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Optimal preventive maintenance in a production inventory system

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We consider a production inventory system that produces a single product type, and inventory is maintained according to an (S, s) policy. Exogenous demand for the product arrives according to a random process. Unsatisfied demands are not back ordered. Such a make-to-stock production inventory policy is found very commonly in discrete part manufacturing industry, e.g., automotive spare parts manufacturing. It is assumed that the demand arrival process is Poisson. Also, the unit production time, the time between failures, and the repair and maintenance times are assumed to have general probability distributions. We conjecture that, for any such system, the down time due to failures can be reduced through preventive maintenance resulting in possible increase in the system performance. We develop a mathematical model of the system, and derive expressions for several performance measures. One such measure (cost benefit) is used as the basis for optimal determination of the maintenance parameters. The model application is explained via detailed study of 21 variants of a numerical example problem. The optimal maintenance policies (obtained using a numerical search technique) vary widely depending on the problem parameters. Plots of the cost benefit versus the system characteristic parameters (such as, demand arrival rate, failure rate, production rate, etc.) reveal the parameter sensitivities. The results show that the actual values of the failure and maintenance costs, and their ratio are significant in determining the sensitivities of the system parameters.

1. Introduction

A discrete part production inventory system is considered in this paper. The system produces a single product type to satisfy an exogenous demand process. To hedge against the uncertainties in both the production and the demand processes, provision for a finished inventory buffer between the system and the demands is kept. Demands that arrive when the inventory buffer is empty are not back ordered and are, therefore, lost. (The assumption here is that the customers are not loyal and thus use alternate manufacturing facilities to meet their needs).

Apart from the randomness of the production and the demand arrival processes, the factors that further complicate the system operation are the failure process and the repair process. It seems logical that some form of preventive maintenance of the system may improve the performance of the production process by avoiding unscheduled and usually long interruptions caused by the system failures. However, since preventive maintenance also interrupts the production process, the important question is: should the system be maintained? If so, how often? In this paper, we address the issue of determining the optimal level of preventive maintenance that maximizes the system performance. The measure of system performance considered for optimization is the average cost benefit ($/unit time) due to maintenance, which is a function of the service level (% of satisfied demands) and other cost parameters of the system. A schematic diagram depicting a system (as described above) is given in Fig. 1.

An alternative popular concept of hardware maintenance is known as *Reliability Centered Maintenance* (RCM). The plans derived from RCM greatly extend the useful life, prevent a decrease of reliability and/or deterioration of safety, and reduce support cost as well as the Life Cycle Cost (LCC) [1]. The RCM philosophy considers the following modes of maintenance:

1. (1) Hard-Time Replacement (HTR), where components are replaced after fixed time intervals.
2. (2) On-Condition Maintenance (OCM), where maintenance plans are made based on periodic inspections and evaluations.
3. (3) Condition Monitoring (CM), where the hardware condition is monitored continuously through instrumentation.

In recent years, the Condition Monitoring (CM) based preventive maintenance strategy has gained widespread

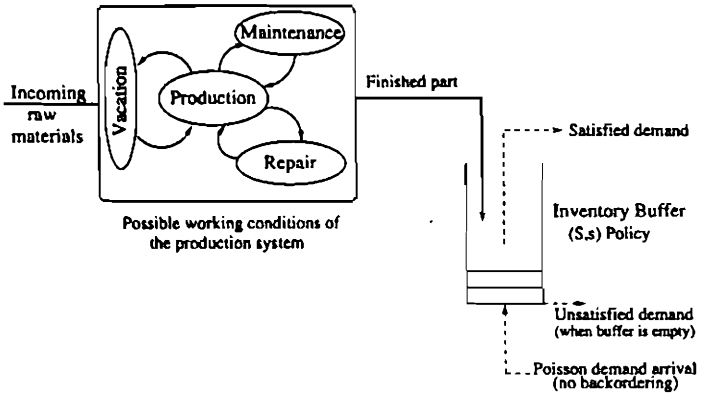

Fig. 1. A single product production-inventory system with exogenous demand.

acceptance. A survey of industrial plants employing more than 100 workers showed that 80.1% had a proactive or predictive maintenance program in place [2]. These programs employed lubricant analysis (78.9%), vibration monitoring (78.3%), infrared thermography (65.2%), ultrasonic monitoring (54.3%), wear particle analysis (43.1%), and other technologies (1.2%). As the number of usable technologies for CM increase, the need for an effective way to integrate them becomes increasingly important. Researchers in this field are working toward matching the technologies to the common modes of failures, and developing optimal combinations (packages) of CM technologies for different industrial preventive maintenance applications [3–5].

In developing CM-based preventive maintenance policies, besides new technology development for condition monitoring, attention should also be paid to characterizing the relationships between the conditions and equipment failure, and also correlations, if any, among the various conditions. Moreover, the existing CM-based maintenance policies can be further improved when they are developed in consideration with the system conditions (not just the machine conditions). For example, certain machine conditions, which are not significant enough in themselves to warrant a maintenance, when considered in conjunction with other system conditions (such as, high inventory status, availability of the maintenance crew, etc.), may need a preventive maintenance.

The preventive maintenance model for a production inventory system, as developed in this paper, uses information on the system conditions (such as, finished product demand, inventory position, and the costs of repair and maintenance), and a continuous probability distribution characterizing the machine failure process. For some machines, it may not be possible (with existing technologies) to effectively monitor their conditions for predicting failure. In such a case, using probability distribution to characterize the failure process is, perhaps, the only alternative. When it is possible to monitor, substituting the machine failure distribution with the observed machine conditions (e.g., vibration, wear, lubricant quality, etc.), our model can be expanded to include CM at the cost of increased computational complexity. However, with the emergence of simulation based stochastic approximation approaches, such as learning automata [6], and reinforcement learning [7], it seems possible that the issue of increased computational complexity can be effectively addressed.

Make-to-stock type production inventory systems are found very commonly in discrete part manufacturing systems, e.g., automotive spare parts manufacturing. The specific characteristics of the system that are considered here are as follows. The time required for manufacture of each part (M) is a random variable having a general probability distribution. The unit demand arrival process is Poisson. The buffer inventory is maintained according to a (S, s) policy; according to which the system stops production when the buffer inventory reaches S (such a period of *inaction* is often referred to as a *server vacation* in the queuing literature), and the production resumes when the inventory drops to s. The system is prone to failures, where the time between failures (T) and time to repair (R) are random variables having general probability distributions. During its vacation, the system stays in cold standby mode and does not age or fail. Thus, the system's vacation is terminated only by a drop in the inventory to level s. Preventive maintenance decisions are made only at the completion epoch of a part, and they depend on both the current inventory level and the production count (number of parts made by the system since the last repair/maintenance). We also consider that the time required for maintenance is a generally distributed random variable, and a maintained system is as good as a repaired one.

Note that, the consideration of the demand arrival process as a Poisson process is a modeling necessity. This helps to preserve the semi-Markov structure of the system without the need to expand the system state space to infinity. However, when the demand arrivals take place from a large customer base, the Poisson assumption can be technically justified.

There are relatively few papers in the current literature that consider the reliability aspect of production inventory systems. These papers can be classified into two groups depending on the type of systems considered: (1) continuous production systems; and (2) discrete part production systems. We first review the papers that deal with continuous production. Meyer *et al.* [8] have considered a system with a constant demand rate, random failure and repair processes, limited inventory, and no back ordering. A similar model was considered by Parthasarathy and Shafarali [9]. Posner and Berg [10] have considered an unreliable production facility for which they obtained the steady-state distribution of the inventory level, which was then used to compute the system service level. Berg *et al.* [11] have considered a system having multiple machines each of which produce (continuously and uniformly over time at a fixed rate) the same type of item. They used the level-crossing analysis technique to compute system performance measures such as production rate, machine utilization, and fraction of demand satisfied.

Shafarali [12] presents an excellent treatment of a single machine discrete part production inventory system. The system considered in Shafarali's work is somewhat similar to ours except that it assumes exponential distributions for the production time and time between failures, and does not consider preventive maintenance. Hsu and Tapiero [13] analyzed a $M/G/1$ queue type job shop as a renewal process where jobs arrive according to a Poisson process and the service time has a general probability distribution. They considered finding the optimal preventive maintenance interval n (number of items produced between a maintenance/repair and the subsequent maintenance) that minimizes the average cost (of maintenance, additional waiting, and lost capacity). Hsu and Tapiero implemented their results on a $M/M/1$ type numerical example problem. We also note here that a broad class of systems studied in the literature under the purview of *single server queues with homogeneous customer* (as in the papers of Morese [14], Prabhu [15], Yao and Buzacott [16], to name a few) are analogous to the job shop type production system studied in Hsu and Taipero [13]. We conclude from our review of the literature that, the consideration of preventive maintenance on a discrete production inventory system, as presented in our paper, is new.

In this paper we develop a semi-regenerative model for the proposed production inventory system with preventive maintenance. The model is developed through characterization of the stochastic processes that underlie the system. Using the properties of the probability structure of these stochastic processes, the desired measures of system performances are developed. Such performance measures are used as a basis to optimally determine the maintenance criteria, which can be stated as: *if the current inventory is i and the production count (number of products made since the last repair/maintenance) is at least N_i, then maintain the machine.*

The remainder of this article is organized as follows. In Section 2 we outline the probability model. Using the properties of the probability model, we develop expressions for the measures of system performance in Section 3. A numerical example is analyzed in detail in Section 4. Some algebraic details of the example problem are placed in the Appendix. Concluding remarks are placed in Section 5.

2. Mathematical model

We first identify the stochastic processes that account for all the random events, namely, production, demand arrival, vacation, failure, repair, and preventive maintenance. We obtain a probability structure on those stochastic processes which are then exploited to obtain system performance measures.

Let the system state be denoted by a 3-tuple (w, i, c) where w indicates the *system status*, i indicates the *inventory status*, and c indicates the *production count* since the last repair or preventive maintenance. The system state space, denoted by E, is given as $E = \sum_{i=0}^{S} N_i + 2(S+1)$, since $w = \{0, 1, 2\}$ (where 0 = working; 1 = preventive maintenance; 2 = repair), $0 \leq i \leq S$, and $0 \leq c \leq N_i$.

We note that the system state changes at those time epochs when any of the following events occur: production completion, demand arrival, system failure (with which the start of repair coincides), repair completion, and maintenance completion (maintenance always begins at a production completion epoch). The start of production epochs always coincide with the other epochs mentioned above. We shall refer to these epochs as *system state change epochs*. Between two such consecutive epochs the system state remains the same.

Let $\hat{X}_k$ denote the system state immediately following $\hat{T}_k$, the time of the kth system state change, where $k \in \mathcal{N}$ with $\mathcal{N}$ denoting the set of positive integers. Also, let Y_t denote the system state at any time t, and $\mathbf{R}^+$ denotes the positive real line. Define $\mathcal{Y} = \{Y_t : t \in \mathbf{R}^+\}$, where for almost all realizations ϕ of $\mathcal{Y}$, $Y_t(\phi) = \hat{X}_k(\phi)$, whenever $\hat{T}_k(\phi) \leq t < \hat{T}_{k+1}(\phi)$.

Now consider the set, H, of epochs corresponding to the beginning and completion of production, repair, and maintenance activities of the system (H is a subset of the system state change epochs described before). Let T_m be the time of the mth epoch of H, where $m \in \mathcal{N}$, and the system state immediately following T_m is X_m. Define $\mathbf{X} = \{X_m : m \in \mathcal{N}\}$, $\mathbf{T} = \{T_m : m \in \mathcal{N}\}$. Then we have $(\mathbf{X}, \mathbf{T}) = \{(X_m, T_m) : m \in \mathcal{N}\}$. We observe that the processes $(\mathbf{X}, \mathbf{T})$, $\mathbf{X}$, and $\mathcal{Y}$ have certain useful properties when the following conditions are satisfied:

$$P\{X_{m+1} = j | X_0, \ldots, X_m; T_0, \ldots, T_m, T_{m+1}\} = P\{X_{m+1} = j | X_m; T_{m+1} - T_m\}, \tag{1}$$

and

$$P\{T_{m+1} - T_m \leq t | X_0, \ldots, X_m; T_0, \ldots, T_m\} = P\{T_{m+1} - T_m \leq t | X_m\}. \tag{2}$$

The properties are: (1) $(\mathbf{X}, \mathbf{T})$ is Markov renewal on $E \times \mathbf{R}^+$; (2) $\mathbf{X}$ forms a Markov chain on E, and (3) if $\sup_m\{T_m : m \in \mathcal{N}\} = \infty$ almost surely, $\mathcal{Y}$ is semi-regenerative on E. The proofs of these results follow easily from similar arguments used (in a different context) by Das and Wortman [17]. The elements of the semi-Markov kernel $\mathbf{Q} = \{Q(i,j,t) : i, j \in E; t \in \mathbf{R}^+\}$ of $(\mathbf{X}, \mathbf{T})$ are obtained from the following expression using the conditions (1) and (2).

$$Q(i,j,t) = \int_0^t P\{X_{m+1} = j | X_m = i, (T_{m+1} - T_m) = u\} \, dP \times \{T_{m+1} - T_m \leq u | X_m = i\}. \quad (3)$$

Using (3) we obtain the one step transition probability matrix **P** of the Markov chain **X** as

$$\mathbf{P} = \{P(i,j) : P(i,j) = \lim_{t \to \infty} Q(i,j,t)\}.$$

Note that the conditions (1) and (2) on which the above mentioned properties 1, 2, and 3 depend, always hold for the type of systems considered here.

3. System performance measures

In what follows, we expose the computational formulae for various performance measures of the production inventory system, such as: (1) service level of the product; (2) average level of inventory in the system; (3) system productivity; and (4) a cost based measure (defined later).

Define Φ_j, for $j \in E$, as the limiting probability of the system being in state j at any time t, i.e.,

$$\Phi_j = \lim_{t \to \infty} P\{Y_t = j\}. \quad (4)$$

It follows from the stationary behavior of the semi-regenerative processes [18] that the limiting relationship in (4) can be written as

$$\Phi_j = \frac{1}{\pi \times \psi} \sum_{l \in E} \pi(l) \times \int_0^\infty P\{Y_t = j | T_1 > t, X_0 = l\} P\{T_1 > t | X_0 = l\} \, dt, \quad (5)$$

where, $\pi = \{\pi(j) : j \in E\}$ is the stationary probability distribution on the Markov chain **X**, and $\psi = \{\psi(j); j \in E\}$ is the vector of the mean sojourn time of the $(\mathbf{X}, \mathbf{T})$ process with $\psi(j) = E[T_1 | X_0 = j]$. ($E[\cdot]$ denotes an expected value.)

3.1. *Service level*

Service level (Θ) is defined as the average percentage of the demands that are satisfied by the production inventory process (recall that, since back ordering is not allowed, the unsatisfied demands leave the system). Let $A_i \subset E$, for $i \in S$, denote the subset of the system states having inventory status i, such that $\cup A_i = E$. Then we have that

$$\Theta = 1 - \sum_{j \in A_0} \Phi_j. \quad (6)$$

3.2. *Average inventory level*

The average inventory level in the system (Ω) is given by

$$\Omega = \sum_{i \in S} i \sum_{j \in A_i} \Phi_j. \quad (7)$$

3.3. *System productivity*

We define the productivity of the system (Γ) as the percentage of time the system is producing (i.e., the system is not in repair, maintenance, or vacation). Define a subset $B \subset E$ such that $B = \{(w, i, c) : w = 0, i < S\}$. We have that

$$\Gamma = \sum_{j \in B} \Phi_j. \quad (8)$$

In a similar manner as above, we can also obtain the percentages of time the system is under repair, under maintenance, and on vacation.

3.4. *Cost model*

Let G denote the net cost benefit (in dollars per unit time) due to the implementation of a preventive maintenance policy. Considering only the cost elements that are directly related to the maintenance policy, we define G as:

G = additional revenue per unit time from increased service level (R_s) + savings in repair cost per unit time (S_r) − cost of maintenanceper unit time (M_c). (9)

The additional revenue per unit time (R_s) may be construed as, for example, the additional receipt due to higher demand service level minus the additional cost of demand service calculated on a per unit time basis. Any maintenance policy will essentially reduce the number of repairs performed on the system per unit time resulting in repair cost saving. (Note here that *repair cost saving* implies only the cost savings of labor and supplies. Reduction in system down-time due to reduced failures is reflected in the service level). The cost of maintenance, thus, should also include only the labor and supplies.

To develop expressions for R_s, S_r, and M_c, we adopt the following notation.

$\Theta_\infty(\Theta_{n_i})$ = service level without (with) preventive maintenance;
c_d = net receipt minus the cost per unit additional demand serviced;
γ = demand arrival rate;
c_r = cost per repair;

c_m = cost per maintenance;
$E[T]$ = mean time between failures (hence, repairs) without maintenance;
$E[T_r]$ = mean time between repairs with maintenance;
$E[R]$ = mean time needed for each repair;
$E[T_m]$ = mean time between maintenance;
$E[K]$ = mean time needed for each maintenance.

Let $E_m \subset E$ ($E_r \subset E$) denote the subset of the system state space indicating that the system is under maintenance (repair). Clearly, $E_m = \{(1,\cdot,\cdot)\}$ and $E_r = \{(2,\cdot,\cdot)\}$. Also let $F(a,b)$ denote the mean first passage transition time of the Markov renewal process $(\mathbf{X},\mathbf{T})$ from state $a \in E$ to state $b \in E$. Then we have that,

$$E[T_m] = \sum_{j\in E_m} \frac{\Phi_j F(j,E_m)}{\sum_{i\in E_m} \Phi_i}, \tag{10}$$

where, for $j \in E_m$, we have that

$$F(j,E_m) = \sum_{k\notin E_m} P(j,k)F(k,E_m) + \psi(j). \tag{11}$$

In a similar manner as above, we can write the following.

$$E[T_r] = \sum_{j\in E_r} \frac{\Phi_j F(j,E_r)}{\sum_{i\in E_r} \Phi_i}, \tag{12}$$

where, for $j \in E_r$, we have that

$$F(j,E_r) = \sum_{k\notin E_r} P(j,k)F(k,E_r) + \psi(j). \tag{13}$$

We then obtain the components of the cost benefit (G) as follows.

$$R_s = \gamma c_d\ (\Theta_{n_i} - \Theta_\infty), \tag{14}$$

$$S_r = c_r \left[\frac{1}{E[T]} - \frac{1}{E[T_r]}\right], \tag{15}$$

$$M_c = \frac{c_m}{E[T_m]}. \tag{16}$$

We also note that the quantity $G/(\gamma C_d \Theta_\infty)$ gives the cost benefit as percentage of the profit derived from the system without maintenance.

4. Optimal maintenance policy selection

The selection of the optimal maintenance policy $\{N_i : 0 \le i \le S\}$ can be accomplished by solving the following optimization model.

$$\text{Maximize } G$$

$$\text{Subject to } N_i \ge 0, \quad \text{for all } 0 \le i \le S.$$

For not having a closed form expression for the objective function G, we have used numerical techniques in solving the optimization model for the example problems.

5. Numerical example problem

We consider a fairly general example problem as a vehicle for providing further details on the solution procedure. Also presented are numerical results that provide further insight to the problem. The characteristics of the example problem are given next. We note that the choices of the types of probability distributions for the example problem (except for demand arrival, which is Poisson) are somewhat arbitrary and do not imply any limitations of the modeling process. The choice of gamma distribution, in particular, is motivated by: (1) the fact that by varying its parameters it can be made to describe a wide variety of data distributions; and (2) its property that *sum of gamma is gamma*, which provides a unique computational convenience.

- Demand arrival process is Poisson (γ).
- Production time (M) for each product has a Gamma (d, λ) distribution.
- Time to system failure (T) has a Gamma (k, μ) distribution.
- Time required for preventive maintenance (K) has a Uniform (a, b) distribution.
- Time required for system repair (R) has a Gamma (r, δ) distribution.
- The system does not age during its vacation (non-production period).
- Buffer inventory is maintained by ($S = 3$, $s = 2$) policy.
- The arriving demands have a batch size of 1.

For the given parameters, the system state space of the problem is given as

$(0,0,0)$, $(0,0,1)$, $(0,0,2)$, ..., $(0,0,N_1-1)$,
$[(0,0,N_1),\ldots,(0,0,N_0)]$,
$(0,1,0)$, $(0,1,1)$, $(0,1,2)$, ..., $(0,1,N_1-1)$,
$(0,2,0)$, $(0,2,1)$, $(0,2,2)$, ..., $(0,2,N_2-1)$,
$(0,3,0)$, $(0,3,1)$, $(0,3,2)$, ..., $(0,3,N_3-1)$,
$(1,0,0)$, $(1,1,0)$, $(1,2,0)$, $(1,3,0)$,
$(2,0,0)$, $(2,1,0)$, $(2,2,0)$, $(2,3,0)$.

Note from the state space that, N_0, N_1, N_2, and N_3 indicate the optimum production counts (corresponding to inventory levels of $0,\ldots,3$ respectively) at which maintenance is to be carried out. Intuitively it is clear that $N_0 \ge N_1 \ge N_2 \ge N_3$ (since when the inventory is low, it may be prudent to hold off maintenance longer). We also note that the minimum inventory at a production completion epoch is one (1), and maintenance decisions are always made at the production completion epochs. Hence, the system (as described by the Markov chain $\mathbf{X}$) never visits the states $(0,0,c)$, for all c, at a production completion epoch. These states are only visited by the semi-regenerative process $\mathscr{Y}$ whenever inventory is completely depleted in between two production completion epochs. Also, since the system status changes immediately to maintenance as soon as the system reaches $(0,1,N_1)$, the system states given by $(0,0,N_1),\ldots,(0,0,N_0)$ (placed within the square brackets in the complete state space listed above) are never visited by the semi-regenerative process ($\mathscr{Y}$) either, and hence these states need not be considered in our analysis. Thus the question of optimizing N_0 does not arise. Some samples of one step system state transitions are shown in Fig. 2.

More algebraic details on the solution procedure are given in the Appendix. In what follows, we discuss the numerical results.

5.1. *Optimization search space*

We discuss the issues related to the nature and size of the search space of the example problem, of which the parameters are as specified in Fig. 3. Note that the average production time/unit and the average time between demand arrivals are both 10 time units. The average time between failures is 100, and the average repair time is 200 time units.

We want to choose the optimum production count for every inventory level (N_i), at which to maintain the system so that the net cost benefit (G) is maximized. Hence, for the problem (with $S = 3$), the search space is over three parameters (N_1, N_2 and N_3). As discussed earlier, N_0 need not be considered. The possible values of N_1, N_2, and N_3 can range from 1 to ∞. However, in practice we need to consider a much smaller search space. Consider the plot shown in Fig. 4(a) which displays the probability that the system fails after a production count of c. We see that for $c \geq 25$ units, the probability tends to zero signifying that the system will almost surely fail when it reaches a production count of 25 units. Thus the optimal maintenance levels have to be less than or equal to 25 units. Hence, we can restrict the search for the optimal values of N_i's between 0 and 25 for this example problem.

The cross-sections across the search space are shown in Fig. 4(b–d). Figure 4(b and c) show the variation of the

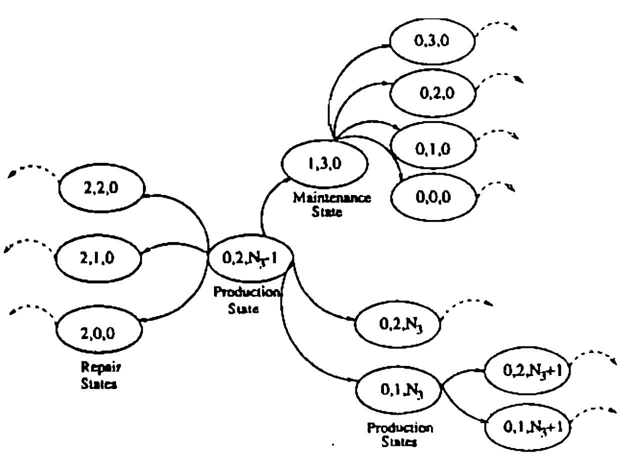

Fig. 2. Sample one-step state transitions of the underlying Markov chain.

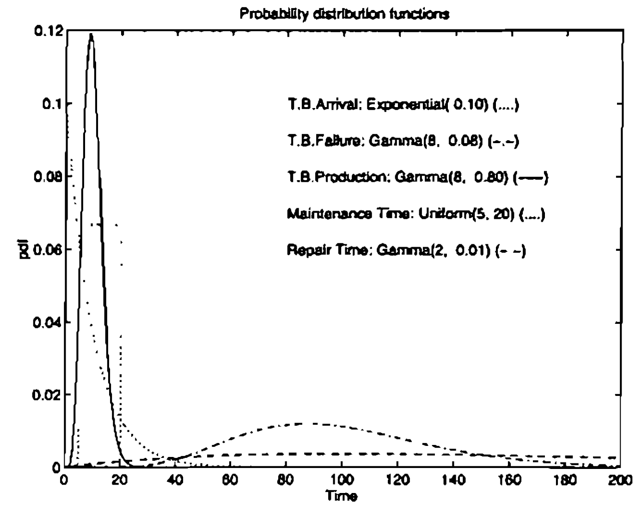

Fig. 3. Parameters of the probability density functions for the numerical example problem.

expected cost benefit as the two maintenance levels are varied keeping the third constant. Observe that the search space is smooth. There are no multiple local optima. Also notice that variation of the service level is the largest with N_3.

Figure 4(d) reaffirms our earlier assertion that the cost benefit is not affected by the choice of N_0. This fact is further illustrated in Fig. 5 where the stationary probabilities for the states (0, 0, c), with $c \geq N_1$, are zero.

5.2. *Optimization*

Next, we consider optimization of the cost benefit function to obtain the optimum maintenance levels. Since it was evident from Fig. 4(b–d) that the search space is smooth, we used a well known gradient search algorithm (method of steepest ascent) as follows. Starting at an initial point, the search proceeds along the direction of maximum gradient until a maximum point is reached, which is then taken as the new starting point. The process continues until a stopping criterion is met. To ensure that the optimum point is indeed the global optimum, we considered ten different starting points for each optimization. The solution converged to the same point for all of the starting points; this was, of course, very much expected since the plots of the cost benefit function (G) was found to be of convex nature.

The optimal values of N_i's are found to be $N_1 = 6$, $N_2 = 5$, $N_3 = 5$. The maximum net cost benefit per unit time of system operation (G) is \$0.0194.

5.3. *The effect of preventive maintenance on service level*

It is interesting to study the variation of just the service level with preventive maintenance. Figure 6 shows the

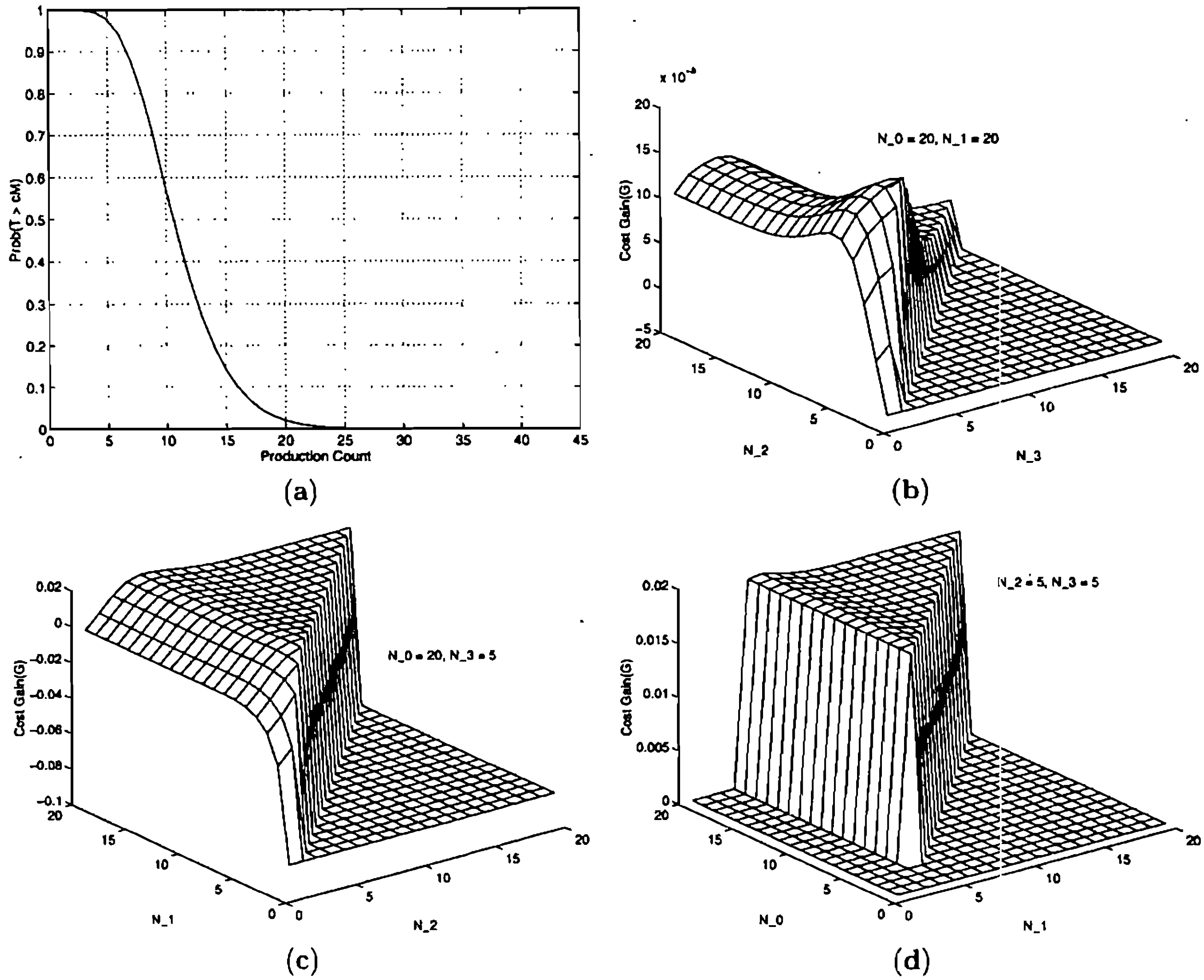

Fig. 4. (a) Probability that time to failure exceeds production of c items. The cross sectional plots of the cost benefit function are displayed in (b), (c), and (d). The maintenance counts N_0 and N_1 are fixed at 20 for (b). The plots in (c) and (d) are with fixed (N_0, N_3) and (N_2, N_3), respectively.

plot of service levels with decreasing levels of maintenance (i.e., with increasing production counts at which maintenance is done); for this purpose, we have chosen the same values of the production counts (N) irrespective of inventory levels. Note that as $N \to \infty$, in which case preventive maintenance is nonexistent, the service level tends to 0.3053. As expected, the service level starts from a low value for a small value of N, which implies too frequent preventive maintenance and the resulting loss of production. As N increases to improve the balance between interruptions due to maintenance and failure, the service level increases to its highest level and then falls until reaching the asymptotic level. The decrease in service level with increasing N is a direct result of the insufficiency of maintenance and, therefore, increase in production losses due to failures. We also notice that the maximum possible service level obtained with maintenance is 0.6487, which represents a more than two-fold increase compared to that with no maintenance. Though the actual service level values (shown in Fig. 6) are dependent on the problem parameters, the nature of the curve shows a general trend.

5.4. *Sensitivities of system parameters*

In this section we study the sensitivities of the input parameters that dictate the following: demand arrivals, time to failure, production time, maintenance time, and the repair time. Table 1 shows the various combinations of input parameters considered for the sensitivity study. The bold faced numbers denote the values of the parameters which are different from the base system (which appears on the first line of the table). For example, systems 2 and 3 have lower demand arrival rates than the base system (system 1). The systems 4 and 5 have higher demand arrival rates than the base system. Similarly, systems 14 and

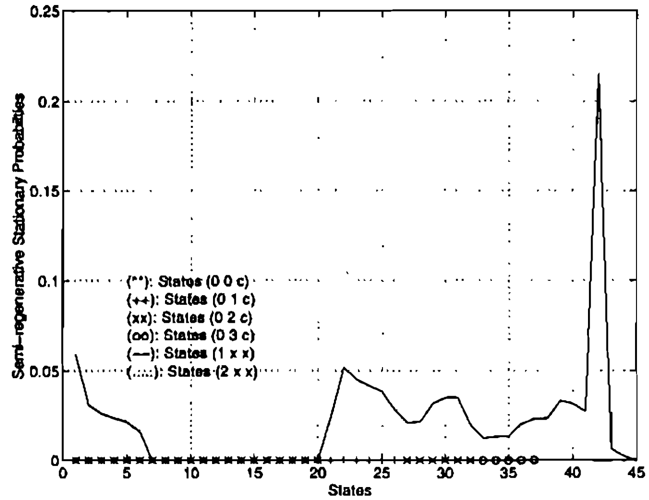

Fig. 5. Stationary distribution (solid plot) of the $\mathscr{Y}$ process on the system states (a total of 45 states labeled and shown on x-axis) when $N_0 = 20, N_1 = 6, N_2 = 6$, and $N_3 = 5$.

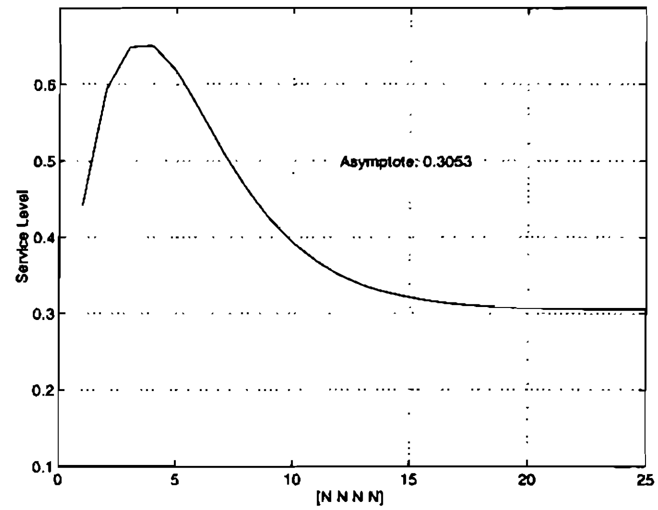

Fig. 6. The change in service level with maintenance counts, when $N_1 = N_2 = N_3 = N$.

15 have shorter maintenance times compared to the base system, whereas, systems 16 and 17 have higher maintenance times than the base system. The last column of Table 1 shows the service levels without preventive maintenance (i.e., Θ_∞). For each of the 21 system parameter sets we consider three different cost structures: (1) $C_d = 1, C_r = 5, C_m = 2$; (2) $C_d = 0.5, C_r = 10, C_m = 2$; and (3) $C_d = 1, C_r = 100, C_m = 10$. The results are enumerated in Tables 2 to 4.

Since the term *cost benefit* encompasses the effects of service level, maintenance cost, and repair cost, we studied the sensitivities of the input parameters with respect to their corresponding cost benefits via the plots shown in Fig. 7(a–e).

We make the following observations from the sensitivity plots.

- Figure 7(a) shows that for higher values of the ratio of repair to maintenance costs (as in cost structure

Table 1. Input parameters for the numerical example problem

System	*Time bet. demands* (γ)	*Time to failure* (κ, μ)	*Product. time/unit* (d, λ)	*Maint. time* (a, b)	*Repair time* (r, δ)	*Serv. lev. w/o maint.*
1	1/10	(8, 0.08)	(8, 0.8)	(5, 20)	(2, 0.01)	0.3051
2	**1/5**	(8, 0.08)	(8, 0.8)	(5, 20)	(2, 0.01)	0.1625
3	**1/7**	(8, 0.08)	(8, 0.8)	(5, 20)	(2, 0.01)	0.2231
4	**1/15**	(8, 0.08)	(8, 0.8)	(5, 20)	(2, 0.01)	0.4181
5	**1/20**	(8, 0.08)	(8, 0.8)	(5, 20)	(2, 0.01)	0.5072
6	1/10	**(4, 0.10)**	(8, 0.8)	(5, 20)	(2, 0.01)	0.1553
7	1/10	**(4, 0.08)**	(8, 0.8)	(5, 20)	(2, 0.01)	0.1838
8	1/10	**(4, 0.01)**	(8, 0.8)	(5, 20)	(2, 0.01)	0.5775
9	1/10	**(8, 0.008)**	(8, 0.8)	(5, 20)	(2, 0.01)	0.6986
10	1/10	(8, 0.08)	**(4, 0.8)**	(5, 20)	(2, 0.01)	0.4979
11	1/10	(8, 0.08)	**(4, 0.4)**	(5, 20)	(2, 0.01)	0.3055
12	1/10	(8, 0.08)	**(4, 0.1)**	(5, 20)	(2, 0.01)	0.0754
13	1/10	(8, 0.08)	**(8, 0.08)**	(5, 20)	(2, 0.01)	0.0374
14	1/10	(8, 0.08)	(8, 0.8)	**(2, 10)**	(2, 0.01)	0.3051
15	1/10	(8, 0.08)	(8, 0.8)	**(2, 15)**	(2, 0.01)	0.3051
16	1/10	(8, 0.08)	(8, 0.8)	**(20, 30)**	(2, 0.01)	0.3051
17	1/10	(8, 0.08)	(8, 0.8)	**(20, 40)**	(2, 0.01)	0.3051
18	1/10	(8, 0.08)	(8, 0.8)	(5, 20)	**(1, 0.05)**	0.7108
19	1/10	(8, 0.08)	(8, 0.8)	(5, 20)	**(2, 0.04)**	0.5847
20	1/10	(8, 0.08)	(8, 0.8)	(5, 20)	**(4, 0.01)**	0.1862
21	1/10	(8, 0.08)	(8, 0.8)	(5, 20)	**(3, 0.001)**	0.0307

Table 2. Optimum maintenance levels and other performance measures for example systems when $C_d = 1, C_r = 5, C_m = 2$. The third column shows the cost benefit as a percentage of profit without maintenance

System	*Cost benefit*		*Service level*	*System productivity*	*Average inventory*	*Opt. maint. lev.* (N_1, N_2, N_3)
	(G)	(%)				
1	0.0194	64	0.6059	0.6003	1.0246	(6, 5, 5)
2	0.0196	60	0.3117	0.6150	0.3845	(6, 6, 5)
3	0.0199	63	0.4460	0.6304	0.6255	(6, 5, 5)
4	0.0169	60	0.7473	0.4924	1.5649	(6, 6, 5)
5	0.0146	58	0.8035	0.3960	1.8515	(6, 6, 6)
6	0.0002	1	0.1957	0.1920	0.3233	(4, 4, 4)
7	0.0021	12	0.2671	0.2585	0.4480	(4, 4, 4)
8	0.0131	23	0.7323	0.7334	1.3100	(21, 19, 13)
9	0.0102	15	0.7962	0.7998	1.4416	(63, 59, 41)
10	0.0322	65	0.8566	0.4306	1.9793	(11, 10, 9)
11	0.0182	60	0.5699	0.5612	1.0000	(6, 6, 5)
12	0.000	0	0.0841	0.3519	0.1046	(5, 3, 2)
13	0.0059	158	0.0374	0.5697	0.0374	(4, 3, 2)
14	0.0219	72	0.6528	0.6478	1.1453	(5, 5, 5)
15	0.0208	68	0.6393	0.6344	1.1050	(5, 5, 5)
16	0.0157	51	0.5345	0.5281	0.8802	(6, 6, 5)
17	0.0141	46	0.5082	0.5010	0.8265	(7, 6, 5)
18	0.0139	20	0.7390	0.7216	1.3132	(8, 8, 6)
19	0.0154	26	0.6902	0.6769	1.2091	(7, 7, 6)
20	0.0203	109	0.5545	0.5509	0.9381	(5, 5, 4)
21	0.0136	443	0.3429	0.3424	0.5645	(4, 4, 3)

Table 3. Optimum maintenance levels and other performance measures for example systems when $C_d = 0.5, C_r = 10, C_m = 2$. The third column shows the cost benefit as a percentage of profit without maintenance

System	*Cost benefit*		*Service level*	*System productivity*	*Average inventory*	*Opt. maint. lev.* (N_1, N_2, N_3)
	(G)	(%)				
1	0.0166	109	0.6059	0.6003	1.0246	(6, 5, 5)
2	0.0147	91	0.3117	0.6150	0.3845	(6, 5, 5)
3	0.0155	94	0.4307	0.6071	0.6160	(6, 6, 5)
4	0.0182	131	0.7720	0.5103	1.5936	(6, 5, 5)
5	0.0197	155	0.8566	0.4249	1.9688	(5, 5, 5)
6	0.0004	47	0.1742	0.1716	0.2896	(5, 5, 5)
7	0.0019	21	0.2671	0.2585	0.4480	(4, 4, 4)
8	0.0118	40	0.7258	0.7268	1.2979	(19, 18, 15)
9	0.0092	26	0.7956	0.7992	1.4403	(57, 54, 42)
10	0.0287	115	0.8397	0.4218	1.9335	(11, 10, 10)
11	0.0158	104	0.5917	0.5848	1.0187	(6, 5, 5)
12	0.0007	18	0.0910	0.3807	0.1083	(5, 1, 1)
13	0.0118	732	0.0374	0.5697	0.0374	(4, 3, 2)
14	0.0172	112	0.6528	0.6478	1.1453	(5, 5, 5)
15	0.0169	110	0.6248	0.6190	1.0811	(6, 5, 5)
16	0.0153	100	0.5348	0.5281	0.8802	(6, 6, 5)
17	0.0157	102	0.5036	0.4966	0.8259	(6, 6, 6)
18	0.0458	128	0.7198	0.7129	1.2198	(6, 5, 5)
19	0.0355	122	0.6982	0.6917	1.1814	(6, 5, 5)
20	0.0096	104	0.5150	0.5103	0.8709	(6, 5, 5)
21	0.0015	98	0.1746	0.1730	0.2953	(6, 5, 5)

3), the cost benefit is sensitive to the time between demand arrivals. Thus, a higher rate of cost benefit, with increase in demand, can be achieved by efficient maintenance, when the maintenance cost is high and the repair cost is significantly higher than the maintenance cost.

Table 4. Optimum maintenance levels and other performance measures for example systems when $C_d = 1.0, C_r = 100, C_m = 10$. The third column shows the cost benefit as a percentage of profit without maintenance

System	*Cost benefit*		*Service level*	*System productivity*	*Average inventory*	*Opt. maint. lev.* (N_1, N_2, N_3)
	(G)	(%)				
1	0.1646	540	0.6178	0.6131	1.0442	(5, 5, 5)
2	0.1441	444	0.3371	0.6690	0.4079	(5, 5, 5)
3	0.1512	475	0.4625	0.6557	0.6475	(5, 5, 5)
4	0.1903	683	0.7773	0.5142	1.6042	(5, 5, 5)
5	0.2130	840	0.8566	0.4249	1.9687	(5, 5, 5)
6	0.0332	214	0.2748	0.2747	0.3915	(3, 1, 1)
7	0.0489	266	0.3535	0.3490	0.5335	(3, 2, 2)
8	0.1016	176	0.7324	0.7341	1.2969	(15, 14, 14)
9	0.0696	100	0.7992	0.8030	1.4445	(47, 45, 35)
10	0.2352	472	0.8610	0.4330	1.9742	(27, 9, 9)
11	0.1589	520	0.6021	0.5960	1.0360	(5, 5, 5)
12	0.0882	1170	0.1494	0.6342	0.1521	(1, 1, 1)
13	0.1182	3160	0.0374	0.5697	0.0374	(4, 3, 2)
14	0.1607	527	0.6528	0.6478	1.1453	(5, 5, 5)
15	0.1621	531	0.6393	0.6344	1.1050	(5, 5, 5)
16	0.1714	562	0.5511	0.5469	0.8850	(5, 5, 5)
17	0.1730	567	0.5255	0.5215	0.8351	(5, 5, 5)
18	0.6075	855	0.6959	0.6942	1.1464	(4, 4, 4)
19	0.4554	780	0.6880	0.6864	1.1326	(4, 4, 4)
20	0.0620	333	0.5371	0.5330	0.9077	(5, 5, 5)
21	0.0002	7	0.0345	0.0328	0.0623	(14, 14, 14)

- It can be observed from Fig. 7(b) that at very low values of time to failure, we can achieve very little gain with maintenance. However, as the time between failures increases, up to a certain point, effective maintenance strategies can ensure significant increases in cost benefit. As the time between failures increases beyond a point (when the system can be interpreted to be very reliable), the cost benefit from maintenance steadily decreases.
- As the production time per unit increases, the profit earned from increased productivity decreases, and, as a result, the cost benefit decreases. (Refer to Fig. 7(c).)
- Figure 7(d) shows that, for cost structures 1 and 2, the cost benefit decreases with increase in maintenance time. For cost structure 3, since the repair to maintenance cost ratio is high we get an increasing rate of cost benefit, even when the maintenance time increases. However, further increases in the maintenance time should lower the cost benefit.
- We note from Fig. 7(e) that, with increasing repair time, the cost benefit has a slightly increasing and then decreasing trend for cost structure 1. The increase in cost benefit could be attributed to the fact that the repair and maintenance cost are not far apart, and thus the increase in repair time makes a failure more critical and thus maintenance more attractive. For cost structures 2 and 3, the cost benefit steadily decreases with increasing repair time, which is quite expected.

The following conclusions can be drawn from the above observations. Sensitivity of the input parameters are magnified when the costs of repair and maintenance, and their ratio are high. The sensitivities of the parameters, such as failure rate, production rate, and the repair rate are nonlinear, and change significantly over the parameter ranges. Hence, the graphs, as shown in Fig. 7(a–e) could be useful in assessing the relative gains and losses from a preventive maintenance program with the changes in the associated parameters.

5.5. *Effect of S and s levels*

For a production inventory system, as studied here, an important design issue is the optimal selection of the S and s levels, together with optimal maintenance conditions, that maximize the system performance. However, in this paper, we have focused only on the optimal maintenance part assuming that the values of S and s are fixed. To examine the effects of changes in the inventory levels (S and s) on the optimal preventive maintenance policies, we considered another set of values of these parameters ($S = 2$ and $s = 1$). The results obtained for all of the 21 systems (listed in Table 1) with the cost structure 1, are presented in Table 5.

The cost benefit, service level, and the system productivity values in Table 5 are (in almost all cases) distinctively lower than those for $S = 3$ and $s = 2$ (refer to Table 2). This indicates that, perhaps, higher (but, not

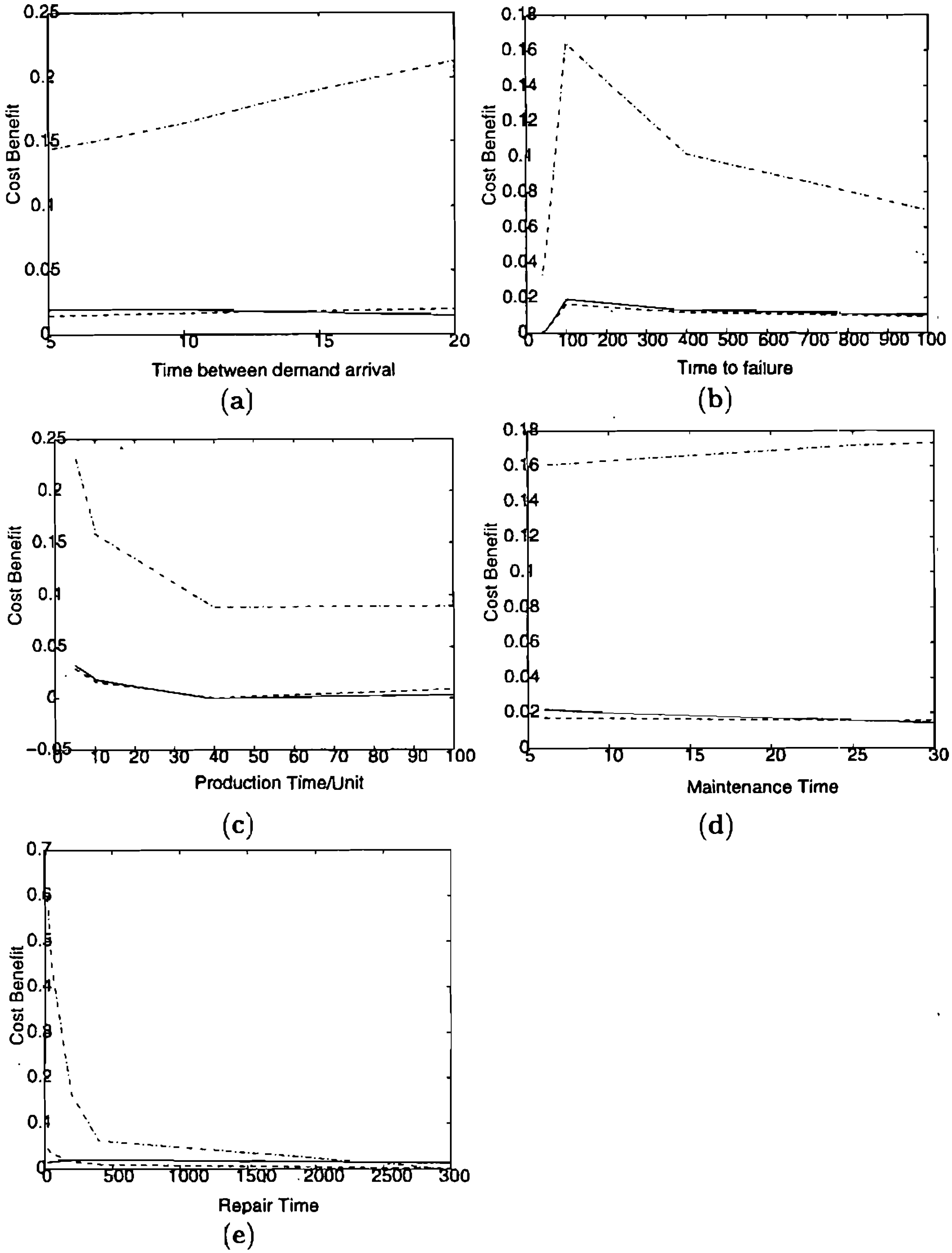

Fig. 7. Sensitivity plots for the input parameters. The solid plots are for cost structure 1 ($C_d = 1, C_r = 5, C_m = 2$), the dashed plots are for cost structure 2 ($C_d = 0.5, C_r = 10, C_m = 2$), and the dot-dashed plots are for cost structure 3 ($C_d = 1.0, C_r = 100, C_m = 10$).

lower) inventory levels than $S = 3$ and $s = 2$ constitute a better inventory policy. We expect that with higher S and s values, the frequency of machine vacation will be lead less to higher productivity and higher service level resulting in higher cost benefit. We limited our numerical experimentations to fairly small values of S to limit the size of the state space and the number of decision variables. In a related paper by Das *et al.* [7], we have studied systems with much larger inventory values (e.g., $S = 30$) using an approximation method, and used the optimal results obtained from this paper for benchmarking purposes.

Table 5. Optimum maintenance levels and other performance measures for example systems when $C_d = 1.0, C_r = 5, C_m = 2$ with $s = 1$ and $S = 2$. The third column shows the cost benefit as a percentage of profit without maintenance.

System	*Cost benefit*		*Service level*		*System productivity*	*Average inventory*	*Opt. maint. lev.* (N_1, N_2)
	(G)	(%)	*w main.*	*w/o main.*			
1	0.0182	63	0.5435	0.2892	0.5372	0.7426	(6, 5)
2	0.0194	112	0.3068	0.1584	0.6059	0.3555	(6, 5)
3	0.0192	90	0.4136	0.2145	0.5837	0.5172	(6, 5)
4	0.0162	41	0.6889	0.3915	0.4544	1.0364	(6, 5)
5	0.0146	31	0.7760	0.4721	0.3841	1.2384	(6, 5)
6	0.0010	7	0.1886	0.1492	0.1852	0.2602	(4, 4)
7	0.0028	16	0.2550	0.1766	0.2469	0.3516	(4, 4)
8	0.0117	22	0.6560	0.5283	0.6570	0.9052	(23, 15)
9	0.0087	14	0.7067	0.6290	0.7100	0.9770	(65, 45)
10	0.0298	63	0.7901	0.4712	0.3971	1.2614	(11, 9)
11	0.0172	59	0.5313	0.2894	0.5238	0.7358	(6, 5)
12	0.0000	0	0.0889	0.0742	0.3609	0.1064	(3, 3)
13	0.0036	96	0.0374	0.0374	0.5697	0.0374	(3, 1)
14	0.0201	70	0.5933	0.2892	0.5888	0.8130	(5, 5)
15	0.0192	66	0.5560	0.2892	0.5496	0.7646	(6, 5)
16	0.0136	47	0.4546	0.2892	0.4465	0.6171	(7, 6)
17	0.0148	51	0.4893	0.2892	0.4824	0.6621	(7, 5)
18	0.0172	27	0.6584	0.6320	0.6427	0.9143	(8, 7)
19	0.0170	32	0.6258	0.5290	0.6147	0.8621	(7, 6)
20	0.0186	103	0.4962	0.1802	0.4924	0.6712	(5, 5)
21	0.0127	416	0.3078	0.0305	0.3071	0.4115	(4, 4)

6. Conclusions

A discrete production inventory system with fairly general characteristics has been considered here. The primary objective of the study was to determine when to perform preventive maintenance, if any, on the system so as to improve the system performance. The mathematical model of the system provided a useful tool for deriving the expressions for system performance measures. The performance measures considered were service level, average inventory, system productivity, and cost benefit due to maintenance. It was demonstrated, through a numerical example problem, how the cost based measure can be used as a basis for determining optimal level of preventive maintenance. The cost benefit function for such problems was found to be convex (as indicated by the plots of the function and its subsequent optimization). As a result we were able to use the steepest ascent search method to locate the optimum point (i.e., optimum parameters for preventive maintenance).

The randomness involved in various operational aspects of such systems makes them fairly difficult to analyze. Furthermore, our assumptions of general probability distributions for all of the associated random variables (except the time between demand arrivals) made the analysis of the system more involved. In the absence of closed form for the performance measures, we used numerical methods to evaluate them. MATLAB was used for computation. A copy of the complete program can be requested from the authors.

As evident from the numerical results, the modeling approach provides a useful tool for studying the effects of various system parameters on the overall system performance. We also note that, since $E = \sum_{i=0}^{S} N_i + 2(S+1)$, the computational effort needed to study the system increases with both S and the time to machine failure (which causes the search space of the N_i's to increase). Hence, for problems with larger state spaces, approximate solution methods are useful. (One such method can be found in Das *et al.* [7], where the maintenance problem, as considered here, and its multiproduct extension are modeled as Semi Markov Decision Problems (SMDPs). The SMDPs are stochastically approximated using a relatively new simulation-based approach, called reinforcement learning). However, the exact analysis presented here, in addition to providing necessary insight for developing approximation procedures, serves as optimality benchmark for the approximate solutions.

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References

[1] Anderson, R.T. and Neri, L. (1990) *Reliability-Centered Maintenance: Management and Engineering Methods*, Elsevier Applied Science, London.

[2] Young, K. (1995) Integrating predictive maintenance, *Maintenance Technology*, June, Applied Technology Publications, Inc.

[3] PdMA Corp. (1997) Motor circuit analyzer plays key role at refinery. *Maintenance Technology*, January, Applied Technology Publications, Inc.

[4] PdMA Corp. (1997) ARCO expands condition monitoring program, *Maintenance Technology*, September.

[5] Nicholas, Jr, J.R. (1994) Motor circuit and current signature analyses. *AIPE Facilities, Journal of Plant and Facilities Management and Engineering*, July/August.

[6] Wheeler, R. and Narendra, K. (1986) Decentralized learning in finite Markov chains. *IEEE Transactions On Automatic Control*, **31**(6), 373–376.

[7] Das, T.K., Gosavi, A., Mahadevan, S. and Marchalleck, N. (1999) Solving semi-markov decision problems using average reward reinforcement learning. *Management Science*, (in press).

[8] Meyer, R.R., Rothkopf, M.H. and Smith, S.A. (1979) Reliability and inventory in a production-storage system. *Management Science*, **25**, 799–807.

[9] Parthasarathy, P.R. and Shafarali, M. (1987) On the unutilized capacity of a production-storage system. *Operations Research*, **6**, 233–238.

[10] Posner, M.J.M. and Berg, M. (1989) Analysis of a production-inventory system with unreliable production facility. *Operations Research Letters*, **8**, 339–345.

[11] Berg, M., Posner, M.J.M. and Zhao, H. (1994) Production-inventory systems with unreliable machines. *Operations Research*, **42**, 111–118.

[12] Shafarali, M. (1984) On a continuous review production-inventory problem. *Operations Research Letters*, **3**, 199–204.

[13] Hsu, L.F. and Tapiero, C.S. (1987) Maintenance of an unreliable M/G/1 queue like job shop. *Queuing Systems*, **2**, 333–350.

[14] Morse, P.M. (1963) *Queues, Inventories, and Maintenance*, John Wiley, NY.

[15] Prabhu, N.U. (1965) *Queues and Inventories*, John Wiley, NY.

[16] Yao, D.D. and Buzacott, J.A. (1985) Modeling the performance of flexible manufacturing systems. *International Journal of Production Research*, **23**, 945–959.

[17] Das, T.K. and Wortman, M.A. (1993) Analysis of asymmetric patrolling repairman systems. *European Journal of Operations Research*, **64**, 45–60.

[18] Çinlar, E. (1995) *Introduction to Stochastic Processes*, Prentice-Hall, Englewood Cliffs, NJ.

Appendix A. Sample calculations

One-step transitions of the Markov chain

Following are the possible one step transitions for which $P(i,j) > 0$; for $i,j \in E$:

$i = (0,0,c) \longrightarrow j = (0,1,c+1)$; for $c < N(1) - 1$,
$i = (0,0,c) \longrightarrow j = (2,0,0)$; for $c = 0,\dots,N_0 - 2$,
$i = (0,0,c) \longrightarrow j = (1,1,0)$; for $c = N_1 - 1,\dots,N_0 - 1$,
$i = (0,0,N_0 - 1) \longrightarrow j = (2,0,0)$,
$i = (0,1,c) \longrightarrow j = (0,1,c+1)$; for $c = 0,\dots,N_1 - 2$,
$i = (0,1,c) \longrightarrow j = (0,2,c+1)$; for $c = 0,\dots,N_2 - 2$,
$i = (0,1,c) \longrightarrow j = (2,1,0)$; for $c = 0,\dots,N_1 - 1$,
$i = (0,1,c) \longrightarrow j = (2,0,0)$; for $c = 0,\dots,N_1 - 1$,
$i = (0,1,N_1 - 1) \longrightarrow j = (1,0,0)$,
$i = (0,1,c) \longrightarrow j = (1,2,0)$; for $c = N_2 - 1,\dots,N_1 - 1$,
$i = (0,2,c) \longrightarrow j = (0,1,c+1)$; for $c = 0,\dots,\min\{N_2 - 2, N_1 - 2\}$,
$i = (0,2,c) \longrightarrow j = (0,2,c+1)$; for $c = 0,\dots,N_2 - 2$,
$i = (0,2,c) \longrightarrow j = (0,3,c+1)$; for $c = 0,\dots,N_3 - 2$,
$i = (0,2,N_3 - 1) \longrightarrow j = (1,3,0)$,
$i = (0,2,c) \longrightarrow j = (2,2,0)$; for $c = 0,\dots,N_2 - 1$,
$i = (0,2,c) \longrightarrow j = (2,1,0)$; for $c = 0,\dots,N_2 - 1$,
$i = (0,2,c) \longrightarrow j = (2,0,0)$; for $c = 0,\dots,N_2 - 1$,
$i = (0,2,N_2 - 1) \longrightarrow j = (1,2,0)$,
$i = (0,2,N_3 - 1) \longrightarrow j = (1,3,0)$,
$i = (0,3,c) \longrightarrow j = (0,2,c)$; for $c = 0,\dots,N_3 - 1$,
$i = (1,c,0) \longrightarrow j = (0,k,0)$; for $k = 0,\dots,c$, and $c = 0,\dots,3$,
$i = (2,c,0) \longrightarrow j = (0,k,0)$; for $k = 0,\dots,c$, and $c = 0,\dots,3$.

Sample one-step transition probabilities P(i, j) of Markov chain

To obtain the transition probabilities, probability distributions on the sojourn times $(T_{m+1} - T_m)$ are needed. Let $Z_c = (T_{m+1} - T_m | X_m) = (0,\cdot,c)$. Then we have that

$$Z_c = \min\{(T - cM | T > cM), M\},$$

and hence,

$$P[Z_c > t] = \frac{P[(T - cM) > t]P[M > t]}{P[T > cM]},$$

where

$$P[(T - cM) > t] = E[P(T > t + m | cM = m)], = \int_0^\infty \{1 - F_T(t + m)\} f_{cM}(m)\,\mathrm{d}m$$

and $F_T(\cdot)$ and $f_{cM}(\cdot)$ are the CDF of T (gamma(k,μ)) and pdf of cM (gamma(cd,λ)) respectively.

Then we have that

$$P[Z_c > t] = \int_0^\infty \frac{\{1 - F_T(t + m)\}}{P[T > cM]} f_{cM}(m)\,\mathrm{d}m[1 - F_M(t)],$$

and

$$P[Z_c \le t] = 1 - P[Z_c > t].$$

Recall that

$$P(i,j) = Q(i,j,\infty) = \int_0^\infty P\{X_{m+1} = j|X_m = i, T_{m+1} - T_m = t\} \times \frac{d}{dt} P\{T_{m+1} - T_m \le t|X_m = i\}\, dt.$$

Now, for $i = (0,0,c)$ and $j = (0,1,c+1)$, where $c = 0, \ldots, N_1 - 2$, we have

$$P(i,j) = \int_0^\infty P\{T > (c+1)M|T > cM\} \times \frac{d}{dt}\{1 - P[Z_c > t]\}\, dt.$$

For $i = (0,1,c)$ and $j = (0,1,c+1)$, where $c = 0, \ldots, N_1 - 2$, we have

$$P(i,j) = \int_0^\infty (1 - e^{-\gamma t}) P\{T > (c+1)M|T > cM\} \times \frac{d}{dt}\{1 - P[Z_c > t]\}\, dt.$$

For $i = (0,2,c)$ and $j = (0,1,c+1)$, where $c = 0, \ldots, \min\{N_2 - 2, N_1 - 2\}$, we have

$$P(i,j) = \int_0^\infty (1 - e^{-\gamma t} - \gamma t e^{-\gamma t}) P\{T > (c+1)M|T > cM\} \times \frac{d}{dt}\{1 - P[Z_c > t]\}\, dt.$$

For $i = (1,2,0) \longrightarrow j = (0,0,0)$, we have

$$P(i,j) = \int_a^b (1 - e^{-\gamma t} - \gamma t e^{-\gamma t}) \frac{1}{b-a}\, dt.$$

For $i = (2,2,0) \longrightarrow j = (0,1,0)$, we have

$$P(i,j) = \int_0^\infty \gamma t e^{-\gamma t}\, \frac{\delta^r t^{r-1} e^{-\delta t}}{\Gamma(r)}\, dt.$$

Sample limiting probabilities ϕ of the semi-regenerative process

For $c = 0, \ldots, (N_2 - 1)$, we have

$$\phi_i = \lim_{t\to\infty} P\{Y_t = i = (0,0,c)\} = \frac{1}{\pi \times \psi}\Bigg[\pi(0,0,c) \int_0^\infty P[Z_c > t]\, dt + \pi(0,1,c) \int_0^\infty (1 - e^{-\gamma t}) P[Z_c > t]\, dt + \pi(0,2,c) \int_0^\infty (1 - e^{-\gamma t} - \gamma t e^{-\gamma t}) P[Z_c > t]\, dt\Bigg].$$

For $c = 0, \ldots, (N_2 - 1)$, we have

$$\phi_i = \lim_{t\to\infty} P\{Y_t = i = (0,1,c)\} = \frac{1}{\pi \times \psi}\Bigg[\pi(0,1,c) \int_0^\infty e^{-\gamma t} P[Z_c > t]\, dt + \pi(0,2,c) \int_0^\infty \gamma t e^{-\gamma t} P[Z_c > t]\, dt\Bigg]$$

For $i = (1,0,0)$, we have

$$\phi_i = \frac{1}{\pi \times \psi}\Bigg[\pi(1,1,0)\left\{\int_0^a (1 - e^{-\gamma t})\, dt + \int_a^b (1 - e^{-\gamma t}) \frac{b-t}{b-a}\, dt\right\} + \pi(1,2,0)\left\{\int_0^a (1 - e^{-\gamma t} - \gamma t e^{-\gamma t})\, dt + \int_a^b (1 - e^{-\gamma t} - \gamma t e^{-\gamma t}) \frac{b-t}{b-a}\, dt\right\} + \pi(1,3,0)\left\{\int_0^a \left(1 - e^{-\gamma t} - \gamma t e^{-\gamma t} - \frac{(\gamma t)^2}{2} e^{-\gamma t}\right) dt + \int_a^b \left(1 - e^{-\gamma t} - \gamma t e^{-\gamma t} - \frac{(\gamma t)^2}{2} e^{-\gamma t}\right) \frac{b-t}{b-a}\, dt\right\}\Bigg].$$

For $i = (2,1,0)$, we have

$$\phi_i = \frac{1}{\pi \times \psi}\Bigg[\pi(2,1,0) \int_0^\infty e^{-\gamma t} \int_t^\infty \frac{\delta^r k^{r-1} e^{-\delta k}}{\Gamma(r)}\, dk\, dt + \pi(2,2,0) \int_0^\infty \gamma t e^{-\gamma t} \int_t^\infty \frac{\delta^r k^{r-1} e^{-\delta k}}{\Gamma(r)}\, dk\, dt + \pi(2,3,0) \int_0^\infty \frac{(\gamma t)^2}{2} e^{-\gamma t} \int_t^\infty \frac{\delta^r k^{r-1} e^{-\delta k}}{\Gamma(r)}\, dk\, dt\Bigg].$$

Proceeding in a similar manner, the remaining elements of **P** and ϕ_i can be computed. Having these, calculation of the cost and other performance measures are fairly straightforward.

Biographies

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