Economics of Planning **35:** 271–292, 2002.

Optimal Growth with Public Capital and Public Services

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Received 20 June 2001; accepted in revised form 22 May 2002

Abstract. We characterize optimal economic growth in an endogenous growth model in which production requires public capital (a stock) and public services (a flow) in addition to private capital and labor. We analyze the comparative static effects of changes in the fundamental technological and preference parameters of the model on the optimal values of several variables, such as the optimal rate of growth and the optimal allocation of resources among consumption, the provision of public services, and investment in public and private capital. We show that the general optimal path converges in finite time to the balanced growth optimal path. We relate our paper to important contributions to the existing literature by obtaining them as special cases of our model.

JEL Classification: E62, H54, O40

Key words: endogenous growth, public capital, public services, social optimum, transitional dynamics

1. Introduction

In this paper, we characterize optimal economic growth in an endogenous growth model in which there are two public goods that we call public capital and public services. We use this model to study how governments should balance the need to provide services that would give an immediate boost to the nation's productivity against the need to invest in assets that would raise productivity only in the future. We analyze the comparative static effects of changes in the fundamental technological and preference parameters of the model on the optimal values of several variables, such as the optimal rate of growth and the optimal allocation of resources among consumption, the provision of public services, and investment in public and private capital. We show that the general optimal path converges in finite time to the balanced growth optimal path. We relate our paper to important contributions to the existing literature by obtaining them as special cases of our model.

Our view of public services is the same as that in the seminal contribution of Barro (1990). Public services are non-accumulatable (or, perishable) public goods that are essential to production and imperfect substitutes to other productive resources. Such services may include the maintenance of law and order and transportation networks. Our view of public capital is the same as that in the important contribution of Futagami et al. (1993) in which public capital is an accumulatable public good that is essential to production and an imperfect substitute to other productive resources. Public capital would include the stock of transportation networks and the stock of freely available scientific knowledge.[1]

Barro (1990) includes public services but not public capital in his model of endogenous growth, whereas Futagami et al. (1993) include public capital but not public services. We include both public services and public capital to explore the trade-off that governments routinely face between spending on public services, which would make the present better at the expense of the future, and spending on the accumulation of public capital, which would make the future better at the expense of the present. A government's dilemma, in other words, is similar to that faced by the representative agent in the canonical Ramsey–Cass–Koopmans model between consumption and the accumulation of capital. Even apart from this similarity, however, it would be interesting to study how the changes in shares of the two components of public spending are related to growth and social welfare and to then characterize the optimal fiscal policies.

With these goals in view, we develop, in section 2, a general model in which public capital, public services and private capital are inputs in the production process. We explore the balanced growth optimal path in some detail, even though it imposes stringent restrictions on the economy's initial endowments of public and private capital, because, as we show also in section 2, the general optimal path for arbitrary initial endowments merges in finite time with the balanced growth optimal path.[2]

In section 3, we develop the Barro (1990) and Futagami et al. (1993) models as extreme cases of our more general model. We show that more can be learned from our general model about the behavior of, for instance, the optimal rate of growth than can be inferred from a simple interpolation of the corresponding results for the extreme cases. This, therefore, confirms the value added by our generalization of the two earlier papers. Most importantly, as we pointed out earlier, those two papers have absolutely nothing to say about the public sector's present-versus-future trade-off. We also develop a model with public capital and public services, but no *private* capital. This special case shows that long-run growth is possible without private capital. And, as in Barro (1990), this special case has no transitional dynamics. Section 4 concludes the paper.

2. A model of optimal growth

Consider an economy that is run by an ideal social planner. It has one final good, Y, and

$$Y_t = F(g_{st}, g_{ft}, K_t, L_t) \tag{1}$$

is the amount of Y that can be produced at time t with g_{st} units of an accumulatable public good (hereafter, public capital, a *stock* variable), g_{ft} units of a non-accumulatable (or, perishable) public good (hereafter, public services, a *flow* variable), K_t units of an accumulatable private good (hereafter, private capital, a stock variable), and L_t units of homogeneous labor.

Let L_t, the total amount of labor available to – and employed by – the social planner at time t, be unity (i.e., $L_t = 1$ for all t). With this assumption, all quantities represent both total and per capita quantities. In particular, we will denote the per capita quantities of the final good and the private capital good at time t by y_t and k_t, respectively. Note that $Y_t = y_t$ and $K_t = k_t$.

One unit of the final good can be used to: (i) increase the stock of private capital, k_t, by one unit, (ii) increase the stock of public capital, g_{st}, by one unit, (iii) provide one unit of public services, g_{ft}, or (iv) provide one unit of consumption, c_t. Therefore, if τ is the constant fraction of y_t that is used for purposes (ii) and (iii), it follows that

$$\dot{g}_{st} = \tau \cdot y_t - g_{ft} \tag{2}$$

and

$$\dot{k}_t = (1 - \tau) \cdot y_t - c_t. \tag{3}$$

The nation's utility is

$$U = \int_0^\infty \mathrm{e}^{-\rho t} \ln c_t \, \mathrm{d}t, \tag{4}$$

where $\rho > 0$ is the rate of time preference.

The social planner's task is to choose c_t, g_{ft}, and τ to maximize utility subject to Equations (2) and (3); the requirement – $\infty > \dot{k}_t, \dot{g}_{st} \geqslant 0$ – that investments in private and public capital be non-negative and finite; and the initial condition that the amounts, k_0 and g_{s0}, of the two capital goods that the economy starts out with are given.[3]

This maximization problem's present value Hamiltonian is $H_t = \mathrm{e}^{-\rho t} \ln c_t + \lambda_t \cdot [(1-\tau) \cdot y_t - c_t] + \mu_t \cdot [\tau \cdot y_t - g_{ft}]$, where λ_t and μ_t are the present value shadow prices of private and public capital, respectively.[4] The corresponding Lagrangean is $\mathbf{L}_t = H_t + p_t^k \cdot [(1-\tau) \cdot y_t - c_t] + p_t^g \cdot [\tau \cdot y_t - g_{ft}]$, where p_t^k and p_t^g are the Lagrangean multipliers for the $\dot{k}_t \geqslant 0$ and $\dot{g}_{st} \geqslant 0$ constraints, respectively. The first order conditions are: $\partial \mathbf{L}_t / \partial c_t = 0$, $\partial \mathbf{L}_t / \partial g_{ft} = 0$, $\partial \mathbf{L}_t / \partial \tau = 0$; $\partial \mathbf{L}_t / \partial g_{st} = -\dot{\mu}_t$ and $\partial \mathbf{L}_t / \partial k_t = -\dot{\lambda}_t$; Equations (2) and (3); the complementary slackness conditions

$$p_t^k \geqslant 0, \quad (1 - \tau) \cdot y_t - c_t \geqslant 0, \quad p_t^k \cdot [(1 - \tau) \cdot y_y - c_t] = 0 \tag{5}$$

$$p_t^g \geqslant 0, \quad \tau \cdot y_t - g_{ft} \geqslant 0, \quad p_t^g \cdot [\tau \cdot y_t - g_{ft}] = 0; \tag{6}$$

and the transversality conditions

$$\lim_{t\to\infty} \lambda_t \cdot k_t = \lim_{t\to\infty} \mu_t \cdot g_{st} = 0. \tag{7}$$

In addition to Equations (2), (3), (5) and (6), these first-order conditions imply, respectively,

$$\frac{\mathrm{e}^{-\rho t}}{c_t} = \lambda_t + p_t^k, \tag{8}$$

$$(\lambda_t + p_t^k) \cdot (1-\tau) \cdot \frac{\partial y_t}{\partial f_{ft}} + (\mu_t + p_t^g) \cdot \left[\tau \cdot \frac{\partial y_t}{\partial g_{ft}} - 1\right] = 0, \tag{9}$$

$$\lambda_t + p_t^k = \mu_t + p_t^g, \tag{10}$$

$$(\lambda_t + p_t^k) \cdot (1-\tau) \cdot \frac{\partial y_t}{\partial g_{st}} + (\mu_t + p_t^g) \cdot \tau \cdot \frac{\partial y_t}{\partial g_{st}} = -\dot{\mu}_t, \tag{11}$$

$$(\lambda_t + p_t^k) \cdot (1-\tau) \cdot \frac{\partial y_t}{\partial k_t} + (\mu_t + p_t^g) \cdot \tau \cdot \frac{\partial y_t}{\partial k_t} = -\dot{\lambda}_t. \tag{12}$$

Note that Equations (9) and (10) imply

$$\frac{\partial y_t}{\partial g_{ft}} = 1. \tag{13}$$

2.1. THE BALANCED GROWTH OPTIMAL OUTCOME

In this subsection, we will describe the solution to the social planner's problem for which $\dot{k}_t > 0$ and $\dot{g}_{st} > 0$ for all t. For this case, Equations (5) and (6) imply $p_t^k = p_t^g = 0$ for all t. This, in turn, implies, by way of Equation (10), that $\lambda_t = \mu_t$ and, therefore, $\dot{\lambda}_t = \dot{\mu}_t$ along the optimal path. Therefore, Equations (11) and (12) imply

$$-\frac{\dot{\mu}_t}{\mu_t} = -\frac{\dot{\lambda}_t}{\lambda_t} = \frac{\partial y_t}{\partial g_{st}} = \frac{\partial y_t}{\partial k_t}.$$

This last equation and Equation (8) together imply

$$\rho + \frac{\dot{c}_t}{c_t} = \frac{\partial y_t}{\partial g_{st}} = \frac{\partial y_t}{\partial k_t}. \tag{14}$$

For simplicity, let the technology available to the social planner, Equation (1), obey the Cobb-Douglas form:

$$Y_t = (g_{st}^{\alpha} \cdot g_{ft}^{1-\alpha})^{1-\beta} \cdot K_t^{\beta} \cdot L_t^{1-\beta},$$

$0 < \alpha, \beta < 1$. In per capita terms, this technology can be written as

$$y_t = (g_{st}^{\alpha} \cdot g_{ft}^{1-\alpha})^{1-\beta} \cdot k_t^{\beta}. \tag{15}$$

Note that when $\alpha = 0$, public capital, g_{st}, is absent from the production function as in Barro (1990) and when $\alpha = 1$, public services, g_{ft}, is absent from the production function as in Futagami et al. (1993). Not surprisingly, as we will show in section 3, the Barro (1990) and Futagami et al. (1993) results can be obtained as special cases of our model by substituting $\alpha = 0$ and $\alpha = 1$, respectively.

Let $m_t \equiv k_t/g_{st}$ and $g_{sft} \equiv g_{st}/g_{ft}$. Then Equation (13) and the second equality in Equation (14) together imply

$$g_{sft} = [\alpha^{\beta} \cdot (1-\alpha)^{-1} \cdot \beta^{-\beta} \cdot (1-\beta)^{\beta-1}]^{\frac{1}{1-(1-\alpha)(1-\beta)}} \equiv g_{sf}, \tag{16}$$

$$m_t = \alpha^{-1} \cdot \beta \cdot (1-\beta)^{-1} \equiv m. \tag{17}$$

The contour plot of g_{sf} is given in Figure 1. It shows that g_{sf} is increasing in α and hump-shaped in β.

Note that both m_t and g_{sft} are constant along the optimal path. Therefore, k_t, g_{st} and g_{ft} must all grow at the same rate, γ_t. And, by Equation (15), γ_t will also be the growth rate of y_t. In short,

$$\frac{\dot{y}_t}{y_t} = \frac{\dot{k}_t}{k_t} = \frac{\dot{g}_{st}}{g_{st}} = \frac{\dot{g}_{ft}}{g_{ft}} = \gamma_t. \tag{18}$$

It is straightforward to show that the constancy of m_t and g_{sft} also implies the constancy of the marginal products $\partial y_t/\partial g_{st}$ and $\partial y_t/\partial k_t$. The first equality in Equation (14) then implies that $\dot{c}_t/c_t$, the growth rate of consumption, is constant along the optimal path as well. Let this constant growth rate of consumption be γ_c.

Also, Equation (2), which implies $\dot{g}_{st}/g_{st} = \tau \cdot (y_t/g_{st}) - (g_{ft}/g_{st})$, and Equation (18), which implies that y_t/g_{st} and g_{ft}/g_{st} are both constant, together imply that γ_t is a constant. So, let $\gamma_t = \gamma$.

Similarly, Equation (3), which implies $\dot{k}_t/k_t = (1-\tau) \cdot (y_t/k_t) - (c_t/k_t)$, and Equation (18), which implies that y_t/k_t is constant, and the previous paragraph's result that $\dot{k}_t/k_t = \gamma_t = \gamma$ together imply that c_t/k_t must be constant along the optimal path. Therefore, we have

$$\frac{\dot{y}_t}{y_t} = \frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t} = \frac{\dot{g}_{st}}{g_{st}} = \frac{\dot{g}_{ft}}{g_{ft}} = \gamma, \tag{19}$$

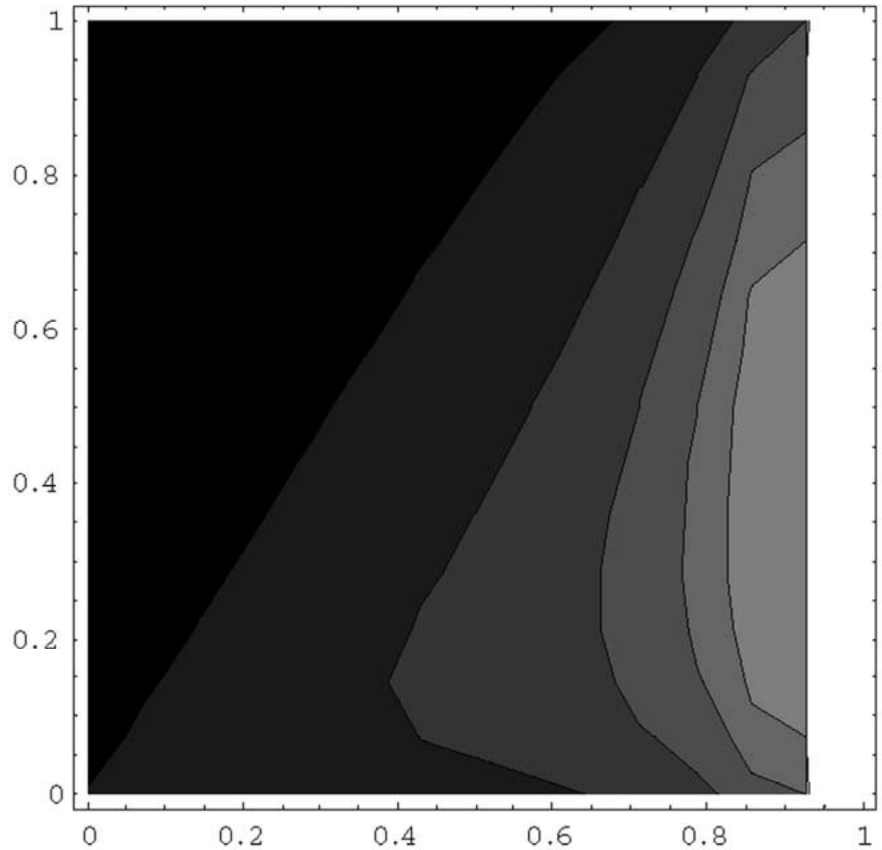

Figure 1. The contour plot of g_{sf}, which is given by Equation (16). Note that α is measured on the horizontal axis and β on the vertical axis. Lighter shaded areas represent level sets of higher value.

where, by Equation (14),

$$\gamma = \frac{\partial y_t}{\partial g_{st}} - \rho = \frac{\partial y_t}{\partial k_t} - \rho.$$

It is straightforward to then show that

$$\gamma = \frac{\alpha}{1-\alpha} \cdot g_{sf}^{-1} - \rho. \tag{20}$$

Given that our goal is to find an expression for γ exclusively in terms of only α, β and ρ, which are the fundamental exogenous parameters of our model, note that although g_{sf} in Equation (20) is *endogenous* we can easily substitute for it using Equation (16) and thereby obtain the desired expression for γ. Once γ is expressed exclusively in terms of α, β and ρ, it is clear that it is inversely related to the rate of time preference, ρ. Moreover, it can be shown that γ is U-shaped in α for any values of ρ and β and U-shaped in β for any values of ρ and α. To do this, let $\hat{\gamma} \equiv \ln(\gamma + \rho)$. It can be checked that

$$\frac{\partial \hat{\gamma}}{\partial \alpha} = \frac{1-\beta}{[1-(1-\alpha)(1-\beta)]^2}\{\beta \ln \alpha - \ln(1-\alpha) \\ - \beta \ln \beta - (1-\beta)\ln(1-\beta)\}.$$

It can be checked that this derivative is negative near $\alpha = 0$ and positive near $\alpha = 1$, thereby indicating that $\hat{\gamma}$ is U-shaped in α. Since $\hat{\gamma}$ is a positive transformation of γ, the latter must also be U-shaped with respect to α. Similarly, we get

$$\frac{\partial \hat{\gamma}}{\partial \beta} = \frac{\alpha \ln \beta - \alpha \ln \alpha - (1-\alpha)\ln(1-\alpha) - \ln(1-\beta)}{[1-(1-\alpha)(1-\beta)]^2}$$

and can show that γ is U-shaped in β. The *Mathematica* contour plot of γ (for $\rho = 0.02$) in Figure 2 confirms the above conclusions.

Equation (2) implies

$$\tau = \frac{\dot{g}_{st} + g_{ft}}{y_t} = [1 + \gamma \cdot g_{sf}] \cdot \frac{g_{ft}}{y_t}. \tag{21}$$

Using Equations (13), (15), (19) and (20), we get

$$\tau = (1-\beta) \cdot [1 - (1-\alpha) \cdot \rho \cdot g_{sf}]. \tag{22}$$

Once Equation (16) is substituted into Equation (22) to derive an expression for τ exclusively in terms of α, β and ρ, it becomes clear that the optimal share of total public spending to total output, τ, is negatively related to the rate of time preference, ρ. The contour plot of τ (for $\rho = 0.02$) in Figure 3 shows further that τ is negatively related to β. As for the dependence of τ on α, it is decreasing in α at high values of β and U-shaped at smaller values such as $\beta = 0.05$.

Let us now look at how the government allocates the total spending on public goods between $\dot{g}_{st}$, its investment in the stock of the accumulatable public good, and g_{ft}, its provision of the non-accumulatable public good. Let $\theta_t \tau y_t = \dot{g}_{st}$ and $(1-\theta_t)\tau y_t = g_{ft}, 0 < \theta_t < 1$. (Note that these two equations imply the government budget constraint (2).) From this, we get

$$\frac{\theta_t}{1-\theta_t} = \frac{\dot{g}_{st}}{g_{st}} \cdot \frac{g_{st}}{g_{ft}} = \gamma_t g_{sft},$$

which implies

$$\theta_t = \frac{\gamma_t g_{sft}}{1 + \gamma_t g_{sft}}, \tag{23}$$

Equation (20) then implies

$$\theta_t = 1 - \frac{1}{(1-\alpha)^{-1} - \rho \cdot g_{sf}} \equiv \theta. \tag{24}$$

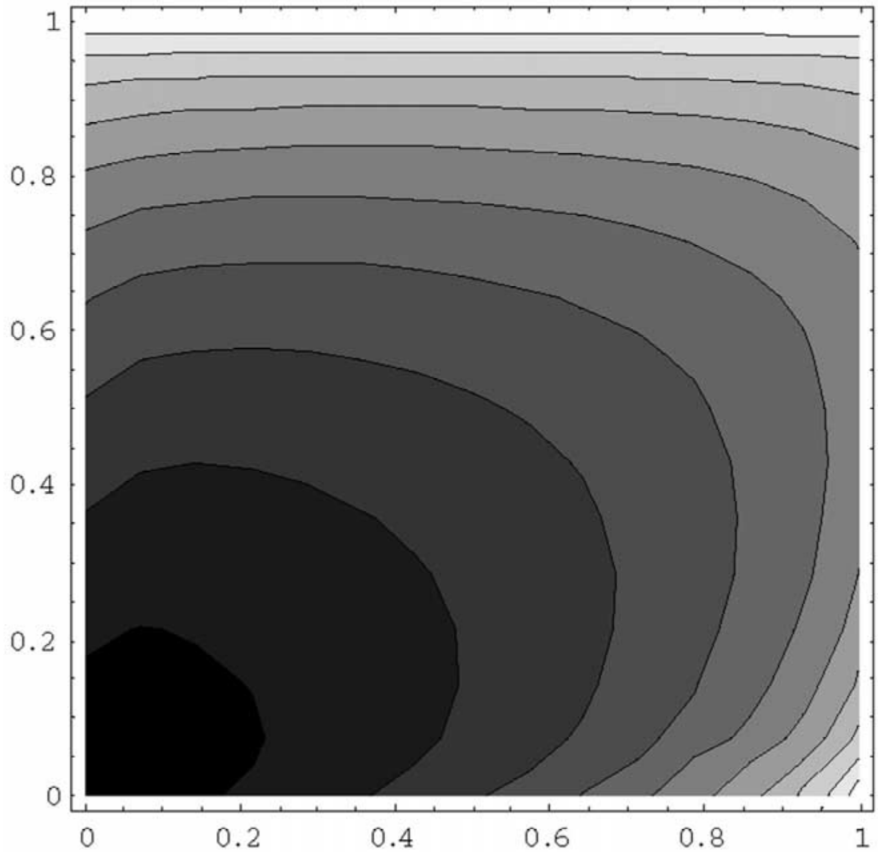

Figure 2. The contour plot of γ, which is given by Equations (20) and (16), for $\rho = 0.02$. Note that α is measured on the horizontal axis and β on the vertical axis. Lighter shaded areas represent level sets of higher value.

When Equation (16) is substituted into Equation (24) to derive an expression for θ exclusively in terms of α, β and ρ, it can be shown that the rate of time preference, ρ, is negatively related to θ. Moreover, the contour plot of θ (for ρ = 0.02) in Figure 4 shows a direct relationship with α and a U-shaped relationship with β.

Finally, let s_t be the fraction of $(1 - \tau) \cdot y_t$ that is used for the accumulation of private capital. That is,

$$s_t = \frac{\dot{k}_t}{(1-\tau) \cdot y_t} = \frac{\dot{k}_t / k_t}{(1-\tau) \cdot (y_t / k_t)}, \tag{25}$$

which gives

$$s_t = 1 - \frac{\alpha - \alpha\beta + \beta}{\alpha \cdot (1-\alpha)^{-1} \beta \cdot g_{sf}^{-1} \cdot \rho^{-1} + \alpha \cdot (1-\beta)} \equiv s. \tag{26}$$

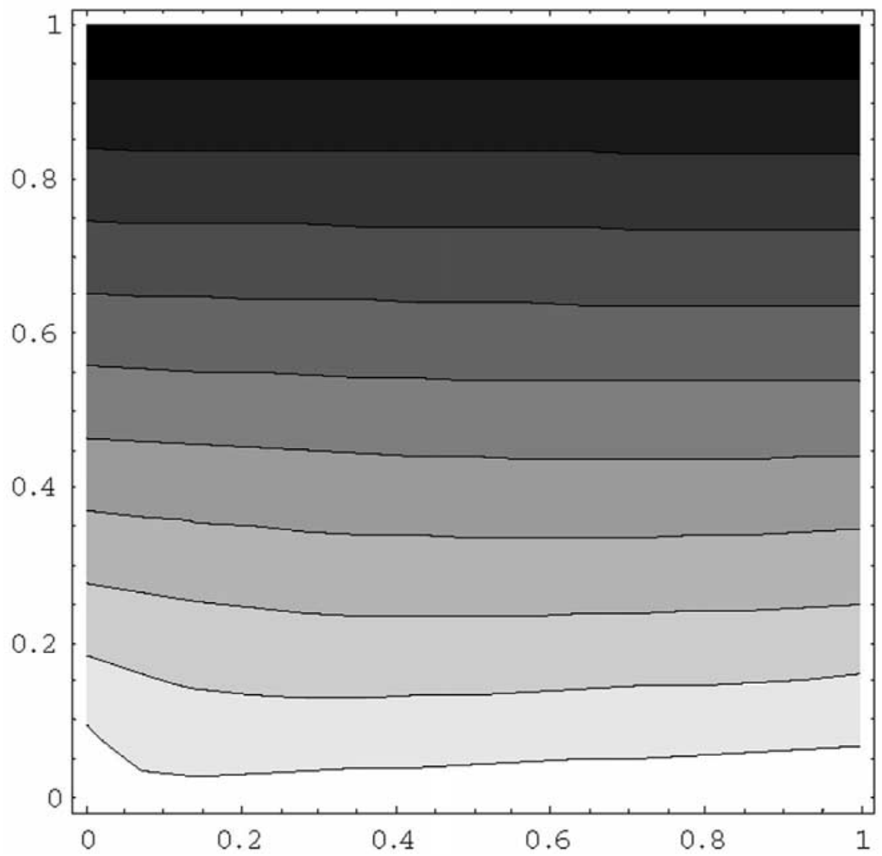

Figure 3. The contour plot of τ, which is given by Equations (22) and (16), for $\rho = 0.02$. Note that α is measured on the horizontal axis and β on the vertical axis. Lighter shaded areas represent level sets of higher value.

Again, when Equation (16) is substituted into Equation (26) to derive an expression for s exclusively in terms of α, β and ρ, it follows that, ρ, the rate of time preference is negatively related to s, the optimal share of total non-public spending that is used for the accumulation of private capital. The contour plot of s (for ρ = 0.02) in Figure 5 shows a U-shaped relationship with α and a direct relationship with β.

Finally, recall from Equation (17) that $m_t \equiv k_t/g_{st} = \alpha^{-1} \cdot \beta \cdot (1-\beta)^{-1} \equiv m$ for all $t \geqslant 0$. This implies that this subsection's simplifying assumption – that $\dot{k}_t > 0$ and $\dot{g}_{st} > 0$ all along the optimal path – is satisfied by only the balanced growth optimal path, which, in turn, imposes the condition that k_0 and g_{s0} must be such that $m_0 \equiv k_0/g_{s0} = \alpha^{-1} \cdot \beta \cdot (1-\beta)^{-1} \equiv m$. Such a condition can be satisfied only by pure coincidence.

In general, m_0 will not equal m. And, given that investments in private and public capital must be non-negative and finite or, equivalently, that jumps in the

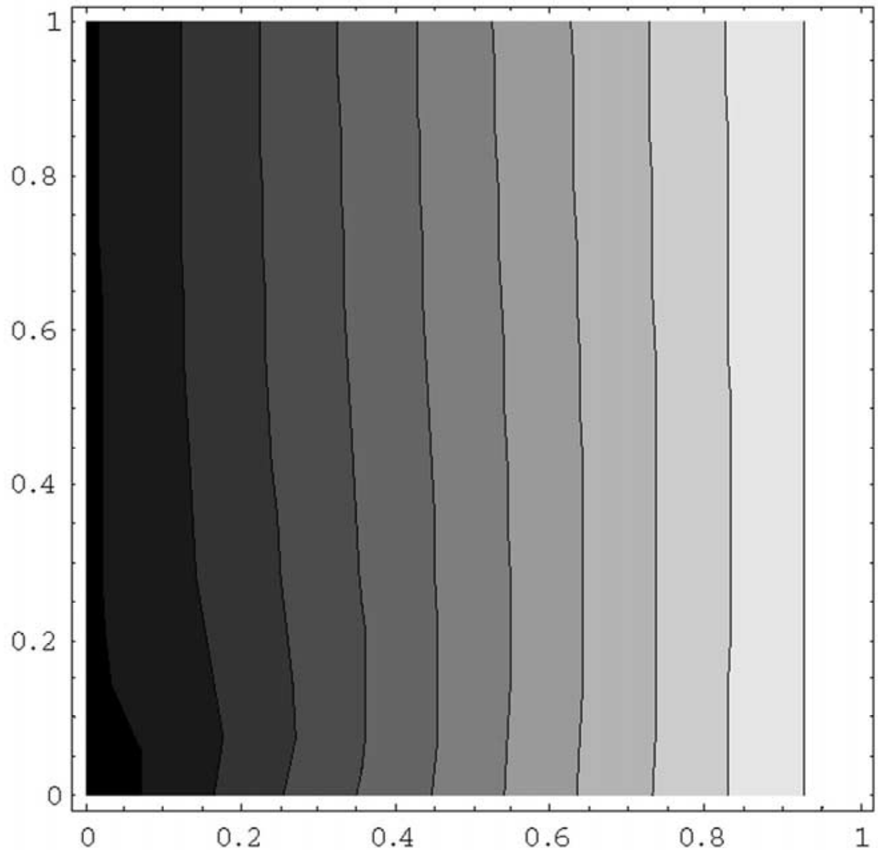

Figure 4. The contour plot of θ, which is given by Equations (24) and (16), for $\rho = 0.02$. Note that α is measured on the horizontal axis and β on the vertical axis. Lighter shaded areas represent level sets of higher value.

state variables are not allowed, the optimal path will *not* coincide with the balanced growth optimal path. Therefore, in the following subsection, we will describe the optimal outcome for the general case in which $m_0 \neq m$. Recalling that we have shown that an optimal path with $\dot{k}_t > 0$ and $\dot{g}_{st} > 0$ for all t is possible only when $m_0 = m$, note that it follows that when $m_0 \neq m$, either $\dot{k}_t$ or $\dot{g}_{st}$ will be zero for part or all of the optimal path.

2.2. TRANSITIONAL DYNAMICS

As was just stated, we will now describe the general solution to the social planner's problem – see the paragraph following Equation (4)—for arbitrary values of k_0 and g_{s0}. Even though the optimal outcome coincides with the balanced growth optimal outcome of section 2.1 only in the exceptional case of $k_0/g_{s0} = \alpha^{-1} \cdot \beta \cdot (1 - \beta)^{-1}$, we will show that the balanced growth optimal outcome is more significant than

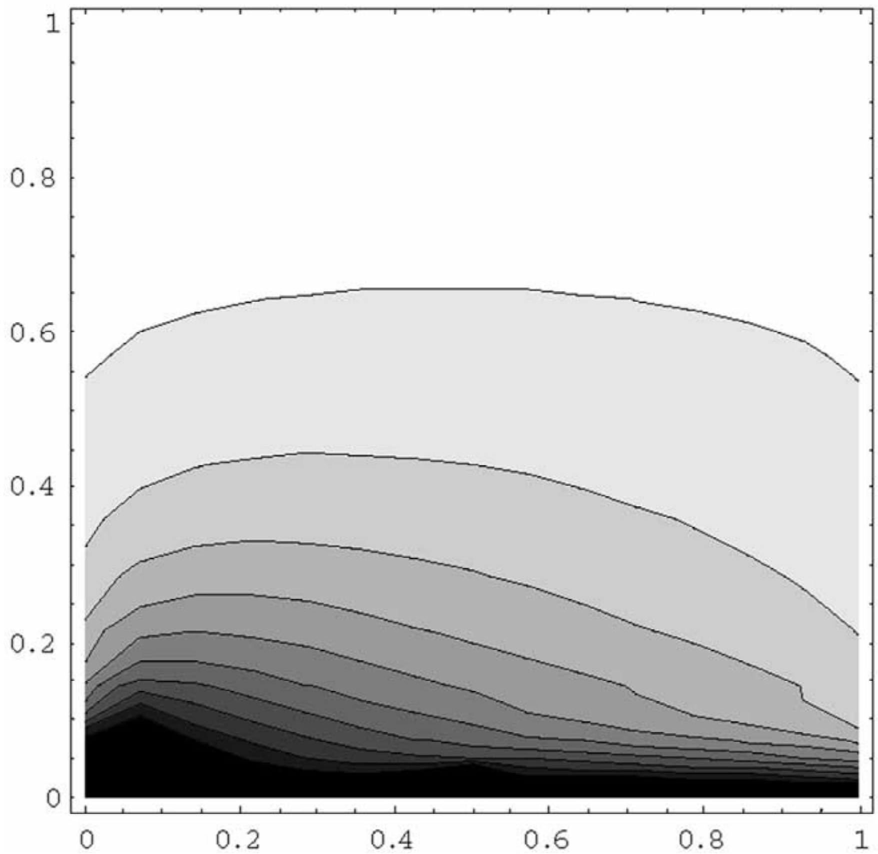

Figure 5. The contour plot of s, which is given by Equations (26) and (16), for $\rho = 0.02$. Note that α is measured on the horizontal axis and β on the vertical axis. Lighter shaded areas represent level sets of higher value.

one might think. Specifically, we will show that *all optimal paths merge with the balanced optimal growth path in a finite amount of time*. We will restate the social planner's problem in a form that, while equivalent, will enable a direct proof of global convergence. This global convergence result will then enable us to restate the social planner's problem yet again as a salvage value problem that can then be solved directly.

Note that Equation (13) is true for any optimal outcome and not just the balanced growth optimal outcome of section 2.1. Therefore, we can use Equation (13) to restate the social planner's problem without g_{ft}. Equations (13) and (15) imply that

$$y_t = \left[[(1-\alpha)(1-\beta)]^{(1-\alpha)(1-\beta)} \cdot g_{st}^{\alpha\cdot(1-\beta)} \cdot k^{\beta} \right]^{\frac{1}{1-(1-\alpha)(1-\beta)}} \equiv y_t(k_t, g_{st}) \quad (27)$$

along the optimal path. Now let us define

$$y_{nt}(k_t, g_{st}) \equiv y_t(k_t, g_{st}) - g_{ft} = [1 - (1-\alpha)(1-\beta)] \cdot y_t(k_t, g_{st}) \tag{28}$$

as output net of public services, noting that the second equality follows from Equation (13).

Let $\eta_t \equiv (\dot{k}_t + \dot{g}_{st})/y_{nt}$ be the proportion of y_{nt} that is used for (public *and* private) investment. It can be checked that

$$\eta_t = \frac{s_t \cdot (1-\tau) + \theta_t \cdot \tau}{1 - (1-\theta_t) \cdot \tau} \tag{29}$$

Also, let $\phi_t \equiv \dot{k}_t/(\dot{k}_t + \dot{g}_{st})$ be the proportion of all investment spending that is invested in private capital. It can be checked that

$$\phi_t = \frac{s_t \cdot (1-\tau)}{s_t \cdot (1-\tau) + \theta_t \cdot \tau}. \tag{30}$$

The social planner's problem can now be restated as follows: choose η_t and ϕ_t to maximize U in Equation (4) subject to

$$c_t = (1-\eta_t) \cdot y_{nt}(k_t, g_{st}), \tag{31}$$

$$\dot{k}_t = \phi_t \cdot \eta_t \cdot y_{nt}(k_t, g_{st}), \tag{32}$$

$$\dot{g}_{st} = (1-\phi_t) \cdot \eta_t \cdot y_{nt}(k_t, g_{st}), \tag{33}$$

$$0 \leqslant \eta_t, \quad \phi_t \leqslant 1$$

and the given initial values of k_0 and g_{s0}.

Note that k_t and g_{st} matter in this problem only through their effect on $y_{nt}(k_t, g_{st})$. So, for given values of k_t and g_{st} and any choice of η_t, the social planner will choose ϕ_t solely to maximize $\dot{y}_{nt}$, the rate of increase of y_{nt}. By Equation (28), this amounts to choosing ϕ_t solely to maximize

$$\dot{y}_t(k_t, g_{st}) = \frac{\partial y_t(k_t, g_{st})}{\partial k_t} \cdot \dot{k}_t + \frac{\partial y_t(k_t, g_{st})}{\partial g_{st}} \cdot \dot{g}_{st}.$$

Clearly, $\phi_t = 1$ or, equivalently, $\dot{g}_{st} = 0$ is optimal when $\partial y_t(k_t, g_{st})/\partial k_t > \partial y_t(k_t, g_{st})/\partial g_{st}$ or, equivalently, $m_t \equiv k_t/g_{st} < \alpha^{-1} \cdot \beta \cdot (1-\beta)^{-1} \equiv m$. Similarly, $\phi_t = 0$ or, equivalently, $\dot{k}_t = 0$ is optimal when $\partial y_t(k_t, g_{st})/\partial k_t < \partial y_t(k_t, g_{st})/\partial g_{st}$ or, equivalently, $m_t > m$. Note that $m_t \neq m$ is equivalent to $\partial y_t(k_t, g_{st})/\partial k_t \neq \partial y_t(k_t, g_{st})/\partial g_{st}$. Since the sole consideration in optimally allocating total investment between public and private capital is the consequent effect on total output, the social planner steers all investment to the capital good with the higher marginal productivity.

As for the $m_t = m$ case, we have seen in section 2.1 that the optimal solution will coincide with the balanced growth optimal path from t onwards. Therefore, it follows that $\phi_t = \phi$, where ϕ is obtained by substituting for s_t, τ and θ_t in Equation (30) using Equations (16), (22), (24) and (26).

In short, the social planner will, in the general optimal outcome, allocate total investment, $\eta_t y_{nt}$, between private and public capital as follows:

$$\phi_t = \begin{cases} 0, & m_t > m \\ \phi, & m_t = m \\ 1, & m_t < m \end{cases} \tag{34}$$

Equation (34) implies that m_t will decrease along the optimal path when $m_t > m$, stay unchanged when $m_t = m$ and increase when $m_t < m$. In other words, m_t will approach m, its balanced growth optimal value, along the optimal path no matter what its initial value, k_0/g_{s0}, happens to be.

The question then is whether m_t will become *equal* to m in finite time or merely approach ever closer to m at all t without ever actually reaching it. Recall that γ is the balanced optimal growth rate as defined by Equations (20) and (16). It can be shown that *if $\gamma > 0$, then m_t will become equal to m in finite time. In other words, if the balanced optimal growth rate is positive, the optimal path for arbitrary k_0 and g_{s0} will merge with the balanced growth optimal path in finite time.*[5] (The proof is straightforward and given in the appendix.) We will consider only the $\gamma > 0$ case and, therefore, only the case in which all optimal paths converge to the balanced growth optimal path.

2.2.1. *When $m_0 > m$*

For this case, Equation (34) implies that $\dot{k}_t = 0$ and $\dot{g}_{st} > 0$ for $0 \leqslant t < T$, where T is the time at which the optimal path merges with the balanced growth optimal path. Note that $k_T = k_0$ and that $g_{sT} = k_0/m$ because $m_T = m$.

Let $V(k, g_s)$ be the representative agent's lifetime utility – see U in Equation (4)—associated with the balanced growth optimal outcome for given initial endowments, k and g_s, of private and public capital. It is then straightforward to show that

$$V(k_t, g_{st}) = \frac{\gamma}{\rho^2} + \frac{\ln c_t}{\rho} \tag{35}$$

where γ is determined in terms of α, β and ρ by Equations (20) and (16) and $c_t = (1 - s)(1 - t)y_t$ is determined in terms of α, β, ρ, k_t and g_{st} by Equations (16), (22), (26) and (27). In other words, the lifetime utility associated with the balanced growth outcome from time t onwards is $V(k_t, g_{st})$ and it is determined entirely in terms of α, β, ρ, k_t and g_{st}.

Since all optimal paths converge to the balanced optimal growth path in finite time, the social planner's problem can be restated as follows: choose c_t, g_{ft} and T to maximize

$$\int_0^T e^{-\rho t} \ln c_t dt + e^{-\rho T} V(k_T, g_{sT})$$

subject to Equations (15) and

$$\dot{g}_{st} = y_t - c_t - g_{ft} \tag{36}$$

as well as the boundary condition that $k_t = k_0$ for $0 \leqslant t \leqslant T$, and that k_0, g_{s0} and $g_{sT} = k_0/m$ are all given.

This is an optimal control problem with free terminal time and fixed terminal values for the state variable g_{st} and the salvage term $V(k_T, g_{sT})$. The present value Hamiltonian is $H_t = \mathrm{e}^{-\rho t} \ln c_t + \mu_t \cdot (y_t - g_{ft} - c_t)$ and the first order necessary conditions are: Equation (36), $\partial H_t/\partial c_t = \partial H_t/\partial g_{ft} = 0$, $-\dot{\mu}_t = \partial H_t/\partial g_{st}$, and the transversality condition $H_t + \partial(\mathrm{e}^{-\rho t} \cdot V(k_T, g_{sT}))/\partial t = 0$ at $t = T$, the date at which the transition phase ends.[6]

The condition $\partial H_t/\partial g_{ft} = 0$ yields Equation (13) and the conditions $\partial H_t/\partial c_t = 0$ and $-\dot{\mu}_t = \partial H_t/\partial g_{st}$ together yield the first equality of Equation (14). By using Equation (13) or, equivalently,

$$g_{ft} = (1-\alpha) \cdot (1-\beta) \cdot y_t(k_t, g_{st})$$

to eliminate g_{ft} from Equation (36) and the first equality of Equation (14), we obtain two differential equations in only two unknowns, c_t and g_{st}, and their time-derivatives. Therefore, these two differential equations, together with the boundary conditions and the transversality condition, fully determine the general optimal outcome for the $m_0 > m$ case.

Recall from Equation (27) that $y_t = y_t(k_t, g_{st})$ and, therefore, $\partial y_t/\partial g_{st} = \partial y_t(k_t, g_{st})/\partial g_{st}$ all along the optimal path. It is straightforward to show that $\partial y_t(k_t, g_{st})/\partial g_{st}$ is increasing in k_t and decreasing in g_{st}. Since k_t is constant and g_{st} is increasing during the transition period, $0 \leqslant t < T$, $\partial y_t/\partial g_{st}$ is decreasing during the transition period. This implies, by the first inequality in Equation (14), that the growth rate of consumption, $\dot{c}_t/c_t$, decreases during the transition phase of the optimal path till it reaches the balanced optimal growth rate of γ.

The transversality condition can be used to determine T, the time it takes the optimal path to merge with the balanced growth optimal path. It yields

$$\mathrm{e}^{-\rho T} \ln c_T + \mu_T \cdot (y_T - g_{fT} - c_T) - \mathrm{e}^{-\rho T} \cdot \rho \cdot V(k_T, g_{sT}) = 0$$

The optimality condition $\partial H_t/\partial c_t = 0$ implies $\bar{\mu}_T \equiv \mathrm{e}^{\rho T} \cdot \mu_T = 1/c_T$, which then turns the transversality condition to

$$\ln c_T + \frac{y_{nT}(k_T, g_{sT}) - c_T}{c_T} - \rho \cdot V(k_T, g_{sT}) = 0$$

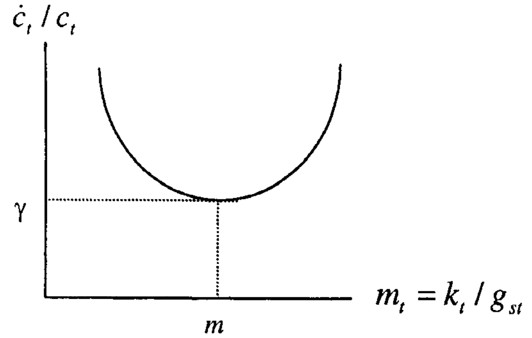

Figure 6. The optimal growth rate of consumption, for arbitrary k_t/g_{st}.

This equation determines c_T and, therefore, $\bar{\mu}_T$ in terms of the known parameters of the model. Also, it can be shown that the present value co-state variable is equal to the marginal lifetime present discounted utility of the corresponding state variable; i.e.,

$$\mu_T = \frac{\partial V(k_T, g_{sT})}{\partial g_{sT}},$$

which determines μ_T in terms of the known parameters of our model.[7] Therefore, T can then be determined because $\ln \bar{\mu}_T = \rho \cdot T + \ln \mu_T$ and both $\bar{\mu}_T$ and μ_T are determined.

2.2.2. *When $m_0 < m$*

The solution to the social planner's problem for this case can be analyzed along the lines of the $m_0 > m$ case in section 2.2.1. Now $g_{st} = g_{s0}$ for $0 \leqslant t < T$, k_t increases from k_0 to $k_T = mg_{s0}$, and the optimal path once again merges with the balanced growth optimal path at $t = T$, which can be determined implicitly as in section 2.2.1. The growth rate of consumption is in this case given by

$$\rho + \frac{\dot{c}_t}{c_t} = \frac{\partial y_t}{\partial k_t}.$$

Both $\partial y_t/\partial k_t$ and, therefore, $\dot{c}_t/c_t$ decrease over time during the transition phase. At $t = T$, $\dot{c}_t/c_t$ reaches the balanced optimal growth rate of γ. The behavior of the growth rate of consumption for the $m_0 < m$ and $m_0 > m$ cases can be consolidated and expressed as follows: the farther m_t is from its balanced growth value, m, the higher is $\gamma_{ct} \equiv \dot{c}_t/c_t$ – see Figure 6.

This completes our discussion of the transition phase of the optimal outcome for arbitrary k_0 and g_{s0}. At the end of the transition phase, the optimal outcome merges with the balanced growth optimal outcome of section 2.1.

3. Special cases: limiting cases of the general model

In this section we will consider the special cases in which the parameter α in the production function for the final good – see Equation (15) – takes on the values of $\alpha = 0$ and $\alpha = 1$. Our model reduces to Barro (1990) when $\alpha = 0$ and to Futagami et al. (1993) when $\alpha = 1$. As for β, the other parameter in the production function, we will consider the $\beta = 0$ case to show, among other things, that long run growth is possible in the absence of private capital.[8]

3.1. THE BARRO (1990) MODEL AS A LIMITING CASE

As the Barro model has only public services and not public capital, we set $\alpha = 0$ in the production function given by Equation (15). This gives

$$y_t = g_{ft}^{1-\beta} \cdot k_t^{\beta}. \tag{15B}$$

The social planner's problem is now to maximize U in Equation (4) subject to Equations (15B) and

$$\dot{k}_t = y_t - c_t - g_{ft}. \tag{3B}$$

The present value Hamiltonian is $H_{Bt} = \mathrm{e}^{-\rho \cdot t} \ln c_t + \lambda_{Bt} \cdot [y_t - c_t - g_{ft}]$. The first-order conditions are given by (a) $\partial H_{Bt}/\partial c_t = 0$, (b) $\partial H_{Bt}/\partial g_{ft} = 0$ and (c) $\partial H_{Bt}/\partial k_t = -\dot{\lambda}_{Bt}$.

Condition (b) implies Equation (13), which now takes the form

$$g_{ft} = (1-\beta) y_t \tag{13B}$$

Conditions (a) and (c) give us

$$\rho + \frac{\dot{c}_t}{c_t} = \frac{\partial y_t}{\partial k_t}. \tag{14B}$$

Using Equations (13B) and (15B), we obtain from Equation (14B) an expression for the optimal growth rate in the Barro model:

$$\gamma_B = \beta \cdot (1-\beta)^{(1-\beta)/\beta} - \rho. \tag{20B}$$

Note that (20B) is obtainable from our general model, simply by plugging in $\alpha = 0$ in Equation (20).[9] It can be shown that γ_B is increasing in β.

Equation (13B) implies that the proportion of total output that is used for the provision of public services in Barro's model is:

$$\tau_{\mathrm{B}} = (1-\beta). \tag{22B}$$

Once again, note that (22B) is obtainable from our general model, simply by plugging in $\alpha = 0$ in Equation (22).

Finally, the optimal saving rate can be obtained, using (20B) and (22B) in the expression for the saving rate:

$$s_{\mathrm{B}} = 1 - \frac{\rho}{\beta \cdot (1-\beta)^{(1-\beta)/\beta}}. \tag{26B}$$

Once again, note that (26B) is obtainable from our general model, simply by plugging in $\alpha = 0$ in Equation (26). It is clear that s_{B} is increasing in γ_{B} (and, therefore, β) and decreasing in ρ.

3.2. THE FUTAGAMI ET AL. (1993) MODEL AS A LIMITING CASE

As the Futagami model has only public capital and not public services, we set $\alpha = 1$ in the production function given by Equation (15). This gives

$$y_t = g_{st}^{1-\beta} \cdot k_t^{\beta} \tag{15F}$$

The social planner's problem is now to maximize U in Equation (4) subject to Equations (3), (15F) and the new government budget constraint

$$\dot{g}_{st} = \tau y_t. \tag{2F}$$

The present value Hamiltonian is $H_{Ft} = \mathrm{e}^{-\rho \cdot t} \ln c_t + \lambda_t \cdot [(1-\tau) \cdot y_t - c_t] + \mu_{Ft} \cdot [\tau \cdot y_t]$. The first-order conditions are given by: (a) $\partial H_{Ft}/\partial c_t = 0$, (b) $\partial H_{Ft}/\partial \tau = 0$, (c) $\partial H_{Ft}/\partial g_{st} = -\dot{\mu}_{Ft}$ and (d) $\partial H_{Ft}/\partial k_t = -\dot{\lambda}_{Ft}$.

From conditions (a)–(d) we obtain Equation (14), which, by way of Equation (15F), leads to an expression for the optimal growth rate in the Futagami model:[10]

$$\gamma_F = \beta^{\beta} \cdot (1-\beta)^{(1-\beta)} - \rho. \tag{20F}$$

Note that (20F) is obtainable from our general model, simply by plugging in $\alpha = 1$ in Equation (20).

The optimal tax rate for the Futagami model[11] can be found from (2F), using (20F) and (15F):

$$\tau_F = (1-\beta) \cdot [1 - \rho \cdot \beta^{-\beta} . (1-\beta)^{(\beta-1)}]. \tag{22F}$$

Once again, note that (22F) is obtainable from our general model, simply by plugging in $\alpha = 1$ in Equation (22). As for our general model, the optimal tax rate in the model with public capital is *not* independent of ρ. This is because when the government chooses the tax rate in a model with public capital, it is essentially choosing between current consumption versus deferred consumption (i.e., diversion of resources towards public capital accumulation), and this does involve the rate of time preference.

Finally, the optimal saving rate can be obtained, using (20F) and (22F) in the expression for the savings rate:

$$s_F = 1 - \frac{\rho}{\beta^{(1+\beta)} \cdot (1-\beta)^{(1-\beta)} + \rho . (1-\beta)}. \tag{26F}$$

Once again, note that (26F) is obtainable from our general model, simply by plugging in $\alpha = 1$ in Equation (26). It can be shown that s_F is increasing in β.

3.3. THE GENERAL MODEL AND THE EXTREME CASES

As we have said before, since Barro (1990) incorporates public services but not public capital and Futagami et al. (1993) has public capital but not public services, neither paper can say anything definitive about how governments ought to allocate resources between the provision of public services and the accumulation of public capital. The only question that remains is whether we can "get by" with educated guesses, based on these two papers, about how a more general model that includes both public capital and public services would behave.

It is straightforward to show that the balanced optimal growth rate is lower in Barro (1990) than in Futagami et al. (1993) – i.e., $\gamma_B < \gamma_F$ for any β and ρ; compare Equations (20B) and (20F) – and that the share of output that is spent on public goods is higher in Barro (1990) than in Futagami et al. (1993) – i.e., $\tau_B > \tau_F$ for any β and ρ; compare Equations (22B) and (22F). Since $\alpha = 0$ in our general model yields Barro (1990) and $\alpha = 1$ yields Futagami et al. (1993), one might be tempted to infer that in our general model, an increase in α would cause an increase in γ and a decrease in τ. But, as is shown in section 2.1, γ is actually U-shaped in α for all values of β and that τ is also U-shaped in α for cases such as $\rho = 0.02$ and $\beta = 0.05$ (see also Figures 2 and 3). This implies that more can be learned from our general model about γ and τ, to take just two variables, than one could infer from Barro (1990) and Futagami et al. (1993). This may be seen as a justification for our construction of a general model that encompasses Barro (1990) and Futagami et al. (1993) because not even the qualitative behavior of the general model – namely the signs of some important comparative static derivatives – can be guessed by interpolating the corresponding results from the two extreme cases. (Had γ and τ been monotonic in α in our model, one could have correctly guessed that $\partial\gamma/\partial\alpha > 0$ and $\partial\tau/\partial\alpha < 0$ in our general model by simply noting that $\gamma_B < \gamma_F$ and $\tau_B > \tau_F$ and that $\alpha = 0$ gets us Barro (1990) and $\alpha = 1$ gets us Futagami et al. (1993).)

3.4. MODEL WITHOUT PRIVATE CAPITAL: $\beta = 0$ WITH $0 < \alpha < 1$

In this section we consider the extreme case of $\beta = 0$, which is a basic model of growth without private capital. Although it is a special case of the model with private capital, this model provides some interesting insights into the functioning of a centrally planned economy where all capital is owned by the state. As we will see, many of the results of this sub-section are obtainable from the previous sub-section simply by setting $\beta = 0$.

Putting $\beta = 0$ in (15) yields a production function

$$y_t = g_{st}^{\alpha} g_{ft}^{1-\alpha} \tag{15$'$}$$

The government budget constraint remains as in (2), but the private budget constraint (with $\dot{k}_t = 0$) simply reduces to

$$c_t = (1-\tau)y_t \tag{3$'$}$$

The social planner chooses τ and g_{ft} to maximize U in Equation (4), subject to Equations (15′), (2) and (3′) and g_{s0}, the given initial endowment of public capital.

This maximization problem's present value Hamiltonian is $H_t = \mathrm{e}^{-\rho t} \ln[(1-\tau)y_t] + \mu_t \cdot [\tau y_t - g_{ft}]$. The first order conditions are: (a) $\partial H_t/\partial \tau = 0$, (b) $\partial H_t/\partial g_{ft} = 0$ and (c) $\partial H_t/\partial g_{st} = -\dot{\mu}_t$. Conditions (a) and (b) imply

$$\frac{\partial y_t}{\partial g_{ft}} = (1-\alpha)g_{sft}^{\alpha} = 1 \tag{13$'$}$$

Conditions (a) and (c) imply

$$\frac{\dot{c}_t}{c_t} = -\frac{\dot{\mu}_t}{\mu_t} - \rho = \frac{\partial y_t}{\partial g_{st}} - \rho = \alpha g_{sft}^{\alpha-1} - \rho \tag{14$'$}$$

Equations (16′) and (14′) imply

$$\gamma_t = \gamma = \frac{\alpha}{1-\alpha} g_{sf}^{-1} - \rho = \alpha \cdot (1-\alpha)^{(1-\alpha)/\alpha} - \rho \tag{20$'$}$$

It is straightforward to show that γ is decreasing in ρ, the discount rate, and increasing in α.

Note that Equation (3′) implies that γ is also the optimal growth rate of GDP; i.e., $\gamma_t = \dot{y}_t/y_t$. So *long-run growth is possible despite the absence of private capital*. Clearly, the expression for γ in terms of g_{sf} is the same as in (20): only g_{sf} is given by (16′) instead of (16).

The optimal tax rate in this model is obtainable using (16′), (2), (15′) and (20′):

$$\tau = 1 - \rho \cdot (1-\alpha) \cdot g_{sf} = 1 - \rho \cdot (1-\alpha)^{\frac{\alpha-1}{\alpha}} \tag{22$'$}$$

It is straightforward to check that the expression for τ (in terms of g_{sf}) is obtainable from (22) by putting $\beta = 0$, although g_{sf} is now given by (16′).

In this model, it is clear that y, c, g_s and g_f all grow at the same constant rate, γ, for all t. In other words, the optimality conditions impose no restrictions on the admissible values of g_{s0} and, therefore, the economy is always on the balanced growth path and *there are no transitional dynamics*.

4. Conclusion

The premise of this paper has been that governments routinely juggle the needs of public spending for the provision of public services, which are essentially a short-term concern, with the need to accumulate public capital goods, which is a future-oriented issue, and that it is important to get the mix right. The derivation of the necessary conditions for the associated allocation problem has, therefore, been this paper's central objective. We also used the aforementioned necessary conditions to derive the (comparative static) effects of changes in the economy's fundamental parameters (such as the elasticities of output with respect to public capital, public services and private capital, and the consumers' rate of time preference) on the planner's allocation decisions.

Appendix

Let $m_0 > m$, as in section 2.2.1. We know from Equation (34) that m_t will decrease over time along the transition phase of the optimal path. The question that was left unresolved in section 2.2.1 was whether m_t would become *equal* to m – its balanced growth value; see Equation (17) – in finite time or forever approach closer to m without ever reaching it. In this appendix, we will show that *if the balanced optimal growth rate – see γ in Equation (20) – is positive, m_t will reach m in finite time*.

Let us assume, to the contrary, that the optimal solution never merges with the balanced growth optimal path and that, therefore, $m_t > m$ for all t. Then, Equation (34) implies that the optimal path would have $\dot{g}_{st} > 0$ and $k_t = k_0$ for all t. In this case, the social planner's problem can be restated as the choice of c_t and g_{ft} to maximize lifetime utility – i.e., U in Equation (4) – subject to Equation (36) and the boundary conditions that k_0 and g_{s0} are given.

The first-order necessary conditions are: Equation (13), the first equality of Equation (14) and Equation (36). This is a basic Ramsey-Cass-Koopmans optimal growth problem for which, as is well known, the long run rate of growth is zero.[12] By setting $\dot{c}_t = \dot{g}_{st} = 0$ in the necessary conditions, we get

$$\left(\lim_{t\to\infty} m_t\right)^{\beta} \equiv m_{\infty}^{\beta} = \alpha^{(1-\alpha)(1-\beta)-1} \cdot (1-\alpha)^{-(1-\alpha)(1-\beta)} \cdot (1-\beta)^{-1} \cdot \rho^{1-(1-\alpha)(1-\beta)}$$

By our assumption, $m_{\infty}^{\beta} > m^{\beta} = \beta^{\beta} \cdot (1-\beta)^{-\beta} \cdot \alpha^{-\beta}$. It can be checked, however, that, by Equations (16) and (20), $\gamma > 0$ implies $m_{\infty}^{\beta} < m^{\beta}$, a contradiction. Therefore, $\gamma > 0$ implies the convergence of the optimal growth path to the balanced optimal growth path in finite time.

A similar argument applies for the $m_0 < m$ case. QED

Notes

1. For more on these issues, see Aschauer (1989), Easterly and Rebelo (1993), and Turnovsky and Fisher (1995).

2. This global convergence result is a familiar feature of the literature on optimal growth with two state variables. For more on this literature, which focuses on the Uzawa-Lucas model with physical and human capital, see Barro and Sala-i-Martin (1995, chapter 5, appendix 5B), Mulligan and Sala-i-Martin (1993) and Caballé and Santos (1993).

3. A variable name with a dot on top indicates that variable's time derivative.

4. See Arrow and Kurz (1970, chapter II, section 6, Proposition 7). See Barro and Sala-i-Martin (1995, Appendix on Mathematical Methods, section 1.3.9) on the necessity of the transversality condition. It can be checked that the Hamiltonian is strictly concave in the state and control variables. Consequently, any outcome that satisfies the necessary conditions is the unique optimal outcome. That is, the necessary conditions also imply sufficiency and uniqueness. See Seierstad and Sydsaeter (1977) and Beavis and Dobbs (1990, p. 328 and Theorem 7.10).

5. Continuing our discussion of the literature on optimal growth with two state variables in the Introduction – see note 2 – it should be pointed out that, unlike our analysis in this section, this literature does not analyze transitional dynamics for the case in which consumption and investments in the two capital goods can be transformed into each other according to a linear point-in-time technology. Mulligan and Sala-i-Martin (1993) deal with this difficulty by allowing infinite rates of investment or, equivalently, "jumps" in their state variables – see their "Interesting Result 8" and section VIIa. This is something that we have specifically ruled out given our conception of the natures of our two capital goods. It is hard to think of an economic scenario that could justify a jump in our capital goods except perhaps a massive infusion of foreign aid or a major disaster and in neither case would these jumps be under the government's control.

6. See Kamien and Schwartz (1991, Part II, Section 7, p. 160). The applicable necessary conditions are their conditions a, b, c and d(v).

7. See Kamien and Schwartz (1991, Part II, Section 4, Equation (7)).

8. When $\beta = 1$, our model reduces to the classic AK model without public goods – see Barro and Sala-i-Martin (1995, section 4.1).

9. In all three special cases considered in this section the existence of the balanced growth optimal path can be established as in section 2.1.

10. Note, though, that this expression is not provided explicitly by Futagami et al. (1993).

11. Futagami et al. (1993) do not derive an expression for the tax rate for the social optimum, which is given here by Equation (22F). The focus of their paper is to demonstrate that in a model with public capital (unlike Barro (1990)), the welfare-maximizing tax rate for the market economy is different from (i.e., lower than) the growth-maximizing tax rate, due to the existence of transitional dynamics.

12. See Arrow and Kurz (1970, chapter III).

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