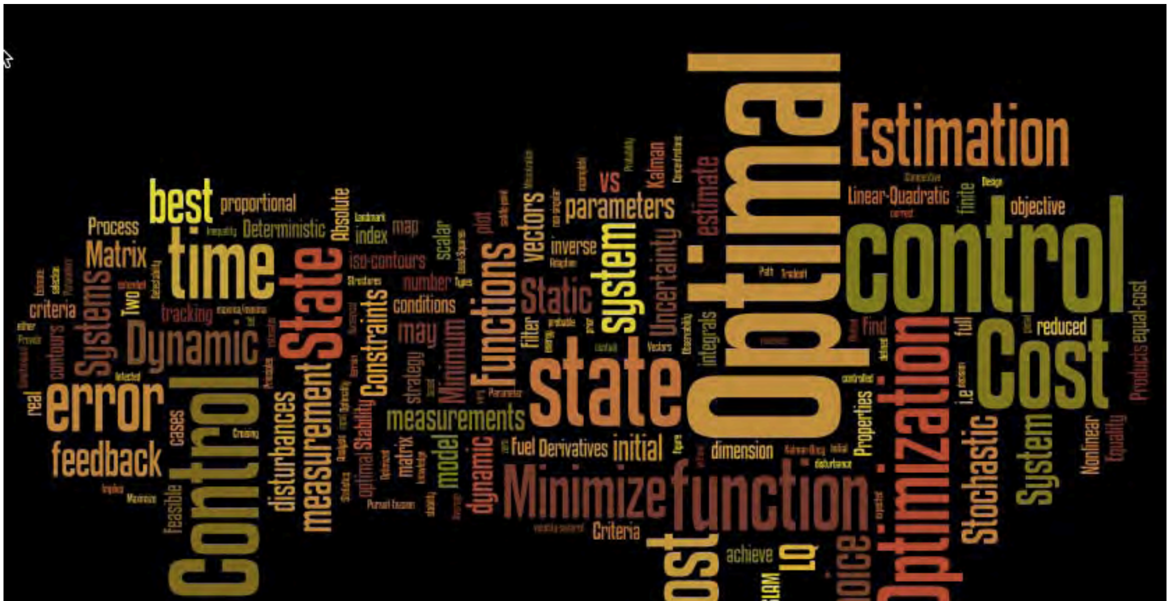
Optimal Control and Estimation

MAE 546, Princeton University
Robert Stengel, 2018



http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Preliminaries

Tuesday and Thursday, 3-4:20 pm
Room 306, Friend Center

- *GRADING*
  - *Class participation: 15%*
  - *5-min quizzes: 10%*
  - *Homework assignments: 35%*
  - *Final Paper: 40%*

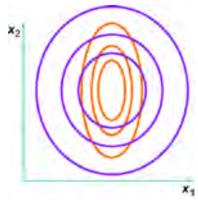
- *Reference*
  - *R. Stengel, Optimal Control and Estimation, Dover, 1994*
  - *Various journal papers and book chapters*
- *Resources*
  - *Blackboard*
    - *https://blackboard.princeton.edu/webapps/login*
  - *Course Home Page, Syllabus, and Links*
    - *www.princeton.edu/~stengel/MAE546.html*
  - *Engineering Library, including Databases and e-Journals*
    - *http://library.princeton.edu/catalogs/articles.php*
- *Assignments will be submitted using the notation and symbols of this course*

Syllabus - 1

Week	Tuesday	Thursday
====	=======	========
1	Overview and Preliminaries Functions	Minimization of Static Cost
2	Principles for Optimal Control of Dynamic Systems	Principles for Optimal Control Part 2
3	Path Constraints and Numerical Optimization	Minimum-Time and -Fuel Optimization
4	Linear-Quadratic (LQ) Control	Dynamic System Stability
5	Linear-Quadratic Regulators	Cost Functions and Controller Structures
6	LQ Control System Design	Modal Properties of LQ Systems
	MID-TERM BREAK	

Syllabus - 2

Week	Tuesday	Thursday
====	=======	========
7	Spectral Properties of LQ Systems	Singular-Value Analysis
8	Probability and Statistics	Least-Squares Estimation for Static Systems
9	Propagation of Uncertainty in Dynamic Systems	Kalman Filter
10	Kalman-Bucy Filter	Nonlinear State Estimation
11	Nonlinear State Estimation	Adaptive State Estimation
12	Stochastic Optimal Control Control	Linear-Quadratic-Gaussian
	READING PERIOD	Final Paper due on "Dean's Date"

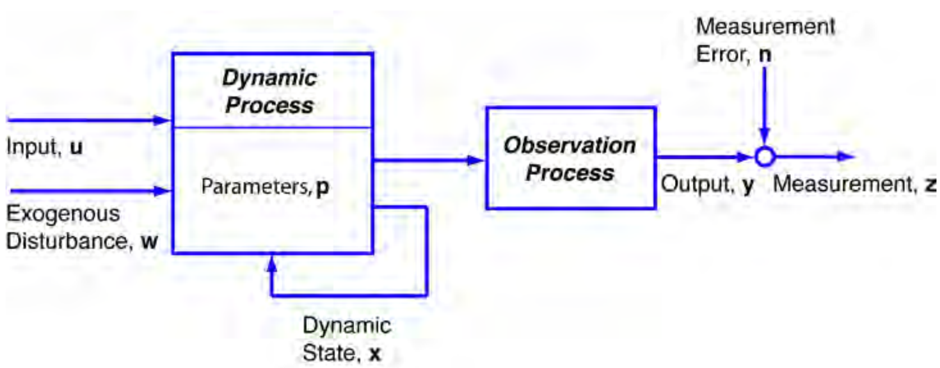


Typical Optimization Problems

- Minimize the probable error in an estimate of the dynamic state of a system
- Maximize the probability of making a correct decision
- Minimize the time or energy required to achieve an objective
- Minimize the regulation error in a controlled system

- *Estimation*
- *Control*

Dynamic Systems



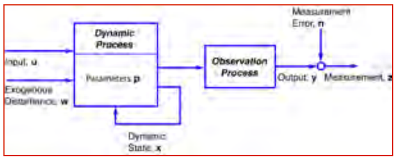
Dynamic Process: Current state depends on prior state

x	= dynamic state
u	= input
w	= exogenous disturbance
p	= parameter
t or *k*	= time or event index

Observation Process: Measurement may contain error or be incomplete

y	= output (error-free)
z	= measurement
n	= measurement error

All of these quantities are vectors



Mathematical Models of Dynamic Systems

Dynamic Process: Current state depends on prior state

x	= dynamic state
u	= input
w	= exogenous disturbance
p	= parameter
t	= time index

Observation Process: Measurement may contain error or be incomplete

y	= output (error-free)
z	= measurement
n	= measurement error

Continuous-time dynamic process: Vector Ordinary Differential Equation

$$\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t), \mathbf{p}(t), t]$$

$$t = \text{ time, s}$$

Output Transformation

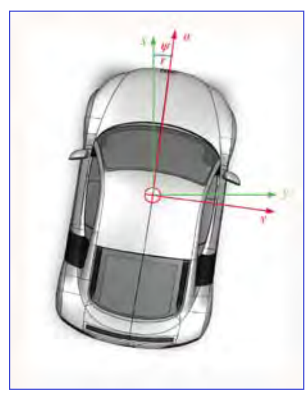
$$\mathbf{y}(t) = \mathbf{h}[\mathbf{x}(t), \mathbf{u}(t)]$$

Measurement with Error

$$\mathbf{z}(t) = \mathbf{y}(t) + \mathbf{n}(t)$$

Example: Lateral Automobile Dynamics

Constant forward (axial) velocity, *u*
No rigid-body rolling motion



State Vector

$$\mathbf{x} = \begin{bmatrix} v \\ r \\ y \\ \psi \end{bmatrix} = \begin{bmatrix} \text{Side velocity, m/s} \\ \text{Yaw angle rate, rad/s} \\ \text{Lateral position, m} \\ \text{Yaw angle, rad} \end{bmatrix}$$

Parameter Vector

$$\mathbf{p} = \begin{bmatrix} m \\ I_{zz} \\ \partial Y/\partial v \\ \partial Y/\partial \psi_{steer} \\ \dots \\ \partial N/\partial v \\ \partial N/\partial \psi_{steer} \\ \dots \end{bmatrix} = \begin{bmatrix} \text{mass, kg} \\ \text{Lateral moment of inertia, N-m} \\ \text{Side force sensitivity to side velocity, N/(m/s)} \\ \text{Side force sensitivity to steering angle, N/rad} \\ \dots \\ \text{Yawing moment sensitivity to side velocity, N-m/(m/s)} \\ \text{Yawing moment sensitivity to steering angle, N-m/rad} \\ \dots \end{bmatrix}$$

Control and Disturbance Vectors

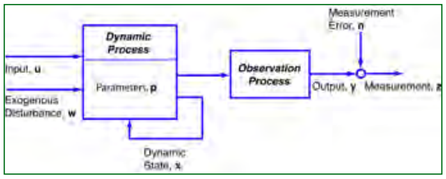
$$\mathbf{u} = \psi_{steer} = \text{ Steering angle, rad}$$

$$\mathbf{w} = \begin{bmatrix} v_{wind} \\ f_{road} \end{bmatrix} = \begin{bmatrix} \text{Crosswind, m/s} \\ \text{Side force on front wheel, N} \end{bmatrix}$$

Output and Measurement Vectors

$$\mathbf{y} = \begin{bmatrix} y \\ \psi \end{bmatrix} = \begin{bmatrix} \text{Lateral position, m} \\ \text{Yaw angle, rad} \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} y_{measured} \\ \psi_{measured} \end{bmatrix} = \begin{bmatrix} y + error \\ \psi + error \end{bmatrix}$$



Lateral Automobile Dynamics Example

Dynamic Process

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{v}(t) \\ \dot{r}(t) \\ \dot{y}(t) \\ \dot{\psi}(t) \end{bmatrix} = \begin{bmatrix} \dfrac{Y(\mathbf{x},\mathbf{u},\mathbf{w})}{m} \\ \dfrac{N(\mathbf{x},\mathbf{u},\mathbf{w})}{I_{yy}} \\ u\sin\psi + v\cos\psi \\ r \end{bmatrix}$$

Observation Process

$$\mathbf{y} = \begin{bmatrix} y \\ \psi \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 + n_1 \\ y_2 + n_2 \end{bmatrix}$$

Discrete-Time Models of Dynamic Systems

Dynamic Process: Current state depends on prior state

- x = dynamic state
- u = input
- w = exogenous disturbance
- p = parameter
- *t* = time index

Observation Process: Measurement may contain error or be incomplete

- y = output (error-free)
- z = measurement
- n = measurement error

Discrete-time dynamic process: Vector Ordinary Difference Equation

$$\mathbf{x}_{k+1} = \mathbf{f}_k[\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k, \mathbf{p}_k, k]$$

$$k = \text{ time index, -}$$
$$(t_{k+1} - t_k) = \text{ time interval, s}$$

Output Transformation

$$\mathbf{y}_k = \mathbf{h}_k[\mathbf{x}_k, \mathbf{u}_k]$$

Measurement with Error

$$\mathbf{z}_k = \mathbf{y}_k + \mathbf{n}_k$$

Approximate Discrete-Time Lateral Automobile Dynamics Example

Approximate Dynamic Process (Rectangular Integration)

$$\mathbf{x}_{k+1} = \begin{bmatrix} v_{k+1} \\ r_{k+1} \\ y_{k+1} \\ \psi_{k+1} \end{bmatrix} \approx \begin{bmatrix} \dfrac{Y_k\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right)}{m} \\ \dfrac{N_k\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right)}{I_{yy}} \\ u_k \sin\psi_k + v_k \cos\psi_k \\ r_k \end{bmatrix} \left(t_{k+1} - t_k\right)$$

Observation Process

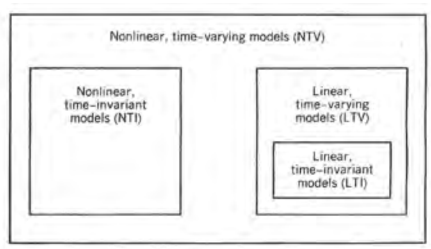
$$\mathbf{y}_k = \begin{bmatrix} y_k \\ \psi_k \end{bmatrix} = \begin{bmatrix} y_{1_k} \\ y_{2_k} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1_k} \\ x_{2_k} \\ x_{3_k} \\ x_{4_k} \end{bmatrix}$$

$$\mathbf{z}_k = \begin{bmatrix} z_{1_k} \\ z_{2_k} \end{bmatrix}$$

$$= \begin{bmatrix} y_{1_k} + n_{1_k} \\ y_{2_k} + n_{2_k} \end{bmatrix}$$

Dynamic System Model Types



- **NTV**
- **NTI**

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t), \mathbf{p}(t), t\right]$$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)\right]$$

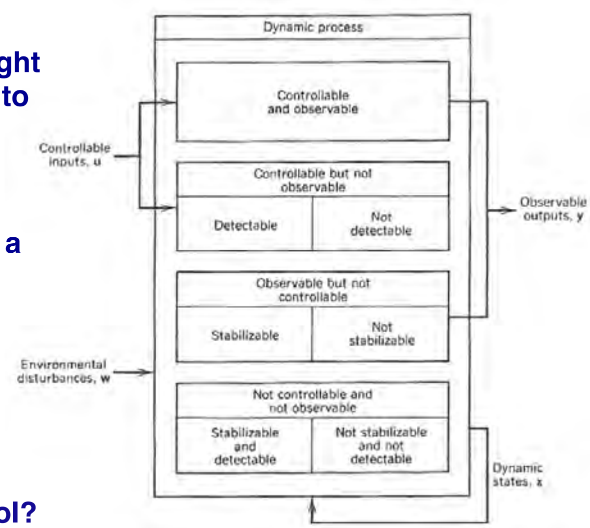
- **LTV**
- **LTI**

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{L}(t)\mathbf{w}(t)$$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t) + \mathbf{L}\,\mathbf{w}(t)$$

Controllability and Observability

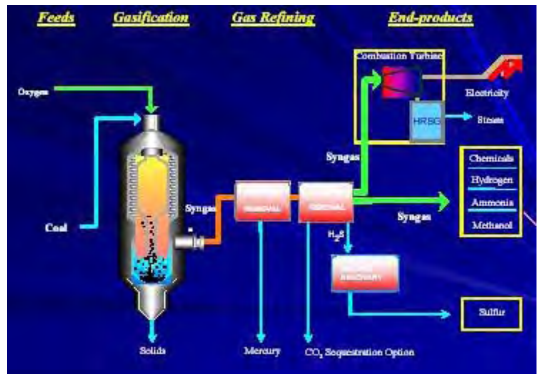
- Controllability: State can be brought from an arbitrary initial condition to zero in finite time by the use of control
- Observability: Initial state can be derived from measurements over a finite time interval
- Subsets of the system may be either, both, or neither
- Effects of Stability
  - Stabilizability
  - Detectability
- Blocks subject to feedback control?



Introduction to Optimization

Optimization Implies Choice

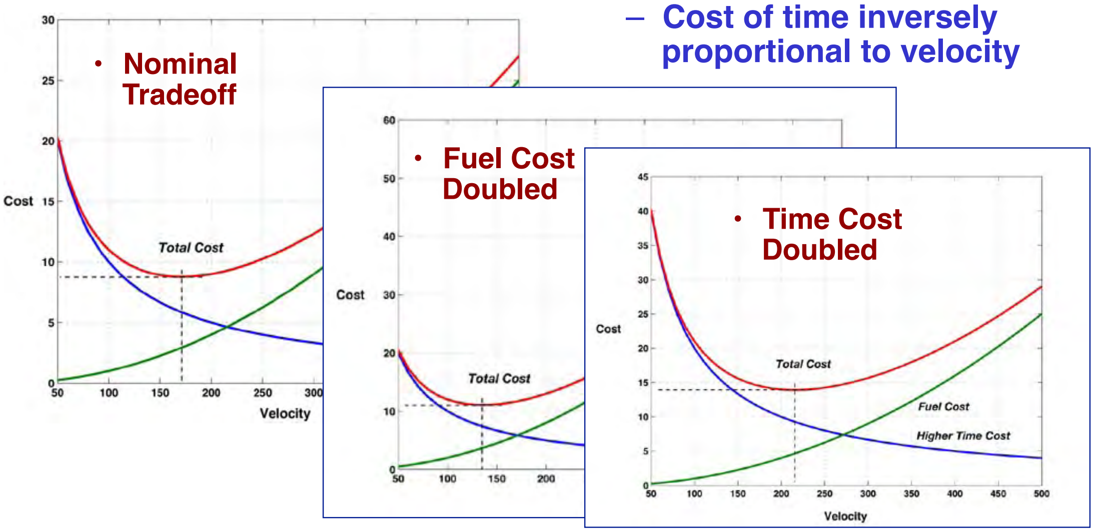
- Choice of best strategy
- Choice of best design parameters
- Choice of best control history
- Choice of best estimate
- Optimization is provided by selection of best control variable(s)



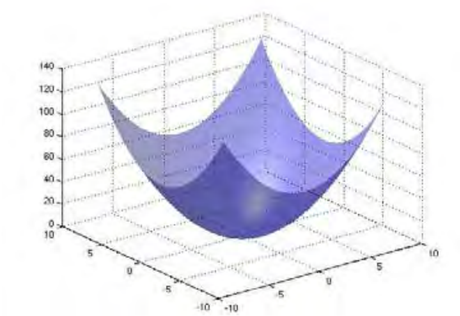
Tradeoff Between Two Cost Factors

Minimum-Cost Cruising Speed

- Fuel cost proportional to velocity-squared
- Cost of time inversely proportional to velocity



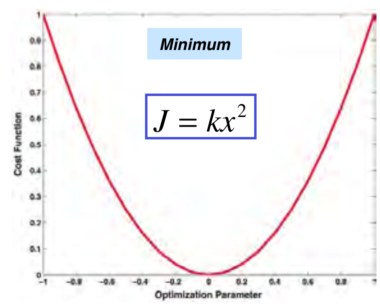
Desirable Characteristics of a Cost Function, *J*

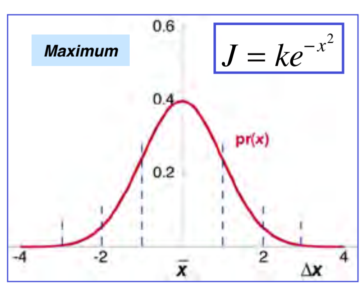


- **Scalar**
- **Clearly defined (preferably unique) maximum or minimum**
  - **Local**
  - **Global**
- **Preferably positive-definite (i.e., always a positive number)**

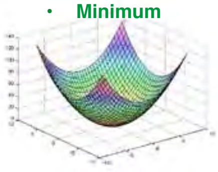
Criteria for Optimization

- **Names for criteria**
  - **Figure of merit**
  - **Performance index**
  - **Utility function**
  - **Value function**
  - **Cost function, *J***
    - **Optimal cost function = *J****
    - **Optimal control = u***
- **Different criteria lead to different optimal solutions**
- **Types of Optimality Criteria**
  - **Absolute**
  - **Regulatory**
  - **Feasible**



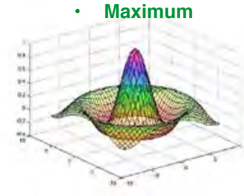


- Minimum

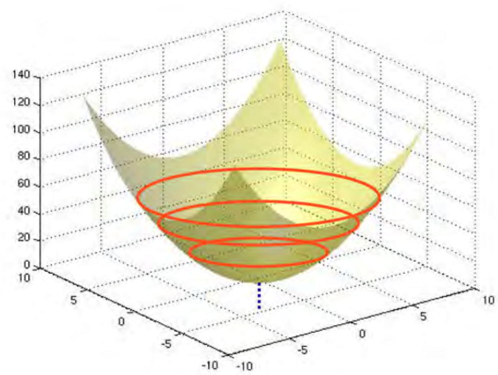


Cost Functions with Two Control Parameters

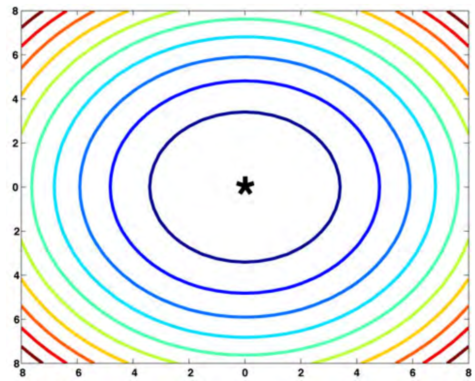
- Maximum

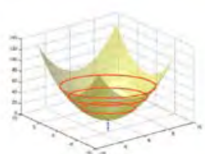


- 3-D plot of equal-cost contours (iso-contours)

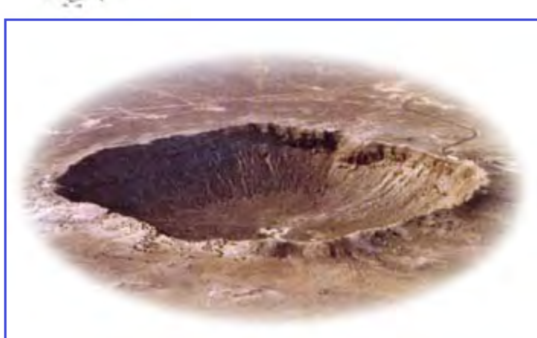


- 2-D plot of equal-cost contours (iso-contours)

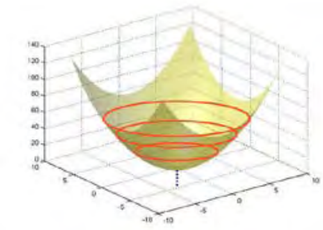




Real-World Topography

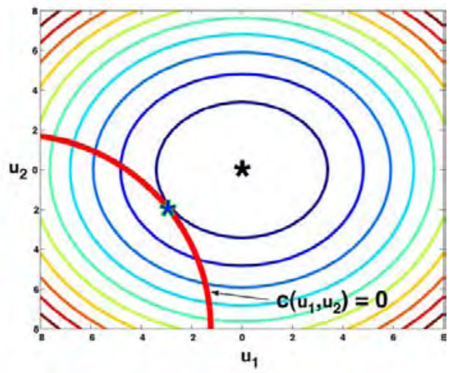


Local vs. global maxima/minima



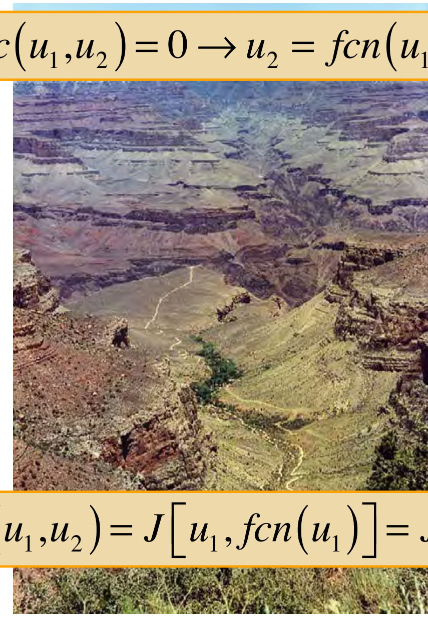
Cost Functions with Equality Constraints

- Must stay on the trail



- Equality constraint may allow control dimension to be reduced

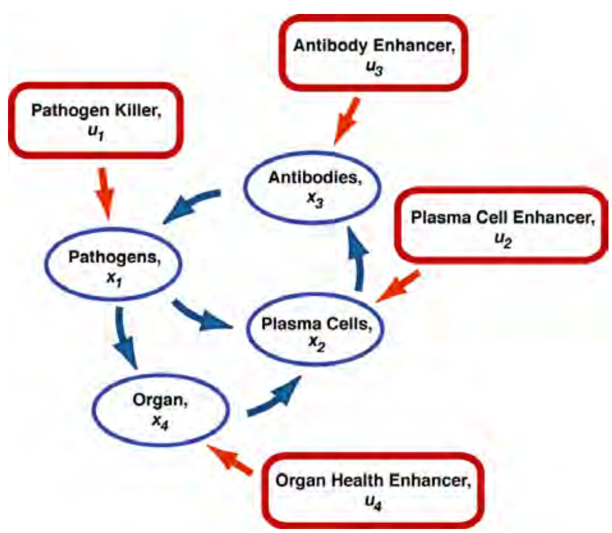
$$c(u_1,u_2)=0 \rightarrow u_2 = fcn(u_1)$$



$$J(u_1,u_2)=J\left[u_1,fcn(u_1)\right]=J'(u_1)$$

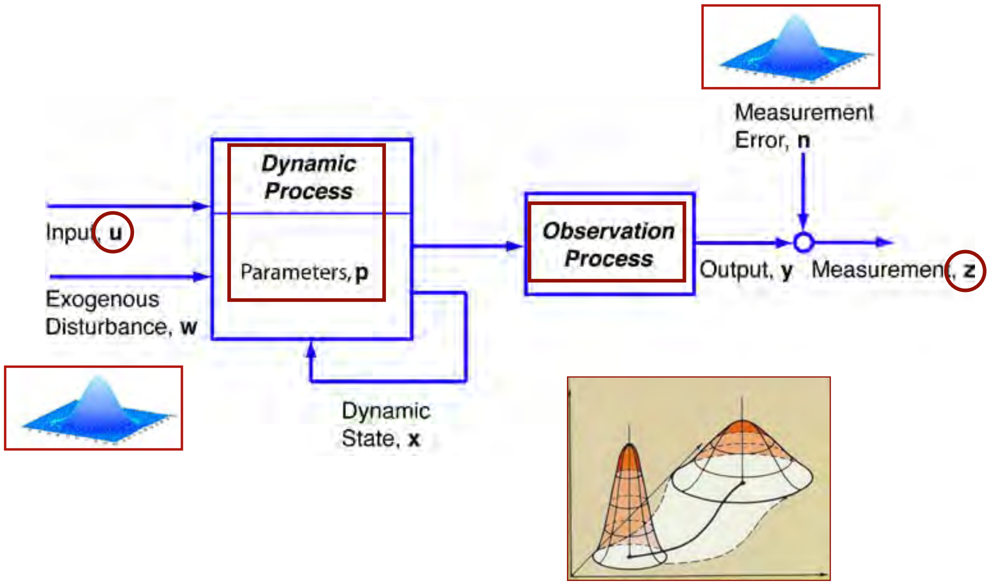
Example: Minimize Concentrations of Bacteria, Infected Cells, and Drug Usage

- x_1 = Concentration of a pathogen, which displays antigen
- x_2 = Concentration of plasma cells, which are carriers and producers of antibodies
- x_3 = Concentration of antibodies, which recognize antigen and kill pathogen
- x_4 = Relative characteristic of a damaged organ [0 = healthy, 1 = dead]

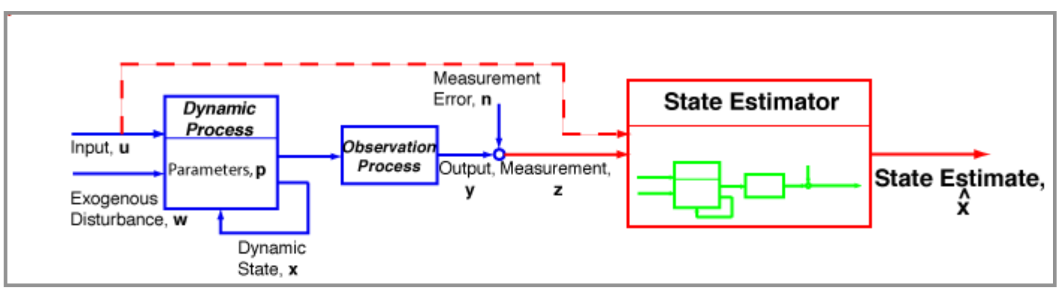


What is a reasonable cost function to minimize?

Optimal Estimate of the State, x, Given Uncertainty



Optimal State Estimation



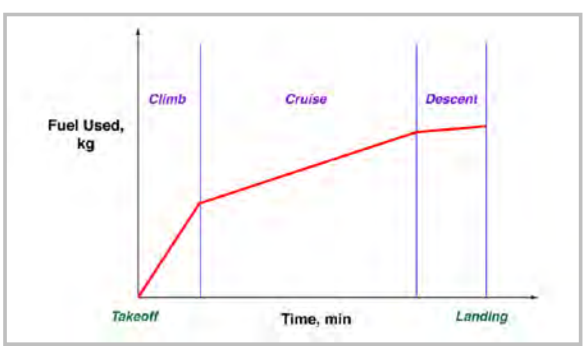
- **Goals**
  - **Minimize effects of measurement error on knowledge of the state**
  - **Reconstruct full state from reduced measurement set ($r \leq n$)**
  - **Average redundant measurements ($r \geq n$) to estimate the full state**
- **Method**
  - **Provide *optimal balance* between *measurements* and estimates based on the *dynamic model* alone**

Typical Problems in Optimal Control and Estimation

Minimize an Absolute Criterion

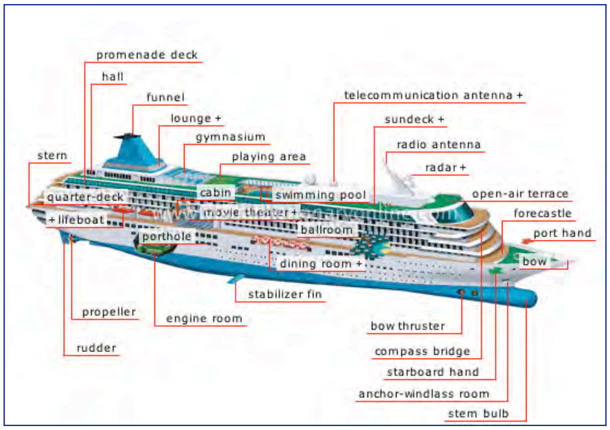
- Achieve a specific objective
  - Minimum time
  - Minimum fuel
  - Minimum financial cost
- to achieve a goal

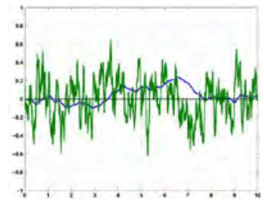
- What is the control variable?



Optimal System Regulation

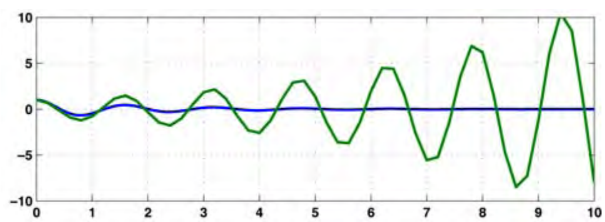
Find feedback control gains that minimize tracking error in presence of random disturbances

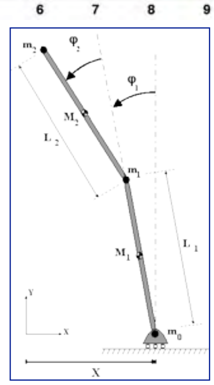




Feasible Control Logic

- **Find feedback control structure that guarantees stability (i.e., that keeps Δx from diverging)**

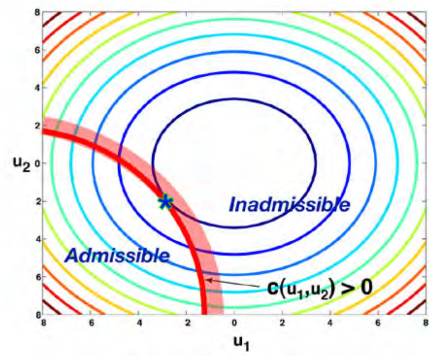




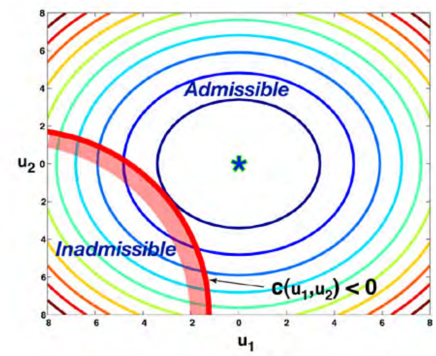
http://www.youtube.com/watch?v=8HDDzKxNMEY

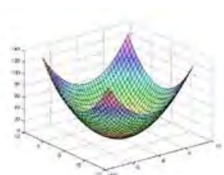
Cost Functions with Inequality Constraints

- Must stay to the left of the trail



- Must stay to the right of the trail

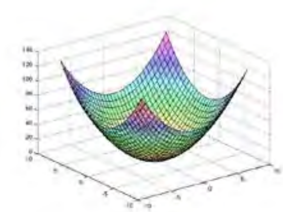




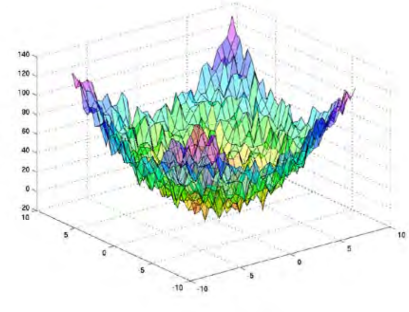
Static vs. Dynamic Optimization

- **Static**
  - Optimal state, x*, and control, u*, are fixed, i.e., they do not change over time
    - $J^* = J(x^*, u^*)$
    - Functional minimization (or maximization)
    - Parameter optimization
- **Dynamic**
  - Optimal state and control vary over time
    - $J^* = J[x^*(t), u^*(t)]$
    - Optimal trajectory
    - Optimal feedback strategy
- **Optimized cost function, *J**, is a scalar, real number in both cases**

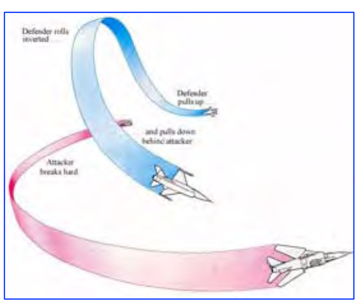
Deterministic vs. Stochastic Optimization



- **Deterministic**
  - **System model, parameters, initial conditions, and disturbances are known without error**
  - **Optimal control operates on the system with certainty**
    - $J^* = J(x^*, u^*)$
- **Stochastic**
  - **Uncertainty in**
    - **system model**
    - **parameters**
    - **initial conditions**
    - **disturbances**
    - **resulting cost function**
  - **Optimal control minimizes the expected value of the cost:**
    - ***Optimal cost*** $= E\{J[x^*, u^*]\}$



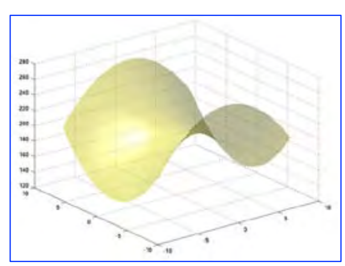
- **Cost function is a scalar, real number in both cases**



Example: Pursuit-Evasion: Competitive Optimization Problem

- **Pursuer's goal: minimize final miss distance**
- **Evader's goal: maximize final miss distance**

- **"Minimax" (saddle-point) cost function**

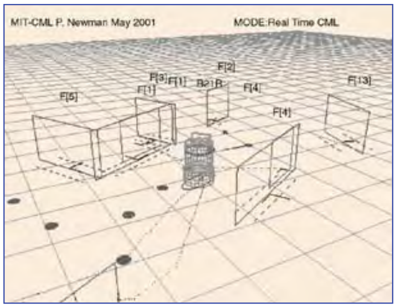


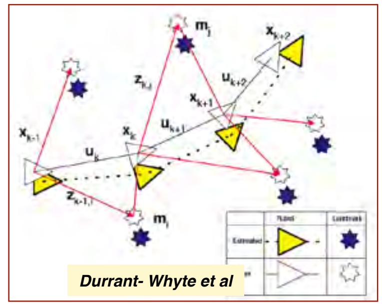
- **Optimal control laws for pursuer and evader**

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{u}_P(t) \\ \mathbf{u}_E(t) \end{bmatrix} = -\begin{bmatrix} \mathbf{C}_P(t) & \mathbf{C}_{PE}(t) \\ \mathbf{C}_{EP}(t) & \mathbf{C}_E(t) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_P(t) \\ \hat{\mathbf{x}}_E(t) \end{bmatrix}$$

Example of a *differential game*, Isaacs (1965), Bryson & Ho (1969)

Example: Simultaneous Location and Mapping (SLAM)





- Build or update a local map within an unknown environment
  - Stochastic map, defined by mean and covariance
  - SLAM Algorithm = State estimation with extended Kalman filter
  - Landmark and terrain tracking

Next Time: Minimization of Static Cost Functions

Reading: Optimal Control and Estimation (OCE): Chapter 1, Section 2.1

Supplemental Material

Math Review

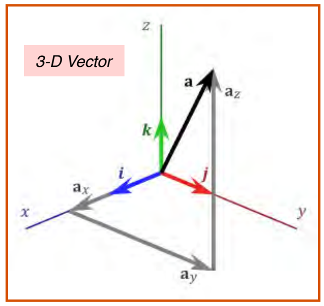
- Scalars and Vectors
- Matrices, Transpose, and Trace
- Sums and Multiplication
- Vector Products
- Matrix Products
- Derivatives, Integrals, and Identity Matrix
- Matrix Inverse

Scalars and Vectors

- **Scalar: usually lower case:** ***a, b, c, …, x, y, z***
- **Vector: usually bold or with underbar: x or x̲**
  - **Ordered set**
  - **Column of scalars**
  - **Dimension = *n* x 1**

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \; ; \quad \mathbf{y} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

3 x 1 *4 x 1*



- **Transpose**: interchange rows and columns

$$\mathbf{x}^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

1 x 3

Matrices and Transpose

- **Matrix**:
  - Usually bold capital or capital: **F** or F
  - Dimension = (*m* x *n*)

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \\ l & m & n \end{bmatrix}$$

4 x 3

- **Transpose**:
  - Interchange rows and columns

$$\mathbf{A}^T = \begin{bmatrix} a & d & g & l \\ b & e & h & m \\ c & f & k & n \end{bmatrix}$$

3 x 4

Trace of a Square Matrix

$$\text{Trace of } \mathbf{A} = \sum_{i=1}^{n} a_{ii}$$

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}; \quad Tr(\mathbf{A}) = a + e + i$$

Sums and Multiplication by a Scalar

- **Operands must be conformable**
- **Conformable vectors and matrices are added term by term**

$$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \; ; \; \mathbf{z} = \begin{bmatrix} c \\ d \end{bmatrix} \; ; \; \mathbf{x} + \mathbf{z} = \begin{bmatrix} (a+c) \\ (b+d) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \; ; \; \mathbf{B} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \; ; \; \mathbf{A} + \mathbf{B} = \begin{bmatrix} (a_1+b_1) & (a_2+b_2) \\ (a_3+b_3) & (a_4+b_4) \end{bmatrix}$$

- **Multiplication of vector by scalar is**
  - **associative**
  - **commutative**
  - **distributive**

$$a\mathbf{x} = \mathbf{x}a = \begin{bmatrix} ax_1 \\ ax_2 \\ ax_3 \end{bmatrix}$$

$$a\mathbf{x}^T = \begin{bmatrix} ax_1 & ax_2 & ax_3 \end{bmatrix}$$

Vector Products

- **Inner (dot) product of vectors produces a scalar**

$$\mathbf{x}^T\mathbf{x} = \mathbf{x} \bullet \mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1^2 + x_2^2 + x_3^2)$$

$(1 \times m)(m \times 1) = (1 \times 1)$

- **Outer product of vectors produces a matrix**

$$\mathbf{x}\mathbf{x}^T = \mathbf{x} \otimes \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 \end{bmatrix}$$

$(m \times 1)(1 \times m) = (m \times m)$

Matrix Products

- **Matrix-vector product transforms one vector into another**

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \\ l & m & n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 + cx_3 \\ dx_1 + ex_2 + fx_3 \\ gx_1 + hx_2 + kx_3 \\ lx_1 + mx_2 + nx_3 \end{bmatrix}$$

$(n \times 1) = (n \times m)(m \times 1)$

- **Matrix-matrix product produces a new matrix**

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \; ; \; \mathbf{B} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \; ; \; \mathbf{AB} = \begin{bmatrix} (a_1b_1 + a_2b_3) & (a_1b_2 + a_2b_4) \\ (a_3b_1 + a_4b_3) & (a_3b_2 + a_4b_4) \end{bmatrix}$$

$(n \times m) = (n \times l)(l \times m)$

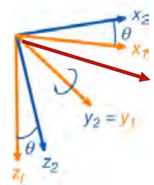
Examples

- **Inner product**

$$\mathbf{x}^T\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (1+4+9) = 14 = (length)^2$$

- **Rotation of expression for velocity vector through pitch angle**

$$\mathbf{y} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_2 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_1 = \begin{bmatrix} v_{x_1}\cos\theta + v_{z_1}\sin\theta \\ v_{y_1} \\ -v_{x_1}\cos\theta + v_{z_1}\sin\theta \end{bmatrix}$$



- **Matrix product**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}$$

Vector Transformation Example

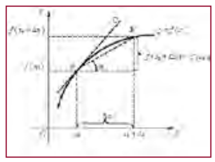
$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 2 & 4 & 6 \\ 3 & -5 & 7 \\ 4 & 1 & 8 \\ -9 & -6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (2x_1 + 4x_2 + 6x_3) \\ (3x_1 - 5x_2 + 7x_3) \\ (4x_1 + x_2 + 8x_3) \\ (-9x_1 - 6x_2 - 3x_3) \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$(n \times 1) = (n \times m)(m \times 1)$$

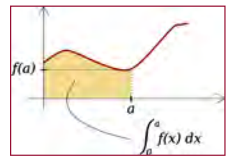
Derivatives and Integrals of Vectors

- Derivatives and integrals of vectors are vectors of derivatives and integrals

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{bmatrix}$$



$$\int \mathbf{x}\, dt = \begin{bmatrix} \int x_1\, dt \\ \int x_2\, dt \\ \int x_3\, dt \end{bmatrix}$$



Matrix Identity and Inverse

- **Identity matrix**: no change when it multiplies a conformable vector or matrix

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{y} = \mathbf{I}\mathbf{y}$$

- A non-singular square matrix multiplied by its inverse forms an identity matrix

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\begin{aligned}\mathbf{A}\mathbf{A}^{-1} &= \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

Matrix Inverse

- A non-singular square matrix multiplied by its inverse forms an identity matrix

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

- The inverse allows a reverse transformation of vectors of equal dimension

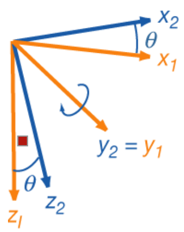
$$\mathbf{y} = \mathbf{A}\mathbf{x}; \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{y}; \quad \dim(\mathbf{x}) = \dim(\mathbf{y}) = (n \times 1); \quad \dim(\mathbf{A}) = (n \times n)$$

$$[\mathbf{A}]^{-1} = \frac{\mathrm{Adj}(\mathbf{A})}{|\mathbf{A}|} = \frac{\mathrm{Adj}(\mathbf{A})}{\det \mathbf{A}} \quad \frac{(n \times n)}{(1 \times 1)}$$

$$= \frac{\mathbf{C}^T}{\det \mathbf{A}}; \quad \mathbf{C} = \textit{matrix of cofactors}$$

Cofactors are signed minors of ***A***

ij^{th} *minor of* ***A*** *is the determinant of* ***A*** *with the* i^{th} *row and* j^{th} *column removed*



Matrix Inverse Example

Transformation

$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_1$$

Inverse Transformation

$$\mathbf{x}_1 = \mathbf{A}^{-1}\mathbf{x}_2$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_1 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_2$$