Applied Mathematics

Optimal Control of Partial Differential Equations

Theory, Methods and Applications

Fredi Tröltzsch

Graduate Studies in Mathematics

Volume 112

American Mathematical Society

http://dx.doi.org/10.1090/gsm/112

Optimal Control of Partial Differential Equations

Theory, Methods and Applications

Optimal Control of Partial Differential Equations

Theory, Methods and Applications

Fredi Tröltzsch

Translated by Jürgen Sprekels

Graduate Studies
in Mathematics

Volume 112

American Mathematical Society
Providence, Rhode Island

Originally published in German by Friedr. Vieweg & Sohn Verlag, 65189 Wiesbaden, Germany, under the title: "Fredi Tröltzsch: Optimale Steuerung partieller Differentialgleichungen." 1. Auflage (1st edition). © Friedr. Vieweg & Sohn Verlag/GWV Fachverlage GmbH, Wiesbaden, 2005

Translated by Jürgen Sprekels

2000 *Mathematics Subject Classification*. Primary 49–01, 49K20, 35J65, 35K60, 90C48, 35B37.

For additional information and updates on this book, visit
www.ams.org/bookpages/gsm-112

Library of Congress Cataloging-in-Publication Data

Tröltzsch, Fredi, 1951–
[Optimale Steuerung partieller Differentialgleichungen. English]
Optimal control of partial differential equations : theory, methods and applications / Fredi Tröltzsch.
p. cm. — (Graduate studies in mathematics : v. 112)
Includes bibliographical references and index.
ISBN 978-0-8218-4904-0 (alk. paper)
1. Control theory. 2. Differential equations, Partial. 3. Mathematical optimization. I. Title.

QA402.3.T71913 2010
515′.642—dc22

2009037756

---

10 9 8 7 6 5 4 3 2 1 15 14 13 12 11 10

To my wife Silvia

Contents

Preface to the English edition xi

Preface to the German edition xiii

Chapter 1. Introduction and examples 1
§1.1. What is optimal control? 1
§1.2. Examples of convex problems 3
§1.3. Examples of nonconvex problems 7
§1.4. Basic concepts for the finite-dimensional case 9

Chapter 2. Linear-quadratic elliptic control problems 21
§2.1. Normed spaces 21
§2.2. Sobolev spaces 24
§2.3. Weak solutions to elliptic equations 30
§2.4. Linear mappings 40
§2.5. Existence of optimal controls 48
§2.6. Differentiability in Banach spaces 56
§2.7. Adjoint operators 60
§2.8. First-order necessary optimality conditions 63
§2.9. Construction of test examples 80
§2.10. The formal Lagrange method 84
§2.11. Further examples * 89
§2.12. Numerical methods 91
§2.13. The adjoint state as a Lagrange multiplier * 106
§2.14. Higher regularity for elliptic problems 111
§2.15. Regularity of optimal controls 114

§2.16. Exercises 116

Chapter 3. Linear-quadratic parabolic control problems 119
§3.1. Introduction 119
§3.2. Fourier's method in the spatially one-dimensional case 124
§3.3. Weak solutions in $W_2^{1,0}(Q)$ 136
§3.4. Weak solutions in $W(0,T)$ 141
§3.5. Parabolic optimal control problems 153
§3.6. Necessary optimality conditions 156
§3.7. Numerical methods 166
§3.8. Derivation of Fourier expansions 171
§3.9. Linear continuous functionals as right-hand sides * 175
§3.10. Exercises 177

Chapter 4. Optimal control of semilinear elliptic equations 181
§4.1. Preliminary remarks 181
§4.2. A semilinear elliptic model problem 182
§4.3. Nemytskii operators 196
§4.4. Existence of optimal controls 205
§4.5. The control-to-state operator 211
§4.6. Necessary optimality conditions 215
§4.7. Application of the formal Lagrange method 220
§4.8. Pontryagin's maximum principle * 224
§4.9. Second-order derivatives 226
§4.10. Second-order optimality conditions 231
§4.11. Numerical methods 257
§4.12. Exercises 263

Chapter 5. Optimal control of semilinear parabolic equations 265
§5.1. The semilinear parabolic model problem 265
§5.2. Basic assumptions for the chapter 268
§5.3. Existence of optimal controls 270
§5.4. The control-to-state operator 273
§5.5. Necessary optimality conditions 277
§5.6. Pontryagin's maximum principle * 285
§5.7. Second-order optimality conditions 286
§5.8. Test examples 298

§5.9. Numerical methods 308
§5.10. Further parabolic problems * 313
§5.11. Exercises 321

Chapter 6. Optimization problems in Banach spaces 323
§6.1. The Karush–Kuhn–Tucker conditions 323
§6.2. Control problems with state constraints 338
§6.3. Exercises 353

Chapter 7. Supplementary results on partial differential equations 355
§7.1. Embedding results 355
§7.2. Elliptic equations 356
§7.3. Parabolic problems 366

Bibliography 385

Index 397

Preface to the English edition

In addition to correcting some misprints and inaccuracies in the German edition, some parts of this book were revised and expanded. The sections dealing with gradient methods were shortened in order to make space for primal-dual active set strategies; the exposition of the latter now leads to the systems of linear equations to be solved. Following the suggestions of several readers, a derivation of the associated Green's functions is provided, using Fourier's method. Moreover, some references are discussed in greater detail, and some recent references on the numerical analysis of state-constrained problems have been added.

The sections marked with an asterisk may be skipped; their contents are not needed to understand the subsequent sections. Within the text, the reader will find formulas in framed boxes. Such formulas contain either results of special importance or the partial differential equations being studied in that section.

I am indebted to all readers who have pointed out misprints and supplied me with suggestions for improvements—in particular, Roland Herzog, Markus Müller, Hans Josef Pesch, Lothar v. Wolfersdorf, and Arnd Rösch. Thanks are also due to Uwe Prüfert for his assistance with the LaTeX typesetting. In the revision of the results on partial differential equations, I was supported by Eduardo Casas and Jens Griepentrog; I am very grateful for their cooperation. Special thanks are due to Jürgen Sprekels for his careful and competent translation of this textbook into English. His suggestions have left their mark in many places. Finally, I have to thank Mrs. Jutta Lohse for her careful proofreading of the English translation.

Berlin, July 2009

F. Tröltzsch

Preface to the German edition

The mathematical optimization of processes governed by partial differential equations has seen considerable progress in the past decade. Ever faster computational facilities and newly developed numerical techniques have opened the door to important practical applications in fields such as fluid flow, microelectronics, crystal growth, vascular surgery, and cardiac medicine, to name just a few. As a consequence, the communities of numerical analysts and optimizers have taken a growing interest in applying their methods to optimal control problems involving partial differential equations; at the same time, the demand from students for this expertise has increased, and there is a growing need for textbooks that provide an introduction to the fundamental concepts of the corresponding mathematical theory.

There are a number of monographs devoted to various aspects of the optimal control of partial differential equations. In particular, the comprehensive text by J. L. Lions [**Lio71**] covers much of the theory of linear equations and convex cost functionals. However, the interest in the class notes of my lectures held at the technical universities in Chemnitz and Berlin revealed a clear demand for an introductory textbook that also includes aspects of nonlinear optimization in function spaces.

The present book is intended to meet this demand. We focus on basic concepts and notions such as:

- Existence theory for linear and semilinear partial differential equations
- Existence of optimal controls

- Necessary optimality conditions and adjoint equations
- Second-order sufficient optimality conditions
- Foundation of numerical methods

In this connection, we will always impose constraints on the control functions, and sometimes also on the state of the system under study. In order to keep the exposition to a reasonable length, we will not address further important subjects such as controllability, Riccati equations, discretization, error estimates, and Hamilton–Jacobi–Bellman theory.

The first part of the textbook deals with convex problems involving quadratic cost functionals and linear elliptic or parabolic equations. While these results are rather standard and have been treated comprehensively in [**Lio71**], they are well suited to facilitating the transition to problems involving semilinear equations. In order to make the theory more accessible to readers having only minor knowledge of these fields, some basic notions from functional analysis and the theory of linear elliptic and parabolic partial differential equations will also be provided.

The focus of the exposition is on nonconvex problems involving semilinear equations. Their treatment requires new techniques from analysis, optimization, and numerical analysis, which to a large extent can presently be found only in original papers. In particular, fundamental results due to E. Casas and J.-P. Raymond concerning the boundedness and continuity of solutions to semilinear equations will be needed.

This textbook is mainly devoted to the analysis of the problems, although numerical techniques will also be addressed. Numerical methods could easily fill another book. Our exposition is confined to brief introductions to the basic ideas, in order to give the reader an impression of how the theory can be realized numerically. Much attention will be paid to revealing hidden mathematical difficulties that, as experience shows, are likely to be overlooked.

The material covered in this textbook will not fit within a one-term course, so the lecturer will have to select certain parts. One possible strategy is to confine oneself to elliptic theory (linear-quadratic and nonlinear), while neglecting the chapters on parabolic equations. This would amount to concentrating on Sections 1.2–1.4, 2.3–2.10, and 2.12 for linear-quadratic theory, and on Sections 4.1–4.6 and 4.8–4.10 for nonlinear theory. The chapters devoted to elliptic problems do not require results from parabolic theory as a prerequisite.

Alternatively, one could select the linear-quadratic elliptic theory and add Sections 3.3–3.7 on linear-quadratic parabolic theory. Further topics can also be covered, provided that the students have a sufficient working knowledge of functional analysis and partial differential equations.

The sections marked with an asterisk may be skipped; their contents are not needed to understand the subsequent sections. Within the text, the reader will find formulas in framed boxes. Such formulas contain either results of special importance or the partial differential equations being studied in that section.

During the process of writing this book, I received much support from many colleagues. M. Hinze, P. Maaß, and L. v. Wolfersdorf read various chapters, in parts jointly with their students. W. Alt helped me with the typographical aspects of the exposition, and the first impetus to writing this textbook came from T. Grund, who put my class notes into a first LaTeX version. My colleagues C. Meyer, U. Prüfert, T. Slawig, and D. Wachsmuth in Berlin, and my students I. Neitzel and I. Yousept, proofread the final version. I am indebted to all of them. I also thank Mrs. U. Schmickler-Hirzebruch and Mrs. P. Rußkamp of Vieweg-Verlag for their very constructive cooperation during the preparation and implementation of this book project.

Berlin, April 2005

F. Tröltzsch

Bibliography

[ABHN01] C. Arendt, C. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Birkhäuser, Basel, 2001.

[ACT02] N. Arada, E. Casas, and F. Tröltzsch, *Error estimates for the numerical approximation of a semilinear elliptic control problem*, Comput. Optim. Appl. **23** (2002), 201–229.

[Ada78] R. A. Adams, *Sobolev Spaces*, Academic Press, Boston, 1978.

[AEFR00] N. Arada, H. El Fekih, and J.-P. Raymond, *Asymptotic analysis of some control problems*, Asymptot. Anal. **24** (2000), 343–366.

[AH01] K. Afanasiev and M. Hinze, *Adaptive control of a wake flow using proper orthogonal decomposition*, Shape Optimization and Optimal Design, Lect. Notes Pure Appl. Math., vol. 216, Marcel Dekker, 2001, pp. 317–332.

[Alt99] H. W. Alt, *Lineare Funktionalanalysis*, Springer, Berlin, 1999.

[Alt02] W. Alt, *Nichtlineare Optimierung*, Vieweg, Wiesbaden, 2002.

[AM84] W. Alt and U. Mackenroth, *On the numerical solution of state constrained coercive parabolic optimal control problems*, Optimal Control of Partial Differential Equations (K.-H. Hoffmann and W. Krabs, eds.), Int. Ser. Numer. Math., vol. 68, Birkhäuser, 1984, pp. 44–62.

[AM89] ———, *Convergence of finite element approximations to state constrained convex parabolic boundary control problems*, SIAM J. Control Optim. **27** (1989), 718–736.

[AM93] W. Alt and K. Malanowski, *The Lagrange–Newton method for nonlinear optimal control problems*, Comput. Optim. Appl. **2** (1993), 77–100.

[Ant05] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, SIAM, Philadelphia, 2005.

[App88] J. Appell, *The superposition operator in function spaces – A survey*, Expo. Math. **6** (1988), 209–270.

[AR97] J.-J. Alibert and J.-P. Raymond, *Boundary control of semilinear elliptic equations with discontinuous leading coefficients and unbounded controls*, Numer. Funct. Anal. Optim. **18** (1997), no. 3–4, 235–250.

[ART02] N. Arada, J.-P. Raymond, and F. Tröltzsch, *On an augmented Lagrangian SQP method for a class of optimal control problems in Banach spaces*, Comput. Optim. Appl. **22** (2002), 369–398.

[AT81] N. U. Ahmed and K. L. Teo, *Optimal Control of Distributed Parameter Systems*, North Holland, New York, 1981.

[AT90] F. Abergel and R. Temam, *On some control problems in fluid mechanics*, Theor. Comput. Fluid Dyn. **1** (1990), 303–325.

[AZ90] J. Appell and P. P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge University Press, Cambridge, 1990.

[Bal65] A. V. Balakrishnan, *Optimal control problems in Banach spaces*, SIAM J. Control **3** (1965), 152–180.

[Bar93] V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems*, Academic Press, Boston, 1993.

[BBEFR03] F. Ben Belgacem, H. El Fekih, and J.-P. Raymond, *A penalized Robin approach for solving a parabolic equation with nonsmooth Dirichlet boundary condition*, Asymptot. Anal. **34** (2003), 121–136.

[BC91] F. Bonnans and E. Casas, *Une principe de Pontryagine pour le contrôle des systèmes semilinéaires elliptiques*, J. Differential Equations **90** (1991), 288–303.

[BC95] ———, *An extension of Pontryagin's principle for state-constrained optimal control of semilinear elliptic equations and variational inequalities*, SIAM J. Control Optim. **33** (1995), 274–298.

[BDPDM92] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter, *Representation and Control of Infinite Dimensional Systems, Vol. I*, Birkhäuser, Basel, 1992.

[BDPDM93] ———, *Representation and Control of Infinite Dimensional Systems, Vol. II*, Birkhäuser, Basel, 1993.

[Ber82] D. M. Bertsekas, *Projected Newton methods for optimization problems with simple constraints*, SIAM J. Control Optim. **20** (1982), 221–246.

[Bet01] J. T. Betts, *Practical Methods for Optimal Control Using Nonlinear Programming*, SIAM, Philadelphia, 2001.

[BIK99] M. Bergounioux, K. Ito, and K. Kunisch, *Primal-dual strategy for constrained optimal control problems*, SIAM J. Control Optim. **37** (1999), 1176–1194.

[Bit75] L. Bittner, *On optimal control of processes governed by abstract functional, integral and hyperbolic differential equations*, Math. Methods Oper. Res. **19** (1975), 107–134.

[BK01] A. Borzi and K. Kunisch, *A multigrid method for optimal control of time-dependent reaction diffusion processes*, Fast Solution of Discretized Optimization Problems (K. H. Hoffmann, R. Hoppe, and V. Schulz, eds.), Int. Ser. Numer. Math., vol. 138, Birkhäuser, 2001, pp. 513–524.

[BK02a] M. Bergounioux and K. Kunisch, *On the structure of the Lagrange multiplier for state-constrained optimal control problems*, Systems Control Lett. **48** (2002), 16–176.

[BK02b] ———, *Primal-dual active set strategy for state-constrained optimal control problems*, Comput. Optim. Appl. **22** (2002), 193–224.

[BKR00] R. Becker, H. Kapp, and R. Rannacher, *Adaptive finite element methods for optimal control of partial differential equations: Basic concepts*, SIAM J. Control Optim. **39** (2000), 113–132.

[Bor03] A. Borzi, *Multigrid methods for parabolic distributed optimal control problems*, J. Comput. Appl. Math. **157** (2003), 365–382.

[BP78] V. Barbu and Th. Precupanu, *Convexity and Optimization in Banach Spaces*, Editura Academiei, Bucharest, and Sijthoff & Noordhoff, Leyden, 1978.

[BR01] R. Becker and R. Rannacher, *An optimal control approach to a posteriori error estimation in finite element methods*, Acta Numer. **10** (2001), 1–102.

[Bra07] D. Braess, *Finite Elements: Theory, Fast Solvers, and Applications in Elasticity Theory,* 4th Edition, Cambridge University Press, Cambridge, 2007.

[BS94] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, New York, 1994.

[BS96] M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions*, Springer, New York, 1996.

[BS00] F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, New York, 2000.

[But69] A. G. Butkovskii, *Distributed Control Systems*, American Elsevier, New York, 1969.

[But75] ______, *Methods for the Control of Distributed Parameter Systems* (in Russian), Isd. Nauka, Moscow, 1975.

[Car67] H. Cartan, *Calcul Différentiel. Formes Différentielles*, Hermann, Paris, 1967.

[Cas86] E. Casas, *Control of an elliptic problem with pointwise state constraints*, SIAM J. Control Optim. **4** (1986), 1309–1322.

[Cas92] ______, *Introduccion a las Ecuaciones en Derivadas Parciales*, Universidad de Cantabria, Santander, 1992.

[Cas93] ______, *Boundary control of semilinear elliptic equations with pointwise state constraints*, SIAM J. Control Optim. **31** (1993), 993–1006.

[Cas95] ______, *Optimality conditions for some control problems of turbulent flow*, Flow Control (New York) (M. D. Gunzburger, ed.), Springer, 1995, pp. 127–147.

[Cas97] ______, *Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations*, SIAM J. Control Optim. **35** (1997), 1297–1327.

[CDlRT08] E. Casas, J. C. De los Reyes, and F. Tröltzsch, *Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints*, SIAM J. Optim. **12** (2008), no. 2, 616–643.

[CH91] Z. Chen and K. H. Hoffmann, *Numerical solutions of the optimal control problem governed by a phase field model*, Estimation and Control of Distributed Parameter Systems (W. Desch, F. Kappel, and K. Kunisch, eds.), Int. Ser. Numer. Math., vol. 100, Birkhäuser, 1991, pp. 79–97.

[Cia78] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[CM02a] E. Casas and M. Mateos, *Second order sufficient optimality conditions for semilinear elliptic control problems with finitely many state constraints*, SIAM J. Control Optim. **40** (2002), 1431–1454.

[CM02b] ______, *Uniform convergence of the FEM. Applications to state constrained control problems*, J. Comput. Appl. Math. **21** (2002), 67–100.

[CMT05] E. Casas, M. Mateos, and F. Tröltzsch, *Error estimates for the numerical approximation of boundary semilinear elliptic control problems*, Comput. Optim. Appl. **31** (2005), 193–220.

[Con90] J. B. Conway, *A Course in Functional Analysis*, Wiley & Sons, Warsaw/Dordrecht, 1990.

[Cor07] J. M. Coron, *Control and Nonlinearity*, American Mathematical Society, Providence, 2007.

[CR06] E. Casas and J.-P. Raymond, *Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations*, SIAM J. Control Optim. **45** (2006), 1586–1611.

[CT99] E. Casas and F. Tröltzsch, *Second-order necessary optimality conditions for some state-constrained control problems of semilinear elliptic equations*, Appl. Math. Optim. **39** (1999), 211–227.

[CT02] ______, *Second-order necessary and sufficient optimality conditions for optimization problems and applications to control theory*, SIAM J. Optim. **13** (2002), 406–431.

[CTU96] E. Casas, F. Tröltzsch, and A. Unger, *Second order sufficient optimality conditions for a nonlinear elliptic control problem*, Z. Anal. Anwendungen (ZAA) **15** (1996), 687–707.

[CTU00] ______, *Second order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations*, SIAM J. Control Optim. **38** (2000), no. 5, 1369–1391.

[Deu04] P. Deuflhard, *Newton Methods for Nonlinear Problems. Affine Invariance and Adaptive Algorithms*, Springer Series in Computational Mathematics, vol. 35, Springer, Berlin, 2004.

[DHPY95] A. L. Dontchev, W. W. Hager, A. B. Poore, and B. Yang, *Optimality, stability, and convergence in nonlinear control*, Appl. Math. Optim. **31** (1995), 297–326.

[DT09] V. Dhamo and F. Tröltzsch, *Some aspects of reachability for parabolic boundary control problems with control constraints*, accepted for publication in: Comput. Optim. Appl. (2009).

[Dun98] J. C. Dunn, *On second order sufficient optimality conditions for structured nonlinear programs in infinite-dimensional function spaces*, Mathematical Programming with Data Perturbations (A. Fiacco, ed.), Marcel Dekker, 1998, pp. 83–107.

[ET74] I. Ekeland and R. Temam, *Analyse Convexe et Problèmes Variationnels*, Dunod, Gauthier-Villars, 1974.

[ET86] K. Eppler and F. Tröltzsch, *On switching points of optimal controls for coercive parabolic boundary control problems*, Optimization **17** (1986), 93–101.

[Eva98] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, 1998.

[Fat99] H. O. Fattorini, *Infinite Dimensional Optimization and Control Theory*, Cambridge University Press, Cambridge, 1999.

[Fri64] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, 1964.

[Fur99] A. V. Fursikov, *Optimal Control of Distributed Systems. Theory and Applications*, American Mathematical Society, Providence, 1999.

[Gal94] P. G. Galdi, *An Introduction to the Navier–Stokes Equations*, Springer, New York, 1994.

[GGZ74] H. Gajewski, K. Gröger, and K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie-Verlag, Berlin, 1974.

[GKT92] H. Goldberg, W. Kampowsky, and F. Tröltzsch, *On Nemytskij operators in L_p-spaces of abstract functions*, Math. Nachr. **155** (1992), 127–140.

[GM99] M. D. Gunzburger and S. Manservisi, *The velocity tracking problem for Navier–Stokes flows with bounded distributed controls*, SIAM J. Control Optim. **37** (1999), 1913–1945.

[GM00] ———, *Analysis and approximation of the velocity tracking problem for Navier–Stokes flows with distributed control*, SIAM J. Numer. Anal. **37** (2000), 1481–1512.

[GMW81] P. E. Gill, W. Murray, and M. H. Wright, *Practical Optimization*, Academic Press, London, 1981.

[Goe92] M. Goebel, *Continuity and Fréchet-differentiability of Nemytskij operators in Hölder spaces*, Monatsh. Math. **113** (1992), 107–119.

[GR01] T. Grund and A. Rösch, *Optimal control of a linear elliptic equation with a supremum-norm functional*, Optim. Methods Softw. **15** (2001), 299–329.

[GR05] C. Grossmann and H.-G. Roos, *Numerische Behandlung partieller Differentialgleichungen*, Teubner, Wiesbaden, 2005.

[Gri85] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.

[Gri02] J. A. Griepentrog, *Linear elliptic boundary value problems with non-smooth data: Campanato spaces of functionals*, Math. Nachr. **243** (2002), 19–42.

[Gri07a] ———, *Maximal regularity for nonsmooth parabolic problems in Sobolev–Morrey spaces*, Adv. Differential Equations **12** (2007), 1031–1078.

[Gri07b] ———, *Sobolev–Morrey spaces associated with evolution equations*, Adv. Differential Equations **12** (2007), 781–840.

[GS77] K. Glashoff and E. W. Sachs, *On theoretical and numerical aspects of the bang-bang principle*, Numer. Math. **29** (1977), 93–113.

[GS80] W. A. Gruver and E. W. Sachs, *Algorithmic Methods in Optimal Control*, Pitman, London, 1980.

[GT93] H. Goldberg and F. Tröltzsch, *Second-order sufficient optimality conditions for a class of nonlinear parabolic boundary control problems*, SIAM J. Control Optim. **31** (1993), 1007–1025.

[GT97a] ———, *On a SQP-multigrid technique for nonlinear parabolic boundary control problems*, Optimal Control: Theory, Algorithms and Applications (Dordrecht) (W. W. Hager and P. M. Pardalos, eds.), Kluwer Academic Publishers, 1997, pp. 154–177.

[GT97b] C. Grossmann and J. Terno, *Numerik der Optimierung*, Teubner, Stuttgart, 1997.

[Gun95] M. D. Gunzburger (ed.), *Flow Control*, Springer, New York, 1995.

[Gun03] ———, *Perspectives in Flow Control and Optimization*, SIAM, Philadelphia, 2003.

[GW76] K. Glashoff and N. Weck, *Boundary control of parabolic differential equations in arbitrary dimensions: Supremum-norm problems*, SIAM J. Control Optim. **14** (1976), 662–681.

[HDMRS09] R. Haller-Dintelmann, C. Meyer, J. Rehberg, and A. Schiela, *Hölder continuity and optimal control for nonsmooth elliptic problems*, Appl. Math. Optim. **60** (2009), 397–428.

[Hei96] M. Heinkenschloss, *Projected sequential quadratic programming methods*, SIAM J. Optim. **6** (1996), 373–417.

[Hei97] ———, *The numerical solution of a control problem governed by a phase field model*, Optim. Methods Softw. **7** (1997), 211–263.

[Heu08] H. Heuser, *Lehrbuch der Analysis, Teil 2*, Vieweg+Teubner, Wiesbaden, 2008.

[Hin99] M. Hinze, *Optimal and instantaneous control of the instationary Navier–Stokes equations*, Habilitation thesis, Technische Universität Berlin, 1999.

[Hin05] ———, *A variational discretization concept in control constrained optimization: The linear-quadratic case*, Comput. Optim. Appl. **30** (2005), 45–63.

[HJ92] K.-H. Hoffmann and L. Jiang, *Optimal control problem of a phase field model for solidification*, Numer. Funct. Anal. Optim. **13** (1992), 11–27.

[HK01] M. Hinze and K. Kunisch, *Second order methods for optimal control of time-dependent fluid flow*, SIAM J. Control Optim. **40** (2001), 925–946.

[HK04] ———, *Second order methods for boundary control of the instationary Navier–Stokes system*, Z. Angew. Math. Mech. (ZAMM) **84** (2004), 171–187.

[HP57] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, Colloquium Publications, vol. 31, American Mathematical Society, Providence, 1957.

[HPUU09] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich, *Optimization with PDE Constraints*, Mathematical Modelling: Theory and Applications, vol. 23, Springer, Berlin, 2009.

[HS94] M. Heinkenschloss and E. W. Sachs, *Numerical solution of a constrained control problem for a phase field model*, Control and Estimation of Distributed Parameter Systems (W. Desch, F. Kappel, and K. Kunisch, eds.), Int. Ser. Numer. Math., vol. 118, Birkhäuser, 1994, pp. 171–188.

[HT99] M. Heinkenschloss and F. Tröltzsch, *Analysis of the Lagrange-SQP-Newton method for the control of a phase field equation*, Control Cybernet. **28** (1999), no. 2, 178–211.

[HV01] M. Heinkenschloss and L. N. Vicente, *Analysis of inexact trust-region SQP algorithms*, SIAM J. Optim. **12** (2001), 283–302.

[HV03] D. Hömberg and S. Volkwein, *Control of laser surface hardening by a reduced-order approach using proper orthogonal decomposition*, Math. Comput. Modelling **38** (2003), 1003–1028.

[IK96] K. Ito and K. Kunisch, *Augmented Lagrangian-SQP methods for nonlinear optimal control problems of tracking type*, SIAM J. Control Optim. **34** (1996), 874–891.

[IK00] ———, *Augmented Lagrangian methods for nonsmooth, convex optimization in Hilbert spaces*, Nonlinear Anal., Theory Methods Appl. **41** (2000), 591–616.

[IK08] ———, *Lagrange Multiplier Approach to Variational Problems and Applications*, SIAM, Philadelphia, 2008.

[Iof79] A. D. Ioffe, *Necessary and sufficient conditions for a local minimum. 3: Second order conditions and augmented duality*, SIAM J. Control Optim. **17** (1979), 266–288.

[IT79] A. D. Ioffe and V. M. Tihomirov, *Theory of Extremal Problems*, North-Holland, Amsterdam, 1979.

[Jah94] J. Jahn, *Introduction to the Theory of Nonlinear Optimization*, Springer, Berlin, 1994.

[JK81] D. S. Jerison and C. E. Kenig, *The Neumann problem on Lipschitz domains*, Bull. Amer. Math. Soc., New Ser. **4** (1981), 203–207.

[KA64] L. V. Kantorovich and G. B. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1964.

[Kar77] A. Karafiat, *The problem of the number of switches in parabolic equations with control*, Ann. Pol. Math. **XXXIV** (1977), 289–316.

[Kel99] C. T. Kelley, *Iterative Methods for Optimization*, SIAM, Philadelphia, 1999.

[KK02] D. Klatte and B. Kummer, *Nonsmooth Equations in Optimization: Regularity, Calculus, Methods and Applications*, Kluwer Academic Publishers, Dordrecht, 2002.

[Kno77] G. Knowles, *Über das Bang-Bang-Prinzip bei Kontrollproblemen aus der Wärmeleitung*, Arch. Math. **29** (1977), 300–309.

[KR02] K. Kunisch and A. Rösch, *Primal-dual active set strategy for a general class of constrained optimal control problems*, SIAM J. Optim. **13** (2002), 321–334.

[Kra95] W. Krabs, *Optimal Control of Undamped Linear Vibrations*, Heldermann, Lemgo, 1995.

[Kre78] E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley, New York, 1978.

[KS80] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[KS92] F.-S. Kupfer and E. W. Sachs, *Numerical solution of a nonlinear parabolic control problem by a reduced SQP method*, Comput. Optim. Appl. **1** (1992), 113–135.

[KS94] C. T. Kelley and E. W. Sachs, *Multilevel algorithms for constrained compact fixed point problems*, SIAM J. Sci. Comput. **15** (1994), 645–667.

[KS95] ———, *Solution of optimal control problems by a pointwise projected Newton method*, SIAM J. Control Optim. **33** (1995), 1731–1757.

[KV01] K. Kunisch and S. Volkwein, *Galerkin proper orthogonal decomposition methods for parabolic problems*, Numer. Math. **90** (2001), 117–148.

[KV02] ———, *Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics*, SIAM J. Numer. Anal. **40** (2002), 492–515.

[KV07] K. Kunisch and B. Vexler, *Constrained Dirichlet boundary control in L^2 for a class of evolution equations*, SIAM J. Control Optim. **46** (2007), 1726–1753.

[KZPS76] M. A. Krasnoselskii, P. P. Zabreiko, E. I. Pustylnik, and P. E. Sobolevskii, *Integral Operators in Spaces of Summable Functions*, Noordhoff, Leyden, 1976.

[Las02] I. Lasiecka, *Mathematical Control Theory of Coupled PDEs*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 75, SIAM, Philadelphia, 2002.

[Lio69] J. L. Lions, *Quelques Méthodes des Résolution des Problèmes aux Limites non Linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[Lio71] ———, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer, Berlin, 1971.

[LLS94] J. E. Lagnese, G. Leugering, and E. J. P. G. Schmidt, *Modeling, Analysis and Control of Dynamic Elastic Multi-Link Structures*, Birkhäuser, Boston, 1994.

[LM72] J. L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems and Applications*, vol. 1–3, Springer, Berlin, 1972.

[LS74] L. A. Lusternik and V. J. Sobolev, *Elements of Functional Analysis*, Wiley & Sons, New York, 1974.

[LS00] F. Leibfritz and E. W. Sachs, *Inexact SQP interior point methods and large scale optimal control problems*, SIAM J. Control Optim. **38** (2000), 272–293.

[LS07] C. Lefter and J. Sprekels, *Control of a phase field system modeling non-isothermal phase transitions*, Adv. Math. Sci. Appl. **17** (2007), 181–194.

[LSU68] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, 1968.

[LT00a] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories. I: Abstract Parabolic Systems*, Cambridge University Press, Cambridge, 2000.

[LT00b] ______, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories. II: Abstract Hyperbolic-like Systems over a Finite Time Horizon*, Cambridge University Press, Cambridge, 2000.

[LU73] O. A. Ladyzhenskaya and N. N. Ural'ceva, *Linear and Quasilinear Equations of Elliptic Type* (in Russian), Izd. Nauka, Moscow, 1973.

[Lue69] D. G. Luenberger, *Optimization by Vector Space Methods*, Wiley, New York, 1969.

[Lue84] ______, *Linear and Nonlinear Programming*, Addison Wesley, Reading, Massachusetts, 1984.

[LY95] X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Birkhäuser, Boston, 1995.

[Mac81] U. Mackenroth, *Time-optimal parabolic boundary control problems with state constraints*, Numer. Funct. Anal. Optim. **3** (1981), 285–300.

[Mac82] ______, *Convex parabolic boundary control problems with state constraints*, J. Math. Anal. Appl. **87** (1982), 256–277.

[Mac83a] ______, *On a parabolic distributed optimal control problem with restrictions on the gradient*, Appl. Math. Optim. **10** (1983), 69–95.

[Mac83b] ______, *Some remarks on the numerical solution of bang-bang type optimal control problems*, Numer. Funct. Anal. Optim. **5** (1983), 457–484.

[Mal81] K. Malanowski, *Convergence of approximations vs. regularity of solutions for convex, control-constrained optimal control problems*, Appl. Math. Optim. **8** (1981), 69–95.

[Mau81] H. Maurer, *First and second order sufficient optimality conditions in mathematical programming and optimal control*, Math. Program. Study **14** (1981), 163–177.

[MM00] H. Maurer and H. D. Mittelmann, *Optimization techniques for solving elliptic control problems with control and state constraints. I: Boundary control*, J. Comput. Appl. Math. **16** (2000), 29–55.

[MM01] ______, *Optimization techniques for solving elliptic control problems with control and state constraints. II: Distributed control*, J. Comput. Appl. Math. **18** (2001), 141–160.

[MR04] C. Meyer and A. Rösch, *Superconvergence properties of optimal control problems*, SIAM J. Control Optim. **43** (2004), 970–985.

[MRT06] C. Meyer, A. Rösch, and F. Tröltzsch, *Optimal control of PDEs with regularized pointwise state constraints*, Comput. Optim. Appl. **33** (2006), 209–228.

[MS82] J. Macki and A. Strauss, *Introduction to Optimal Control Theory*, Springer, Berlin, 1982.

[MS00] B. Maar and V. Schulz, *Interior point multigrid methods for topology optimization*, Structural Optimization **19** (2000), 214–224.

[MT02] H. D. Mittelmann and F. Tröltzsch, *Sufficient optimality in a parabolic control problem*, Trends in Industrial and Applied Mathematics (Dordrecht) (A. H. Siddiqi and M. Kocvara, eds.), Kluwer Academic Publishers, 2002, pp. 305–316.

[MT06] C. Meyer and F. Tröltzsch, *On an elliptic optimal control problem with pointwise mixed control-state constraints*, Recent Advances in Optimization. Proceedings of the 12th French-German-Spanish Conference on Optimization held in Avignon, September 20–24, 2004 (A. Seeger, ed.), Lect. Notes Econ. Math. Syst., vol. 563, Springer, 2006, pp. 187–204.

[MZ79] H. Maurer and J. Zowe, *First- and second-order conditions in infinite-dimensional programming problems*, Math. Program. **16** (1979), 98–110.

[Nec67] J. Nečas, *Les Méthodes Directes en Théorie des Equations Elliptiques*, Academia, Prague, 1967.

[Nit09] R. Nittka, *Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains*, arXiv:0906.5285v1, June 2009.

[NPS09] I. Neitzel, U. Prüfert, and T. Slawig, *Strategies for time-dependent PDE control with inequality constraints using an integrated modeling and simulation environment*, Numer. Algorithms **50** (2009), 241–269.

[NST06] P. Neittaanmäki, J. Sprekels, and D. Tiba, *Optimization of Elliptic Systems: Theory and Applications*, Springer, Berlin, 2006.

[NT94] P. Neittaanmäki and D. Tiba, *Optimal Control of Nonlinear Parabolic Systems: Theory, Algorithms, and Applications*, Marcel Dekker, New York, 1994.

[NW99] J. Nocedal and S. J. Wright, *Numerical Optimization*, Springer, New York, 1999.

[Paz83] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[PBGM62] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, Wiley, New York, 1962.

[Pen82] J.-P. Penot, *On regularity conditions in mathematical programming*, Math. Program. Study **19** (1982), 167–199.

[Pol97] E. Polak, *Optimization: Algorithms and Consistent Approximations*, Applied Math. Sciences, vol. 124, Springer, New York, 1997.

[Rob80] S. M. Robinson, *Strongly regular generalized equations*, Math. Oper. Res. **5** (1980), 43–62.

[Rös04] A. Rösch, *Error estimates for parabolic optimal control problems with control constraints*, Z. Anal. Anwendungen (ZAA) **23** (2004), 353–376.

[Rou02] T. Roubíček, *Optimization of steady-state flow of incompressible fluids*, Analysis and Optimization of Differential Systems (Boston) (V. Barbu, I. Lasiecka, D. Tiba, and C. Varsan, eds.), Kluwer Academic Publishers, 2002, pp. 357–368.

[RT00] J.-P. Raymond and F. Tröltzsch, *Second order sufficient optimality conditions for nonlinear parabolic control problems with state constraints*, Discrete Contin. Dyn. Syst. **6** (2000), 431–450.

[RT03] T. Roubíček and F. Tröltzsch, *Lipschitz stability of optimal controls for the steady state Navier–Stokes equations*, Control Cybernet. **32** (2003), 683–705.

[RW08] A. Rösch and D. Wachsmuth, *Numerical verification of optimality conditions*, SIAM J. Control Optim. **47** (2008), no. 5, 2557–2581.

[RZ98] J.-P. Raymond and H. Zidani, *Pontryagin's principle for state-constrained control problems governed by parabolic equations with unbounded controls*, SIAM J. Control Optim. **36** (1998), 1853–1879.

[RZ99] ______, *Hamiltonian Pontryagin's principles for control problems governed by semilinear parabolic equations*, Appl. Math. Optim. **39** (1999), 143–177.

[Sac78] E. W. Sachs, *A parabolic control problem with a boundary condition of the Stefan–Boltzmann type*, Z. Angew. Math. Mech. (ZAMM) **58** (1978), 443–449.

[Sch79] K. Schittkowski, *Numerical solution of a time-optimal parabolic boundary-value control problem*, J. Optimization Theory Appl. **27** (1979), 271–290.

[Sch80] E. J. P. G. Schmidt, *The bang-bang principle for the time-optimal problem in boundary control of the heat equation*, SIAM J. Control Optim. **18** (1980), 101–107.

[Sch89] ______, *Boundary control for the heat equation with non-linear boundary condition*, J. Differential Equations **78** (1989), 89–121.

[Spe93] P. Spellucci, *Numerische Verfahren der nichtlinearen Optimierung*, Birkhäuser, Basel, 1993.

[Sta65] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier, Grenoble **15** (1965), 189–258.

[SW98] V. Schulz and G. Wittum, *Multigrid optimization methods for stationary parameter identification problems in groundwater flow*, Multigrid Methods V (W. Hackbusch and G. Wittum, eds.), Lect. Notes Comput. Sci. Eng., vol. 3, Springer, 1998, pp. 276–288.

[SZ92] J. Sprekels and S. Zheng, *Optimal control problems for a thermodynamically consistent model of phase-field type for phase transitions*, Adv. Math. Sci. Appl. **1** (1992), 113–125.

[Tan79] H. Tanabe, *Equations of Evolution*, Pitman, London, 1979.

[Tem79] R. Temam, *Navier–Stokes Equations*, North-Holland, Amsterdam, 1979.

[Tib90] D. Tiba, *Optimal Control of Nonsmooth Distributed Parameter Systems*, Lect. Notes Math., vol. 1459, Springer, Berlin, 1990.

[Tri95] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, J. A. Barth, Heidelberg-Leipzig, 1995.

[Trö84a] F. Tröltzsch, *The generalized bang-bang principle and the numerical solution of a parabolic boundary-control problem with constraints on the control and the state*, Z. Angew. Math. Mech. (ZAMM) **64** (1984), 551–557.

[Trö84b] ———, *Optimality Conditions for Parabolic Control Problems and Applications*, Teubner Texte zur Mathematik, vol. 62, Teubner, Leipzig, 1984.

[Trö99] ———, *On the Lagrange-Newton-SQP method for the optimal control of semilinear parabolic equations*, SIAM J. Control Optim. **38** (1999), 294–312.

[Trö00] ———, *Lipschitz stability of solutions of linear-quadratic parabolic control problems with respect to perturbations*, Dyn. Contin. Discrete Impulsive Syst. **7** (2000), 289–306.

[TS64] A. N. Tychonov and A. A. Samarski, *Partial Differential Equations of Mathematical Physics, Vol. I*, Holden-Day, San Francisco, 1964.

[TT96] D. Tiba and F. Tröltzsch, *Error estimates for the discretization of state constrained convex control problems*, Numer. Funct. Anal. Optim. **17** (1996), 1005–1028.

[TW06] F. Tröltzsch and D. Wachsmuth, *Second-order sufficient optimality conditions for the optimal control of Navier–Stokes equations*, ESAIM: Control Optim. Calc. Var. **12** (2006), 93–119.

[Ung97] A. Unger, *Hinreichende Optimalitätsbedingungen 2. Ordnung und Konvergenz des SQP-Verfahrens für semilineare elliptische Randsteuerprobleme*, Ph.D. thesis, Technische Universität Chemnitz, 1997.

[UU00] M. Ulbrich and S. Ulbrich, *Superlinear convergence of affine-scaling interior point Newton methods for infinite-dimensional nonlinear problems with pointwise bounds*, SIAM J. Control Optim. **38** (2000), 1938–1984.

[UUH99] M. Ulbrich, S. Ulbrich, and M. Heinkenschloss, *Global convergence of trust-region interior-point algorithms for infinite-dimensional nonconvex minimization subject to pointwise bounds*, SIAM J. Control Optim. **37** (1999), 731–764.

[Vex07] B. Vexler, *Finite element approximation of elliptic Dirichlet optimal control problems*, Numer. Funct. Anal. Optim. **28** (2007), 957–973.

[Vol01] S. Volkwein, *Optimal control of a phase-field model using proper orthogonal decomposition*, Z. Angew. Math. Mech. (ZAMM) **81** (2001), 83–97.

[vW76] L. v. Wolfersdorf, *Optimal control for processes governed by mildly nonlinear differential equations of parabolic type I*, Z. Angew. Math. Mech. (ZAMM) **56** (1976), 531–538.

[vW77] ———, *Optimal control for processes governed by mildly nonlinear differential equations of parabolic type II*, Z. Angew. Math. Mech. (ZAMM) **57** (1977), 11–17.

[War06] M. Warma, *The Robin and Wentzell–Robin Laplacians on Lipschitz domains*, Semigroup Forum **73** (2006), no. 1, 10–30.

[Wer97] D. Werner, *Funktionalanalysis*, Springer, Berlin, 1997.

[Wlo82] J. Wloka, *Partielle Differentialgleichungen*, Teubner, Leipzig, 1982.

[Wlo87] ———, *Partial Differential Equations*, Cambridge University Press, Cambridge, 1987.

[Wou79] A. Wouk, *A Course of Applied Functional Analysis*, Wiley, New York, 1979.

[Wri93] S. J. Wright, *Primal-Dual Interior-Point Methods*, SIAM, Philadelphia, 1993.

[WS04] M. Weiser and A. Schiela, *Function space interior point methods for PDE constrained optimization*, Proc. Appl. Math. Mech. (PAMM) **4** (2004), 43–46.

[Yos80] K. Yosida, *Functional Analysis*, Springer, New York, 1980.

[Zei86] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-point Theorems*, Springer, New York, 1986.

[Zei90a] ______, *Nonlinear Functional Analysis and its Applications II/A: Linear Monotone Operators*, Springer, New York, 1990.

[Zei90b] ______, *Nonlinear Functional Analysis and its Applications II/B: Nonlinear Monotone Operators*, Springer, New York, 1990.

[Zei95] ______, *Applied Functional Analysis and its Applications. Main Principles and their Applications*, Springer, New York, 1995.

[ZK79] J. Zowe and S. Kurcyusz, *Regularity and stability for the mathematical programming problem in Banach spaces*, Appl. Math. Optim. **5** (1979), 49–62.

Index

active set strategy, 101, 106

Banach space, 23
bang-bang
  control, 80
  principle, 133
bilinear form, 31, 165, 227
bisection method, 257
Bochner integral, 143
Borel measure
  regular, 342
boundary condition
  inhomogeneous Dirichlet, 39
  Neumann, 82, 113, 124, 125, 190
  of the third kind, 34
  Robin, 34
  Stefan–Boltzmann, 7, 8, 220, 223, 276, 283, 299
boundary control, 4
boundary observation, 55, 154
boundedness condition, 197
  of order k, 199

c (generic constant), 36
$C[a,b]$, 22
$C([a,b],X)$, 142
$C_0^\infty(\Omega)$, 25
Carathéodory condition, 197, 204
chain rule, 60
cone
  τ-critical, 296
  convex, 324
  critical, 245
  dual, 324
conormal, 37
constraint
  strongly active, 233, 251, 290, 296
constraint qualification, 330
  Zowe–Kurcyusz, 330
control
  admissible, 49
  bang-bang, 70
  distributed, 5
  locally optimal, 207, 271
  optimal, 49, 207, 271
control-to-state operator, 50
convergence
  strong, 22
  weak, 44
cost functional, 4

ds, 4
D_i, D_x, 11
∂_ν, 31
$\partial_{\nu_\mathcal{A}}$, 37
derivative
  Fréchet, 59
  Gâteaux, 56
  weak, 27
descent direction, 92
differential operator
  elliptic, 37
  in divergence form, 37, 163
Dirac measure, 344
directional derivative, 56
distribution
  vector-valued, 145
domain, 25
  Lipschitz, 26
  of class $C^{k,1}$, 26

E_Y, 50
ellipticity
uniform, 37
embedding
compact, 356
continuous, 355
equation
adjoint, 13, 67, 76, 122, 159, 163, 166, 216, 219, 279, 342
generalized, 259
semilinear, 7
semilinear elliptic, 7, 8
semilinear parabolic, 8, 9
error analysis, 106

finite difference method, 96
first-order condition
sufficient, 250
formulation
weak, 31
Fréchet derivative
continuous, 203
first-order, 275
of a Nemytskii operator, 202, 204
second-order, 226, 229, 230, 238, 242, 288
function
Green's, 126, 136, 381
measurable vector-valued, 142
vector-valued, 141
functional, 40
convex, 47
linear, 32, 40
reduced, 50
strictly convex, 47
weakly lower semicontinuous, 47

Γ, 3
Gelfand triple, 147
gradient, 11, 58
reduced, 13, 73, 77, 88
gradient method
conditioned, 92
projected, 95, 167, 308
growth condition, 204
quadratic, 231, 237, 248, 250, 254, 255, 257, 291, 296

$H^k(\Omega)$, $H_0^k(\Omega)$, 28
$H^s(\Gamma)$, 112
Hamiltonian function, 285
heat source, 4
Hilbert space, 24

index
conjugated, 43
inequality
Cauchy–Schwarz, 23
Friedrichs, 33
generalized Friedrichs, 35
generalized Poincaré, 35
Hölder, 43
Poincaré, 35
integral operator, 40, 62
adjoint, 129
integration by parts, 27, 148

Karush–Kuhn–Tucker
conditions, 18
system, 73, 351

$L^p(a,b;X)$, 143
$L^p(E)$, 24
$\mathcal{L}(U,V)$, 41
Lagrangian function, 15, 17, 87, 89, 120, 221, 325, 336
Lax–Milgram lemma, 32
Lipschitz condition
local, of order k, 199
Lipschitz continuity
local, 197
Lipschitz domain, 26

$M(\bar{\Omega})$, 340, 366
main theorem
on monotone operators, 185
mapping
continuous, 40
mass matrix, 104
maximum condition, 226
maximum norm, 111
minimization of, 345
maximum principle
Pontryagin's, 226, 286
minimum principle, 70, 78
weak, 70
multiplier
Lagrange, 15, 17, 73, 85, 110, 325, 328–330, 351
multiplier rule, 15, 331

ν, 4
Navier–Stokes equations
nonstationary, 9, 316
stationary, 8
Nemytskii operator, 196, 197
Neumann problem, 190
Newton method, 257–259
projected, 106
norm
axioms, 21
of a linear operator, 41
normal derivative, 31

observation operator, 154, 162

operator
  adjoint, 61
  bounded, 40
  continuous, 40
  convex, 325
  dual, 61
  monotone, 184
optimality system, 14, 67, 73, 161, 343, 351
optimization problem
  quadratic, in Hilbert space, 50

partial ordering, 324
phase field model, 9, 313
Poisson's equation, 30
problem
  reduced, 11, 98, 168
projection formula, 70, 78, 131, 133, 161, 163, 217, 281, 284

Q, 5

regularization parameter, 4, 155
remainder, 237
  in integral form, 235, 236

S, 50
Σ, 5
saddle point, 325
second-order condition
  necessary, 246, 247
  sufficient, 247–249, 254, 255, 257, 262, 291, 296, 300, 305
semigroup, 136
set
  convex, 47
  strongly active, 251, 253, 290, 296
  weakly sequentially closed, 47
Slater condition, 326, 329
  linearized, 332
Sobolev space, 28
Sobolev–Slobodetskii space, 112
solution
  generalized, 127
  weak, 31, 140, 164, 186, 191, 266, 267, 317, 366
space
  bidual, 43
  complete, 23
  dual, 42
  normed, 21
  reflexive, 43
SQP method, 258, 259, 262, 309
state
  adjoint, 13, 67
  optimal, 207
state constraint
  integral, 302
  pointwise, 111, 337, 339, 346, 349
state equation, 3
step function, 98, 142, 168
step size
  Armijo, 96, 257
  exact, 94
  for bisection method, 95, 257
stiffness matrix, 104
superconductivity, 7, 217, 281
surface element, 4
surface measure
  Lebesgue's, 27
switching point, 134

τ, 29, 355
test function, 31, 139, 141
theorem
  Taylor's, 227
  Browder–Minty, 185
  Rellich's, 356
  Riesz representation, 42
trace operator, 29
  continuity of, 356
trace theorem, 29
two-norm discrepancy, 235, 254, 256, 296

V-elliptic, 107
variational equality, 140, 164
variational formulation, 31, 267
variational inequality, 12, 63, 64, 215, 216, 219, 278, 280, 283
  in Hilbert space, 63
  pointwise, 68

$W(0,T)$, 146
$W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$, 28
$W_2^{1,0}(Q)$, 138
$W_2^{1,1}(Q)$, 138
weakly
  convergent, 44
  sequentially closed, 46
  sequentially compact, 46
  sequentially continuous, 45

y_t, 6

Optimal control theory is concerned with finding control functions that minimize cost functions for systems described by differential equations. The methods have found widespread applications in aeronautics, mechanical engineering, the life sciences, and many other disciplines.

Photograph by Elke Weiß

This book focuses on optimal control problems where the state equation is an elliptic or parabolic partial differential equation. Included are topics such as the existence of optimal solutions, necessary optimality conditions and adjoint equations, second-order sufficient conditions, and main principles of selected numerical techniques. It also contains a survey on the Karush-Kuhn-Tucker theory of nonlinear programming in Banach spaces.

The exposition begins with control problems with linear equations, quadratic cost functions and control constraints. To make the book self-contained, basic facts on weak solutions of elliptic and parabolic equations are introduced. Principles of functional analysis are introduced and explained as they are needed. Many simple examples illustrate the theory and its hidden difficulties. This start to the book makes it fairly self-contained and suitable for advanced undergraduates or beginning graduate students.

Advanced control problems for nonlinear partial differential equations are also discussed. As prerequisites, results on boundedness and continuity of solutions to semilinear elliptic and parabolic equations are addressed. These topics are not yet readily available in books on PDEs, making the exposition also interesting for researchers.

Alongside the main theme of the analysis of problems of optimal control, Tröltzsch also discusses numerical techniques. The exposition is confined to brief introductions into the basic ideas in order to give the reader an impression of how the theory can be realized numerically. After reading this book, the reader will be familiar with the main principles of the numerical analysis of PDE-constrained optimization.

ISBN 978-0-8218-4904-0

9 780821 849040

GSM/112

For additional information and updates on this book, visit

www.ams.org/bookpages/gsm-112

AMS *on the* Web
www.ams.org