Optimal PMU Placement for Modeling Power Grid Observability with Mathematical Programming Methods

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Summary

Phasor measurement units (PMUs) can obtain synchronized voltage and current phasors to increase the accuracy of state estimation results. Optimal PMU placement (OPP) reduces the required number of PMUs to make the system fully observable. In this paper, two mathematical programming formulations, which are mixed integer linear programming (MILP) and nonlinear programming (NLP), for power grid observability modeling to solve the OPP problem are presented. Power flow and zero injection measurement modeling along with restricted communication facilities, PMU failure, and limited channel capacity contingencies are investigated. MILP zero injection formulation is improved to overcome the observability redundancy and optimality drawbacks. A new formulation for nonlinear programming-based PMU placement for zero injection measurement is proposed. MILP and NLP methods are compared to illustrate the advantages and disadvantages of each method. The comparison and proposed formulations are examined on IEEE 14-, 57-, 118-, 300-bus test systems and a large 2383-bus Polish system to demonstrate their effectiveness.

Keywords: Observability, linear programming, nonlinear programming, phasor measurement units, PMU placement

1. Introduction

The power system is required to have a real-time monitoring of the system operating conditions to enhance its security. In power grids, the measured bus voltage, currents, real and reactive power are collected by remote terminal units at each substation. Those measurements are sent to a control center. A control center then conducts state estimation to determine the best estimates of system's state variables (every node's voltage magnitude and phase angle). Most recently, phasor measurement unit (PMU)-based sensors are used to collect time-stamped measurements from global positioning system (GPS) [1, 2]. PMU sensors obtain synchronized voltage and current phasors measurements at a faster rate of (30 $\sim$ 120 Hz) [3]. Hence, PMUs can give a superior situation awareness of the power grid. By installing a PMU at one bus, it

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can obtain the bus voltage phasor and all current phasors of the branches connected to that bus [4]. However, placing PMU sensors at all buses of the system can be expensive and uneconomical. Therefore, optimal PMU placement (OPP) problem should be solved to make the system entirely observable by installing less PMU devices at specific buses.

Heuristic and mathematical programming techniques are used to solve the OPP problem. The heuristic technique is based on search process to obtain the OPP. There are several heuristic-based techniques that have been studied in the literature. A graph theory and simulated annealing algorithm to obtain the minimum number of PMU sensors are developed in [5]. Then other heuristic-based approaches have been proposed such as simulated annealing with Tabu search [6], spanning tree [7], genetic algorithm [8], nondominated sorting genetic [9], Tabu search genetic [10], particle swarm optimization approach [11], and recursive Tabu search [12]. An immunity genetic algorithm [13] and binary particle swarm optimization [14, 15] are used to solve the OPP problem.

Heuristic-based OPP does not guarantee a global optimum solution. Hence, two major mathematical programming approaches are developed in the literature: MILP and NLP. While MILP formulations guarantee a global optimum solution, NLP formulations provide several local minimum solutions.

Integer linear programming (ILP) to obtain the OPP is introduced in [2, 4]. Several algorithms and techniques considering integer linear programming and contingency-constrained PMU placement are developed in [16, 17, 18, 19, 20, 21, 22]. In [23], ILP is used with auxiliary variable to find the OPP in case of zero injection. The same method considering conventional measurements is developed in [24]. Zero injection redundancy limitation and global optimal solution considering mutual buses are presented in [25]. Reference [26] proposes an integer quadratic programming approach. A weighted least square algorithm using nonlinear observability constraint is presented in [27]. Nonlinear programming (NLP) formulations are introduced in [28]. This type of formulations has been explored under several contingencies in [29, 30]. In [31], MILP and NLP comparison is conducted using a simple system, and limitation of zero injection formulation for NLP is discussed. However, zero injection formulation in nonlinear programming-based PMU placement has not been properly solved in the literature.

In this paper, modeling power grid observability to solve the OPP problem is implemented using two different approaches which are MILP and NLP. Power flows, zero injections, restricted communication facilities, PMU failure, and limited channel capacity are discussed. A new nonlinear programming formulation for zero injection is proposed. The proposed formulation is examined by validating the results with the MILP formulation. MILP and NLP comparison is conducted to illustrate their advantages and disadvantages.

The main contributions of this paper can be summarized as follows. First, a new effective zero injection formulation in nonlinear programming is proposed and validated to provide minimum number of PMUs compared to other papers. Second, MILP zero injection formulation is improved to solve the observability redundancy and optimality drawbacks. Third, two mathematical programming methods are compared under several contingencies applied to different IEEE test systems, and the proposed zero injection formulations are evaluated on a large 2383-bus Polish system.

The rest of this paper is organized as follows. Sections 2 and 3 present MILP and NLP formulations. Section 4 proposes the effective power flow and zero injection measurement formulations and investigates the aforementioned contingencies. Section 5 concludes the paper.

2. Mixed Integer Linear Programming

Power system state estimation with a DC power flow is analyzed in this paper, and (1) presents the linear measurement function which maps the state to the measurement.

$$z = Hx + e \tag{1}$$

where z represents the measurement vector, H is the measurement matrix, x is the state variable vector, and e is the error measurement vector. The state variables are the voltage phase angle for each bus in the power system. The PMUs can obtain the measurements including the voltage phase angle (θ_i) of Bus i and the power flow from Bus i to Bus j, where $j \in ad_i$ represents the adjacent buses to Bus i. Thus, the PMU will measure θ_i, and θ_j can be obtained as the power flow P_{ij} is measurable. Therefore, Bus i with its adjacent buses are observable when a PMU is installed only at Bus i.

In other words, Bus i itself can be observable with at least a single PMU placed at this bus or one of its adjacent buses. This is can be represented by the following inequality:

$$f_i(x) = x_i + \sum_{j \in ad_i} x_j \geq 1 \tag{2}$$

where $f_i(x)$ is the observability function for Bus i, x_i is a binary decision variable to install a PMU at Bus i ($x_i = 1$) or not ($x_i = 0$), and x_j is the binary decision variable for the buses adjacent to that bus.

The OPP for the IEEE 14-bus system (shown in Fig. 1) [32] is formulated as MILP and solved by MATLAB's *intlinprog* function. The OPP result indicates that only four PMUs can be installed on buses

2, 8, 10, and 13 to make the system entirely observable. Generalized MILP for the OPP problem can be expressed as [4]:

$$\min_{x} \quad \sum_{k=1}^{N} w_k \; x_k \tag{3a}$$

$$\text{subject to:} \quad Ax \geq B \tag{3b}$$

$$x_i \in \{0, 1\}, i = 1, \cdots, N \tag{3c}$$

where x_i is the binary decision, and w_k is the PMU placement cost. It is assumed that the PMUs have the placement cost $w_i = 1$ making the PMU placement cost minimization equivalent to the number of PMUs minimization. Entries of A and the B matrices are:

$$a(i,j) = \begin{cases} 1, \text{ Bus } i \text{ and Bus } j \text{ are connected} \\ 1, \ i \text{ is equal to } j \\ 0. \text{ Otherwise} \end{cases}$$

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$$

3. Nonlinear Programming

OPP problem can be formulated and solved using nonlinear programming (NLP) which based on sequential quadratic programming (SQP) [27, 28]. NLP method can produce more than one solution to the OPP problem, while the MILP formulation can provide a single solution.

For NLP, x_i is considered as a continuous decision variable rather a binary variable as the MILP formulation. Therefore, x_i is forced to be 1 or 0 by the following constraint: $x_i(x_i - 1) = 0$.

The quadratic objective function, which represents the overall PMU placement cost, is minimized by the NLP formulation subjected to nonlinear equality constraints. The decision variables are 0 and 1 which indicate the lower and upper bounds of the problem formulation. Hence, the nonlinear constraints can assure the complete observability of the system [27, 28].

The NLP formulation for the OPP problem can be expressed as:

$$\min_{x} \quad J(x) = x^T W x = \sum_{k=1}^{N} w_k \; x_k^2 \tag{4a}$$

$$\text{s.t.:} \quad g_i(x) = (\; 1 - x_i) \prod_{j \in ad_i} (\; 1 - x_j) \;\; = 0 \tag{4b}$$

$$0 \leq x_i \leq 1, \quad \text{for all } i \in \boldsymbol{S} \tag{4c}$$

where $J(x)$ represents the OPP objective function, x^T is the transposed vector of x, W is the diagonal weight matrix, ad_i indicates the adjacent buses of Bus i, and $\boldsymbol{S}$ represents the system buses set.

This NLP formulation is a nonconvex optimization problem since a number of local minimum solutions can result in using the nonlinear equality constraints [28], and it is solved by sequential quadratic programming (SQP) algorithm. As a consequence, several solutions for the OPP problem can be obtained by choosing different initial conditions x.

The IEEE 14-bus system [32] as shown in Fig. 1 is used as an example to solve the OPP problem with the NLP formulation, and the NLP and MILP solutions are compared to each other. It is assumed that the weight of all PMUs is $w_i = 1$ to make the installation cost minimization equivalent to the number of PMUs minimization. The MATLAB's function $fmincon$ is used to solve this nonconvex optimization problem with NLP formulation and SQP solver.

The NLP obtains various solutions to the OPP problem based on the initial points x. Therefore, the initial points are programmed to be random numbers in the feasible region between the upper and lower bounds of one and zero. As a result, optimal solutions are found after several iterations:

$$x = \begin{bmatrix} 0\;1\;0\;0\;0\;0\;0\;1\;0\;1\;0\;0\;1\;0 \end{bmatrix}^T, \; x = \begin{bmatrix} 0\;1\;0\;0\;0\;0\;1\;0\;0\;1\;0\;0\;1\;0 \end{bmatrix}^T, \; x = \begin{bmatrix} 0\;1\;0\;0\;0\;0\;1\;0\;0\;0\;1\;0\;1\;0 \end{bmatrix}^T,$$

$$x = \begin{bmatrix} 0\;1\;0\;0\;0\;1\;0\;1\;1\;0\;0\;0\;0\;0 \end{bmatrix}^T, \text{ and } \; x = \begin{bmatrix} 0\;1\;0\;0\;0\;1\;1\;0\;1\;0\;0\;0\;0\;0 \end{bmatrix}^T$$

The above solutions indicate that the OPP buses are the following five sets: $\{2, 8, 10, 13\}$, $\{2, 7, 10, 13\}$, $\{2, 7, 11, 13\}$, $\{2, 6, 8, 9\}$, or $\{2, 6, 7, 9\}$. Note that the first optimal solution is the same as the MILP solution.

Thus, NLP is another effective algorithm to solve the OPP problem by obtaining various optimal solutions to select from. MILP and NLP comparison for the OPP problem is conducted using four different systems which are IEEE 14-, 57-, 118-, and 300-bus systems [32] as shown in Table 1.

4. OPP Case Studies

4.1. Power Flow Measurements

Suppose that Branch ij in the system has a meter to measure the power flow. In the case that one of the state variables of Bus i or j (θ_i or θ_j) is measured, the state variable of the other bus can be provided since the power flow (P_{ij}) is known.

4.1.1. MILP and NLP Approaches

With the information of the power flows, the observability constraints must be reformulated to find the optimal solution. If the power flow measurement on Branch k of Bus i and Bus j is known, the MILP constraints are modified to be a joint observability constraint as follows [4, 18].

$$f_{flow,k} = f_i + f_j \geq 1 \tag{5}$$

Constraint (5) indicates that if Bus i or Bus j is observable, the other bus can also be observable since the power flow of Branch k is given.

Similarly, the observability constraints of the nonlinear programming must be reformulated to find the optimal solution with the measured power flow. Therefore, the observability constraints of Bus i and Bus j are modified to be a joint observability constraint as the following [29]:

$$g_{flow,k} = g_i g_j = 0 \tag{6}$$

Constraint (6) can result in high orders since several terms of $(1 - x_i)$ can be produced with multiplying the constraints of Bus i and Bus j [29]. Hence, the resulted terms with high orders will be treated as a first order term since this constraint has a zero right hand side.

4.1.2. Power Flow Measurement Example

Suppose that the power flow measurements for the IEEE 14-bus system (Fig. 1) [32] are on branches $2-3$, $3-4$, $6-11$, $6-12$, and $7-8$. For Branch $2-3$, the joint constraint (5) is applied since there are power flows on branches $2-3$ and $3-4$ as the following:

$$\begin{aligned} f_{flow,2-3,3-4} &= f_2 + f_3 + f_4 \geq 1 \\ &= x_1 + 3x_2 + 3x_3 + 3x_4 + 2x_5 + x_7 + x_9 \geq 1 \end{aligned}$$

Joint constraint $f_{flow,2-3,3-4}$ means that only one of the buses 2, 3, and 4 must be observable to make the other buses observable since the power flows are known. Hence, this joint constraint meets the minimum requirement of installing at least a single PMU at one of those buses or at the buses adjacent to them. The other branches are formulated in a similar way to obtain the joint constraints.

For the nonlinear programming, joint constraint (6) is applied to the three observability functions due to the power flows on branches $2-3$ and $3-4$ as follows.

$$\begin{aligned} g_{flow,2-3,3-4} &= g_2 g_3 g_4 = 0 \\ &= (1-x_1)(1-x_2)(1-x_3)(1-x_4)(1-x_5)(1-x_7)(1-x_9) = 0 \end{aligned}$$

Table 2 shows the results of the OPP for power flow case. The number of PMUs in this case is reduced due to the power flow meters. Table 3 presents the location of the power flow measurement branches.

4.2. Zero Injection Measurements

A four-bus system as illustrated in Fig. 2 is used to easily demonstrate the zero injection case. Assume that Bus ℓ has a zero injection measurement. The power injection and the voltage phase angles of the four buses are related to each other as the following:

$$P_{inj,l} = \frac{\theta_l - \theta_i}{X_{li}} + \frac{\theta_l - \theta_j}{X_{lj}} + \frac{\theta_l - \theta_k}{X_{lk}} = 0 \tag{7}$$

If the power injection and three of the phase angles are known, the fourth phase angle can be measured. Therefore, three buses have to be observable to make the fourth one observable with the help of the installed PMU and the zero injection at Bus ℓ. This requirement is formulated using the MILP as [4]:

$$f_{inj,l} = f_i + f_j + f_k + f_l \geq 3 \tag{8}$$

When one of the observability functions (f_i, f_j, f_k, or f_l) equals to zero, then the joint constraint (8) meets the zero injection requirement. Nevertheless, this joint constraint may result in two drawbacks. First, adding the observability functions can produce a redundancy for certain buses which can make the constraint (8) satisfied with two zero observability functions [25]. A six-bus system (shown in Fig. 3) is employed to explain the drawbacks of the constraint (8). Assume that Bus 2 has a zero injection measurement. Three buses (1, 3, and 5) are adjacent to Bus 2. Thus, the MILP constraints for this system will be as follows.

$$
\begin{aligned}
f_{inj,2} &= f_1 + f_2 + f_3 + f_5 \geq 3 \\
&= 3x_1 + 4x_2 + 2x_3 + x_4 + 3x_5 + 2x_6 \geq 3 \\
f_4 &= x_3 + x_4 \geq 1, \quad f_6 = x_1 + x_3 + x_6 \geq 1
\end{aligned}
$$

From the above constraints, the OPP can be on buses 3 and 4 (i.e. $f_1 = 0$, $f_2 = 1$, $f_3 = 2$, $f_4 = 2$, $f_5 = 0$, and $f_6 = 1$) which leaves buses 1 and 5 unobservable. Note that the two buses 1 and 5 cannot be observable even with the help of zero injection measurement since two out of four buses are unobservable. Therefore, the system complete observability is not guaranteed in some configuration. Recently, it has been clarified in [25] that f_i cannot be guaranteed to be 0 or 1 which is the main reason of this limitation. Thus, the authors propose a formulation to keep the right hand side equals to 1 which can solve the redundant observability of some buses. Then the joint constraint (8) is reformulated as follows.

$$
f_{inj,l} = \begin{cases} f_i + f_j \geq 1, \quad f_i + f_k \geq 1, \quad f_i + f_l \geq 1, \quad f_j + f_k \geq 1, \quad f_j + f_l \geq 1, \quad f_k + f_l \geq 1 \end{cases} \tag{9}
$$

The observability constraint (9) guarantees complete observability since it can be satisfied if at most one of the observability constraints (f_i, f_j, f_k, or f_l) is zero. Solving the same problem using (9), the optimal PMU placement can be on buses 3 and 6 which leaves Bus 5 unseen by the PMUs but can be observable with the help of zero injection measurement at Bus 2.

In addition to the redundant observability, the joint constraint (8) cannot obtain the optimum solution if there are two or more zero injections with mutual buses [25]. Assume that there are zero injections at Bus 1 and Bus 3 in the six-bus system (shown in Fig. 3). The adjacent buses to Bus 1 are buses 2, 5, and 6, while the adjacent buses to Bus 3 are buses 2, 4, and 6. In this case, Bus 2 and Bus 6 are mutual buses, then MILP constraints using (8) can as the following:

$$
\begin{aligned}
f_{inj,1} &= f_1 + f_2 + f_5 + f_6 \geq 3 \\
&= 4x_1 + 3x_2 + 2x_3 + 3x_5 + 2x_6 \geq 3
\end{aligned}
$$

$$
\begin{aligned}
f_{inj,3} &= f_2 + f_3 + f_4 + f_6 \geq 3 \\
&= 2x_1 + 2x_2 + 4x_3 + 2x_4 + x_5 + 2x_6 \geq 3
\end{aligned}
$$

Theses constraints can be satisfied using at least two PMUs (e.g. placement at Bus 1 and Bus 3), while this problem can be satisfied using only one PMU at Bus 2. Note that by placing a single PMU at Bus 2, Bus 4 and Bus 6 are unseen by the PMU but can be observable with the zero injections at Bus 1 and Bus 3. Therefore, the optimal solution may not be provided using the joint constraint (8). This problem can be solved using (9) with some modification [25]. Suppose that Bus 1 and Bus 3 have zero injections in the six-bus system (Fig. 3). The MILP constraints using (9) will be as follows.

$$f_{inj,1} = \begin{cases} f_1 + f_2 \geq 1, \quad f_1 + f_5 \geq 1, \quad f_1 + f_6 \geq 1, \quad f_2 + f_5 \geq 1, \quad f_2 + f_6 \geq 1, \quad f_5 + f_6 \geq 1 \end{cases}$$

$$f_{inj,3} = \begin{cases} f_2 + f_3 \geq 1, \quad f_2 + f_4 \geq 1, \quad f_2 + f_6 \geq 1, \quad f_3 + f_4 \geq 1, \quad f_3 + f_6 \geq 1, \quad f_4 + f_6 \geq 1 \end{cases}$$

Then mutual observability functions in the left-hand side of $f_{inj,1}$ and $f_{inj,3}$ have to be merged. In this paper, the MILP formulation is improved by solving the optimality limitation with less constraints as the following:

$$f_{inj,1\ and\ 3} = \begin{cases} f_1 + f_2 + f_3 \geq 1, \quad f_1 + f_2 + f_4 \geq 1, \quad f_1 + f_3 + f_6 \geq 1, \quad f_1 + f_4 + f_6 \geq 1, \\ f_2 + f_3 + f_5 \geq 1, \quad f_2 + f_4 + f_5 \geq 1, \quad f_3 + f_5 + f_6 \geq 1, \quad f_4 + f_5 + f_6 \geq 1, \\ f_1 + f_5 \geq 1, \quad f_3 + f_4 \geq 1, \quad f_2 + f_6 \geq 1 \end{cases}$$

These constraints can obtain the OPP by placing a single PMU at Bus 2 after merging the mutual observability functions. With zero injections at Bus 1 and Bus 3, observability can be assured for Bus 4 and Bus 6. The number of constraints in [25] to solve the six-bus system is 16 compared to 11 in this paper. This reduction in the number of the constraints is significant for solving large systems.

For the NLP formulation, the equivalent has not been addressed adequately. The zero injection joint constraint for NLP in [29] is not equivalent to (8) or (9). It indicates that the zero injection bus and its adjacent buses are observable if one of them is observable. As a consequence, this constraint can result in unobservable buses. In this paper, the equivalent in nonlinear programming formulation is proposed.

Once Bus ℓ has a zero injection measurement (Fig. 2), for this particular case, then we need at least any

3 buses among all 4 buses to be observable to guarantee a complete observability.

That is, the following six constraints should be satisfied.

$$g_{inj,l} = \begin{cases} g_i g_j = 0, \quad g_i g_k = 0, \quad g_i g_l = 0, \quad g_j g_k = 0, \quad g_j g_l = 0, \quad g_k g_l = 0 \end{cases} \tag{10}$$

Suppose that Bus 2 has a zero injection in the six-bus system (shown in Fig. 3). Buses 1, 3, and 5 are adjacent buses to Bus 2. The NLP constraints will be as follows.

$$g_{inj,2} = \begin{cases} g_1 g_2 = 0, \quad g_1 g_3 = 0, \quad g_1 g_5 = 0, \quad g_2 g_3 = 0, \quad g_2 g_5 = 0, \quad g_3 g_5 = 0 \end{cases}$$

$$g_4 = (1 - x_3)(1 - x_4) = 0$$

$$g_6 = (1 - x_1)(1 - x_3)(1 - x_6) = 0$$

Then an optimal solution can be achieved by installing PMUs on Bus 3 and Bus 6 (i.e. $g_1 = 0$, $g_2 = 0$, $g_3 = 0$, $g_4 = 0$, $g_5 = 1$, and $g_6 = 0$) which makes Bus 5 cannot be seen by the PMUs but can be observable with the help of Bus 2 zero injection measurement.

Then suppose that Bus 1 and Bus 3 have zero injections in the same aforementioned system. The NLP constraints with the mutual buses will be as follows.

$$g_{inj,1 \ and \ 3} = \begin{cases} g_1 g_2 g_3 = 0, \quad g_1 g_2 g_4 = 0, \quad g_1 g_3 g_6 = 0, \quad g_1 g_4 g_6 = 0, \\ g_2 g_3 g_5 = 0, \quad g_2 g_4 g_5 = 0, \quad g_3 g_5 g_6 = 0, \quad g_4 g_5 g_6 = 0, \\ g_1 g_5 = 0, \quad g_3 g_4 = 0, \quad g_2 g_6 = 0 \end{cases}$$

From these constraints, the optimal PMU placement can be at Bus 2. Note that buses 4 and 6 are observable with the zero injection measurements at buses 1 and 3.

Thus, both MILP and NLP joint constraints for zero injection measurements can be satisfied if at most one of the observability constraints (zero injection bus or its adjacent buses constraints) is zero. Also, these joint constraints guarantee complete observability and optimal solution to the problem. Table 4 and Table 5 show the location of the zero injections and the results of the zero injection case for MILP and NLP, respectively.

4.3. Limited Communication Facility

PMUs need to communicate with the control center through data links at the substations to provide the measurements of synchronized voltage and current phasors. Therefore, a substation with limited communication facility can obstruct the PMU placement. With restricted communication problem, the PMU placement cost will be increased [33]. Hence, the placement cost w_i for MILP and NLP will be increased for any bus with limited communications. As a result, this high placement cost can omit the limited communication buses from the optimal solution [29]. Suppose that Bus 2 and Bus 9 have limited communication facilities on the IEEE 14-bus system (shown in Fig. 1) [32]. Then the placement costs of Bus 2 and Bus 9 are increased to be $w_i = 10^9$. Table 6 shows the results of the limited communication facility case.

4.4. Single PMU Failure

Even though the PMUs have a high reliability, there is a chance of a single PMU failure. To assure the complete observability of the system, main and backup sets are obtained. The optimal PMU solution without taking the PMU failure into account is the main set, whereas the backup set is generated in case of a PMU failure. The right hand side of the MILP constraints can be modified to be two to let each bus observed by two PMUs [16]. Instead, the main set terms x_i and x_j of the MILP constraints can be removed to generate the backup set. Likewise, the main set terms $(1 - x_i)$ and $(1 - x_j)$ of the NLP constraints are removed to provide the backup set [29]. Therefore, the buses in the main set will not be selected again, and the backup set will assure the complete observability of the system when a one PMU fails.

IEEE 14-bus system (Fig. 1) main set is obtained as in Section 2 and Section 3, and then MILP and NLP main set can be the following: $\{2, 8, 10, 13\}$. Therefore, all of the terms x_2, x_8, x_{10}, and x_{13} of MILP constraints are ignored to generate the backup set. Similarly, all of the terms $(1-x_2)$, $(1-x_8)$, $(1-x_{10})$, and $(1 - x_{13})$ are removed from the NLP constraints.

After solving the problem, the resulted backup set for the MILP is $\{1, 4, 6, 7, 9\}$, whereas the backup sets for the NLP formulation are $\{1, 4, 6, 7, 9\}$, $\{1, 3, 6, 7, 9\}$, $\{3, 5, 6, 7, 9\}$, or $\{4, 5, 6, 7, 9\}$. From Table 7, we can see that the single PMU failure case would double the total minimum number of PMUs due to the backup set.

4.5. Limited PMU Channel Capacity

The OPP has been solved supposing that all PMUs have enough channels to make all adjacent buses observable. In reality, PMUs are made to have a different number of channels with different prices [19]. In this section, the OPP is analyzed in case that we have PMUs with limited channel capacity.

Let's assume that the number of adjacent buses to Bus i (m_i) is larger than the PMU channel capacity (c). The number of line combinations ($C^c_{m_i}$) is given as follows [29, 30].

$$C^c_{m_i} = \frac{m_i!}{c!(m_i - c)!} \tag{11}$$

Then the observability constraints are changed for both MILP and NLP to meet the possible line combinations. Note that if the number of adjacent buses to Bus i is less than or equal to the number of channel capacity, the observability constraint of Bus i is kept the same.

Let's assume that we have PMUs with limited channel capacity where $c = 3$ for the 14-bus system (Fig. 1). Then the observability constraints are changed as follows.

At Bus 1:

The adjacent buses are 2 and 5 which means that $m_1 = 2$ and $c > m_1$. Thus, we have enough channels for this bus, and the constraints f_1 and g_1 are kept the same.

At Bus 2:

The adjacent buses are 1, 3, 4, and 5 which means that $m_2 = 4$ and $c < m_2$. Thus, the number of line combinations is 4, and they are $\{2-1, 2-3, 2-4\}, \{2-1, 2-3, 2-5\}, \{2-1, 2-4, 2-5\}$, and $\{2-3, 2-4, 2-5\}$. Then the observability constraint for Bus 2 is changed as follows.

For MILP:

$f_{2,1} = x_2 + x_1 + x_3 + x_4 \geq 1, \quad f_{2,2} = x_2 + x_1 + x_3 + x_5 \geq 1$

$f_{2,3} = x_2 + x_1 + x_4 + x_5 \geq 1, \quad f_{2,4} = x_2 + x_3 + x_4 + x_5 \geq 1$

For NLP:

$g_{2,1} = (1 - x_2)(1 - x_1)(1 - x_3)(1 - x_4) = 0, \quad g_{2,2} = (1 - x_2)(1 - x_1)(1 - x_3)(1 - x_5) = 0$

$g_{2,3} = (1 - x_2)(1 - x_1)(1 - x_4)(1 - x_5) = 0, \quad g_{2,4} = (1 - x_2)(1 - x_3)(1 - x_4)(1 - x_5) = 0$

Then the process is repeated for the rest of buses to make sure that each constraint has only three adjacent buses. Table 8 shows the limited channel capacity case results.

4.6. Remarks on OPP Problem Simulation Results

MILP and NLP comparison is conducted using different IEEE test case systems. Five case studies, which are Power flows, zero injections, limited communications, PMU failure, and limited PMU channels, are formulated using MILP and NLP approaches. A new formulation for zero injection using NLP is presented and examined. MATLAB's *intlinprog* function is used to solve the MILP, while NLP is solved by MATLAB's *fmincon* function with SQP solver. The initial values are chosen as random numbers in the feasible region. In a large-scale system, the initial values should be designed carefully to make the NLP converge to the minimum point. The total number of the initial values should not exceed 45% of total number of buses. Then some of the initial values can be designed with different random numbers to achieve several solutions. The nonlinear constraints tolerance can be varied from 10^{-4} to 10^{-12} to get the least number of PMUs. From Table 1, we can see that NLP obtains the least number of PMUs as same as MILP. NLP can also provide several solutions to the OPP problem. One of the NLP optimal sets matches the MILP solution. On the other hand, the computational time of the MILP is less than the NLP. Table 9 presents the average CPU time for both MILP and NLP on different IEEE systems. From Table 2 and Table 5, it can be seen that the number of PMUs in both methods is reduced to be less than the general case because of the power flow measurements and zero injection measurements, respectively. On the contrary, more PMUs are resulted in the restricted communication and PMU failure cases as shown in Table 6 and Table 7. A backup set is generated for the single PMU failure case which would increase the PMU installation cost. It should be noted that the number of PMUs would be reduced if power flows and zero injections are considered in this case. To validate the effectiveness of the NLP zero injection formulation, a comparison of several algorithms results for zero injection case is shown in Table 10. For further analysis, the proposed zero injection formulations for MILP and NLP are evaluated on a large 2383-bus Polish system provided by MATPOWER [34] as can be seen from Table 11. Therefore, MILP and NLP approaches are effective to work out the OPP problem, and they can provide the same results.

5. Conclusion

Power grid observability modeling to tackle the OPP problem is presented using two approaches. MILP and NLP formulations for the OPP problem are demonstrated for complete observability. Nonlinear programming has an advantage of providing several optimal solutions compared to the MILP method. However, mixed integer linear programming has less CPU time compared to the nonlinear programming. MILP zero injection formulation is enhanced to solve the redundancy and optimality limitations. A new zero injection formulation for nonlinear programming is developed. Power flows, zero injections, limited communication facilities, PMU failure, and limited channel capacity case studies are demonstrated for the two methods. MILP and NLP advantages and disadvantages are discussed.

List of Symbols

z	Measurement vector
H	Measurement matrix
e	Error vector
j	Adjacent bus to Bus i
ad_i	Set of buses adjacent to Bus i
θ_i, θ_j	Voltage phase angles of buses i and j
P_{ij}	Real power flow measurement from Bus i to Bus j
$P_{inj,l}$	Real power injection measurement at Bus l
X_{ij}	Reactance of the line $i - j$
f_i	Observability constraint function of Bus i for MILP
g_i	Observability constraint function of Bus i for NLP
x	PMU placement vector
x_i	PMU placement binary decision variable of Bus i
A	Binary connectivity matrix
B	Vector of all-ones
T	Transpose operator
$J(x)$	Objective function of vector x
W	Diagonal cost matrix
w_i	PMU Installation cost at Bus i
$\boldsymbol{S}$	Set of system buses
$C^c_{m_i}$	Number of line combinations
c	Channel capacity
m_i	Number of adjacent buses to Bus i

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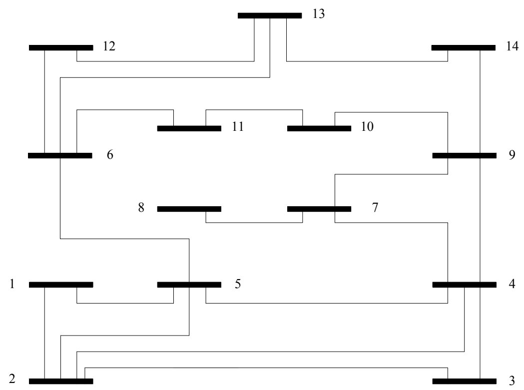

Figure 1: IEEE 14-bus system.

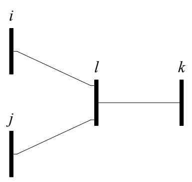

Figure 2: Four-bus system.

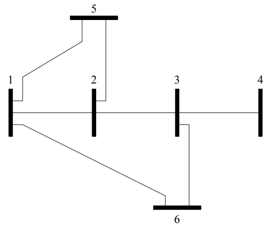

Figure 3: Six-bus system.

Table 1: OPP results using MILP and NLP/SQP.

IEEE Test System	Number of PMUs		Number of NLP Solutions
	MILP	NLP	
14-bus	4	4	5
57-bus	17	17	19
118-bus	32	32	10
300-bus	87	87	8

Table 2: Power flow measurements case results.

IEEE Test System	Number of Flow Branches	Number of PMUs		Number of NLP Solutions
		MILP	NLP	
14-bus	5	3	3	11
57-bus	40	6	6	5
118-bus	31	24	24	5
300-bus	43	81	81	4

Table 3: Branches of power flow measurements.

IEEE Test System	Branches of Flow Measurements
14-bus	2-3,3-4,6-11,7-8,6-12
57-bus	1-2,1-15,1-16,1-17,3-15,4-5,4-6,4-18,7-29, 29-52,8-9,9-10,10-12,10-51,12-13,51-50, 11-41,11-43,41-42,42-56,14-46,47-46, 19-20,20-21,22-38,38-37,38-44,38-48, 49-38,23-24,24-25,24-26,27-26,28-27, 30-31,32-34,34-35,36-35,40-36,53-54
118-bus	1-3,3-5,6-7,8-9,11-13,16-17,20-21,23-25, 23-32,32-114,27-28,34-43,35-36,41-42, 47-46,49-50,50-57,51-52,56-58,60-62, 65-68,68-116,71-73,76-77,77-82,82-83, 86-87,90-91,95-96,99-100,110-112
300-bus	1-3,3-4,6-7,8-11,11-13,15-16,21-22,24-25, 25-26,32-35,37-38,40-68,68-174,46-47, 50-51,55-56,70-71,77-84,84-86,95-103, 108-112,120-125,136-138,145-265, 156-157,160-166,166-167,173-198, 198-216,216-220,182-190,184-185, 200-202,208-209,88-235,64-239,2-248, 17-252,109-263,270-292,270-296, 269-288,294-300

Table 4: Zero injection measurement locations.

IEEE Test System	Zero Injection Measurement Buses
14-bus	7
57-bus	4,7,11,21,22,24,26,34,36,37,39,40,45,46,48
118-bus	5,9,30,37,38,63,64,68,71,81
300-bus	17,58,233,256,294

Table 5: Zero injection case results.

IEEE Test System	Number of Zero Injections	Number of PMUs		Number of NLP Solutions
		MILP	NLP	
14-bus	1	3	3	1
57-bus	15	11	11	6
118-bus	10	28	28	4
300-bus	5	82	82	2

Table 6: Limited communication facility case results.

IEEE Test System	Limited Communication Buses	Number of PMUs		Number of NLP Solutions
		MILP	NLP	
14-bus	2,9	5	5	11
57-bus	1,4,9,15	17	17	10
118-bus	2,9,11,12,17	35	35	8
300-bus	2,9,11,64,111, 277,299,300	92	92	4

Table 7: Single PMU failure results.

IEEE Test System	Number of PMUs		Number of NLP Solutions
	MILP	NLP	
14-bus	9	9	4
57-bus	35	35	4
118-bus	75	75	2
300-bus	221	221	2

Table 8: Limited channel capacity case results.

IEEE Test System	Number of Channels	Number of PMUs		Number of NLP Solutions
		MILP	NLP	
14-bus	3	4	4	1
57-bus	4	17	17	4
118-bus	6	32	32	3
300-bus	7	87	87	3

Table 9: MILP and NLP CPU time comparison.

Case	IEEE Test System	CPU Time (s)	
		MILP	NLP
None	14-bus	0.0313	0.1563
	57-bus	0.0469	0.9375
	118-bus	0.0781	9.9063
	300-bus	0.0938	49.7656
Power Flow Measurements	14-bus	0.0313	0.0781
	57-bus	0.0469	0.4844
	118-bus	0.0625	6.7500
	300-bus	0.0938	43.3750
Zero Injection Measurements	14-bus	0.0313	0.0938
	57-bus	0.0469	0.8423
	118-bus	0.0781	6.7969
	300-bus	0.1094	43.2344
Limited Communication Facility	14-bus	0.0313	0.0781
	57-bus	0.0469	0.6406
	118-bus	0.0625	6.3906
	300-bus	0.0938	44.5469
Single PMU Failure	14-bus	0.0313	0.0781
	57-bus	0.0469	0.8438
	118-bus	0.0625	6.7969
	300-bus	0.0938	45.4063
Limited Channel Capacity	14-bus	0.0313	0.0938
	57-bus	0.0469	13.2188
	118-bus	0.0781	25.6875
	300-bus	0.1250	64.5313

Table 10: Comparison results of zero injection using different methods.

Method	IEEE Test System		
	14-bus	57-bus	118-bus
ILP [2]	3	12	29
TS [10]	3	13	-
GA [8]	3	12	29
NSG [9]	-	-	29
PSO [11]	3	11	28
ILP [17]	3	13	29
SA [5]	3	11	-
ILP [16]	3	14	29
ILP [23]	3	11	28
ILP [25]	3	11	28
NLP [29]	3	13	29
Proposed MILP	**3**	**11**	**28**
Proposed NLP	**3**	**11**	**28**

Table 11: OPP results for a large 2383-bus Polish system.

Case	Bus Location	Number of PMUs		Number of NLP Solutions	CPU Time (s)	
		MILP	NLP		MILP	NLP
None	—	746	746	8	0.9531	2.1607×10^3
Zero Injection Measurements	43,220,1185,1486 1871,2054,2086, 2196,2259,2285	740	740	7	1.0156	2.1393×10^3