OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **4756**

Further Methods for Advanced Mathematics (FP2)

Monday **16 JANUARY 2006** Morning 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions in Section A and **one** question from Section B.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 4 printed pages.

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Section A (54 marks)

Answer all the questions

1 **(a)** A curve has polar equation $r = a\cos 3\theta$ for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, where a is a positive constant.

(i) Sketch the curve, using a continuous line for sections where $r > 0$ and a broken line for sections where $r < 0$. [3]

(ii) Find the area enclosed by one of the loops. [5]

(b) Find the exact value of $\displaystyle\int_0^{\frac{3}{4}} \frac{1}{\sqrt{3-4x^2}}\,\mathrm{d}x$. [5]

(c) Use a trigonometric substitution to find $\displaystyle\int_0^{1} \frac{1}{(1+3x^2)^{\frac{3}{2}}}\,\mathrm{d}x$. [5]

2 In this question, θ is a real number with $0 < \theta < \frac{1}{6}\pi$, and $w = \frac{1}{2}\mathrm{e}^{3\mathrm{j}\theta}$.

(i) State the modulus and argument of each of the complex numbers

$$w, \quad w^* \quad \text{and} \quad \mathrm{j}w.$$

Illustrate these three complex numbers on an Argand diagram. [6]

(ii) Show that $(1+w)(1+w^*) = \frac{5}{4} + \cos 3\theta$. [4]

Infinite series C and S are defined by

$$C = \cos 2\theta - \tfrac{1}{2}\cos 5\theta + \tfrac{1}{4}\cos 8\theta - \tfrac{1}{8}\cos 11\theta + \ldots,$$

$$S = \sin 2\theta - \tfrac{1}{2}\sin 5\theta + \tfrac{1}{4}\sin 8\theta - \tfrac{1}{8}\sin 11\theta + \ldots.$$

(iii) Show that $C = \dfrac{4\cos 2\theta + 2\cos\theta}{5 + 4\cos 3\theta}$, and find a similar expression for S. [8]

3 The matrix $\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 6 \\ 2 & 2 & -4 \end{pmatrix}$.

(i) Show that the characteristic equation for **M** is $\lambda^3 + 6\lambda^2 - 9\lambda - 14 = 0$. [3]

(ii) Show that –1 is an eigenvalue of **M**, and find the other two eigenvalues. [3]

(iii) Find an eigenvector corresponding to the eigenvalue –1. [3]

(iv) Verify that $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$ are eigenvectors of **M**. [3]

(v) Write down a matrix **P**, and a diagonal matrix **D**, such that $\mathbf{M}^3 = \mathbf{PDP}^{-1}$. [3]

(vi) Use the Cayley-Hamilton theorem to express $\mathbf{M}^{-1}$ in the form $a\mathbf{M}^2 + b\mathbf{M} + c\mathbf{I}$. [3]

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(a)** Solve the equation

$$\sinh x + 4\cosh x = 8,$$

giving the answers in an exact logarithmic form. [6]

(b) Find the exact value of $\int_0^2 e^x \sinh x \, dx$. [4]

(c) **(i)** Differentiate $\operatorname{arsinh}(\frac{2}{3}x)$ with respect to x. [2]

(ii) Use integration by parts to show that $\int_0^2 \operatorname{arsinh}(\frac{2}{3}x)\, dx = 2\ln 3 - 1$. [6]

[Turn over

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 A curve has equation $y = \dfrac{x^3 - k^3}{x^2 - 4}$, where k is a positive constant and $k \neq 2$.

(i) Find the equations of the three asymptotes. [3]

(ii) Use your graphical calculator to obtain rough sketches of the curve in the two separate cases $k < 2$ and $k > 2$. [4]

(iii) In the case $k < 2$, your sketch may not show clearly the shape of the curve near $x = 0$. Use calculus to show that the curve has a minimum point when $x = 0$. [5]

(iv) In the case $k > 2$, your sketch may not show clearly how the curve approaches its asymptote as $x \longrightarrow +\infty$. Show algebraically that the curve crosses this asymptote. [2]

(v) Use the results of parts **(iii)** and **(iv)** to produce more accurate sketches of the curve in the two separate cases $k < 2$ and $k > 2$. These sketches should indicate where the curve crosses the axes, and should show clearly how the curve approaches its asymptotes. The presence of stationary points should be clearly shown, but there is no need to find their coordinates. [4]

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4756

Further Methods for Advanced Mathematics (FP2)

Tuesday **6 JUNE 2006** Afternoon 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions in Section A and **one** question from section B.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 4 printed pages.

Section A (54 marks)

Answer all the questions

1 **(a)** A curve has polar equation $r = a(\sqrt{2} + 2\cos\theta)$ for $-\frac{3}{4}\pi \leqslant \theta \leqslant \frac{3}{4}\pi$, where a is a positive constant.

(i) Sketch the curve. [2]

(ii) Find, in an exact form, the area of the region enclosed by the curve. [7]

(b) **(i)** Find the Maclaurin series for the function $\mathrm{f}(x) = \tan(\frac{1}{4}\pi + x)$, up to the term in x^2. [6]

(ii) Use the Maclaurin series to show that, when h is small,

$$\int_{-h}^{h} x^2 \tan(\tfrac{1}{4}\pi + x)\,\mathrm{d}x \approx \tfrac{2}{3}h^3 + \tfrac{4}{5}h^5.$$ [3]

2 **(a)** **(i)** Given that $z = \cos\theta + \mathrm{j}\sin\theta$, express $z^n + \dfrac{1}{z^n}$ and $z^n - \dfrac{1}{z^n}$ in simplified trigonometric form. [2]

(ii) By considering $\left(z - \dfrac{1}{z}\right)^4 \left(z + \dfrac{1}{z}\right)^2$, find A, B, C and D such that

$$\sin^4\theta\cos^2\theta = A\cos 6\theta + B\cos 4\theta + C\cos 2\theta + D.$$ [6]

(b) **(i)** Find the modulus and argument of $4 + 4\mathrm{j}$. [2]

(ii) Find the fifth roots of $4 + 4\mathrm{j}$ in the form $r\mathrm{e}^{\mathrm{j}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

Illustrate these fifth roots on an Argand diagram. [6]

(iii) Find integers p and q such that $(p + q\mathrm{j})^5 = 4 + 4\mathrm{j}$. [2]

3 **(i)** Find the inverse of the matrix $\begin{pmatrix} 4 & 1 & k \\ 3 & 2 & 5 \\ 8 & 5 & 13 \end{pmatrix}$, where $k \neq 5$. [6]

(ii) Solve the simultaneous equations

$$\begin{aligned} 4x + y + 7z &= 12 \\ 3x + 2y + 5z &= m \\ 8x + 5y + 13z &= 0 \end{aligned}$$

giving x, y and z in terms of m. [5]

(iii) Find the value of p for which the simultaneous equations

$$\begin{aligned} 4x + y + 5z &= 12 \\ 3x + 2y + 5z &= p \\ 8x + 5y + 13z &= 0 \end{aligned}$$

have solutions, and find the general solution in this case. [7]

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that

$$1 + 2\sinh^2 x = \cosh 2x.$$ [3]

(ii) Solve the equation

$$2\cosh 2x + \sinh x = 5,$$

giving the answers in an exact logarithmic form. [6]

(iii) Show that $\int_0^{\ln 3} \sinh^2 x \, dx = \frac{10}{9} - \frac{1}{2}\ln 3$. [5]

(iv) Find the exact value of $\int_3^5 \sqrt{x^2 - 9} \, dx$. [4]

[Question 5 is printed overleaf.]

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 A curve has parametric equations

$$x = \theta - k\sin\theta, \quad y = 1 - \cos\theta,$$

where k is a positive constant.

(i) For the case $k = 1$, use your graphical calculator to sketch the curve. Describe its main features. [4]

(ii) Sketch the curve for a value of k between 0 and 1. Describe briefly how the main features differ from those for the case $k = 1$. [3]

(iii) For the case $k = 2$:

(A) sketch the curve; [2]

(B) find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ; [2]

(C) show that the width of each loop, measured parallel to the x-axis, is

$$2\sqrt{3} - \frac{2\pi}{3}.$$ [5]

(iv) Use your calculator to find, correct to one decimal place, the value of k for which successive loops just touch each other. [2]

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RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT **4756/01**

MATHEMATICS (MEI)

Further Methods for Advanced Mathematics (FP2)

TUESDAY 16 JANUARY 2007

Morning
Time: 1 hour 30 minutes

Additional materials:
- Answer booklet (8 pages)
- Graph paper
- MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions in Section A and **one** question from Section B.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

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- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

 [Turn over

Section A (54 marks)

Answer all the questions

1 **(a)** A curve has polar equation $r = a\mathrm{e}^{-k\theta}$ for $0 \leqslant \theta \leqslant \pi$, where a and k are positive constants. The points A and B on the curve correspond to $\theta = 0$ and $\theta = \pi$ respectively.

(i) Sketch the curve. [2]

(ii) Find the area of the region enclosed by the curve and the line AB. [4]

(b) Find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{3+4x^2}\,\mathrm{d}x$. [5]

(c) **(i)** Find the Maclaurin series for $\tan x$, up to the term in x^3. [4]

(ii) Use this Maclaurin series to show that, when h is small, $\displaystyle\int_h^{4h} \frac{\tan x}{x}\,\mathrm{d}x \approx 3h + 7h^3$. [3]

2 **(a)** You are given the complex numbers $w = 3\mathrm{e}^{-\frac{1}{12}\pi\mathrm{j}}$ and $z = 1 - \sqrt{3}\,\mathrm{j}$.

(i) Find the modulus and argument of each of the complex numbers w, z and $\dfrac{w}{z}$. [5]

(ii) Hence write $\dfrac{w}{z}$ in the form $a + b\mathrm{j}$, giving the exact values of a and b. [2]

(b) In this part of the question, n is a positive integer and θ is a real number with $0 < \theta < \dfrac{\pi}{n}$.

(i) Express $\mathrm{e}^{-\frac{1}{2}\mathrm{j}\theta} + \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}$ in simplified trigonometric form, and hence, or otherwise, show that

$$1 + \mathrm{e}^{\mathrm{j}\theta} = 2\mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}\cos\tfrac{1}{2}\theta.$$

[4]

Series C and S are defined by

$$C = 1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \binom{n}{3}\cos 3\theta + \ldots + \binom{n}{n}\cos n\theta,$$

$$S = \binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \binom{n}{3}\sin 3\theta + \ldots + \binom{n}{n}\sin n\theta.$$

(ii) Find C and S, and show that $\dfrac{S}{C} = \tan\tfrac{1}{2}n\theta$. [7]

3 Let $\mathbf{P} = \begin{pmatrix} 4 & 2 & k \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ (where $k \neq 4$) and $\mathbf{M} = \begin{pmatrix} 2 & -2 & -6 \\ -1 & 3 & 1 \\ 1 & -2 & -2 \end{pmatrix}$.

(i) Find $\mathbf{P}^{-1}$ in terms of k, and show that, when $k = 2$, $\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$. [6]

(ii) Verify that $\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ are eigenvectors of $\mathbf{M}$, and find the corresponding eigenvalues. [4]

(iii) Show that $\mathbf{M}^n = \begin{pmatrix} 4 & -6 & -10 \\ 2 & -3 & -5 \\ 0 & 0 & 0 \end{pmatrix} + 2^{n-1}\begin{pmatrix} -2 & 4 & 4 \\ -3 & 6 & 6 \\ 1 & -2 & -2 \end{pmatrix}$. [8]

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Show that $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$. [5]

(ii) Find $\displaystyle\int_{2.5}^{3.9} \frac{1}{\sqrt{4x^2 - 9}}\,dx$, giving your answer in the form $a \ln b$, where a and b are rational numbers. [5]

(iii) There are two points on the curve $y = \dfrac{\cosh x}{2 + \sinh x}$ at which the gradient is $\frac{1}{9}$.

Show that one of these points is $\left(\ln\left(1 + \sqrt{2}\right), \frac{1}{3}\sqrt{2}\right)$, and find the coordinates of the other point, in a similar form. [8]

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 Cartesian coordinates (x, y) and polar coordinates (r, θ) are set up in the usual way, with the pole at the origin and the initial line along the positive x-axis, so that $x = r\cos\theta$ and $y = r\sin\theta$.

A curve has polar equation $r = k + \cos\theta$, where k is a constant with $k \geqslant 1$.

(i) Use your graphical calculator to obtain sketches of the curve in the three cases

$$k = 1, k = 1.5 \text{ and } k = 4.$$ [5]

(ii) Name the special feature which the curve has when $k = 1$. [1]

(iii) For each of the three cases, state the number of points on the curve at which the tangent is parallel to the y-axis. [2]

(iv) Express x in terms of k and θ, and find $\frac{dx}{d\theta}$. Hence find the range of values of k for which there are just two points on the curve where the tangent is parallel to the y-axis. [4]

The distance between the point (r, θ) on the curve and the point $(1, 0)$ on the x-axis is d.

(v) Use the cosine rule to express d^2 in terms of k and θ, and deduce that $k^2 \leqslant d^2 \leqslant k^2 + 1$. [4]

(vi) Hence show that, when k is large, the shape of the curve is very nearly circular. [2]

RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT **4756/01**

MATHEMATICS (MEI)

Further Methods for Advanced Mathematics (FP2)

THURSDAY 7 JUNE 2007

Morning

Time: 1 hour 30 minutes

Additional materials:

Answer booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions in Section A and **one** question from Section B.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

Section A (54 marks)

Answer all the questions

1 **(a)** A curve has polar equation $r = a(1 - \cos\theta)$, where a is a positive constant.

(i) Sketch the curve. [2]

(ii) Find the area of the region enclosed by the section of the curve for which $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ and the line $\theta = \frac{1}{2}\pi$. [6]

(b) Use a trigonometric substitution to show that $\displaystyle\int_0^1 \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx = \frac{1}{4\sqrt{3}}$. [4]

(c) In this part of the question, $f(x) = \arccos(2x)$.

(i) Find $f'(x)$. [2]

(ii) Use a standard series to expand $f'(x)$, and hence find the series for $f(x)$ in ascending powers of x, up to the term in x^5. [4]

2 **(a)** Use de Moivre's theorem to show that $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$. [5]

(b) **(i)** Find the cube roots of $-2 + 2\mathrm{j}$ in the form $r\mathrm{e}^{\mathrm{j}\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$. [6]

These cube roots are represented by points A, B and C in the Argand diagram, with A in the first quadrant and ABC going anticlockwise. The midpoint of AB is M, and M represents the complex number w.

(ii) Draw an Argand diagram, showing the points A, B, C and M. [2]

(iii) Find the modulus and argument of w. [2]

(iv) Find w^6 in the form $a + b\mathrm{j}$. [3]

3 Let $\mathbf{M} = \begin{pmatrix} 3 & 5 & 2 \\ 5 & 3 & -2 \\ 2 & -2 & -4 \end{pmatrix}$.

(i) Show that the characteristic equation for $\mathbf{M}$ is $\lambda^3 - 2\lambda^2 - 48\lambda = 0$. [4]

You are given that $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$ corresponding to the eigenvalue 0.

(ii) Find the other two eigenvalues of $\mathbf{M}$, and corresponding eigenvectors. [8]

(iii) Write down a matrix $\mathbf{P}$, and a diagonal matrix $\mathbf{D}$, such that $\mathbf{P}^{-1}\mathbf{M}^2\mathbf{P} = \mathbf{D}$. [3]

(iv) Use the Cayley-Hamilton theorem to find integers a and b such that $\mathbf{M}^4 = a\mathbf{M}^2 + b\mathbf{M}$. [3]

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(a)** Find $\int_0^1 \frac{1}{\sqrt{9x^2+16}}\,dx$, giving your answer in an exact logarithmic form. [5]

(b) **(i)** Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that

$$\sinh 2x = 2\sinh x \cosh x.$$ [2]

(ii) Show that one of the stationary points on the curve

$$y = 20\cosh x - 3\cosh 2x$$

has coordinates $(\ln 3, \frac{59}{3})$, and find the coordinates of the other two stationary points. [7]

(iii) Show that $\int_{-\ln 3}^{\ln 3} (20\cosh x - 3\cosh 2x)\,dx = 40.$ [4]

[Question 5 is printed overleaf.]

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 The curve with equation $y = \dfrac{x^2 - kx + 2k}{x + k}$ is to be investigated for different values of k.

(i) Use your graphical calculator to obtain rough sketches of the curve in the cases $k = -2$, $k = -0.5$ and $k = 1$. [6]

(ii) Show that the equation of the curve may be written as $y = x - 2k + \dfrac{2k(k+1)}{x+k}$.

Hence find the two values of k for which the curve is a straight line. [4]

(iii) When the curve is not a straight line, it is a conic.

(A) Name the type of conic. [1]

(B) Write down the equations of the asymptotes. [2]

(iv) Draw a sketch to show the shape of the curve when $1 < k < 8$. This sketch should show where the curve crosses the axes and how it approaches its asymptotes. Indicate the points A and B on the curve where $x = 1$ and $x = k$ respectively. [5]

RECOGNISING ACHIEVEMENT

ADVANCED GCE **4756/01**

MATHEMATICS (MEI)

Further Methods for Advanced Mathematics (FP2)

WEDNESDAY 9 JANUARY 2008 Afternoon

Time: 1 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
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INFORMATION FOR CANDIDATES

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This document consists of **4** printed pages.

Section A (54 marks)

Answer all the questions

1 **(a)** Fig. 1 shows the curve with polar equation $r = a(1 - \cos 2\theta)$ for $0 \leqslant \theta \leqslant \pi$, where a is a positive constant.

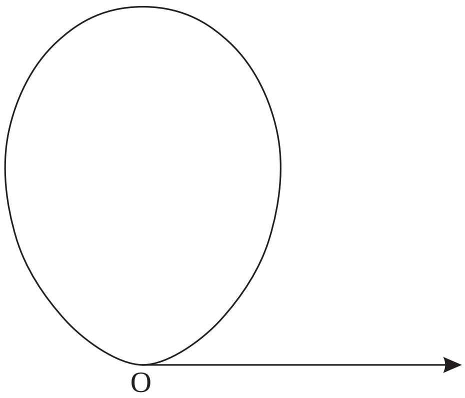

Fig. 1

Find the area of the region enclosed by the curve. [7]

(b) **(i)** Given that $\mathrm{f}(x) = \arctan(\sqrt{3} + x)$, find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$. [4]

(ii) Hence find the Maclaurin series for $\arctan(\sqrt{3} + x)$, as far as the term in x^2. [4]

(iii) Hence show that, if h is small, $\int_{-h}^{h} x \arctan(\sqrt{3} + x)\,\mathrm{d}x \approx \frac{1}{6}h^3$. [3]

2 **(a)** Find the 4th roots of 16j, in the form $r\mathrm{e}^{\mathrm{j}\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$. Illustrate the 4th roots on an Argand diagram. [6]

(b) **(i)** Show that $(1 - 2\mathrm{e}^{\mathrm{j}\theta})(1 - 2\mathrm{e}^{-\mathrm{j}\theta}) = 5 - 4\cos\theta$. [3]

Series C and S are defined by

$$C = 2\cos\theta + 4\cos 2\theta + 8\cos 3\theta + \ldots + 2^n \cos n\theta,$$
$$S = 2\sin\theta + 4\sin 2\theta + 8\sin 3\theta + \ldots + 2^n \sin n\theta.$$

(ii) Show that $C = \dfrac{2\cos\theta - 4 - 2^{n+1}\cos(n+1)\theta + 2^{n+2}\cos n\theta}{5 - 4\cos\theta}$, and find a similar expression for S. [9]

3 You are given the matrix $\mathbf{M} = \begin{pmatrix} 7 & 3 \\ -4 & -1 \end{pmatrix}$.

(i) Find the eigenvalues, and corresponding eigenvectors, of the matrix $\mathbf{M}$. [8]

(ii) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{MP} = \mathbf{D}$. [2]

(iii) Given that $\mathbf{M}^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $a = -\frac{1}{2} + \frac{3}{2} \times 5^n$, and find similar expressions for b, c and d. [8]

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Given that $k \geqslant 1$ and $\cosh x = k$, show that $x = \pm \ln\left(k + \sqrt{k^2 - 1}\right)$. [5]

(ii) Find $\displaystyle\int_1^2 \frac{1}{\sqrt{4x^2 - 1}}\,dx$, giving the answer in an exact logarithmic form. [5]

(iii) Solve the equation $6\sinh x - \sinh 2x = 0$, giving the answers in an exact form, using logarithms where appropriate. [4]

(iv) Show that there is no point on the curve $y = 6\sinh x - \sinh 2x$ at which the gradient is 5. [4]

[Question 5 is printed overleaf.]

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 A curve has parametric equations $x = \dfrac{t^2}{1+t^2}$, $y = t^3 - \lambda t$, where λ is a constant.

(i) Use your calculator to obtain a sketch of the curve in each of the cases

$$\lambda = -1, \quad \lambda = 0 \quad \text{and} \quad \lambda = 1.$$

Name any special features of these curves. [5]

(ii) By considering the value of x when t is large, write down the equation of the asymptote. [1]

For the remainder of this question, assume that λ is positive.

(iii) Find, in terms of λ, the coordinates of the point where the curve intersects itself. [3]

(iv) Show that the two points on the curve where the tangent is parallel to the x-axis have coordinates

$$\left(\frac{\lambda}{3+\lambda}, \pm\sqrt{\frac{4\lambda^3}{27}}\right).$$ [4]

Fig. 5 shows a curve which intersects itself at the point (2, 0) and has asymptote $x = 8$. The stationary points A and B have y-coordinates 2 and -2.

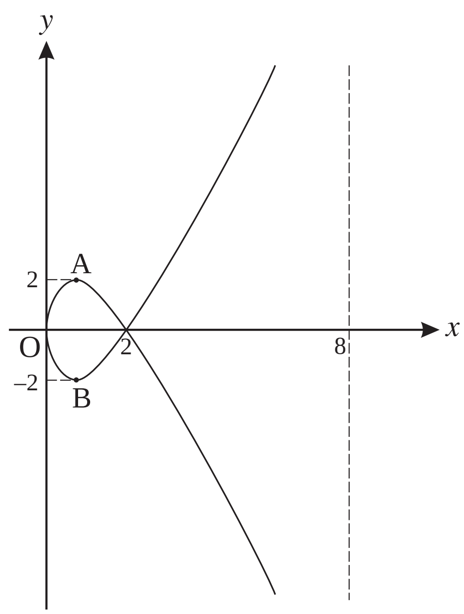

Fig. 5

(v) For the curve sketched in Fig. 5, find parametric equations of the form $x = \dfrac{at^2}{1+t^2}$, $y = b(t^3 - \lambda t)$, where a, λ and b are to be determined. [5]

RECOGNISING ACHIEVEMENT

ADVANCED GCE **4756/01**

MATHEMATICS (MEI)

Further Methods for Advanced Mathematics (FP2)

THURSDAY 15 MAY 2008 Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):

Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions in Section A and **one** question from Section B.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72.**
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

Section A (54 marks)

Answer all the questions

1 **(a)** A curve has cartesian equation $(x^2 + y^2)^2 = 3xy^2$.

(i) Show that the polar equation of the curve is $r = 3\cos\theta\sin^2\theta$. [3]

(ii) Hence sketch the curve. [3]

(b) Find the exact value of $\displaystyle\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,\mathrm{d}x$. [5]

(c) **(i)** Write down the series for $\ln(1+x)$ and the series for $\ln(1-x)$, both as far as the term in x^5. [2]

(ii) Hence find the first three non-zero terms in the series for $\ln\left(\dfrac{1+x}{1-x}\right)$. [2]

(iii) Use the series in part **(ii)** to show that $\displaystyle\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r} = \ln 3$. [3]

2 You are given the complex numbers $z = \sqrt{32}(1+\mathrm{j})$ and $w = 8\left(\cos\frac{7}{12}\pi + \mathrm{j}\sin\frac{7}{12}\pi\right)$.

(i) Find the modulus and argument of each of the complex numbers z, z^*, zw and $\dfrac{z}{w}$. [7]

(ii) Express $\dfrac{z}{w}$ in the form $a + b\mathrm{j}$, giving the exact values of a and b. [2]

(iii) Find the cube roots of z, in the form $r\mathrm{e}^{\mathrm{j}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [4]

(iv) Show that the cube roots of z can be written as

$$k_1 w^*, \quad k_2 z^* \quad \text{and} \quad k_3 \mathrm{j}w,$$

where k_1, k_2 and k_3 are real numbers. State the values of k_1, k_2 and k_3. [5]

3 **(i)** Given the matrix $\mathbf{Q} = \begin{pmatrix} 2 & -1 & k \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ (where $k \neq 3$), find $\mathbf{Q}^{-1}$ in terms of k.

Show that, when $k = 4$, $\mathbf{Q}^{-1} = \begin{pmatrix} -1 & 6 & -1 \\ 1 & -8 & 2 \\ 1 & -5 & 1 \end{pmatrix}$. [6]

The matrix $\mathbf{M}$ has eigenvectors $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$, with corresponding eigenvalues 1, −1 and 3 respectively.

(ii) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{MP} = \mathbf{D}$, and hence find the matrix $\mathbf{M}$. [7]

(iii) Write down the characteristic equation for $\mathbf{M}$, and use the Cayley-Hamilton theorem to find integers a, b and c such that $\mathbf{M}^4 = a\mathbf{M}^2 + b\mathbf{M} + c\mathbf{I}$. [5]

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that

$$\cosh^2 x - \sinh^2 x = 1.$$ [3]

(ii) Solve the equation $4\cosh^2 x + 9\sinh x = 13$, giving the answers in exact logarithmic form. [6]

(iii) Show that there is only one stationary point on the curve

$$y = 4\cosh^2 x + 9\sinh x,$$

and find the y-coordinate of the stationary point. [4]

(iv) Show that $\int_0^{\ln 2} (4\cosh^2 x + 9\sinh x)\,\mathrm{d}x = 2\ln 2 + \frac{33}{8}$. [5]

[Question 5 is printed overleaf.]

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 A curve has parametric equations $x = \lambda \cos\theta - \frac{1}{\lambda}\sin\theta$, $y = \cos\theta + \sin\theta$, where λ is a positive constant.

(i) Use your calculator to obtain a sketch of the curve in each of the cases

$$\lambda = 0.5, \quad \lambda = 3 \quad \text{and} \quad \lambda = 5.$$ [3]

(ii) Given that the curve is a conic, name the type of conic. [1]

(iii) Show that y has a maximum value of $\sqrt{2}$ when $\theta = \frac{1}{4}\pi$. [2]

(iv) Show that $x^2 + y^2 = (1 + \lambda^2) + \left(\frac{1}{\lambda^2} - \lambda^2\right)\sin^2\theta$, and deduce that the distance from the origin of any point on the curve is between $\sqrt{1 + \frac{1}{\lambda^2}}$ and $\sqrt{1 + \lambda^2}$. [6]

(v) For the case $\lambda = 1$, show that the curve is a circle, and find its radius. [2]

(vi) For the case $\lambda = 2$, draw a sketch of the curve, and label the points A, B, C, D, E, F, G, H on the curve corresponding to $\theta = 0, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi, \pi, \frac{5}{4}\pi, \frac{3}{2}\pi, \frac{7}{4}\pi$ respectively. You should make clear what is special about each of these points. [4]

OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE
MATHEMATICS (MEI) 4756
Further Methods for Advanced Mathematics (FP2)

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Friday 9 January 2009
Morning

Duration: 1 hour 30 minutes

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Section A (54 marks)

Answer all the questions

1 **(a)** **(i)** By considering the derivatives of $\cos x$, show that the Maclaurin expansion of $\cos x$ begins

$$1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4.$$ **[4]**

(ii) The Maclaurin expansion of $\sec x$ begins

$$1 + ax^2 + bx^4,$$

where a and b are constants. Explain why, for sufficiently small x,

$$\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4\right)\left(1 + ax^2 + bx^4\right) \approx 1.$$

Hence find the values of a and b. **[5]**

(b) **(i)** Given that $y = \arctan\left(\dfrac{x}{a}\right)$, show that $\dfrac{dy}{dx} = \dfrac{a}{a^2 + x^2}$. **[4]**

(ii) Find the exact values of the following integrals.

(A) $\displaystyle\int_{-2}^{2} \frac{1}{4 + x^2}\,dx$ **[3]**

(B) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{4}{1 + 4x^2}\,dx$ **[3]**

2 **(i)** Write down the modulus and argument of the complex number $e^{j\pi/3}$. **[2]**

(ii) The triangle OAB in an Argand diagram is equilateral. O is the origin; A corresponds to the complex number $a = \sqrt{2}(1 + j)$; B corresponds to the complex number b.

Show A and the two possible positions for B in a sketch. Express a in the form $re^{j\theta}$. Find the two possibilities for b in the form $re^{j\theta}$. **[5]**

(iii) Given that $z_1 = \sqrt{2}e^{j\pi/3}$, show that $z_1^6 = 8$. Write down, in the form $re^{j\theta}$, the other five complex numbers z such that $z^6 = 8$. Sketch all six complex numbers in a new Argand diagram. **[6]**

Let $w = z_1 e^{-j\pi/12}$.

(iv) Find w in the form $x + jy$, and mark this complex number on your Argand diagram. **[3]**

(v) Find w^6, expressing your answer in as simple a form as possible. **[2]**

3 **(a)** A curve has polar equation $r = a\tan\theta$ for $0 \leqslant \theta \leqslant \frac{1}{3}\pi$, where a is a positive constant.

(i) Sketch the curve. **[3]**

(ii) Find the area of the region between the curve and the line $\theta = \frac{1}{4}\pi$. Indicate this region on your sketch. **[5]**

(b) **(i)** Find the eigenvalues and corresponding eigenvectors for the matrix $\mathbf{M}$ where

$$\mathbf{M} = \begin{pmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{pmatrix}.$$ **[6]**

(ii) Give a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{QDQ}^{-1}$. **[3]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(a)** **(i)** Prove, from definitions involving exponentials, that

$$\cosh^2 x - \sinh^2 x = 1.$$ **[2]**

(ii) Given that $\sinh x = \tan y$, where $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$, show that

(A) $\tanh x = \sin y$,

(B) $x = \ln(\tan y + \sec y)$. **[6]**

(b) **(i)** Given that $y = \operatorname{artanh} x$, find $\frac{dy}{dx}$ in terms of x.

Hence show that $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-x^2}\,dx = 2\operatorname{artanh}\tfrac{1}{2}$. **[4]**

(ii) Express $\frac{1}{1-x^2}$ in partial fractions and hence find an expression for $\displaystyle\int \frac{1}{1-x^2}\,dx$ in terms of logarithms. **[4]**

(iii) Use the results in parts **(i)** and **(ii)** to show that $\operatorname{artanh}\frac{1}{2} = \frac{1}{2}\ln 3$. **[2]**

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 The limaçon of Pascal has polar equation $r = 1 + 2a\cos\theta$, where a is a constant.

(i) Use your calculator to sketch the curve when $a = 1$. (You need not distinguish between parts of the curve where r is positive and negative.) **[3]**

(ii) By using your calculator to investigate the shape of the curve for different values of a, positive and negative,

(A) state the set of values of a for which the curve has a loop within a loop,

(B) state, with a reason, the shape of the curve when $a = 0$,

(C) state what happens to the shape of the curve as $a \to \pm\infty$,

(D) name the feature of the curve that is evident when $a = 0.5$, and find another value of a for which the curve has this feature. **[7]**

(iii) Given that $a > 0$ and that a is such that the curve has a loop within a loop, write down an equation for the values of θ at which $r = 0$. Hence show that the angle at which the curve crosses itself is $2\arccos\left(\frac{1}{2a}\right)$.

Obtain the cartesian equations of the tangents at the point where the curve crosses itself. Explain briefly how these equations relate to the answer to part **(ii)**(A). **[8]**

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RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) **4756**

Further Methods for Advanced Mathematics (FP2)

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Friday 5 June 2009
Afternoon

Duration: 1 hour 30 minutes

* 4 7 5 6 *

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Section A (54 marks)

Answer all the questions

1 **(a)** **(i)** Use the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$ to obtain the first three non-zero terms in the Maclaurin series for $\ln\left(\dfrac{1+x}{1-x}\right)$. State the range of validity of this series. **[4]**

(ii) Find the value of x for which $\dfrac{1+x}{1-x} = 3$. Hence find an approximation to $\ln 3$, giving your answer to three decimal places. **[4]**

(b) A curve has polar equation $r = \dfrac{a}{1+\sin\theta}$ for $0 \leqslant \theta \leqslant \pi$, where a is a positive constant. The points on the curve have cartesian coordinates x and y.

(i) By plotting suitable points, or otherwise, sketch the curve. **[3]**

(ii) Show that, for this curve, $r + y = a$ and hence find the cartesian equation of the curve. **[5]**

2 **(i)** Obtain the characteristic equation for the matrix $\mathbf{M}$ where

$$\mathbf{M} = \begin{pmatrix} 3 & 1 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Hence or otherwise obtain the value of $\det(\mathbf{M})$. **[3]**

(ii) Show that -1 is an eigenvalue of $\mathbf{M}$, and show that the other two eigenvalues are not real.

Find an eigenvector corresponding to the eigenvalue -1.

Hence or otherwise write down the solution to the following system of equations. **[9]**

$$\begin{aligned} 3x + y - 2z &= -0.1 \\ -y \qquad &= 0.6 \\ 2x \qquad + z &= 0.1 \end{aligned}$$

(iii) State the Cayley-Hamilton theorem and use it to show that

$$\mathbf{M}^3 = 3\mathbf{M}^2 - 3\mathbf{M} - 7\mathbf{I}.$$

Obtain an expression for $\mathbf{M}^{-1}$ in terms of $\mathbf{M}^2$, $\mathbf{M}$ and $\mathbf{I}$. **[4]**

(iv) Find the numerical values of the elements of $\mathbf{M}^{-1}$, showing your working. **[3]**

3 **(a)** **(i)** Sketch the graph of $y = \arcsin x$ for $-1 \leqslant x \leqslant 1$. **[1]**

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, justifying the sign of your answer by reference to your sketch. **[4]**

(ii) Find the exact value of the integral $\displaystyle\int_0^1 \frac{1}{\sqrt{2-x^2}}\,\mathrm{d}x$. **[3]**

(b) The infinite series C and S are defined as follows.

$$C = \cos\theta + \tfrac{1}{3}\cos 3\theta + \tfrac{1}{9}\cos 5\theta + \ldots$$

$$S = \sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{9}\sin 5\theta + \ldots$$

By considering $C + \mathrm{j}S$, show that

$$C = \frac{3\cos\theta}{5 - 3\cos 2\theta},$$

and find a similar expression for S. **[11]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Prove, from definitions involving exponentials, that

$$\cosh 2u = 2\cosh^2 u - 1.$$ **[3]**

(ii) Prove that $\operatorname{arsinh} y = \ln\left(y + \sqrt{y^2+1}\right)$. **[4]**

(iii) Use the substitution $x = 2\sinh u$ to show that

$$\int \sqrt{x^2+4}\,\mathrm{d}x = 2\operatorname{arsinh}\tfrac{1}{2}x + \tfrac{1}{2}x\sqrt{x^2+4} + c,$$

where c is an arbitrary constant. **[6]**

(iv) By first expressing $t^2 + 2t + 5$ in completed square form, show that

$$\int_{-1}^{1} \sqrt{t^2+2t+5}\,\mathrm{d}t = 2\left(\ln(1+\sqrt{2}) + \sqrt{2}\right).$$ **[5]**

[Question 5 is printed overleaf.]

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 Fig. 5 shows a circle with centre C $(a, 0)$ and radius a. B is the point $(0, 1)$. The line BC intersects the circle at P and Q; P is above the x-axis and Q is below.

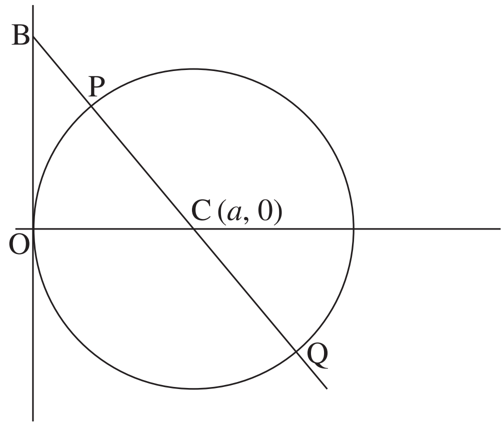

Fig. 5

(i) Show that, in the case $a = 1$, P has coordinates $\left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Write down the coordinates of Q. **[3]**

(ii) Show that, for all positive values of a, the coordinates of P are

$$x = a\left(1 - \frac{a}{\sqrt{a^2 + 1}}\right), \qquad y = \frac{a}{\sqrt{a^2 + 1}}. \qquad (*)$$

Write down the coordinates of Q in a similar form. **[4]**

Now let the variable point P be defined by the parametric equations $(*)$ for all values of the parameter a, positive, zero and negative. Let Q be defined for all a by your answer in part **(ii)**.

(iii) Using your calculator, sketch the locus of P as a varies. State what happens to P as $a \to \infty$ and as $a \to -\infty$.

Show algebraically that this locus has an asymptote at $y = -1$.

On the same axes, sketch, as a dotted line, the locus of Q as a varies. **[8]**

(The single curve made up of these two loci and including the point B is called a *right strophoid*.)

(iv) State, with a reason, the size of the angle POQ in Fig. 5. What does this indicate about the angle at which a right strophoid crosses itself? **[3]**

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ADVANCED GCE

MATHEMATICS (MEI) 4756

Further Methods for Advanced Mathematics (FP2)

Candidates answer on the Answer Booklet

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- 8 page Answer Booklet
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Monday 11 January 2010
Morning

Duration: 1 hour 30 minutes

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Section A (54 marks)

Answer all the questions

1 **(a)** Given that $y = \arctan \sqrt{x}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer in terms of x. Hence show that

$$\int_0^1 \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x = \frac{\pi}{2}.$$ **[6]**

(b) A curve has cartesian equation

$$x^2 + y^2 = xy + 1.$$

(i) Show that the polar equation of the curve is

$$r^2 = \frac{2}{2 - \sin 2\theta}.$$ **[4]**

(ii) Determine the greatest and least positive values of r and the values of θ between 0 and 2π for which they occur. **[6]**

(iii) Sketch the curve. **[2]**

2 **(a)** Use de Moivre's theorem to find the constants a, b, c in the identity

$$\cos 5\theta \equiv a\cos^5\theta + b\cos^3\theta + c\cos\theta.$$ **[6]**

(b) Let

$$C = \cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \cos\left(\theta + \frac{4\pi}{n}\right) + \ldots + \cos\left(\theta + \frac{(2n-2)\pi}{n}\right),$$

and $S = \sin\theta + \sin\left(\theta + \frac{2\pi}{n}\right) + \sin\left(\theta + \frac{4\pi}{n}\right) + \ldots + \sin\left(\theta + \frac{(2n-2)\pi}{n}\right)$,

where n is an integer greater than 1.

By considering $C + \mathrm{j}S$, show that $C = 0$ and $S = 0$. **[7]**

(c) Write down the Maclaurin series for e^t as far as the term in t^2.

Hence show that, for t close to zero,

$$\frac{t}{\mathrm{e}^t - 1} \approx 1 - \tfrac{1}{2}t.$$ **[5]**

3 **(i)** Find the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & a \\ 2 & -1 & 2 \\ 3 & -2 & 2 \end{pmatrix}$$

where $a \neq 4$.

Show that when $a = -1$ the inverse is

$$\frac{1}{5}\begin{pmatrix} 2 & 0 & 1 \\ 2 & 5 & -4 \\ -1 & 5 & -3 \end{pmatrix}.$$ **[6]**

(ii) Solve, in terms of b, the following system of equations. **[5]**

$$\begin{aligned} x + y - z &= -2 \\ 2x - y + 2z &= b \\ 3x - 2y + 2z &= 1 \end{aligned}$$

(iii) Find the value of b for which the equations

$$\begin{aligned} x + y + 4z &= -2 \\ 2x - y + 2z &= b \\ 3x - 2y + 2z &= 1 \end{aligned}$$

have solutions. Give a geometrical interpretation of the solutions in this case. **[7]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Prove, using exponential functions, that

$$\cosh 2x = 1 + 2\sinh^2 x.$$

Differentiate this result to obtain a formula for $\sinh 2x$. **[4]**

(ii) Solve the equation

$$2\cosh 2x + 3\sinh x = 3,$$

expressing your answers in exact logarithmic form. **[7]**

(iii) Given that $\cosh t = \frac{5}{4}$, show by using exponential functions that $t = \pm \ln 2$.

Find the exact value of the integral

$$\int_4^5 \frac{1}{\sqrt{x^2 - 16}}\,dx.$$ **[7]**

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 A line PQ is of length k (where $k > 1$) and it passes through the point $(1, 0)$. PQ is inclined at angle θ to the positive x-axis. The end Q moves along the y-axis. See Fig. 5. The end P traces out a locus.

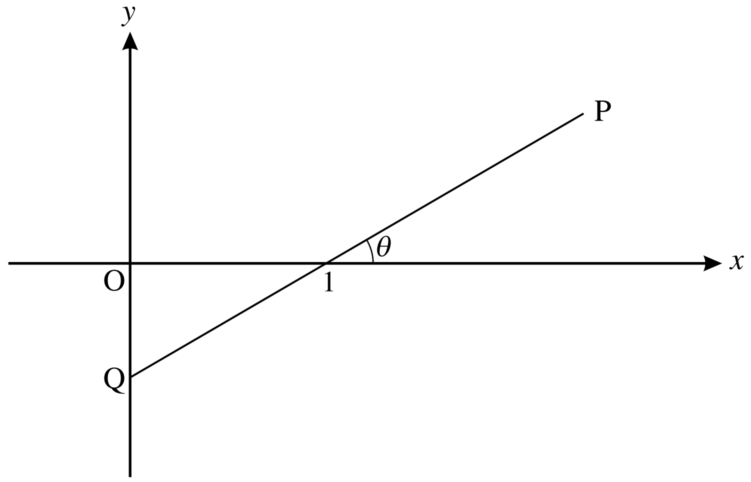

Fig. 5

(i) Show that the locus of P may be expressed parametrically as follows. **[3]**

$$x = k\cos\theta \qquad y = k\sin\theta - \tan\theta$$

You are now required to investigate curves with these parametric equations, where k may take any non-zero value and $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(ii) Use your calculator to sketch the curve in each of the cases $k = 2$, $k = 1$, $k = \frac{1}{2}$ and $k = -1$. **[4]**

(iii) For what value(s) of k does the curve have

(A) an asymptote (you should state what the asymptote is),

(B) a cusp,

(C) a loop? **[3]**

(iv) For the case $k = 2$, find the angle at which the curve crosses itself. **[2]**

(v) For the case $k = 8$, find in an exact form the coordinates of the highest point on the loop. **[3]**

(vi) Verify that the cartesian equation of the curve is

$$y^2 = \frac{(x-1)^2}{x^2}(k^2 - x^2).$$ **[3]**

RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4756

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Candidates answer on the Answer Booklet

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Other Materials Required:
- Scientific or graphical calculator

Friday 11 June 2010
Morning

Duration: 1 hour 30 minutes

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Section A (54 marks)

Answer all the questions

1 **(a)** **(i)** Given that $\mathrm{f}(t) = \arcsin t$, write down an expression for $\mathrm{f}'(t)$ and show that

$$\mathrm{f}''(t) = \frac{t}{(1-t^2)^{\frac{3}{2}}}.$$ **[3]**

(ii) Show that the Maclaurin expansion of the function $\arcsin\left(x + \frac{1}{2}\right)$ begins

$$\frac{\pi}{6} + \frac{2}{\sqrt{3}}x,$$

and find the term in x^2. **[5]**

(b) Sketch the curve with polar equation $r = \dfrac{\pi a}{\pi + \theta}$, where $a > 0$, for $0 \leqslant \theta < 2\pi$.

Find, in terms of a, the area of the region bounded by the part of the curve for which $0 \leqslant \theta \leqslant \pi$ and the lines $\theta = 0$ and $\theta = \pi$. **[6]**

(c) Find the exact value of the integral

$$\int_0^{\frac{3}{2}} \frac{1}{9 + 4x^2}\,\mathrm{d}x.$$ **[5]**

2 **(a)** Given that $z = \cos\theta + \mathrm{j}\sin\theta$, express $z^n + \dfrac{1}{z^n}$ and $z^n - \dfrac{1}{z^n}$ in simplified trigonometric form.

Hence find the constants A, B, C in the identity

$$\sin^5\theta \equiv A\sin\theta + B\sin 3\theta + C\sin 5\theta.$$ **[5]**

(b) **(i)** Find the 4th roots of $-9\mathrm{j}$ in the form $r\mathrm{e}^{\mathrm{j}\theta}$, where $r > 0$ and $0 < \theta < 2\pi$. Illustrate the roots on an Argand diagram. **[6]**

(ii) Let the points representing these roots, taken in order of increasing θ, be P, Q, R, S. The mid-points of the sides of PQRS represent the 4th roots of a complex number w. Find the modulus and argument of w. Mark the point representing w on your Argand diagram. **[5]**

3 **(a)** **(i)** A 3×3 matrix $\mathbf{M}$ has characteristic equation

$$2\lambda^3 + \lambda^2 - 13\lambda + 6 = 0.$$

Show that $\lambda = 2$ is an eigenvalue of $\mathbf{M}$. Find the other eigenvalues. **[4]**

(ii) An eigenvector corresponding to $\lambda = 2$ is $\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$.

Evaluate $\mathbf{M}\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{M}^2\begin{pmatrix} 1 \\ -1 \\ \frac{1}{3} \end{pmatrix}$.

Solve the equation $\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$. **[5]**

(iii) Find constants A, B, C such that

$$\mathbf{M}^4 = A\mathbf{M}^2 + B\mathbf{M} + C\mathbf{I}.$$ **[4]**

(b) A 2×2 matrix $\mathbf{N}$ has eigenvalues -1 and 2, with eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ respectively. Find $\mathbf{N}$. **[6]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Prove, using exponential functions, that

$$\sinh 2x = 2 \sinh x \cosh x.$$

Differentiate this result to obtain a formula for $\cosh 2x$. **[4]**

(ii) Sketch the curve with equation $y = \cosh x - 1$.

The region bounded by this curve, the x-axis, and the line $x = 2$ is rotated through 2π radians about the x-axis. Find, correct to 3 decimal places, the volume generated. (You must show your working; numerical integration by calculator will receive no credit.) **[7]**

(iii) Show that the curve with equation

$$y = \cosh 2x + \sinh x$$

has exactly one stationary point.

Determine, in exact logarithmic form, the x-coordinate of the stationary point. **[7]**

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 In parts **(i)**, **(ii)**, **(iii)** of this question you are required to investigate curves with the equation

$$x^k + y^k = 1$$

for various positive values of k.

(i) Firstly consider cases in which k is a positive even integer.

(A) State the shape of the curve when $k = 2$.

(B) Sketch, on the same axes, the curves for $k = 2$ and $k = 4$.

(C) Describe the shape that the curve tends to as k becomes very large.

(D) State the range of possible values of x and y. **[6]**

(ii) Now consider cases in which k is a positive odd integer.

(A) Explain why x and y may take any value.

(B) State the shape of the curve when $k = 1$.

(C) Sketch the curve for $k = 3$. State the equation of the asymptote of this curve.

(D) Sketch the shape that the curve tends to as k becomes very large. **[6]**

(iii) Now let $k = \frac{1}{2}$.

Sketch the curve, indicating the range of possible values of x and y. **[2]**

(iv) Now consider the modified equation $|x|^k + |y|^k = 1$.

(A) Sketch the curve for $k = \frac{1}{2}$.

(B) Investigate the shape of the curve for $k = \dfrac{1}{n}$ as the positive integer n becomes very large. **[4]**

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RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) **4756**

Further Methods for Advanced Mathematics (FP2)

Candidates answer on the answer booklet.

OCR supplied materials:
- 8 page answer booklet (sent with general stationery)
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Monday 10 January 2011
Morning

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions in Section A and **one** question from Section B.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
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INFORMATION FOR CANDIDATES

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- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Section A (54 marks)

Answer all the questions

1 **(a)** A curve has polar equation $r = 2(\cos\theta + \sin\theta)$ for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{3}{4}\pi$.

(i) Show that a cartesian equation of the curve is $x^2 + y^2 = 2x + 2y$. Hence or otherwise sketch the curve. **[5]**

(ii) Find, by integration, the area of the region bounded by the curve and the lines $\theta = 0$ and $\theta = \frac{1}{2}\pi$. Give your answer in terms of π. **[7]**

(b) **(i)** Given that $f(x) = \arctan\left(\frac{1}{2}x\right)$, find $f'(x)$. **[2]**

(ii) Expand $f'(x)$ in ascending powers of x as far as the term in x^4.

Hence obtain an expression for $f(x)$ in ascending powers of x as far as the term in x^5. **[5]**

2 **(a)** **(i)** Given that $z = \cos\theta + \mathrm{j}\sin\theta$, express $z^n + z^{-n}$ and $z^n - z^{-n}$ in simplified trigonometrical form. **[2]**

(ii) By considering $\left(z + z^{-1}\right)^6$, show that

$$\cos^6\theta = \tfrac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10). \qquad \textbf{[3]}$$

(iii) Obtain an expression for $\cos^6\theta - \sin^6\theta$ in terms of $\cos 2\theta$ and $\cos 6\theta$. **[5]**

(b) The complex number w is $8\mathrm{e}^{\mathrm{j}\pi/3}$. You are given that z_1 is a square root of w and that z_2 is a cube root of w. The points representing z_1 and z_2 in the Argand diagram both lie in the third quadrant.

(i) Find z_1 and z_2 in the form $r\mathrm{e}^{\mathrm{j}\theta}$. Draw an Argand diagram showing w, z_1 and z_2. **[6]**

(ii) Find the product $z_1 z_2$, and determine the quadrant of the Argand diagram in which it lies. **[3]**

3 **(i)** Show that the characteristic equation of the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & -4 & 5 \\ 2 & 3 & -2 \\ -1 & 4 & 1 \end{pmatrix}$$

is $\lambda^3 - 5\lambda^2 + 28\lambda - 66 = 0$. **[4]**

(ii) Show that $\lambda = 3$ is an eigenvalue of $\mathbf{M}$, and determine whether or not $\mathbf{M}$ has any other real eigenvalues. **[4]**

(iii) Find an eigenvector, $\mathbf{v}$, of unit length corresponding to $\lambda = 3$.

State the magnitude of the vector $\mathbf{M}^n\mathbf{v}$, where n is an integer. **[5]**

(iv) Using the Cayley-Hamilton theorem, obtain an equation for $\mathbf{M}^{-1}$ in terms of $\mathbf{M}^2$, $\mathbf{M}$ and $\mathbf{I}$. **[3]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Solve the equation

$$\sinh t + 7\cosh t = 8,$$

expressing your answer in exact logarithmic form. **[6]**

A curve has equation $y = \cosh 2x + 7\sinh 2x$.

(ii) Using part **(i)**, or otherwise, find, in an exact form, the coordinates of the points on the curve at which the gradient is 16.

Show that there is no point on the curve at which the gradient is zero.

Sketch the curve. **[8]**

(iii) Find, in an exact form, the positive value of a for which the area of the region between the curve, the x-axis, the y-axis and the line $x = a$ is $\frac{1}{2}$. **[4]**

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 A curve has parametric equations

$$x = t + a\sin t, \qquad y = 1 - a\cos t,$$

where a is a positive constant.

(i) Draw, on separate diagrams, sketches of the curve for $-2\pi < t < 2\pi$ in the cases $a = 1$, $a = 2$ and $a = 0.5$.

By investigating other cases, state the value(s) of a for which the curve has

(A) loops,

(B) cusps. **[7]**

(ii) Suppose that the point P (x, y) lies on the curve. Show that the point P′ $(-x, y)$ also lies on the curve. What does this indicate about the symmetry of the curve? **[3]**

(iii) Find an expression in terms of a and t for the gradient of the curve. Hence find, in terms of a, the coordinates of the turning points on the curve for $-2\pi < t < 2\pi$ and $a \neq 1$. **[5]**

(iv) In the case $a = \frac{1}{2}\pi$, show that $t = \frac{1}{2}\pi$ and $t = \frac{3}{2}\pi$ give the same point. Find the angle at which the curve crosses itself at this point. **[3]**

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RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) **4756**

Further Methods for Advanced Mathematics (FP2)

Candidates answer on the answer booklet.

OCR supplied materials:
- 8 page answer booklet (sent with general stationery)
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Monday 20 June 2011
Morning

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions in Section A and **one** question from Section B.
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Section A (54 marks)

Answer all the questions

1 **(a)** A curve has polar equation $r = a(1 - \sin\theta)$, where $a > 0$ and $0 \leqslant \theta < 2\pi$.

(i) Sketch the curve. **[2]**

(ii) Find, in an exact form, the area of the region enclosed by the curve. **[7]**

(b) **(i)** Find, in an exact form, the value of the integral $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx$. **[3]**

(ii) Find, in an exact form, the value of the integral $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\left(1+4x^2\right)^{\frac{3}{2}}}\,dx$. **[6]**

2 **(a)** Use de Moivre's theorem to find expressions for $\sin 5\theta$ and $\cos 5\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Hence show that, if $t = \tan\theta$, then

$$\tan 5\theta = \frac{t(t^4 - 10t^2 + 5)}{5t^4 - 10t^2 + 1}.$$ **[6]**

(b) **(i)** Find the 5th roots of $-4\sqrt{2}$ in the form $re^{j\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. **[4]**

These 5th roots are represented in the Argand diagram, in order of increasing θ, by the points A, B, C, D, E.

(ii) Draw the Argand diagram, making clear which point is which. **[2]**

The mid-point of AB is the point P which represents the complex number w.

(iii) Find, in exact form, the modulus and argument of w. **[3]**

(iv) w is an nth root of a real number a, where n is a positive integer. State the least possible value of n and find the corresponding value of a. **[3]**

3 **(i)** Find the value of k for which the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & k \\ 5 & 4 & 6 \\ 3 & 2 & 4 \end{pmatrix}$$

does not have an inverse.

Assuming that k does not take this value, find the inverse of $\mathbf{M}$ in terms of k. **[7]**

(ii) In the case $k = 3$, evaluate

$$\mathbf{M}\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}.$$ **[2]**

(iii) State the significance of what you have found in part **(ii)**. **[2]**

(iv) Find the value of t for which the system of equations

$$\begin{aligned} x - y + 3z &= t \\ 5x + 4y + 6z &= 1 \\ 3x + 2y + 4z &= 0 \end{aligned}$$

has solutions. Find the general solution in this case and describe the solution geometrically. **[7]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Given that $\cosh y = x$, show that $y = \pm\ln\left(x + \sqrt{x^2 - 1}\right)$ and that $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$. **[7]**

(ii) Find $\displaystyle\int_{\frac{4}{5}}^{1} \frac{1}{\sqrt{25x^2 - 16}}\,dx$, expressing your answer in an exact logarithmic form. **[5]**

(iii) Solve the equation

$$5\cosh x - \cosh 2x = 3,$$

giving your answers in an exact logarithmic form. **[6]**

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 In this question, you are required to investigate the curve with equation

$$y = x^m(1-x)^n, \qquad 0 \leqslant x \leqslant 1,$$

for various positive values of m and n.

(i) On separate diagrams, sketch the curve in each of the following cases.

(A) $m = 1, n = 1$,

(B) $m = 2, n = 2$,

(C) $m = 2, n = 4$,

(D) $m = 4, n = 2$. **[4]**

(ii) What feature does the curve have when $m = n$?

What is the effect on the curve of interchanging m and n when $m \neq n$? **[2]**

(iii) Describe how the x-coordinate of the maximum on the curve varies as m and n vary. Use calculus to determine the x-coordinate of the maximum. **[6]**

(iv) Find the condition on m for the gradient to be zero when $x = 0$. State a corresponding result for the gradient to be zero when $x = 1$. **[2]**

(v) Use your calculator to investigate the shape of the curve for large values of m and n. Hence conjecture what happens to the value of the integral $\int_0^1 x^m(1-x)^n \, dx$ as m and n tend to infinity. **[2]**

(vi) Use your calculator to investigate the shape of the curve for small values of m and n. Hence conjecture what happens to the shape of the curve as m and n tend to zero. **[2]**

Friday 13 January 2012 – Morning

A2 GCE MATHEMATICS (MEI)

4756 Further Methods for Advanced Mathematics (FP2)

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4756
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

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INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
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- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

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Section A (54 marks)

Answer all the questions

1 **(a)** A curve has polar equation $r = 1 + \cos\theta$ for $0 \leqslant \theta < 2\pi$.

(i) Sketch the curve. **[2]**

(ii) Find the area of the region enclosed by the curve, giving your answer in exact form. **[6]**

(b) Assuming that x^4 and higher powers may be neglected, write down the Maclaurin series approximations for $\sin x$ and $\cos x$ (where x is in radians).

Hence or otherwise obtain an approximation for $\tan x$ in the form $ax + bx^3$. **[6]**

(c) Find $\displaystyle\int_0^1 \frac{1}{\sqrt{1 - \frac{1}{4}x^2}}\,dx$, giving your answer in exact form. **[4]**

2 **(a)** The infinite series C and S are defined as follows.

$$C = 1 + a\cos\theta + a^2\cos 2\theta + \ldots,$$

$$S = a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \ldots,$$

where a is a real number and $|a| < 1$.

By considering $C + \mathrm{j}S$, show that $C = \dfrac{1 - a\cos\theta}{1 + a^2 - 2a\cos\theta}$ and find a corresponding expression for S. **[8]**

(b) Express the complex number $z = -1 + \mathrm{j}\sqrt{3}$ in the form $r\,\mathrm{e}^{\mathrm{j}\theta}$.

Find the 4th roots of z in the form $r\,\mathrm{e}^{\mathrm{j}\theta}$.

Show z and its 4th roots in an Argand diagram.

Find the product of the 4th roots and mark this as a point on your Argand diagram. **[10]**

3 **(i)** Show that the characteristic equation of the matrix

$$\mathbf{M} = \begin{pmatrix} 3 & -1 & 2 \\ -4 & 3 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

is $\lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$. **[4]**

(ii) Show that $\lambda = 5$ is an eigenvalue of $\mathbf{M}$, and find its other eigenvalues. **[4]**

(iii) Find an eigenvector, $\mathbf{v}$, of unit length corresponding to $\lambda = 5$.

State the magnitudes and directions of the vectors $\mathbf{M}^2\mathbf{v}$ and $\mathbf{M}^{-1}\mathbf{v}$. **[6]**

(iv) Use the Cayley-Hamilton theorem to find the constants a, b, c such that

$$\mathbf{M}^4 = a\mathbf{M}^2 + b\mathbf{M} + c\mathbf{I}.$$ **[4]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Define $\tanh t$ in terms of exponential functions. Sketch the graph of $\tanh t$. **[3]**

(ii) Show that $\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. State the set of values of x for which this equation is valid. **[5]**

(iii) Differentiate the equation $\tanh y = x$ with respect to x and hence show that the derivative of $\operatorname{artanh} x$ is $\frac{1}{1-x^2}$.

Show that this result may also be obtained by differentiating the equation in part **(ii)**. **[5]**

(iv) By considering $\operatorname{artanh} x$ as $1 \times \operatorname{artanh} x$ and using integration by parts, show that

$$\int_0^{\frac{1}{2}} \operatorname{artanh} x \, dx = \frac{1}{4}\ln\frac{27}{16}.$$ **[5]**

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 The points A(–1, 0), B(1, 0) and P(x, y) are such that the product of the distances PA and PB is 1. You are given that the cartesian equation of the locus of P is

$$\left((x+1)^2 + y^2\right)\left((x-1)^2 + y^2\right) = 1.$$

(i) Show that this equation may be written in polar form as

$$r^4 + 2r^2 = 4r^2\cos^2\theta.$$

Show that the polar equation simplifies to

$$r^2 = 2\cos 2\theta.$$ **[4]**

(ii) Give a sketch of the curve, stating the values of θ for which the curve is defined. **[4]**

(iii) The equation in part **(i)** is now to be generalised to

$$r^2 = 2\cos 2\theta + k,$$

where k is a constant.

(A) Give sketches of the curve in the cases $k = 1$, $k = 2$. Describe how these two curves differ at the pole.

(B) Give a sketch of the curve in the case $k = 4$. What happens to the shape of the curve as k tends to infinity? **[7]**

(iv) Sketch the curve for the case $k = –1$.

What happens to the curve as $k \to –2$? **[3]**

THERE ARE NO QUESTIONS WRITTEN ON THIS PAGE.

OCR
RECOGNISING ACHIEVEMENT

Thursday 21 June 2012 – Afternoon

A2 GCE MATHEMATICS (MEI)

4756 Further Methods for Advanced Mathematics (FP2)

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4756
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

4715740612

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

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- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions in Section A and **one** question from Section B.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
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INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

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Turn over

Section A (54 marks)

Answer all the questions

1 **(a)** **(i)** Differentiate the equation $\sin y = x$ with respect to x, and hence show that the derivative of $\arcsin x$ is $\dfrac{1}{\sqrt{1-x^2}}$. **[4]**

(ii) Evaluate the following integrals, giving your answers in exact form.

(A) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{2-x^2}}\,\mathrm{d}x$ **[3]**

(B) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-2x^2}}\,\mathrm{d}x$ **[4]**

(b) A curve has polar equation $r = \tan\theta$, $0 \leqslant \theta < \frac{1}{2}\pi$. The points on the curve have cartesian coordinates (x, y). A sketch of the curve is given in Fig. 1.

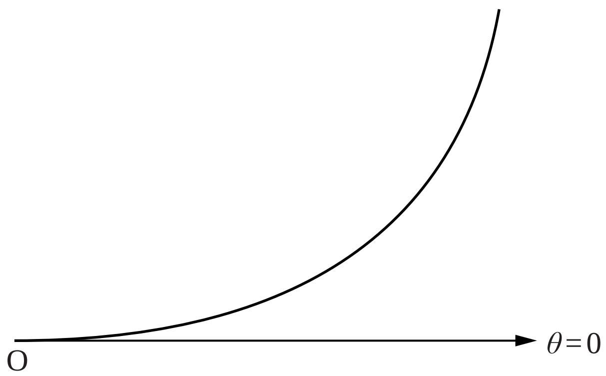

Fig. 1

Show that $x = \sin\theta$ and that $r^2 = \dfrac{x^2}{1-x^2}$.

Hence show that the cartesian equation of the curve is

$$y = \frac{x^2}{\sqrt{1-x^2}}.$$

Give the cartesian equation of the asymptote of the curve. **[7]**

2 **(a)** **(i)** Given that $z = \cos\theta + \mathrm{j}\sin\theta$, express $z^n + \dfrac{1}{z^n}$ and $z^n - \dfrac{1}{z^n}$ in simplified trigonometric form. **[2]**

(ii) Beginning with an expression for $\left(z + \dfrac{1}{z}\right)^4$, find the constants A, B, C in the identity

$$\cos^4\theta \equiv A + B\cos 2\theta + C\cos 4\theta.$$ **[4]**

(iii) Use the identity in part **(ii)** to obtain an expression for $\cos 4\theta$ as a polynomial in $\cos\theta$. **[2]**

(b) **(i)** Given that $z = 4\mathrm{e}^{\mathrm{j}\pi/3}$ and that $w^2 = z$, write down the possible values of w in the form $r\mathrm{e}^{\mathrm{j}\theta}$, where $r > 0$. Show z and the possible values of w in an Argand diagram. **[5]**

(ii) Find the least positive integer n for which z^n is real.

Show that there is no positive integer n for which z^n is imaginary.

For each possible value of w, find the value of w^3 in the form $a + \mathrm{j}b$ where a and b are real. **[5]**

3 **(i)** Find the value of a for which the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & a & 4 \\ 3 & -2 & 2 \end{pmatrix}$$

does not have an inverse.

Assuming that a does not have this value, find the inverse of $\mathbf{M}$ in terms of a. **[7]**

(ii) Hence solve the following system of equations.

$$\begin{aligned} x + 2y + 3z &= 1 \\ -x \qquad\;\; + 4z &= -2 \\ 3x - 2y + 2z &= 1 \end{aligned}$$ **[4]**

(iii) Find the value of b for which the following system of equations has a solution.

$$\begin{aligned} x + 2y + 3z &= 1 \\ -x + 6y + 4z &= -2 \\ 3x - 2y + 2z &= b \end{aligned}$$

Find the general solution in this case and describe the solution geometrically. **[7]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Prove, from definitions involving exponential functions, that

$$\cosh 2u = 2\sinh^2 u + 1. \qquad \textbf{[3]}$$

(ii) Prove that, if $y \geqslant 0$ and $\cosh y = u$, then $y = \ln\left(u + \sqrt{(u^2 - 1)}\right)$. **[4]**

(iii) Using the substitution $2x = \cosh u$, show that

$$\int \sqrt{4x^2 - 1}\,\mathrm{d}x = ax\sqrt{4x^2 - 1} - b\,\mathrm{arcosh}\,2x + c,$$

where a and b are constants to be determined and c is an arbitrary constant. **[7]**

(iv) Find $\int_{\frac{1}{2}}^{1} \sqrt{4x^2 - 1}\,\mathrm{d}x$, expressing your answer in an exact form involving logarithms. **[4]**

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 This question concerns curves with polar equation $r = \sec\theta + a$, where a is a constant.

(i) State the set of values of θ between 0 and 2π for which r is undefined. **[2]**

For the rest of the question you should assume that θ takes all values between 0 and 2π for which r is defined.

(ii) Use your graphical calculator to obtain a sketch of the curve in the case $a = 0$. Confirm the shape of the curve by writing the equation in cartesian form. **[3]**

(iii) Sketch the curve in the case $a = 1$.

Now consider the curve in the case $a = -1$. What do you notice?

By considering both curves for $0 < \theta < \pi$ and $\pi < \theta < 2\pi$ separately, describe the relationship between the cases $a = 1$ and $a = -1$. **[6]**

(iv) What feature does the curve exhibit for values of a greater than 1?

Sketch a typical case. **[3]**

(v) Show that a cartesian equation of the curve $r = \sec\theta + a$ is $(x^2 + y^2)(x - 1)^2 = a^2x^2$. **[4]**

OCR
RECOGNISING ACHIEVEMENT

Monday 14 January 2013 – Morning

A2 GCE MATHEMATICS (MEI)

4756/01 Further Methods for Advanced Mathematics (FP2)

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4756/01
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions in Section A and **one** question from Section B.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

Section A (54 marks)

Answer all the questions

1 **(a)** **(i)** Differentiate with respect to x the equation $a\tan y = x$ (where a is a constant), and hence show that the derivative of $\arctan\frac{x}{a}$ is $\frac{a}{a^2 + x^2}$. **[3]**

(ii) By first expressing $x^2 - 4x + 8$ in completed square form, evaluate the integral $\int_0^4 \frac{1}{x^2 - 4x + 8}\,dx$, giving your answer exactly. **[4]**

(iii) Use integration by parts to find $\int \arctan x \, dx$. **[4]**

(b) **(i)** A curve has polar equation $r = 2\cos\theta$, for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Show, by considering its cartesian equation, that the curve is a circle. State the centre and radius of the circle. **[5]**

(ii) Another circle has radius 2 and its centre, in cartesian coordinates, is (0, 2). Find the polar equation of this circle. **[2]**

2 **(a)** **(i)** Show that

$$1 + e^{j2\theta} = 2\cos\theta(\cos\theta + j\sin\theta).$$ **[2]**

(ii) The series C and S are defined as follows.

$$C = 1 + \binom{n}{1}\cos 2\theta + \binom{n}{2}\cos 4\theta + \ldots + \cos 2n\theta$$

$$S = \binom{n}{1}\sin 2\theta + \binom{n}{2}\sin 4\theta + \ldots + \sin 2n\theta$$

By considering $C + jS$, show that

$$C = 2^n \cos^n\theta \cos n\theta,$$

and find a corresponding expression for S. **[7]**

(b) **(i)** Express $e^{j2\pi/3}$ in the form $x + jy$, where the real numbers x and y should be given exactly. **[1]**

(ii) An equilateral triangle in the Argand diagram has its centre at the origin. One vertex of the triangle is at the point representing $2 + 4j$. Obtain the complex numbers representing the other two vertices, giving your answers in the form $x + jy$, where the real numbers x and y should be given exactly. **[6]**

(iii) Show that the length of a side of the triangle is $2\sqrt{15}$. **[2]**

3 You are given the matrix $\mathbf{M} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

(i) Show that the characteristic equation of **M** is

$$\lambda^3 - 13\lambda + 12 = 0.$$ **[3]**

(ii) Find the eigenvalues and corresponding eigenvectors of **M**. **[12]**

(iii) Write down a matrix **P** and a diagonal matrix **D** such that

$$\mathbf{M}^n = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

(You are not required to calculate $\mathbf{P}^{-1}$.) **[3]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Show that the curve with equation

$$y = 3\sinh x - 2\cosh x$$

has no turning points.

Show that the curve crosses the x-axis at $x = \frac{1}{2}\ln 5$. Show that this is also the point at which the gradient of the curve has a stationary value. **[7]**

(ii) Sketch the curve. **[2]**

(iii) Express $(3\sinh x - 2\cosh x)^2$ in terms of $\sinh 2x$ and $\cosh 2x$.

Hence or otherwise, show that the volume of the solid of revolution formed by rotating the region bounded by the curve and the axes through 360° about the x-axis is

$$\pi(3 - \tfrac{5}{4}\ln 5).$$ **[9]**

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 This question concerns the curves with polar equation

$$r = \sec\theta + a\cos\theta, \qquad (*)$$

where a is a constant which may take any real value, and $0 \leqslant \theta \leqslant 2\pi$.

(i) On a single diagram, sketch the curves for $a = 0$, $a = 1$, $a = 2$. **[3]**

(ii) On a single diagram, sketch the curves for $a = 0$, $a = -1$, $a = -2$. **[2]**

(iii) Identify a feature that the curves for $a = 1$, $a = 2$, $a = -1$, $a = -2$ share. **[1]**

(iv) Name a distinctive feature of the curve for $a = -1$, and a different distinctive feature of the curve for $a = -2$. **[2]**

(v) Show that, in cartesian coordinates, equation (*) may be written

$$y^2 = \frac{ax^2}{x-1} - x^2.$$

Hence comment further on the feature you identified in part **(iii)**. **[5]**

(vi) Show algebraically that, when $a > 0$, the curve exists for $1 < x < 1 + a$.

Find the set of values of x for which the curve exists when $a < 0$. **[5]**

OCR RECOGNISING ACHIEVEMENT

Tuesday 18 June 2013 – Morning

A2 GCE MATHEMATICS (MEI)

4756/01 Further Methods for Advanced Mathematics (FP2)

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4756/01
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

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INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

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Section A (54 marks)

1 **(a)** You are given that $f(x) = \dfrac{1}{(1-2x)^2}$.

Find $f'(x)$, $f''(x)$ and $f'''(x)$. Hence obtain the Maclaurin series for $f(x)$ as far as the term in x^3.

By considering the equivalent binomial expansion, give the set of values of x for which the Maclaurin series is valid. **[7]**

(b) A curve has polar equation $r = a\sin 3\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{3}\pi$.

(i) Sketch the curve. **[2]**

(ii) Find, in terms of a, the cartesian coordinates of the point on the curve furthest from the origin. **[4]**

(iii) Find, in terms of a, the area of the region enclosed by the curve. **[5]**

2 **(a)** **(i)** Use de Moivre's theorem to show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$ **[3]**

(ii) Given that $\cos 5\theta = 0$ but $\cos\theta \neq 0$, find in surd form the two possible values for $\cos^2\theta$.

Hence show that $\cos 18° = \left(\dfrac{5+\sqrt{5}}{8}\right)^{\frac{1}{2}}$.

Find, in similar form, an expression for $\sin 18°$. **[6]**

(b) **(i)** Find the cube roots of the complex number $4(\sqrt{3} + \mathrm{j})$ in the form $r\mathrm{e}^{\mathrm{j}\theta}$, where $r > 0$ and $0 < \theta < 2\pi$. Illustrate the roots on an Argand diagram. **[7]**

The points representing the two roots with smallest values of θ are P and Q. The mid-point of PQ is M, and M represents the complex number w.

(ii) Find the argument of w. Write down the smallest positive integer n for which w^n is a real number. **[2]**

3 You are given the matrix $\mathbf{A} = \begin{pmatrix} k & -7 & 4 \\ 2 & -2 & 3 \\ 1 & -3 & -2 \end{pmatrix}$.

(i) Show that when $k = 5$ the determinant of **A** is zero. Obtain an expression for the inverse of **A** when $k \neq 5$. **[7]**

(ii) Solve the matrix equation

$$\begin{pmatrix} 4 & -7 & 4 \\ 2 & -2 & 3 \\ 1 & -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ 1 \\ 2 \end{pmatrix},$$

giving your answer in terms of p. **[5]**

(iii) Find the value of p for which the matrix equation

$$\begin{pmatrix} 5 & -7 & 4 \\ 2 & -2 & 3 \\ 1 & -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ 1 \\ 2 \end{pmatrix}$$

has a solution. Give the general solution in this case and describe it geometrically. **[6]**

Section B (18 marks)

4 **(i)** Prove, using exponential functions, that $\cosh^2 u - \sinh^2 u = 1$. **[2]**

(ii) Given that $y = \operatorname{arsinh} x$, show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}},$$

and that

$$y = \ln\left(x + \sqrt{1+x^2}\right).$$ **[9]**

(iii) Show that

$$\int_0^2 \frac{1}{\sqrt{4+9x^2}}\,dx = \tfrac{1}{3}\ln\left(3+\sqrt{10}\right).$$ **[4]**

(iv) Find, in exact logarithmic form,

$$\int_0^1 \frac{1}{\sqrt{1+x^2}}\operatorname{arsinh} x\,dx.$$ **[3]**

THERE ARE NO QUESTIONS PRINTED ON THIS PAGE.