposed to have arisen from Random Sampling," 'Phil. Mag.,' l., p. 157.
- (16) K. PEARSON and L. N. G. FILON (1898). "Mathematical Contributions to the Theory of Evolution. IV.—On the Probable Errors of Frequency Constants, and on the influence of Random Selection on Variation and Correlation," 'Phil. Trans.,' exci., p. 229.
- (17) R. A. FISHER (1922). "The Interpretation of  $\chi^2$  from Contingency Tables, and the Calculation of P," 'J.R.S.S.,' lxxxv., pp. 87-94.
- (18) K. PEARSON (1915). "On the General Theory of Multiple Contingency, with special reference to Partial Contingency," 'Biom.,' xi., p. 145.
- (19) K. PEARSON (1903). "On the Probable Errors of Frequency Constants," 'Biom.,' ii., p. 273, Editorial.
- (20) W. F. SHEPPARD (1898). "On the Application of the Theory of Error to Cases of Normal Distribution and Correlations," 'Phil. Trans.,' A., xcii., p. 101.
- (21) J. M. KEYNES (1921). "A Treatise on Probability," Macmillan & Co., London.