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CHAPTER

# 5

# Sequences and series

## Objectives

- To explore **sequences of numbers and their difference equations**
- To use a graphics calculator to generate sequences and display graphs
- To recognise **arithmetic sequences**
- To find the terms, **difference equation and number of terms for an arithmetic sequence**
- To calculate the sum of the terms of an **arithmetic series**
- To recognise **geometric sequences**
- To find the terms, **difference equation and number of terms for a geometric sequence**
- To calculate the sum of the terms of a **geometric series**
- To recognise and calculate the sum of the terms in an infinite **geometric series**
- To use **fixed point iteration** to generate convergent sequences and hence solve equations
- To apply sequences and series to solving problems



## 5.1 Introduction to sequences

The following are examples of sequences of numbers:

**A** 1, 3, 5, 7, 9, . . . . . .    **B** 0.1, 0.11, 0.111, 0.1111, . . . . . .

**C** $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots\ldots$    **D** 10, 7, 4, 1, −2, . . . . . .    **E** 0.6, 1.7, 2.8, 3.9, . . . . . .

Note each sequence is a set of numbers with order being important. For some sequences of numbers a rule can be found for getting from one number to the next. For example:

for sequence **A**, a rule is: add 2

for sequence **C**, a rule is: multiply by $\frac{1}{3}$

for sequence **D**, a rule is: subtract 3

for sequence **E**, a rule is: add 1.1



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The numbers of a sequence are called terms. The $n$th term of a sequence is denoted by the symbol $t_n$. So the first term is $t_1$, the 12th term is $t_{12}$ and so on.

A sequence may be defined by specifying a rule which enables each subsequent term to be found using the previous term. In this case, the rule specified is called an **iterative** rule or a **difference equation**. For example:

sequence **A** may be defined by $t_1 = 1, \; t_n = t_{n-1} + 2$

sequence **C** may be defined by $t_1 = \frac{1}{3}, \; t_n = \frac{1}{3}t_{n-1}$

## Example 1

Use the difference equation to find the first four terms of the sequence

$$t_1 = 3, \; t_n = t_{n-1} + 5$$

### Solution

$$\begin{aligned} t_1 &= 3 \\ t_2 &= t_1 + 5 = 8 \\ t_3 &= t_2 + 5 = 13 \\ t_4 &= t_3 + 5 = 18 \end{aligned}$$

The first four terms are 3, 8, 13, 18.

## Example 2

Find the first four terms of the sequence defined by the rule $t_n = 2n + 3$.

### Solution

$$\begin{aligned} t_1 &= 2(1) + 3 = 5 \\ t_2 &= 2(2) + 3 = 7 \\ t_3 &= 2(3) + 3 = 9 \\ t_4 &= 2(4) + 3 = 11 \end{aligned}$$

The first four terms are 5, 7, 9, 11.

## Example 3

Find the difference equation for the following sequence.

$$9, -3, 1, -\frac{1}{3}, \ldots$$

### Solution

$$-3 = -\frac{1}{3} \times 9 \qquad \text{i.e. } t_2 = -\frac{1}{3}t_1$$

$$1 = -\frac{1}{3} \times -3 \qquad \text{i.e. } t_3 = -\frac{1}{3}t_2$$

$$\therefore \quad t_n = -\frac{1}{3}t_{n-1}, \; t_1 = 9$$



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Alternatively a sequence may be defined by a rule that is stated in terms of $n$. For example:

$t_n = 2n$ defines the sequence $t_1 = 2, t_2 = 4, t_3 = 6, t_4 = 8, \ldots$

$t_n = 2^{n-1}$ defines the sequence $t_1 = 1, t_2 = 2, t_3 = 4, t_4 = 8, \ldots$

## Example 4

Find the rule for the $n$th term for the sequence 1, 4, 9, 16 in terms of $n$.

### Solution

$$\begin{aligned} t_1 &= 1 \\ t_2 &= 4 = 2^2 \\ t_3 &= 9 = 3^2 \\ t_4 &= 16 = 4^2 \\ \therefore t_n &= n^2 \end{aligned}$$

## Example 5

At a particular school, the number of students studying General Mathematics increases each year. If in 2006 there are 40 students studying General Mathematics

**a** set up the difference equation if the number is increasing by five students each year

**b** write down an expression for $t_n$ in terms of $n$ for the difference equation found in **a**

**c** find the number of students expected to be doing General Mathematics at the school in 2011.

### Solution

**a** $t_n = t_{n-1} + 5$

**b** $t_1 = 40,$

$$t_2 = t_1 + 5 = 45 = 40 + 1 \times 5$$
$$t_3 = t_2 + 5 = 50 = 40 + 2 \times 5$$

Therefore $t_n = 40 + (n - 1) \times 5$

$t_n = 35 + 5n$

**c** $n = 6$

$t_6 = 40 + 5 \times 5 = 65$

Sixty-five students will study General Mathematics in 2011.

## Example 6

The height of a sand dune is increasing by 10% each year. It is currently 4 m high.

**a** Set up the difference equation that describes the height of the sand dune.

**b** Write down an expression for $t_n$ in terms of $n$ for the difference equation found in **a**.

**c** Find the height of the sand dune seven years from now.

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**Solution**

**a** $t_n = t_{n-1} \times 1.1$

**b** $t_1 = 4$

$t_2 = 4 \times 1.1 = 4.4$

$t_3 = 4 \times (1.1)^2 = 4.84$

Therefore $\quad t_n = 4 \times (1.1)^{n-1}$

**c** Seven years from now implies $n = 8$

$$t_8 = 4 \times (1.1)^7 \approx 7.795$$

The sand dune will be 7.795 m high in 7 years

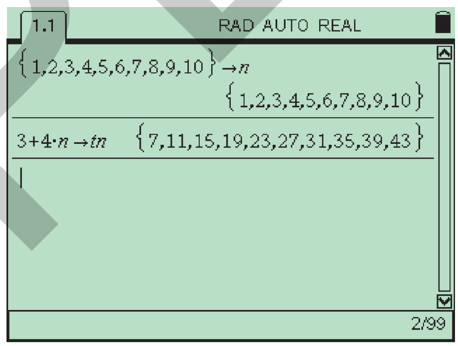
# Using the TI-Nspire

## Sequences defined in terms of *n*

This type of sequence is easiest handled in a **Calculator** application (⌂ 1).

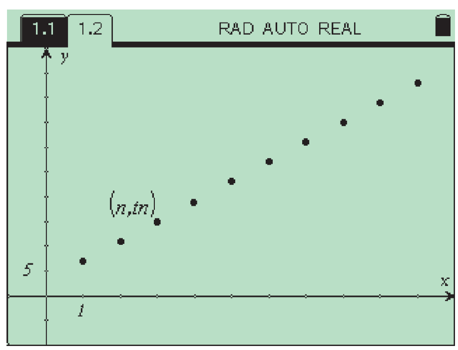
For example, complete as shown to generate the first 10 terms of the sequence of numbers defined by the rule $t_n = 3 + 4n$.

Storing the resulting list will enable the sequence to be graphed.




To graph the sequence, open a **Graphs & Geometry** application (⌂ 2) and graph the sequence as a **Scatter Plot** (menu 3 4), using an appropriate **Window** (menu 4).

Note that it is possible to see the coordinates of the points using **Trace** (menu 5 1).




## Iteratively defined sequences

This type of sequence is easiest handled in a **Lists & Spreadsheet** application (⌂ 3).



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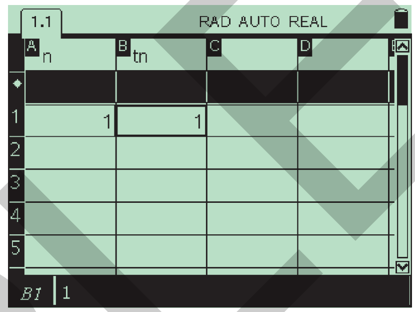
## Example 7

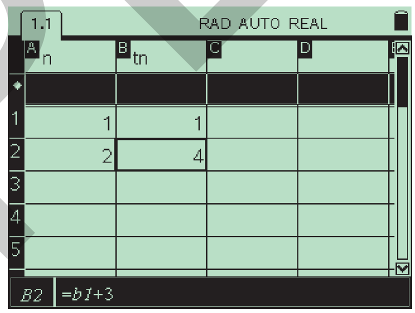
Use a graphics calculator to generate the sequence defined by the difference equation $t_n = t_{n-1} + 3$, $t_1 = 1$ and sketch the graph of the sequence against $n$.

### Solution

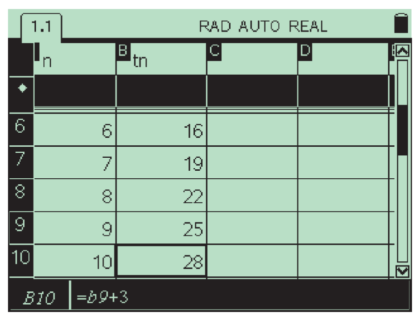
Use the **arrows** (◄ ► ▲ ▼) to name the first two columns *n* and *tn* respectively.
Enter 1 in cell A1 and enter 1 in cell B1.




Enter = a1 + 1 in cell A2 and = b1 + 3 in cell B2.




Highlight the cells A2 and B2 using (caps) and the NavPad, and use **Fill Down** (menu 3 3) to generate the sequence of numbers.




To graph the sequence, open a **Graphs & Geometry** application (home 2) and graph the sequence as a **Scatter Plot** (menu 3 4), using an appropriate **Window** (menu 4).

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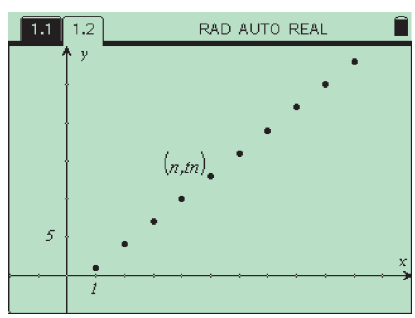
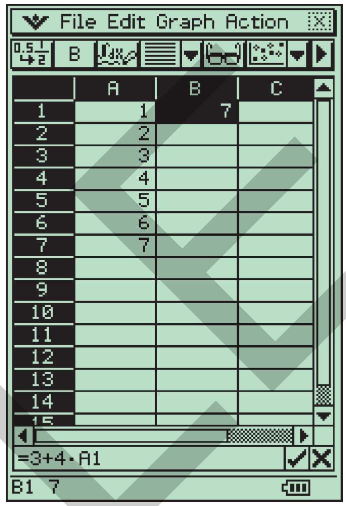


# Using the Casio ClassPad

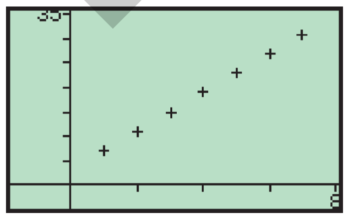
The Casio Classpad spreadsheet is an efficient way to produce and graph the sequence. It is similar to operating in a computer spreadsheet such as Microsoft Excel.

Enter the values for $n$ in column A, then in B1, enter the formula $= 3 + 4{*}A1$.

Highlight cell B2 and the cells below it and tap **Edit, Fill Range** to complete the sequence.

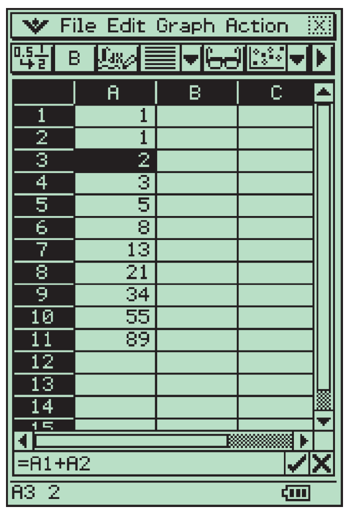



Click the arrow beside [icon], select graph type [icon] and click on this icon to produce the graph.



For an **iteratively defined sequence**, the procedure is similar except the cells in column A are each defined in terms of the cell(s) above.

The entry for the Fibonacci sequence is $A1 = 1$, $A2 = 1$, $A3 = A1 + A2$. Highlight the formula in A3, together with as many cells as required below, and then tap **Edit, Fill Range** to complete the operation.




**Note:** In the screen shown, the formula in cell A3 is displayed in the formula bar at the bottom of the screen.

## Exercise 5A

Example 1

**1** In each of the following an iterative definition for a sequence is given. List the first five terms.

**a** $t_1 = 3, t_n = t_{n-1} + 4$ **b** $t_1 = 5, t_n = 3t_{n-1} + 4$ **c** $t_1 = 1, t_n = 5t_{n-1}$
**d** $t_1 = -1, t_n = t_{n-1} + 2$ **e** $t_{n+1} = 2t_n + t_{n-1}, t_1 = 1, t_2 = 3$

Example 2

**2** Each of the following is a rule for a sequence. In each case find $t_1, t_2, t_3, t_4$.

**a** $t_n = \frac{1}{n}$ **b** $t_n = n^2 + 1$ **c** $t_n = 2n$ **d** $t_n = 2^n$
**e** $t_n = 3n + 2$ **f** $t_n = (-1)^n n^3$ **g** $t_n = 2n + 1$ **h** $t_n = 2 \times 3^{n-1}$

Examples 3, 4

**3** For each of the following sequences

**i** find a possible rule for $t_n$ in terms of $n$
**ii** find the difference equation.

**a** 3, 6, 9, 12 **b** 1, 2, 4, 8 **c** $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$
**d** 3, −6, 12, −24 **e** 4, 7, 10, 13 **f** 4, 9, 14, 19

**4** Consider a sequence for which $t_n = 3n + 1$. Find $t_{n+1}, t_{2n}$.

Example 5

**5** Hamish collects Pokemon cards. He currently has 15 and he adds three to his collection every week.

**a** Set up the difference equation that will generate the number of cards Hamish has in any given week.
**b** Write down an expression for $t_n$ in terms of $n$ for the difference equation found in **a**.
**c** Find the number of cards Hamish should have after another 12 weeks.

Example 6

**6** Isobel can swim 100 m in 94.3 s. She aims to reduce her time by 4% each week.

**a** Set up the difference equation that generates Isobel's time for the 100 m in any given week.
**b** Write down an expression for $t_n$ in terms of $n$ for the difference equation found in **a**.
**c** Find the time in which Isobel expects to be able to complete the 100 m after another 8 weeks.

**7** Stephen is a sheep farmer with a flock of 100 sheep. He wishes to increase the size of his flock by both breeding and buying new stock. He estimates that 80% of his sheep will produce one lamb each year and he intends to buy 20 sheep to add to the flock each year. Assuming no sheep die

**a** write the difference equation for the expected number of sheep at the end of each year (let $t_0 = 100$)
**b** calculate the number of sheep at the end of each of the first five years.

8 Alison invests \$2000 at the beginning of the year. At the beginning of each of the following years, she puts a further \$400 into the account. Compound interest of 6% p.a. is paid on the investment at the end of each year.

- **a** Write down the amount of money in the account at the end of each of the first three years.
- **b** Set up a difference equation to generate the sequence for the investment. (Let $t_1$ be the amount in the investment at the end of the first year.)
- **c** With a calculator or spreadsheet, use the difference equation to find the amount in the account after ten years.

Example 7

9 For each of the following difference equations, use a graphics calculator to find the first six terms of the sequence defined and sketch the graph of these terms against $n$.

- **a** $t_n = t_{n-1} + 3, t_1 = 1$
- **b** $t_n = t_{n-1} - 2, t_1 = 3$
- **c** $t_n = 2t_{n-1}, t_1 = \frac{1}{2}$
- **d** $t_n = \frac{1}{2} t_{n-1}, t_1 = 32$
- **e** $t_n = (t_{n-1})^2, t_1 = 1.1$
- **f** $t_n = \frac{2}{3} t_{n-1}, t_1 = 27$
- **g** $t_n = 2t_{n-1} + 5, t_1 = -1$
- **h** $t_n = 4 - t_{n-1}, t_1 = -3$

10 **a** For a sequence for which $t_n = 2^{n-1}$, find $t_1, t_2, t_3$.

- **b** For a sequence for which $u_n = \frac{1}{2}(n^2 - n) + 1$, find $u_1, u_2, u_3$.
- **c** What do you notice?
- **d** Find $t_4$ and $u_4$.

11 If $S_n = an^2 + bn, a \in R, b \in R$, find $S_1, S_2, S_3$ and $S_{n+1} - S_n$.

12 For the sequence defined by $t_1 = 1, t_{n+1} = \frac{1}{2}\left(t_n + \frac{2}{t_n}\right)$, find $t_2, t_3, t_4$. The terms of the sequence are successive rational approximations of a real number. Can you recognise the number?

13 The Fibonacci sequence is defined by $t_1 = 1, t_2 = 1, t_{n+2} = t_{n+1} + t_n$ $(n \in N)$. Use the rule to find $t_3, t_4, t_5$. Show that $t_{n+2} = 2t_n + t_{n-1}$ $(n \in N\backslash\{1\})$.

## 5.2 Arithmetic sequences

A sequence in which each successive term is found by adding a constant amount to the previous term is called an **arithmetic sequence**. For example, 2, 5, 8, 11. . . . is an arithmetic sequence.

An arithmetic sequence can be defined by a difference equation of the form

$$t_n = t_{n-1} + d, \text{ where } d \text{ is a constant.}$$

If the first term of an arithmetic sequence $t_1 = a$ then the $n$th term of the sequence can also be described by the rule

$$t_n = a + (n-1)d \quad \text{where} \quad a = t_1 \quad \text{and} \quad d = t_n - t_{n-1}$$

where $d$ is the common difference.



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**Example 8**

Find the 10th term of the arithmetic sequence −4, −1, 2, 5 . . .

**Solution**

$$a = -4, d = 3$$

$$\therefore \quad t_n = a + (n-1)d$$

$$t_{10} = -4 + (10-1)3$$

$$t_{10} = 23$$

**Example 9**

A national park has a series of huts along one of its mountain trails. The first hut is 5 km from the start of the trail, the second is 8 km from the start, the third 11 km and so on.

**a** How far from the start of the trail is the sixth hut?

**b** How far is it from the sixth hut to the twelfth hut?

**Solution**

**a** Distances of the huts from the start of the trail form an arithmetic sequence with $a = 5$ and $d = 3$.

For the sixth hut $t_6 = a + 5d$

$$t_6 = 5 + 5 \times 3 = 20$$

The sixth hut is 20 km from the start of the trail

**b** For the twelfth hut $t_{12} = a + 11d$

$$t_{12} = 5 + 11 \times 3 = 38$$

$\therefore$ distance from sixth to the twelfth hut $= t_{12} - t_6 = 38 - 20 = 18$ km

The twelfth hut is 18 km from the sixth hut.

**Example 10**

The 12th term of an arithmetic sequence is 9 and the 25th term is 100. Find $a$ and $d$.

**Solution**

Since $\quad t_n = a + (n-1)d$

$$9 = a + 11d \quad \ldots \quad \boxed{1}$$

$$100 = a + 24d \quad \ldots \quad \boxed{2}$$

Subtract $\boxed{1}$ from $\boxed{2}$

$$91 = 13d$$

$$\therefore \quad d = 7$$

From $\boxed{1}$ we have

$$9 = a + 11(7)$$
$$\therefore \quad a = -68$$

## Arithmetic mean

The **arithmetic mean** of two numbers $a$ and $b$ is defined as $\dfrac{a+b}{2}$.

If the numbers $a$, $c$ and $b$ are consecutive terms of an arithmetic sequence, then

$$c - a = b - c$$
$$\therefore \quad 2c = a + b$$
$$\therefore \quad c = \frac{a+b}{2}$$

i.e. $c$ is the arithmetic mean of $a$ and $b$.

## Exercise 5B

Example 8

**1** For the arithmetic sequence where $t_n = a + (n-1)d$, find the first four terms given that

**a** $a = 0, d = 2$ **b** $a = -3, d = 5$ **c** $a = d = -\sqrt{5}$ **d** $a = 11, d = -2$

**2** Find $a$ and $d$ and hence find the rule of the arithmetic sequence whose first few terms are

**a** 3, 7, 11 **b** $3, -1, -5$ **c** $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{11}{2}$ **d** $5-\sqrt{5}, 5, 5+\sqrt{5}$

**3** **a** An arithmetic sequence has a first term of 5 and a common difference of $-3$. Find the thirteenth term.

**b** An arithmetic sequence has a first term of $-12$ and a common difference of 4. Find the tenth term.

**c** In an arithmetic sequence $a = 25$ and $d = -2.5$. Find the ninth term.

**d** In an arithmetic sequence $a = 2\sqrt{3}$ and $d = \sqrt{3}$. Find the fifth term.

Example 9

**4** David goes fishing every day for 10 days. On the first day he catches four fish and each day after that he catches two more than the previous day.

**a** How many fish did David catch on the sixth day?

**b** How many fish did he catch on the 10th day?

**c** On which day did he catch 10 fish?

**5** An amphitheatre has 25 seats in row A, 28 seats on row B, 31 seats in row C and so on.

**a** How many seats in row P?

**b** How many seats are there in row X?

**c** Which row has 40 seats in it?

**6** In each of the following, $t_n$ is the $n$th term of an arithmetic sequence.

**a** Find $t_5$ if $t_1 = 6, t_2 = 10$. **b** Find $t_{12}$ if $t_1 = 5, t_2 = 2$.

**c** Find $n$ if $t_1 = 16, t_2 = 13$ and $t_n = -41$.

**d** Find $n$ if $t_1 = 7, t_2 = 11$ and $t_n = 227$.

Example 10

**7** For an arithmetic sequence the first term is 7 and the thirtieth term is $108\frac{1}{2}$. Find the common difference.

**8** The number of people who go to see a movie over a period of a week follows an arithmetic sequence. If on the first day only three people go to the movie but on the sixth day 98 go, find the rule for the sequence and hence determine how many attend on the seventh day.

**9** For an arithmetic sequence, $t_3 = 18$ and $t_6 = 486$, find the rule for the sequence, i.e. find $t_n$.

**10** The number of laps a swimmer swims each week follows an arithmetic sequence. If in the fifth week she swims 24 laps and in the tenth week she swims 39 laps, how many laps did she swim in the fifteenth week?

**11** In an arithmetic sequence, $t_7 = 0.6$ and $t_{12} = -0.4$. Find $t_{20}$.

**12** An arithmetic sequence contains 10 terms. If the first is 4 and the tenth is 30, what are the other eight terms?

**13** The number of goals kicked by a team in the first six games of a season follows an arithmetic sequence. If the team kicked 5 goals in the first game and 15 in the sixth, how many did they score in each of the other four games?

**14** The first term of an arithmetic sequence is $a$. The $m$th term is zero. Find the rule for $t_n$ for the sequence.

**15** For an arithmetic sequence, find $t_6$ if $t_{15} = 3 + 9\sqrt{3}$ and $t_{20} = 38 - \sqrt{3}$.

**16** Find the arithmetic mean of

**a** 8 and 15

**b** $\dfrac{1}{2\sqrt{2}-1}$ and $\dfrac{1}{2\sqrt{2}+1}$

**17** Find $x$ if $3x - 2$ is the arithmetic mean of $5x + 1$ and 11.

**18** If $a$, $4a - 4$ and $8a - 13$ are successive terms of an arithmetic sequence, find $a$.

**19** If $t_x = y$ and $t_y = x$, prove that $t_{x+y}, = 0$ ($t_x$ and $t_y$ are the $x$th and $y$th terms of an arithmetic sequence).

**20** If $a$, $2a$ and $a^2$ are consecutive terms of an arithmetic sequence, find $a (a \neq 0)$.

## 5.3 Arithmetic series

The sum of the terms in a sequence is called a **series**. If the sequence in question is arithmetic, the series is called an **arithmetic series**. The symbol $S_n$ is used to denote the sum of $n$ terms of a sequence.

i.e. $$S_n = a + a + d + a + 2d + \cdots + a + (n-1)d$$

If this sum is written in reverse order, then

$$S_n = a + (n - l)d + a + (n - 2)d + \cdots + a + d + a$$

Adding these two expression together gives

$$2S_n = n[2a + (n - 1)d]$$

$$\therefore \quad S_n = \frac{n}{2}[2a + (n - 1)d]$$

and since the last term $l = t_n = a + (n - 1)d$

$$S_n = \frac{n}{2}(a + l)$$

**Example 11**

A hardware store sells nails in a range of packets containing different numbers of nails. Packet A contains 50 nails, packet B has 75 nails, packet C has 100 and so on.

**a** Find the number of nails in packet J.

**b** Lachlan buys one of each of packets A to J. How many nails in total does Lachlan have?

**c** Assuming he buys one of each packet starting at A, how many packets does he need to buy to have a total of 1100 nails?

**Solution**

**a** $a = 50, d = 25,$

$$t_n = a + (n - 1)d$$

For packet J, $t_{10} = 50 + 9 \times 25$
$= 275$

Packet J contains 275 nails

**b** $a = 50, d = 25$

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

$$S_{10} = \frac{10}{2}(2(50) + 11 \times 25)$$

$$S_{10} = 1625$$

Packets A to J contain a total of 1625 nails

**c** $a = 50, d = 25, S_n = 1100$

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

$$S_n = \frac{n}{2}(2(50) + (n - 1)(25)) = 1100$$

$$n(100 + 25n - 25) = 2200$$

$$25n^2 + 75n - 2200 = 0$$

$$n^2 + 3n - 88 = 0$$

$$(n + 11)(n - 8) = 0$$

$$n = -11 \text{ or } n = 8$$

$$\text{since } n > 0, n = 8$$

If Lachlan buys one of each of the first eight packets (A to H) he will have exactly 1100 nails.

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## Example 12

For the arithmetic sequence 3, 6, 9, 12, . . . , calculate

**a** the sum of the first 25 terms

**b** the number of terms in the series if $S_n = 1395$.

### Solution

**a** $a = 3, d = 3, n = 25$

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ &= \frac{25}{2}[2(3) + (24)(3)] \\ &= 975 \end{aligned}$$

**b** $a = 3, d = 3, S_n = 1395$

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ 1395 &= \frac{n}{2}[2(3) + (n-1)(3)] \\ 2790 &= n[6 + 3n - 3] \\ 2790 &= 3n + 3n^2 \\ 3n^2 + 3n - 2790 &= 0 \\ n^2 + n - 930 &= 0 \\ (n-30)(n+31) &= 0 \\ \therefore \quad n &= 30 \text{ since } n > 0 \end{aligned}$$

$\therefore$ there are 30 terms in the series.

## Example 13

For the arithmetic sequence 27, 23, 19, 15, . . . , −33, find

**a** the number of terms **b** the sum of the terms.

### Solution

**a** $a = 27, d = -4, l = t_n = -33$

$$\begin{aligned} t_n &= a + (n-1)d \\ -33 &= 27 + (n-1)(-4) \\ -60 &= (n-1)(-4) \\ 15 &= n - 1 \\ n &= 16 \end{aligned}$$

There are 16 terms in the series.

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**b** $a = 27, l = t_n = -33, n = 16$

$$S_n = \frac{n}{2}(a + l)$$

$$S_{16} = \frac{16}{2}(27 - 33)$$

$$S_{16} = -48$$

The sum of the terms is $-48$.

## Example 14

The sum of the first 10 terms of an arithmetic sequence is $48\frac{3}{4}$. If the fourth term is $3\frac{3}{4}$, find the first term and the common difference.

### Solution

$$t_4 = a + 3d = 3\frac{3}{4}$$

$$\therefore \quad a + 3d = \frac{15}{4} \qquad \ldots \boxed{1}$$

$$S_{10} = \frac{10}{2}(2a + 9d) = 48\frac{3}{4}$$

$$\therefore \quad 10a + 45d = \frac{195}{4} \qquad \ldots \boxed{2}$$

$$\boxed{1} \times 40 \quad 40a + 120d = 150$$

$$\boxed{2} \times 4 \quad 40a + 180d = 195$$

$$\therefore \quad 60d = 45$$

$$\therefore \quad d = \frac{3}{4}$$

Substitute in $\boxed{1}$ $a + 3\left(\frac{3}{4}\right) = \frac{15}{4}$

$$a = \frac{6}{4}$$

The first term is $1\frac{1}{2}$ and the common difference is $\frac{3}{4}$.

## Exercise 5C

1 For the arithmetic sequences

**a** 8, 13, 18, . . . , find $S_{12}$

**b** $-3.5, -1.5, 0.5, \ldots$, find $S_{10}$

**c** $\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{3}{\sqrt{2}}, \ldots$, find $S_{15}$

**d** $-4, 1, 6, \ldots$, find $S_8$

Example 11

2 Greg goes fishing every day for a week. On the first day he catches seven fish and each day he catches three more than the previous day. How many fish did he catch in total?

3 There are 110 logs to be put in a pile, with 15 logs in the bottom layer, 14 in the next, 13 in the next and so on. How many layers will there be?

**4** Find the sum of the first 16 multiples of 5.

**5** Find the sum of all the even numbers between 1 and 99.

**6** Dora's walking club plans 15 walks for the summer. The first walk is a distance of 6 km and the last walk is distance of 27 km and the distances of each of the walks form an arithmetic sequence,

**a** How far is the eighth walk?
**b** How far does the club plan to walk in the first five walks?

Dora goes away on holiday and misses the 9th, 10th and 11th walks but completes all other walks.

**c** How far does Dora walk in total?

**7** Liz has to proofread 500 pages of a new novel. She plans to do 30 pages on the first day and increase the number by five each day.

**a** How many days will it take her to complete the proofreading?

She has only five days to complete the task. She therefore decides to read 50 pages on the first day and increase the number she reads by a constant amount each day.

**b** By how many should she increase the number she reads each day if she is to meet her deadline?

**8** For the sequence 4, 8, 12, . . . , find $\{n : S_n = 180\}$.

Example 12 **9** The sum of $m$ terms of an arithmetic sequence with first term $-5$ and common difference 4 is 660. Find $m$.

**10** An assembly hall has 50 seats in row A, 54 seats in row B, 58 seats in row C, i.e. there are four more seats in each row.

**a** How many seats in row J?
**b** How many seats are there altogether if the back row is row Z?

If on a particular day the front four rows are reserved for parents (and there is no other seating for parents)

**c** how many parents can be seated
**d** how many students can be seated?

The hall is extended by adding more rows following the same pattern.

**e** If the final capacity of the hall is 3410, how many rows were added?

**11** A new golf club is formed with 40 members in its first year. Each following year the number of new members exceeds the number of retirements by 15. Each member pays $120 p.a. in membership fees. Calculate the amount received from fees in the first 12 years of the club's existence.

Example 14 **12** In an arithmetic sequence, $t_2 = -12$ and $S_{12} = 18$. Find $a$, $d$, $t_6$ and $S_6$.

**13** The sum of the first ten terms of an arithmetic sequence is 120 and the sum of the first twenty terms is 840. Find the sum of the first thirty terms.

Example 13 **14** Evaluate $54 + 48 + 42 + \cdots + -54$.

**15** If $t_6 = 16$ and $t_{12} = 28$, find $S_{14}$.

**16** For an arithmetic sequence, find $t_n$ if:

**a** $t_3 = 6.5,\ S_8 = 67$ **b** $t_4 = \dfrac{6}{\sqrt{5}},\ S_5 = 16\sqrt{5}$

**17** For the sequence with $t_n = bn\ (b \in R)$, find

**a** $t_{n+1} - t_n$ **b** $t_1 + t_2 + \cdots + t_n$

**18** For a sequence where $t_n = 15 - 5n$, find $t_5$ and the sum of the first 25 terms.

**19** An arithmetic sequence has a common difference of $d$. If the sum of 20 terms is 25 times the first term, find, in terms of $d$, the sum of 30 terms.

**20** The sum of the first $n$ terms of a particular sequence is given by $S_n = 17n - 3n^2$.

**a** Find an expression for the sum to $(n - 1)$ terms.
**b** Find an expression for the $n$th term of the sequence.
**c** Show that the corresponding sequence is arithmetic and find $a$ and $d$.

**21** Three consecutive terms of an arithmetic sequence have a sum of 36 and a product of 1428. Find the three terms.

**22** Show that the sum of the first $2n$ terms of an arithmetic sequence is $n$ times the sum of the two middle terms.

## 5.4 Geometric sequences

A sequence in which each successive term is found by multiplying the previous term by a fixed amount is called a **geometric sequence**. For example, 2, 6, 18, 54, . . . is a geometric sequence. A geometric sequence can be defined by an iterative equation of the form

$$t_n = rt_{n-1}, \text{ where } r \text{ is constant}$$

If the first term of a geometric sequence $t_1 = a$ then the $n$th term of the sequence can also be described by the rule

$$t_n = ar^{n-1}, \text{ where } r = \frac{t_n}{t_{n-1}}$$
and $r$ is the common ratio.

**Example 15**

Calculate the tenth term of the sequence 2, 6, 18, . . .

**Solution**

$$a = 2, r = 3$$
$$t_n = ar^{n-1}$$
$$t_{10} = 2 \times 3^{(10-1)}$$
$$= 39\,366$$

**Example 16**

Georgina draws a pattern consisting of a number of similar equilateral triangles. The first triangle has sides of length 4 cm and the side length of each successive triangle is one and a half times the side length of the previous one.

**a** How long is the side length of the fifth triangle?

**b** Which triangle has a side length of $45\frac{9}{16}$ cm?

**Solution**

**a** $a = 4, r = \frac{3}{2}$

$t_n = ar^{n-1}$

$t_5 = ar^4 = 4 \times \left(\frac{3}{2}\right)^4 = 20\frac{1}{4}$

The fifth triangle has a side length of $20\frac{1}{4}$ cm

**b** $a = 4, r = \frac{3}{2}\ t_n = 45\frac{9}{16}$

$t_n = ar^{n-1} = 45\frac{9}{16}$

which implies $4 \times \left(\frac{3}{2}\right)^{n-1} = 45\frac{9}{16}$

$= \frac{729}{16}$

Hence $\left(\frac{3}{2}\right)^{n-1} = \frac{729}{64}$

Recognising that $729 = 3^6$ and $64 = 2^6$

yields $\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^6$

Therefore $n - 1 = 6$

and $n = 7$

The seventh triangle will have a side length of $45\frac{9}{16}$ cm

An application of geometric sequences is **compound interest**. Compound interest is interest calculated at regular intervals on the total of the amount originally invested and the amount accumulated in the previous years.

So \$1000 invested at 10% per annum would grow to

$$1000 + 10\%(1000) = \$1100 \text{ at the end of the first year.}$$

At the end of the second year, it will have grown to

$$(1000 + 10\%(1000)) + 10\%(1000 + 10\%(1000)) = \$1210$$

The value of the investment at the end of each year forms a geometric sequence. In the above example

$$a = 1000, r = 1.1; \text{ i.e. } r = 100\% + 10\%$$

### Example 17

Hamish invests \$2500 at 7% p.a. compounded annually. Find

**a** the value of his investment after 5 years

**b** how long it takes until his investment is worth \$10 000.

**Solution**

$$a = 2500, r = 1.07$$

**a** $t_6 = ar^5$ — $t_n$ is the end of the $(n-1)$th year.

$$= 2500(1.07)^5$$
$$= 3506.38$$

The value of his investment after 5 years is \$3506.38.

**b**

$$t_n = ar^{n-1} = 10\,000$$
$$2500(1.07)^{n-1} = 10\,000$$
$$(1.07)^{n-1} = 4$$
$$\log_{10}(1.07)^{n-1} = \log_{10} 4$$
$$(n-1)\log_{10}(1.07) = \log_{10} 4$$
$$n - 1 = \frac{\log_{10} 4}{\log_{10}(1.07)}$$
$$n = 21.489$$

By the end of the 21st year, his investment will be worth in excess of \$10 000.

**Note:** The number of years can also be found by trial and error or through using the table facility of a graphics calculator.

### Example 18

The third term of a geometric sequence is 10 and the sixth term is 80. Find $r$ and the first term.

**Solution**

$$t_3 = ar^2 = 10 \quad \ldots \quad \boxed{1}$$
$$t_6 = ar^5 = 80 \quad \ldots \quad \boxed{2}$$

Divide $\boxed{2}$ by $\boxed{1}$

$$\frac{ar^5}{ar^2} = \frac{80}{10}$$
$$\therefore \quad r^3 = 8$$
$$\therefore \quad r = 2$$

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Substitute in $\boxed{1}$ to find $a$.

$$a \times 4 = 10$$
$$\therefore \quad a = \frac{5}{2}$$

The first term is $\frac{5}{2}$

## Geometric mean

The **geometric mean** of two numbers $a$ and $b$ is $\sqrt{ab}$.

Note that if three numbers $a$, $c$, $b$ are consecutive members of a geometric sequence

$$\frac{c}{a} = \frac{b}{c} \text{ and } c = \sqrt{ab}$$

## Exercise 5D

1 For a geometric sequence $t_n = ar^{n-1}$, find the first four terms given that

a $a = 3, r = 2$ b $a = 3, r = -2$ c $a = 10\,000, r = 0.1$ d $a = r = 3$

Example 15

2 Find the specified term in each of the following geometric sequences.

a $\frac{15}{7}, \frac{5}{7}, \frac{5}{21}, \ldots$ find $t_6$ b $1, -\frac{1}{4}, \frac{1}{16}, \ldots$ find $t_5$

c $\sqrt{2}, 2, 2\sqrt{2}, \ldots$ find $t_{10}$ d $a^x, a^{x+1}, a^{x+2}, \ldots$ find $t_6$

3 Find the rule for the geometric sequence whose first few terms are

a $3, 2, \frac{4}{3}$ b $2, -4, 8, -16$ c $2, 2\sqrt{5}, 10$

4 For a geometric sequence the first term is 25 and the fifth term is $\frac{16}{25}$. Find the common ratio.

5 A geometric sequence has first term $\frac{1}{4}$ and common ratio 2. Which term of the sequence is 64?

6 If $t_n$ is the $n$th term of the following geometric sequences, find $n$ in each case.

a $2, 6, 18, \ldots$ $t_n = 486$ b $5, 10, 20, \ldots$ $t_n = 1280$

c $768, 384, 192, \ldots$ $t_n = 3$ d $\frac{8}{9}, \frac{4}{3}, 2, \ldots$ $t_n = \frac{27}{4}$

e $-\frac{4}{3}, \frac{2}{3}, -\frac{1}{3}, \ldots$ $t_n = \frac{1}{96}$

Example 16

7 An art collector has a painting that is increasing in value by 8% each year. If the painting is currently valued at \$2500

a how much will it be worth in 10 years

b how many years before its value exceeds \$100 000?

**8** An algal bloom is growing in a lake. The area it covers triples each day.

**a** If it initially covers an area of 10 $m^2$, how many square metres will it cover after 1 week?

**b** If the lake has a total area of 200 000 $m^2$, how long before the entire lake is covered?

**9** A ball is dropped from a height of 2 m and continues to bounce so that it rebounds to $\frac{3}{4}$ of the height from which it previously falls. Find the height it rises to on the fifth bounce.

**10** The Tour de Moravia is a cycling event which lasts for 15 days. On the first day the cyclists must ride 120 km and each successive day they ride 90% of the distance of the previous day.

**a** How far do they ride on the eighth day? **b** On which day do they ride 30.5 km?

**11** A child negotiates a new pocket money deal with her unsuspecting father in which she receives 1 cent on the first day of the month, 2 cents on the second, 4 cents on the third, 8 cents on the fourth and so on . . . until the end of the month. How much would the child receive on the 30th day of the month? (Give your answer to the nearest thousand dollars.)

**12** The number of fish in the breeding tanks of a fish farm follow a geometric sequence. The third tank contains 96 fish and the sixth tank contains 768.

**a** How many fish are in the first tank?

**b** How many fish are in the 10th tank?

Example 18 **13** The 12th term of a geometric sequence is 2 and the fifteenth term is 54. Find the seventh term.

**14** A geometric sequence has $t_2 = \frac{1}{2\sqrt{2}}$ and $t_4 = \sqrt{2}$. Find $t_8$.

**15** The first three terms of a geometric sequence are 4, 8, 16. Find the first term which exceeds 2000.

**16** The first three terms of a geometric sequence are 3, 9, 27. Find the first term in the sequence which exceeds 500.

Example 17 **17** \$5000 is invested at 6% p.a. compounded annually.

**a** Find the value of the investment after 6 years.

**b** Find how long it will take for the original investment to double in value.

**18** How much would need to be invested at 8.5% p.a. compounded annually to yield a return of \$8000 after 12 years?

**19** What annual compound interest rate would be required to triple the value of an investment of \$200 in 10 years?

**20** The number of ‘type A’ apple bugs present in an orchard is estimated to be 40 960 and is reducing in number by 50% each week. At the same time it is estimated that there are

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40 'type B' apple bugs whose number is doubling each week. After how many weeks will there be the same number of each type of bug?

**21** Find the geometric means of

**a** 5 and 720 **b** 1 and 6.25 **c** $\frac{1}{\sqrt{3}}$ and $\sqrt{3}$ **d** $x^2y^3$ and $x^6y^{11}$

**22** The fourth, seventh and sixteenth terms of an arithmetic sequence also form consecutive terms of a geometric sequence. Find the common ratio of the geometric sequence.

## 5.5 Geometric series

The sum of the terms in a geometric sequence is called a **geometric series**. An expression for $S_n$, the sum of $n$ terms of a geometric sequence, can be found using a similar method to that used in the development of a formula for an arithmetic series.

Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad \ldots \boxed{1}$

Then $rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad \ldots \boxed{2}$

Subtract $\boxed{1}$ from $\boxed{2}$

$$rS_n, -S_n = ar^n - a$$

$\therefore$ $S_n(r-1) = a(r^n - 1)$

and
$$S_n = \frac{a(r^n-1)}{r-1}$$

For values of $r$ such that $-1 < r < 1$, it is often more convenient to use the alternative formula

$$S_n = \frac{a(1-r^n)}{1-r}$$

which is obtained by subtracting $\boxed{2}$ from $\boxed{1}$ above.

**Example 19**

Find the sum of the first nine terms of the sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

**Solution**

$$a = \frac{1}{3}, r = \frac{1}{3}, n = 9$$

$$\therefore \quad S_9 = \frac{\frac{1}{3}\left(\left(\frac{1}{3}\right)^9 - 1\right)}{\frac{1}{3} - 1}$$

$$= \frac{-1}{2}\left(\left(\frac{1}{3}\right)^9 - 1\right)$$

$$\approx \frac{1}{2}(0.999949)$$

$$\approx 0.499975$$



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## Example 20

For the geometric sequence 1, 3, 9, . . . , find how many terms must be added together to obtain a sum of 1093.

### Solution

$$a = 1, r = 3, S_n = 1093$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{1(3^n - 1)}{3 - 1} = 1093$$

$$\therefore \quad 3^n - 1 = 1093 \times 2$$

$$\therefore \quad 3^n = 2187$$

Taking logarithms of both sides gives

$$\log_{10} 3^n = \log_{10} 2187$$

$$n \log_{10} 3 = \log_{10} 2187$$

$$n = \frac{\log_{10} 2187}{\log_{10} 3}$$

$$n = 7$$

Seven terms are required to give a sum of 1093.

Trial and error or using the table facility of a graphics calculator will also give the required result.

## Example 21

In the 15-day Tour de Moravia the cyclists must ride 120 km and each successive day they ride 90% of the distance of the previous day.

**a** How far do they ride in total to the nearest km?

**b** After how many days will they have ridden half that distance?

### Solution

**a**

$$a = 120, r = 0.9$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{15} = \frac{120(1 - (0.9)^{15})}{1 - 0.9}$$

$$= 952.93$$

$$\approx 953 \text{ km}$$

**b**

$$a = 120, r = 0.9, S_n = 476.5 \text{ km}$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{120(1 - (0.9)^n)}{1 - 0.9} = 476.5$$

$$\therefore \quad 1 - (0.9)^n = \frac{476.5 \times 0.1}{120} = 0.3971$$
$$\therefore \quad (0.9)^n = 1 - 0.3971$$
$$\therefore \quad (0.9)^n = 0.6029$$

Taking logarithms of both sides gives

$$\log_{10}(0.9)^n = \log_{10}(0.6029)$$
$$n \log_{10}(0.9) = \log_{10}(0.6029)$$
$$n = \frac{\log_{10}(0.6029)}{\log_{10}(0.9)}$$
$$n = 4.8023$$

$\therefore$ on the fifth day they pass the halfway mark.

## Exercise 5E

Example 19

**1** Find the sum specified for each of the following geometric series.

**a** $5 + 10 + 20 + \cdots$, find $S_{10}$ **b** $1 - 3 + 9 - \cdots$, find $S_6$
**c** $-\frac{4}{3} + \frac{2}{3} - \frac{1}{3} + \cdots$, find $S_9$

**2** Find

**a** $2 - 6 + 18 - \cdots + 1458$ **b** $-4 + 8 - 16 + \cdots - 1024$
**c** $6250 + 1250 + 250 + \cdots + 2$

**3** Gerry owns a milking cow. On the first day he milks the cow, it produces 600 mL of milk. On each successive day, the amount of milk increases by 10%.

**a** How much milk does the cow produce on the seventh day?
**b** How much milk does it produce in the first week?

Example 21

**4** An insurance salesman makes \$15 000 commission on sales in his first year. Each year, he increases his sales by 5%.

**a** How much commission would he make in his fifth year?
**b** How much commission would he make in total over 5 years?

**5** On Monday, William spends 20 minutes playing the piano. On Tuesday, he spends 25 minutes playing and on each successive day he increases the time he spends playing in the same ratio.

**a** For how many hours does he play on Friday?
**b** How many hours in total does he play from Monday to Friday?
**c** On which day of the following week will his total time played pass 15 hours?

**6** A ball dropped from a height of 15 m rebounds from the ground to a height of 10 m. With each successive rebound, it rises two-thirds of the height of the previous rebound. What total distance will it have travelled when it strikes the ground for the 10th time?



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**7** Andrew invests \$1000 at 20% simple interest for 10 years. Bianca invests her \$1000 at 12.5% compound interest for 10 years. At the end of 10 years, whose investment is worth more?

**8** For the geometric sequence with $n$th term $t_n$

**a** $t_3 = 20$, $t_6 = 160$, find $S_5$ **b** $t_3 = \sqrt{2}$, $t_8 = 8$, find $S_8$

Example 20

**9** **a** How many terms of the geometric sequence where $t_1 = 1, t_2 = 2, t_3 = 4, \ldots$ must be taken for $S_n = 255$?

**b** Let $S_n = 1 + 2 + 4 + \cdots + 2^{n-1}$. Find $\{n: S_n > 1000\,000\}$.

**10** Find $1 - x^2 + x^4 - x^6 + \cdots + x^{2m}$ ($m$ is even).

## 5.6 Infinite geometric series

If the common ratio of a geometric sequence has a magnitude less than 1, i.e. $-1 < r < 1$, then each successive term of the sequence is closer to zero.

e.g. $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$

When the terms of the sequence are added, the corresponding series

$a + ar + ar^2 + \cdots + ar^{n-1}$ will approach a limiting value,

i.e. as $n \to \infty$, $S_n \to$ a limiting value.

Such a series is called **convergent**.

In Example 19 from the previous section, it was found that for the sequence

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots,$ the sum of the first nine terms, $S_9$, was 0.499975

For the same sequence, $S_{20} = 0.4999999999 \approx 0.5$

Given that $S_n = \dfrac{a(1 - r^n)}{1 - r}$

$\Rightarrow S_n = \dfrac{a}{1 - r} - \dfrac{ar^n}{1 - r}$

as $n \to \infty$, $r^n \to 0$ and hence $\dfrac{ar^n}{1 - r} \to 0$

It follows then that the limit as $n \to \infty$ of $S_n$ is $\dfrac{a}{1 - r}$

So $$S_\infty = \frac{a}{1 - r}$$

This is also referred to as 'the sum to infinity' of the series.

## Example 22

Find the sum to infinity of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

**Solution**

$$r = \frac{1}{2}, a = 1$$

$$\therefore \quad S_\infty = \frac{1}{1 - \frac{1}{2}} = 2$$

## Example 23

A square has a side length of 40 cm. A copy of the square is made so that the area of the copy is 80% of the original. The process is repeated each time with the area of the new square being 80% of the previous one. If this process continues indefinitely, find the total area of all the squares.

**Solution**

Area of first square is $40^2 = 1600$ cm$^2$

$$a = 1600, r = 0.8$$

$$S_\infty = \frac{a}{1 - r}$$

$$\therefore \quad S_\infty = \frac{1600}{1 - 0.8} = 8000 \text{ cm}^2$$



## Example 24

Express the recurring decimal $0.\dot{3}\dot{2}$ as a ratio of two integers.

**Solution**

$$0.\dot{3}\dot{2} = 0.32 + 0.0032 + 0.000032 + \cdots$$

$\therefore \quad a = 0.32, r = 0.01$

and $\quad S_\infty = \frac{0.32}{0.99} = \frac{32}{99}$

i.e. $\quad 0.\dot{3}\dot{2} = \frac{32}{99}$

Skillsheet TEACHER

## Exercise 5F

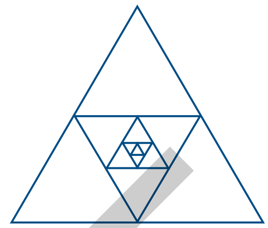
Example 22

**1** Find

**a** $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$ **b** $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$

Example 23

**2** An equilateral triangle has perimeter $p$ cm. The midpoints of the sides are joined to form another triangle, and this process is repeated. Find the perimeter and area of the $n$th triangle, and find the limits as $n \to \infty$ of the sums of perimeters and areas of the first $n$ triangles.



**3** A rocket is launched into the air so that it reaches a height of 200 m in the first second. Each subsequent second it gains 6% less height. Find how high the rocket will climb.

**4** A patient has an infection that, if it exceeds a certain level, will kill him. He is given a drug that will inhibit the spread of the infection. The drug acts in such a way that the level of infection only increases by 65% of the previous day's level. On the first day, the level of infection is measured at 450.
The critical level of infection is 1280. Will the infection kill him?

**5** A man can walk 3 km in the first hour of a journey, but in each succeeding hour walks half the distance covered in the preceding hour. Can he complete a journey of 6 km? Where does this problem cease to be realistic?

**6** A frog standing 10 m from the edge of a pond sets out to jump towards it. Its first jump is 2 m, its second jump is $1\frac{1}{2}$ m, its third jump is $1\frac{1}{8}$ m and so on. Show that the frog will never reach the edge of the pond.

**7** A computer-generated virus acts in such a way that initially it blocks out a third of the area of the screen of an infected computer. On each successive day, it blocks out a further $\frac{1}{3}$ of the area it blocked the previous day. If the virus continues to act unchecked indefinitely, what percentage of the user's screen will eventually be blocked out?

**8** A stone is thrown so that it skips across the surface of a lake. If each skip is 30% less that the previous skip, how long should the first skip be if the total distance travelled by the stone is 40 m?

**9** A ball dropped from a height of 15 m rebounds from the ground to a height of 10 m. With each successive rebound, it rises two-thirds of the height of the previous rebound. If it continues to bounce indefinitely, what is the total distance it will travel?

Example 24

**10** Express each of the following periodic decimals as the ratio of a pair of integers.

**a** $0.\dot{4}$ **b** $0.0\dot{3}$ **c** $10.\dot{3}$ **d** $0.0\dot{3}\dot{5}$ **e** $0.\dot{9}$ **f** $4.\dot{1}$

**11** The sum of the first four terms of a geometric series is 30 and the sum to infinity is 32. Find the first two terms.

**12** Find the third term of a geometric sequence that has a common ratio of $-\frac{1}{4}$ and a sum to infinity of 8.

**13** Find the common ratio of a geometric sequence that has a first term of 5 and a sum to infinity of 15.

## 5.7 Fixed point iteration

The solution(s) to equations may be found using a numerical method that involves generating a sequence of numbers. This method is particularly useful when solving equations for which using an analytic method may be problematic or impossible.

The solution to an equation of the form $f(x) = 0$ may be found by first finding an approximation to the solution and then using this first approximation to generate a better approximation, which is in turn used to produce an even better approximation and so on.

This process of using a previous value to generate the next value is called **iteration**. If the sequence of numbers produced using the iterative process converges to a limit, this limit will be the solution of the equation in question.

Consider the equation $f(x) = 0$. Begin by rewriting the equation in the form $x = g(x)$. If $x_1$ is the initial approximation for the solution to the equation, the second approximation is found by evaluating

$$x_2 = g(x_1).$$

This value is then used to generate a third approximation

$$x_3 = g(x_2), \text{ and so on.}$$

If the sequence is convergent, each successive term will be closer to the actual solution of the original equation i.e., the sequence of numbers generated by the equation $x_n = g(x_{n-1})$ converges to a value which is the solution to the equation $x = g(x)$ (which is of course the solution to the equation $f(x) = 0$).

The iterative process is continued until the value of $x_n$ is equal to $x_{n-1}$ to a pre-determined level of accuracy such as five decimal places. This type of iteration is called **fixed point iteration**.

### Example 25

Write down the first five terms generated by the iterative equation $x_n = \dfrac{x_{n-1}}{5} + 1$, $x_1 = 2$ and hence state if the sequence produced appears to be convergent.

#### Solution

$$x_1 = 2$$
$$x_2 = \frac{2}{5} + 1 = 1.4$$
$$x_3 = \frac{1.4}{5} + 1 = 1.28$$
$$x_4 = \frac{1.28}{5} + 1 = 1.256$$
$$x_5 = \frac{1.256}{5} + 1 = 1.2512$$
$$x_6 = \frac{1.2512}{5} + 1 = 1.25024$$

The sequence generated appears to be convergent.

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**Example 26**

Given that the solution to the equation $f(x) = 0$ where $f(x) = \sqrt{x} - x + 4$ is approximately 6.5, use fixed point iteration to find the solution to the equation correct to five decimal places.

**Solution**

First rearrange the equation into the form $x = g(x)$.

**Note:** There may be a number of ways to re-arrange the original equation producing different functions $g(x)$. The implications of these different forms will be discussed later.

$$\sqrt{x} - x + 4 = 0$$

Therefore $x = \sqrt{x} + 4$ i.e., $g(x) = \sqrt{x} + 4$

Using $x_1 = 6.5$

Therefore $x_2 = \sqrt{x_1} + 4$

$= \sqrt{6.5} + 4$

$= 6.549509757$

$x_3 = \sqrt{x_2} + 4$

$= \sqrt{6.549509757} + 4$

$= 6.559201$

$x_4 = \sqrt{x_3} + 4$

$= \sqrt{6.559201} + 4$

$= 6.561093712$

$x_5 = 6.561463197$

$x_6 = 6.56153532$ ($= 6.56154$ to five decimal places)

$x_7 = 6.561549398$ ($= 6.56155$ to five decimal places)

$x_8 = 6.561552146$ ($= 6.56155$ to five decimal places)

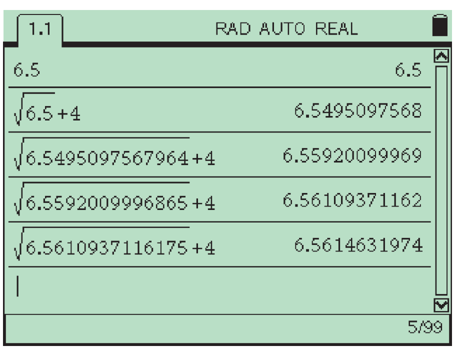
Since the values of $x_8 = x_7$ to the required level of accuracy the iteration process is terminated.

Hence the solution to the equation $\sqrt{x} - x + 4 = 0$ is 6.56155 correct to five decimal places.

## Using the TI-Nspire

A solution using this technique can be found efficiently using the TI-Nspire calculator.

Using the above example, start by entering the initial approximation $x_1 = 6.5$ followed by **enter** (enter). Then enter $\sqrt{\text{ans}} + 4$ and repeatedly press **enter**. The successive terms of the sequence will be generated until the required level of accuracy is achieved.



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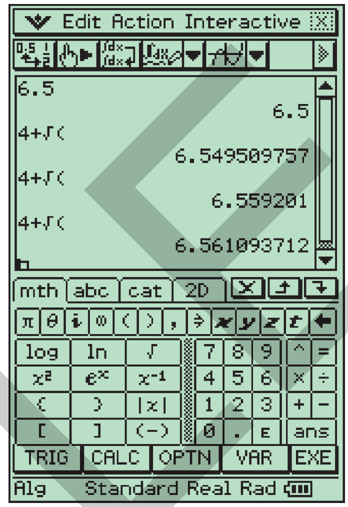
## Using the Casio ClassPad

A solution using this technique can be found efficiently using the CAS calculator. Enter the initial approximation 6.5 in the first entry line.

Then, in the next line, enter $4 + \wp$ as shown. Repeatedly tapping EXE will produce successive terms in the sequence until the desired accuracy is achieved.

**Note:** It is useful to change the display mode from **Standard** to **Decimal** to return answers in the appropriate form.




In the above example a decision was made to rearrange the original equation $\sqrt{x} - x + 4 = 0$ into an equation of the form $x = \sqrt{x} + 4$ so that $g(x) = \sqrt{x} + 4$. An alternative rearrangement could have been used.

Again consider the equation $\sqrt{x} - x + 4 = 0$

i.e. $\sqrt{x} = x - 4$

$x = (x - 4)^2$ i.e., in this case $g(x) = (x - 4)^2$

Again using $x_1 = 6.5$ a solution can be sought using the iterative process

If $x_1 = 6.5$

Therefore $x_2 = (x_1 - 4)^2$
$= (6.5 - 4)^2$
$= 6.25$

$x_3 = (x_2 - 4)^2 = (6.25 - 4)^2 = 5.0625$

$x_4 = (x_3 - 4)^2 = (5.0625 - 4)^2 = 1.12890625$

$x_5 = 8.243179321 \quad x_6 = 18.00457075 \quad x_7 = 196.128002 \quad x_8 = 36913.16914$

It is clear that the sequence of numbers generated by the iterative equation $x_n = (x_{n-1} - 4)^2$ is not convergent and the solution to the equation $\sqrt{x} - x + 4 = 0$ cannot be found using the iterative process with this particular rearrangement.

This method of finding solutions to equations is not universally applicable. It is, however, possible to establish whether the sequence to be generated will converge, therefore producing a solution, by considering the graph of $y = g(x)$.

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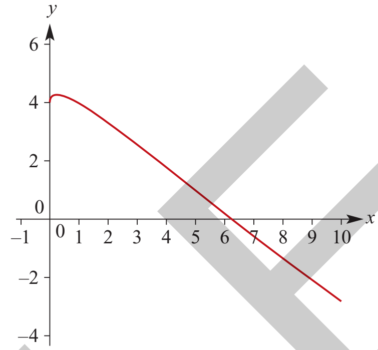
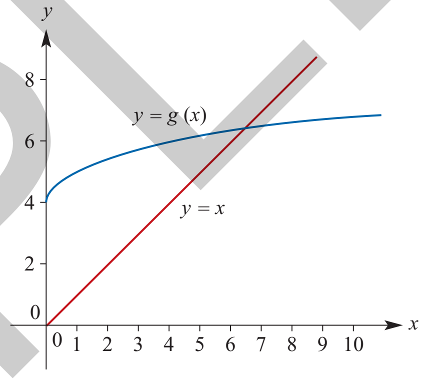
In solving the equation $\sqrt{x} - x + 4 = 0$, two different rearrangements were used to produce different functions.

First consider the graph of $f(x) = \sqrt{x} - x + 4$.

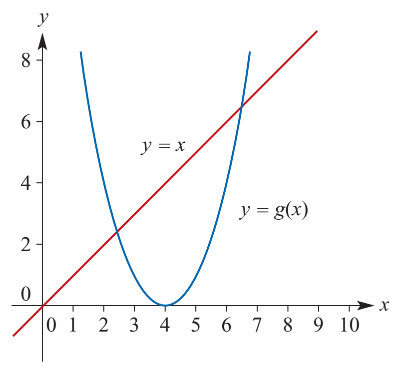
It appears that a solution to the equation $f(x) = 0$ occurs between $x = 6$ and $x = 7$.




Now consider the graphs of $y = x$ and $y = g(x)$, where $g(x) = \sqrt{x} + 4$.




Also consider the graphs of $y = x$ and $y = g(x)$, where $g(x) = (x - 4)^2$.




In both of these pairs of graphs it is clear that a solution to the equation $x = g(x)$ occurs between $x = 6$ and $x = 7$. What is of interest in both cases is the gradient of the graph of $y = g(x)$ in the vicinity of the actual solution.

In the first of the two, the gradient of $y = g(x)$ in the vicinity of $x = 6.5$ is quite small (less than that of the line $y = x$); however in the second the gradient of $y = g(x)$ in the vicinity of $x = 6.5$ is quite large (greater than that of the line $y = x$).

It is the gradient of the function of $y = g(x)$ in the vicinity of the solution which will determine if the iterative process is to be successful.

(If students have studied differential calculus a more rigorous examination of the gradient of $y = g(x)$ may be done, however an informal recognition of the significance of the gradient of $y = g(x)$ is sufficient for students to appreciate that the iterative process will not always succeed.)



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# Exercise 5G

Example 25 **1** **a** Write down the first six terms generated by the following iterative equations

**i** $x_n = \dfrac{x_{n-1}}{4} + 2, x_1 = 3$ **ii** $x_n = x_{n-1}^2 - 3, x_1 = 1$

**iii** $x_n = 3x_{n-1}^2 + 1, x_1 = 2$ **iv** $x_n = \sqrt{x_{n-1} + 2} + 1, x_1 = 3$

**b** Which of the sequences produced in **a** are convergent?

Example 26 **2** Use fixed point iteration to find a solution to the equations $f(x) = 0$. In each question, the initial approximation $x_1$ is given. **Note:** It may be necessary to try more than one re-arrangement of $f(x) = 0$ before a solution can be found successfully.

**a** $f(x) = x^3 + 4x - 3, x_1 = 1$ **b** $f(x) = x^3 + x - 1, x_1 = 1$

**c** $f(x) = \dfrac{x^2}{3} - x - 1, x_1 = -1$ **d** $f(x) = x^4 - x - 2, x_1 = 1$

**e** $f(x) = 2^x - 4x, x_1 = 0.5$ **f** $f(x) = -x + \log_{10} x + 2, x_1 = 5$

**g** $f(x) = 4x \times 2^x - 3, x_1 = 1$ **h** $f(x) = x^3 - 3x + 1, x_1 = 0.5$

MC TEST D&D TEST

## Chapter summary

- **Sequences**
  The $n$th term of a sequence is denoted using the symbol $t_n$.
  A **difference equation** enables each subsequent term to be found using the previous term. A rule specified in this way is said to be defined iteratively.

  e.g. $t_1 = 1, t_n = t_{n-1} + 2$

- A sequence may be defined by a rule that is stated in terms of $n$.

  e.g. $t_n = 2n$

- An **arithmetic sequence** is a sequence where

  $$t_n = a + (n-1)d \text{ with } d = t_n - t_{n-1}$$

  where $a$ is the first term and $d$ is called the common difference.
- The arithmetic mean of two numbers $a$ and $b$ is defined as $\frac{a+b}{2}$
- The sum of the terms in an arithmetic sequence is called an **arithmetic series**.
- The sum to $n$ terms of an arithmetic sequence,

  $$S_n = \frac{n}{2}[2a + (n-1)d]$$
  $$= \frac{n}{2}[a + l] \text{ where } l = \text{the last term } (l = t_n = a + (n-1)d)$$

- A **geometric sequence** is a sequence where

  $$t_n = ar^{n-1} \text{ with } r = \frac{t_n}{t_{n-1}}$$

  $a$ is the first term and $r$ is called the **common ratio**.
- The sum of the terms in a geometric sequence is called a **geometric series**.
- The sum of $n$ terms of a geometric sequence is

  $$S_n = \frac{a(r^n - 1)}{(r-1)} \quad r \neq 1$$
  $$\text{or } S_n = \frac{a(1 - r^n)}{(1-r)}$$

- If $-1 < r < 1$, the sequence is **convergent** and $S_n$ approaches a limiting value.
- The sum to an infinite number of terms (sum to infinity) is denoted by $S_\infty$ and $S_\infty = \frac{a}{1-r}$.
- Fixed point iteration can be used to find the solution(s) to equations of the form $f(x) = 0$ by finding the sequence of numbers generated by the equation $x_n = g(x_{n-1})$, as long as the sequence is convergent. The equation $x_n = g(x_{n-1})$ is found by an appropriate rearrangement of the equation $f(x) = 0$.

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## Multiple-choice questions

**1** The first three terms of the sequence defined by the rule $t_n = 3n + 2$ are
**A** 1, 2, 3 **B** 2, 4, 6 **C** 5, 7, 9 **D** 5, 8, 11 **E** 5, 8, 10

**2** If $t_1 = 3$, $t_{n+1} = t_n + 3$, then $t_4$ is
**A** 4 **B** 12 **C** 9 **D** 15 **E** 14

**3** For the arithmetic sequence 10, 8, 6 . . . $t_{10} =$
**A** $-8$ **B** $-10$ **C** $-12$ **D** 10 **E** 8

**4** For the arithmetic sequence 10, 8, 6 . . . $S_{10} =$
**A** 10 **B** 0 **C** $-10$ **D** 20 **E** $-20$

**5** If 58 is the $n$th term of the arithmetic sequence 8, 13, 18 . . . then $n =$
**A** 12 **B** 11 **C** 10 **D** 5 **E** 3

**6** The sixth term of the geometric sequence 12, 8, $\frac{16}{3}$, . . . is . . .
**A** $\frac{16}{3}$ **B** $\frac{128}{27}$ **C** $\frac{64}{81}$ **D** $\frac{128}{81}$ **E** $\frac{256}{81}$

**7** For the sequence 8, 4, 2, . . . $S_6 =$
**A** $\frac{1}{4}$ **B** $15\frac{1}{2}$ **C** $15\frac{7}{8}$ **D** 15 **E** $15\frac{3}{4}$

**8** For the sequence 8, 4, 2, . . . $S_\infty =$
**A** $\frac{1}{2}$ **B** 0 **C** 16 **D** 4 **E** $\infty$

**9** \$2000 is invested at 5.5% p.a. compounded annually. The value of the investment after 6 years is
**A** \$13 766.10 **B** \$11 162.18 **C** \$2550 **D** \$2613.92 **E** \$2757.69

**10** If $S_\infty = 37.5$ and $r = \frac{1}{3}$, then $a$ equals
**A** $\frac{2}{3}$ **B** 12.5 **C** $16\frac{2}{3}$ **D** 25 **E** 56.25

## Short-answer questions (technology-free)

**1** Find the first six terms of the following sequences
**a** $t_1 = 3, t_n = t_{n-1} - 4$ **b** $t_1 = 5, t_n = 2t_{n-1} + 2$

**2** Find the first six terms of the following sequences
**a** $t_n = 2n$ **b** $t_n = -3n + 2$

**3** Nick invests \$5000 at 5% p.a. compound interest at the beginning of the year. At the beginning of each of the following years he puts a further \$500 into the account.
**a** Write down the amount of money in the account at the end of each of the first two years.
**b** Set up a difference equation to generate the sequence for the investment.

**4** The fourth term of an arithmetic sequence is 19 and the seventh term is 43. Find the 20th term.

**5** In an arithmetic sequence, $t_5 = 0.35$ and $t_9 = 0.15$. Find $t_{14}$.

6 An arithmetic sequence has $t_6 = -24$ and $t_{14} = 6$. Find $S_{10}$.

7 For the arithmetic sequence $-5, 2, 9, \ldots$, find $\{n: S_n = 402\}$.

8 The sixth term of a geometric sequence is 9 and the tenth is 729. Find the fourth term.

9 One thousand dollars is invested at 3.5% p.a. compounded annually. Find the value of the investment after $n$ years.

10 The first term of a geometric sequence is 9 and the third term is 4. Find the possible values for the second and fourth terms.

11 The sum of three consecutive terms of a geometric sequence is 24 and the sum of the next three terms is also 24. Find the sum of the first 12 terms.

12 Find the sum of the first eight terms of a geometric sequence with first term 6 and common ratio $-3$.

13 Find the sum to infinity of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$

14 The numbers $x, x + 4, 2x + 2$ are three successive terms of a geometric sequence. Find the value of $x$.

## Extended-response questions

1 A firm offering a do-it-yourself picture frame kit makes the kit in various sizes. Size 1 contains 0.8 m of 'moulding', size 2 contains 1.5 m, size 3 contains 2.2 m, . . . and so on.
- a Form the sequence of lengths of mouldings.
- b Is the sequence of lengths of moulding an arithmetic sequence?
- c Find the length of moulding contained in the largest kit, size 12.

2 A firm proposes to sell coated seeds in packs containing the following number of seeds: 50, 75, 100, 125,
- a Is this an arithmetic sequence?
- b Find a formula for the $n$th term.
- c Find the number of seeds in the 25th size packet.

3 A number of telegraph poles are to be placed in a straight line between two towns, A and B, which are 32 km apart. The first is placed 5 km from town A, the last is placed 3 km from town B. The poles are placed so that the intervals starting from town A and finishing at town B are $5, 5 - d, 5 - 2d, 5 - 3d, \ldots, 5 - 6d, 3$. There are seven poles. How far is the fifth pole from town A and how far is it from town B?

4 A new, electronic desk-top telephone exchange, for use in large organisations, is available in various sizes.

Size 1 can handle 20 internal lines
Size 2 can handle 36 internal lines
Size 3 can handle 52 internal lines
Size 4 can handle 68 internal lines, and so on . . .
Size $n$ can handle $T_n$ internal lines

*(cont'd)*

**a** Continue the sequence up to $T_8$.

**b** Write down a formula for $T_n$ in terms of $n$.

**c** A customer said he needed an exchange to handle 196 lines. Is there a version of the desk-top exchange which will just do this? If so, which size is it? If not, which is the next largest size?

**5** A firm making nylon thread made it in the following deniers (thicknesses):

$$2, 9, 16, 23, 30, \ldots \text{etc.}$$

**a** Find the denier number, $D_n$, of the firm's $n$th thread in order of increasing thickness.

A request came in for some very heavy 191 denier thread, but this turned out to be one stage beyond the thickest thread made by the firm.

**b** How many different thicknesses did the firm make?

**6** A new house appears to be slipping down a hillside. The first year it slipped 4 mm, the second year 16 mm, the third year 28 mm. If it goes on like this, how far will it slip during the 40th year?

**7** Anna sends 16 Christmas cards the first year, 24 the second year, 32 the next year and so on. How many Christmas cards will she have sent altogether after ten years if she keeps increasing the number sent each year in the same way?

**8** Each time Lee rinses her hair after washing it, the result is to remove a quantity of shampoo from the hair. With each rinsing the quantity of shampoo removed is a tenth of the previous rinse.

**a** If Lee rinses out 90 mg of shampoo with the first rinse, how much will she have washed out altogether after six rinses?

**b** How much shampoo do you think was present in her hair at the beginning?

**9** A prisoner is trapped in an underground cell which is inundated with a sudden rush of water which comes up to a depth of 1 m, a third of the height of the ceiling (3 m). After an hour a second inundation occurs, but this time the water level rises by only $\frac{1}{3}$ m. After a second hour another inundation of water raises the level by $\frac{1}{9}$ m. If this process continues for 6 hours, write down

**a** the amount the water level will rise at the end of the sixth hour,

**b** the total height of the water level then.

If this process continues, do you think the prisoner, who cannot swim, will drown? Why?

**10** After an undetected leak in a storage tank, the staff at an experimental station were subjected to 500 curie hours of radiation the first day, 400 curie hours the second day, 320 the third day and so on.

Find the number of curie hours they were subjected to

**a** on the 14th day  **b** during the first 5 days of the leak.

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**11** A rubber ball is dropped from a height of 81 m. Each time it strikes the ground, it rebounds two-thirds of the distance through which it has fallen.

**a** Find the height the ball reaches after the sixth bounce.

**b** Assuming the ball continues to bounce indefinitely, find the total distance travelled by the ball.

**12** In payment for loyal service to the king, a wise peasant asked to be given one grain of rice for the first square of a chessboard, two grains for the second square, four for the third square and so on for all 64 squares of the board. The king thought this seemed fair and readily agreed, but was horrified when the court mathematician informed him of how many grains of rice he would have to pay the peasant. How many grains of rice did the king have to pay? (Leave your answer in index form.)

**13** **a** In its first month of operation a cement factory, $A$, produces 4000 tonnes of cement. In each successive month, production rises by 250 tonnes per month. This growth in production is illustrated for the first five months in the table shown.

| Month number ($n$) | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Amount of cement produced (tonnes) | 4000 | 4250 | 4500 | 4750 | 5000 |

**i** Find an expression in terms of ($n$) for the amount of cement produced in the $n$th month.

**ii** Find an expression in terms of $n$ for the total amount of cement produced in the first $n$ months.

**iii** In which month is the amount of cement produced 9250 tonnes?

**iv** In month $m$ the amount of cement produced is $T$ tonnes. Find $m$ in terms of $T$

**v** The total amount of cement produced in the first $p$ months is 522 750. Find the value of $p$.

**b** A second factory, $B$, commences production at exactly the same time as the first. In its first month of production it produces 3000 tonnes of cement. In each successive month, production increases by 8%.

**i** Find an expression for the total amount of cement produced by this factory after $n$ months.

**ii** Let $Q_A$ be the total amount of cement produced by factory $A$ in the first $n$ months and $Q_B$ be the total amount of cement produced by factory $B$ in the first $n$ months. Find an expression in terms of $n$ for $Q_B - Q_A$ and find the smallest value of $n$ for which $Q_B - Q_A \geq 0$.

**14** By using fixed point iteration to solve the equation $x^2 - 8 = 0$, find the value of $\sqrt{8}$ correct to five decimal places. Hint: Add $x^2$ to both sides of the equation and then re-arrange to produce an iterative equation of the form $x_n = g(x_{n-1})$.



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