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Paper II(A)-MT 212(A): Discrete Mathematics

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Preface

This text book is an initiative by the Board of Studies-Mathematics(BOS), University of Pune. The book is written in accordance with the new prescribed syllabus of S.Y.B.Sc./S.Y.B.A. Mathematics(2013 Pattern), for the Paper II(A)-MT 212(A): Discrete Mathematics/MG2(First Term). It is based on few chapters of the book "Discrete Mathematics and its Applications" by Kenneth H. Rosen. The syllabus deals with two important topics in Mathematics, Logic and Counting(Combinatorics).

This books consists of a detail introduction at the beginning of each chapter, several illustrative examples and problems for practice with hints and solutions given at the end of each section. A proper understanding of all the illustrative examples would go a long way in making the subject fully comprehensible. The authors have given a suggested list of practical problems to cover different types of problems.

The book is available online at: math.unipune.ac.in

In case of any query/suggestions you may send an email to:
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Syllabus

Paper II(A) MT 212(A):Discrete Mathematics

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Chapter 1

Propositional Logic

Introduction

Logic is the basis of all mathematical reasoning. It has practical applications in areas of computer science as well as to many other fields of study. In mathematics, we must understand what makes up a correct mathematical argument, that is, a proof. Once we prove that a mathematical statement is true, we call it a theorem. A collection of theorems on a topic organize what we know about this topic. To learn a mathematical topic, a person needs to actively construct mathematical arguments on this topic. Moreover, knowing the proof of a theorem often makes it possible to modify the result to fit new situations. Everyone knows that proofs are important throughout mathematics. The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. In this chapter, we will explain what makes up a correct mathematical argument and introduce tools to construct these arguments. These basic tools will help us to develop different proof methods that will enable us to prove many different types of results in the later chapters.

1.1 Basic Concepts in Logic

Our discussion begins with an introduction to the basic building blocks of logic viz., propositions.

Definition 1.1. *A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.*

All the following declarative sentences are propositions.
1. New Delhi, is the capital of India.
2. $2 + 1 = 3$.
3. $2 + 1 = 2$.
Here propositions 1 and 2 are true, whereas 3 is false.
Some sentences that are not propositions are:
1. How are you?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.
Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of the sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

We use letters to denote propositional variables (or statement variables), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are $p, q, r, s, \ldots$. The truth value of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.

Definition 1.2. *The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**.*

p	$\neg p$
T	F
F	T

Table 1.1: Negation

Definition 1.3. ***Compound propositions*** *are new propositions formed from existing propositions using logical operators.*

Definition 1.4. *Let p be a proposition. The* ***negation*** *of p, denoted by $\neg p$, is the statement "It is not the case that p." The proposition $\neg p$ is read "not p." The truth value of the negation of $p, \neg p$, is the opposite of the truth value of p.*

The negation operator constructs a new proposition from a single existing proposition. We will now introduce the logical operators that are used to form new propositions from two or more existing propositions. These logical operators are also called connectives.

Definition 1.5. *Let p and q be propositions. The* ***conjunction*** *of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.*

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 1.2: $p \wedge q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 1.3: $p \vee q$

Table 1.2 displays the truth table of $p \wedge q$. This table has a row for each of the four possible combinations of truth values of p and q. The four rows correspond to the pairs of truth values TT, TF, FT, and FF, where the first truth value in the pair is the truth value of p and the second truth value is the truth value of q. Note that in logic the word "but" sometimes is used instead of "and" in a conjunction. For example, the statement "The sun is shining, but it is raining" is another way of saying "The sun is shining and it is raining."

Definition 1.6. *Let p and q be propositions. The* ***disjunction*** *of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.*

Table 1.3 displays the truth table for $p \vee q$. The use of the connective "or" in a disjunction corresponds to one of the two ways the word "or" is used in English, namely, in an inclusive way. Thus, a disjunction is true when at least one of the two propositions in it is true. Sometimes, we use "or" in an exclusive sense. When the "exclusive or" is used to connect the propositions p and q, the proposition "p or q (but not both)" is obtained.

Definition 1.7. *Let p and q be propositions. The* ***exclusive or*** *of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.*

The truth table for the exclusive or of two propositions is displayed in Table 1.4.

Definition 1.8. *Let p and q be propositions. The* ***conditional statement*** *$p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise.*

In the conditional statement $p \to q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence). The statement $p \to q$ is called a conditional statement because $p \to q$ asserts

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 1.4: Exclusive or

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.5: Conditional Statement

that q is true on the condition that p holds. A conditional statement is also called an implication. The truth table for the conditional statement $p \to q$ is shown in Table 1.5. Note that the statement $p \to q$ is true when both p and q are true and when p is false (no matter what truth value q has). Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \to q$. A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract. For example, a pledge many politicians make when running for office is "If I am elected, then I will lower taxes." It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the campaign pledge. This scenario corresponds to the case when p is true but q is false in $p \to q$.

You will encounter most if not all of the following ways to express this conditional statement:

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"q unless $\neg p$"	"q follows from p"
"a necessary condition for p is q".	

Example 1.1. Let p be the statement "Nilesh learns discrete mathematics"

and q the statement "Nilesh will find a good job." Express the statement $p \to q$ as a statement in English.

Solution: From the definition of conditional statements, we see that when p is the statement "Nilesh learns discrete mathematics" and q is the statement "Nilesh will find a good job," $p \to q$ represents the statement "If Nilesh learns discrete mathematics, then he will find a good job."

There are many other ways to express this conditional statement in English. Among the most natural of these are:

"Nilesh will find a good job when he learns discrete mathematics."

"For Nilesh to get a good job, it is sufficient for him to learn discrete mathematics." and

"Nilesh will find a good job unless he does not learn discrete mathematics." and so on.

1.1.1 Converse, Contrapositive, and Inverse

We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names.

Definition 1.9. *The proposition $q \to p$ is called the* ***converse*** *of $p \to q$. The* ***contrapositive*** *of $p \to q$ is the proposition $\neg q \to \neg p$. The proposition $\neg p \to \neg q$ is called the* ***inverse*** *of $p \to q$.*

From the truth table we can easily check that the truth values of $p \to q$ and $\neg q \to \neg p$ are same. This leads us to the next definition.

Definition 1.10. *When two compound propositions always have the same truth value we call them* ***equivalent****.*

The converse and the inverse of a conditional statement are also equivalent.

Example 1.2. What are the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining."
Solution: Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as "If it is raining, then the home team wins." Consequently, the contrapositive of this conditional statement is "If the home team does not win, then it is not raining." The converse is "If the home team wins, then it is raining." The inverse is "If it is not raining, then the home team does not win." Only the contrapositive is equivalent to the original statement.

We now introduce another way to combine propositions that expresses that two propositions have the same truth value.

Definition 1.11. *Let p and q be propositions. The* ***biconditional statement*** *$p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.*

The truth table for $p \leftrightarrow q$ is shown in Table 1.6. There are some other common ways to express $p \leftrightarrow q$: "p is necessary and sufficient for q" "if p then q, and conversely" "p iff q." The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation "iff" for "if and only if." Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \to q) \wedge (q \to p)$. We

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \leftrightarrow q}$
T	T	T
T	F	F
F	T	F
F	F	T

Table 1.6: Biconditional Statement

have now introduced four important logical connectives–conjunctions, disjunctions, conditional statements, and biconditional statements–as well as negations. We can use these connectives to build up complicated compound propositions involving any number of propositional variables. We can use truth tables to determine the truth values of these compound propositions. We use a separate column to find the truth value of each compound expression that occurs in the compound proposition as it is built up. The truth values of the compound proposition for each combination of truth values of the propositional variables in it is found in the final column of the table.

Example 1.3. Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$.

Solution: Because this truth table involves two propositional variables p and q, there are four rows in this truth table, one for each of the pairs of truth values TT, TF, FT, and FF. The first two columns are used for the truth values of p and q, respectively. In the third column we find the truth value of $\neg q$, needed to find the truth value of $p \vee \neg q$, found in the fourth column. The fifth column gives the truth value of $p \wedge q$. Finally, the truth value of $(p \vee \neg q) \rightarrow (p \wedge q)$ is found in the last column. The resulting truth table is shown in Table 1.7.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Table 1.7: The Truth table.

1.1.2 Precedence of Logical Operators

We can construct compound propositions using the negation operator and the logical operators defined so far. We will generally use parentheses to specify the order in which logical operators in a compound proposition are to be applied. For instance, $(p \vee q) \wedge (\neg r)$ is the conjunction of $p \vee q$ and $\neg r$. However, to reduce the number of parentheses, we specify that the negation operator is applied before all other logical operators. This means that $\neg p \wedge q$ is the conjunction of $\neg p$ and q, namely, $(\neg p) \wedge q$, not the negation of the conjunction of p and q, namely $\neg(p \wedge q)$. Another general rule of precedence is that the conjunction operator takes precedence over the disjunction operator, so that $p \vee q \wedge r$ means $p \vee (q \wedge r)$ rather than $(p \vee q) \wedge r$. Because this rule may be difficult to remember, we will continue to use parentheses so that the order of the disjunction and conjunction operators is clear.

Operator	**Precedence**
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Table 1.8: Precedence of Logical Operators.

Most of the theorems are in the form of conditional statements. To prove theorems sometimes we use contrapositive statement, which is equivalent to the conditional statement. We will use this equivalence in Chapter 3. In the next section we will learn the equivalence of logical statements.

Exercises

1. Which of these sentences are propositions? What are the truth values of those that are propositions?

   a) Mumbai is the capital of Maharashtra. b) $2 + 3 = 5$.

   c) Attempt each of the question. d) $5 + 7 = 10$.

2. What is the negation of each of these propositions?

   a) Satish has more than 10 GB free disk space on his handset.

   b) There is no pollution in Pune city. c) $7 \times 11 \times 13 = 1003$.

3. Let p and q be the propositions

   p : It is below freezing, q : It is snowing.

   Write these propositions using p and q and logical connectives (including negations).

   a) It is below freezing and snowing.

   b) It is below freezing but not snowing.

   c) It is not below freezing and it is not snowing.

   d) It is either snowing or below freezing (or both).

   e) If it is below freezing, it is also snowing.

   f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.

4. Write each of these statements in the form "if p, then q" in English.

   a) It rains whenever the wind blows from the southwest.

   b) The apple trees will bloom if it stays warm for a week.

   c) Maria will go swimming unless the water is too cold.

5. State the converse, contrapositive, and inverse of each of these conditional statements.

   a) If it snows today, I will ski tomorrow.

b) A positive integer is a prime only if it has no divisors other than 1 and itself.

6. How many rows appear in a truth table for each of these compound propositions?
   a) $p \rightarrow \neg p$
   b) $(p \vee \neg r) \wedge (q \vee \neg s)$
   c) $q \vee p \vee \neg s \vee \neg r \vee \neg t \vee u$
   d) $(p \wedge r \wedge t) \leftrightarrow (q \wedge t)$

7. Construct a truth table for each of these compound propositions.
   a) $p \wedge \neg p$
   b) $(p \vee \neg q) \rightarrow q$
   c) $(p \vee q) \rightarrow (p \wedge q)$
   d) $(p \rightarrow q) \rightarrow (q \rightarrow p)$
   e) $(p \vee q) \rightarrow (p \oplus q)$
   f) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$
   g) $(p \rightarrow q) \vee (\neg p \rightarrow q)$
   h) $(p \leftrightarrow q) \vee (\neg p \leftrightarrow q)$
   i) $p \rightarrow (\neg q \vee r)$
   j) $(p \rightarrow q) \wedge (\neg p \rightarrow r)$
   k) $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$

Hints and Solutions of Selected Problems

1. a) Yes, T. b) Yes, T. c) No. d) Yes, F.

2. a) Satish has less than or equal to 10 GB free space on his handset. b)There is pollution in Pune city. c) $7 \times 11 \times 13 \neq 1003$.

3. a) $p \wedge q$ b) $p \wedge \neg q$.

4. Hint: Refer to the 13 ways to express the conditional statement.

5. a) Converse: "I will ski tomorrow only if it snows today."
   Contrapositive: "If I do not ski tomorrow, then it will not have snowed today."
   Inverse: "If it does not snow today, then I will not ski tomorrow."

6. a) 2. b) 16. c) 64.

1.2 Propositional Equivalences

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value. Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments. Note that we will use the term "compound proposition" to refer to an expression formed from propositional variables using logical operators, such as $p \wedge q$. We begin our discussion with a classification of compound propositions according to their possible truth values.

Definition 1.12. *A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a* ***tautology****. A compound proposition that is always false is called a* ***contradiction****. A compound proposition that is neither a tautology nor a contradiction is called a* ***contingency****.*

Tautologies and contradictions are often important in mathematical reasoning. An example of a tautology is $p \vee \neg p$ whereas an example for a contradiction is $p \wedge \neg p$. The following truth table illustrates this.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Table 1.9: Examples of Tautology and Contradiction

1.2.1 Logical Equivalences

Definition 1.13. *The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.*

One way to determine whether two compound propositions are equivalent is to use a truth table.

Example 1.4. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.
Solution: The truth tables for these compound propositions are displayed in Table 1.10. Because the truth values of the compound propositions $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology and that these compound propositions are logically equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Table 1.10: The truth table.

Example 1.5. Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.
Solution: We construct the truth table for these compound propositions in

Table 1.11. Because the truth values of $\neg p \vee q$ and $p \rightarrow q$ agree, they are logically equivalent.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Table 1.11: The truth table.

We will now establish a logical equivalence of two compound propositions involving three different propositional variables p, q, and r. To use a truth table to establish such a logical equivalence, we need eight rows, one for each possible combination of truth values of these three variables. In general, 2^n rows are required in the truth table to establish logical equivalence involving n propositional variables.

Table 1.12 demonstrates that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Table 1.12: The truth table.

Example 1.6. Show that $p \lor (q \land r)$ and $(p \lor q) \land (p \lor r)$ are logically equivalent. This is the distributive law of disjunction over conjunction.
Solution: We construct the truth table for these compound propositions in Table 1.12. Because the truth values of $p \lor (q \land r)$ and $(p \lor q) \land (p \lor r)$ agree, these compound propositions are logically equivalent.

Equivalence	**Name**	**Equivalence**	**Name**
$p \land \mathbf{T} \equiv p$ $p \lor \mathbf{F} \equiv p$	Identity laws	$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor \mathbf{T} \equiv \mathbf{T}$ $p \land \mathbf{F} \equiv \mathbf{F}$	Domination laws	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws	$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws
$\neg(\neg p) \equiv p$	Double negation law	$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws	$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws

Table 1.13: Logical Equivalances

Table 1.13 contains some important equivalences. In these equivalences, **T** denotes the compound proposition that is always true and **F** denotes the compound proposition that is always false. Note that $p_1 \lor p_2 \lor \ldots \lor p_n$ and $p_1 \land p_2 \land \ldots \land p_n$ are well defined whenever $p_1, p_2, \ldots, p_n$ are propositions. Also De Morgan's laws extend to

$$\neg(p_1 \lor p_2 \lor \ldots \lor p_n) \equiv (\neg p_1 \land \neg p_2 \land \ldots \land \neg p_n)$$

and

$$\neg(p_1 \land p_2 \land \ldots \land p_n) \equiv (\neg p_1 \lor \neg p_2 \lor \ldots \lor \neg p_n).$$

Equivalence	
$p \to q \equiv \neg p \vee q$ $p \to q \equiv \neg q \to \neg p$ $p \vee q \equiv \neg p \to q$ $p \wedge q \equiv \neg(p \to \neg q)$ $\neg(p \to q) \equiv p \wedge \neg q$ $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$ $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$ $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$ $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$	Logical Equivalences Involving Conditional Statements
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$	Logical Equivalences Involving Biconditional Statements

Table 1.14: Logical Equivalances

Example 1.7. Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: We will use one of the equivalences in Table 1.13 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$. We have the following equivalences.

$\neg(p \vee (\neg p \wedge q))$	$\equiv$	$\neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
	$\equiv$	$\neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
	$\equiv$	$\neg p \wedge (p \vee \neg q)$	by the double negation law
	$\equiv$	$(\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
	$\equiv$	$\mathbf{F} \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv \mathbf{F}$
	$\equiv$	$(\neg p \wedge \neg q) \vee \mathbf{F}$	by the commutative law for disjunction
	$\equiv$	$\neg p \wedge \neg q$	by the identity law for **F**

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Logical equivalences involving conditional statements and biconditional statements are given in the table 1.14. These equivalences are important as they form basic tools for proving theorems. Few theorems involve "if and only if" $p \leftrightarrow q$. To prove the theorem of this type we use the equivalence $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$. So it is enough to prove the statements $p \rightarrow q$ and $q \rightarrow p$ separately.

Remark 1.1. *A logical equivalence can be proved by using either a truth table or by using a chain of known logical equivalences. Also a tautology can be proved by using either a truth table or by using logical equivalences.*

Example 1.8. Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.
Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **T**. (Note: This could also be done using a truth table.)

$(p \wedge q) \rightarrow (p \vee q)$	$\equiv$	$\neg(p \wedge q) \vee (p \vee q)$	since $p \rightarrow q \equiv \neg p \vee q$
	$\equiv$	$(\neg p \vee \neg q) \vee (p \vee q)$	by the first De Morgan law
	$\equiv$	$(\neg p \vee p) \vee (\neg q \vee q)$	by the associative and commutative laws for disjunction
	$\equiv$	$\mathbf{T} \vee \mathbf{T}$	by the commutative law for disjunction
	$\equiv$	$\mathbf{T}$	by the domination law

Thus we have shown that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Logic has practical applications to the design of computing machines, to the specification of systems, to artificial intelligence, to computer programming, to programming languages, and to other areas of computer science, as well as to many other fields of study. In the next chapter we will introduce the concepts which will help us to express the meaning of statements in mathematics and natural language.

Exercises

1. Use truth tables to verify
   a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$. b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.
   c) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

2. Use De Morgan's laws to find the negation of each of the following statements.
   a) Arun is rich and happy. b) Meera will bicycle or run tomorrow.
   c) Arif is smart and hard working.

3. Show that each of these conditional statements is a tautology by using truth tables.
   a) $(p \wedge q) \to p$.
   b) $p \to (p \vee q)$.
   c) $\neg(p \to q) \to p$.
   d) $[\neg p \wedge (p \vee q)] \to q$.
   e) $[(p \to q) \wedge (q \to r)] \to (p \to r)$.
   f) $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$.

4. Determine whether $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

5. Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.

6. Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

7. Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent.

8. Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

Hints and Solutions of Selected Problems

1. Hint: Remember to take 8 rows in the truth table.

2. a) Arun is not rich, or Arun is not happy. b) Meera will not bicycle tomorrow, and Meera will not run tomorrow.

3. Hint: For tautology all truth values of the compound statement are T.

4. It is a tautology.

5. Use truth table.

6. Each of these is true precisely when p and q have opposite truth values.

7. Use truth table.

8. These are not logically equivalent because when p, q, and r are all false, $(p \to q) \to r$ is false, but $p \to (q \to r)$ is true.

Chapter 2

Predicates and Quantifiers

Introduction

Propositional logic, studied in Chapter 1, cannot adequately express the meaning of all statements in mathematics and in natural language. For example, suppose that we know that "Every computer connected to the university network is functioning properly." No rules of propositional logic allow us to conclude the truth of the statement "MATH5 is functioning properly," where MATH5 is one of the computers connected to the college network. In this chapter, we will introduce a more powerful type of logic called predicate logic. We will see how predicate logic can be used to express the meaning of a wide range of statements in mathematics in ways that permit us to reason and explore relationships between objects. To understand predicate logic, we first need to introduce the concept of a predicate. Afterwards, we will introduce the notion of quantifiers, which enable us to reason with statements that assert that a certain property holds for all objects of a certain type and with statements that assert the existence of an object with a particular property. We will learn nested quantifiers which play an important role in mathematics as well as in computer science.

2.1 Predicates

Statements involving variables, such as "$x > 3$," "$x = y - 4$," "$x - y = 2z$," and "x is a rational number," and "n is a prime number," are often found in mathematical assertions. These statements are neither true nor false when the values of the variables are not specified. In this section, we will discuss the ways that propositions can be produced from such statements. The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. The second part-the **predicate**, "is greater than 3"-refers to a property that the subject of the statement can have. We can denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate "is greater than 3" and x is the variable. The statement $P(x)$ is also said to be the value of the **propositional function** P at x. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value. Consider the following example.

Example 2.1. Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(5)$ and $P(1)$?
Solution: We obtain the statement $P(5)$ by setting $x = 5$ in the statement "$x > 3$." Hence, $P(5)$, which is the statement "$5 > 3$," is true. However, $P(1)$, which is the statement "$1 > 3$," is false.

We can also have statements that involve more than one variable. For instance, consider the statement ''$x = y - 4$." We can denote this statement by $F(x, y)$, where x and y are variables and F is the predicate. When values are assigned to the variables x and y, the statement $F(x, y)$ has a truth value. Hence, F(1, 2) is the statement "$1 = 2 - 4$,"' which is false. The statement $F(4, 8)$ is the proposition "$4 = 8 - 4$," which is true. Similarly, we can let $Q(x, y, z)$ denote the statement "$x - y = 2z$." When values are assigned to the variables x, y, and z, this statement has a truth value. The proposition $Q(1, 2, 3)$ is obtained by setting $x = 1, y = 2$, and $z = 3$ in the statement

$Q(x, y, z)$. We see that $Q(1, 2, 3)$ is the statement "$1 - 2 = 2 \times 3$," which is false. Also note that $Q(3, 1, 1)$, which is the statement "$3 - 1 = 2 \times 1$," is true.

In general, a statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$. A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called an ***n*-place predicate** or a n-ary predicate.

Exercises

1. Let $P(x)$ denote the statement "$x \leq 4$." What are the truth values of the following?
   a) $P(0)$ b) $P(4)$ c) $P(6)$ d) $P(-3)$

2. Let $E(x)$ be the statement "The word x contains the letter 'e'." What are these truth values?
   a) E(mathematics) b) E(teacher) c) E(great) d) E(true)

3. Let $C(x, y)$ denote the statement "x is the capital of y." What are these truth values?
   a) C(New Delhi, India) b) C(Brasilia, Brazil)
   c) C(Sydney, Australia) d) C(New York, United States of America)

4. Let $S(x, y)$ denote the statement "x is square of y."
   a) $S(9, 3)$ b) $S(25, 4)$ c) $S(121, -11)$ d) $S(109, 10.2)$

Hints and Solutions of Selected Problems

1. a) T b) T c) F d) T

2. a) T b) T c) T d) T

3. a) T b) T c) F d) F

2.2 Quantifiers

When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value. However, there is another important way, called quantification, to create a proposition from a propositional function. Quantification expresses the extent to which a predicate is true over a range of elements. In English, the words *all, some, many, none,* and *few* are used in quantifications. Our focus will be on two types of quantification here: universal quantification, which tells us that a predicate is true for every element under consideration, and existential quantification, which tells us that there is one or more element under consideration for which the predicate is true.

Definition 2.1. *The area of logic that deals with predicates and quantifiers is called the* ***predicate calculus****.*

2.2.1 The Universal Quantifier

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the **domain of discourse** (or the **universe of discourse**), often just referred to as the **domain**. Such a statement is expressed using universal quantification. The universal quantification of $P(x)$ for a particular domain is the proposition that asserts that $P(x)$ is true for all values of x in this domain. Note that the domain specifies the possible values of the variable x. The meaning of the universal quantification of $P(x)$ changes when we change the domain. The domain must always be specified when a universal quantifier is used; without it, the universal quantification of a statement is not defined.

Definition 2.2. *The* ***universal quantification*** *of* $P(x)$ *is the statement "*$P(x)$ *for all values of* x *in the domain."*

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the ***universal quantifier****. We read $\forall x P(x)$ as "for all $xP(x)$" or "for every $xP(x)$." An element for which $P(x)$ is false is called a* ***counterexample*** *of $\forall x P(x)$.*

Example 2.2. Let $P(x)$ be the statement "$x + 3 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?
Solution: Because $P(x)$ is true for all real numbers x, the quantification $\forall x P(x)$ is true.

Example 2.3. Let $Q(x)$ be the statement "$x > 0$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?
Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(0)$ is false. That is, $x = 0$ is a counterexample for the statement $\forall x Q(x)$. Thus $\forall x Q(x)$ is false.

Note that a single counterexample is all we need to establish that $\forall x P(x)$ is false. Looking for counterexamples to universally quantified statements is an important activity in the study of mathematics. When all the elements in the domain can be listed say, $x_1, x_2, \ldots, x_n$ it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$, because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

Example 2.4. What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?
Solution: The universal quantification $\forall x(x^2 \geq x)$, where the domain consists of all real numbers, is false. A counterexample is $x = 0.5$(in fact any x satisfying $0 < x < 1$ will be a counterexample). However, if the domain

consists of the integers, $\forall x(x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$.

2.2.2 The Existential Quantifier

Many mathematical statements assert that there is an element with a certain property. Such statements are expressed using existential quantification. With existential quantification, we form a proposition that is true if and only if $P(x)$ is true for at least one value of x in the domain.

Definition 2.3. *The **existential quantification** of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$."*
*We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the **existential quantifier**.*

A domain must always be specified when a statement $\exists x P(x)$ is used. Furthermore, the meaning of $\exists x P(x)$ changes when the domain changes.

Example 2.5. Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?
Solution: Because "$x > 3$" is true when $x = 4$. Thus the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.

Example 2.6. Let $Q(x)$ denote the statement "$x > x + 1$." What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?
Solution: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x Q(x)$, is false.

When all elements in the domain can be listed-say, $x_1, x_2, \ldots, x_n$ the existential quantification $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee$

$\ldots \vee P(x_n)$, because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

2.3 Quantifiers with Restricted Domains

An abbreviated notation is often used to restrict the domain of a quantifier. In this notation, a condition a variable must satisfy is included after the quantifier. This is illustrated in the following example.

Example 2.7. What do the statements $\forall x < 0(x^2 > 0), \forall y \neq 0(y^3 \neq 0)$, and $\exists z > 0(z^2 = 2)$ mean, where the domain in each case consists of the real numbers?
Solution: The statement $\forall x < 0(x^2 > 0)$ states that for every real number x with $x < 0, x^2 > 0$. That is, it states "The square of a negative real number is positive." This statement is the same as $\forall x(x < 0 \to x^2 > 0)$. The statement $\forall y \neq 0(y^3 \neq 0)$ states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$. That is, it states "The cube of every nonzero real number is nonzero." Note that this statement is equivalent to $\forall y(y \neq 0 \to y^3 \neq 0)$. Finally, the statement $\exists z > 0(z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$. That is, it states "There is a positive square root of 2." This statement is equivalent to $\exists z(z > 0 \wedge z^2 = 2)$.

2.3.1 Precedence of Quantifiers and Binding Variables

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus. For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$. In other words, it means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.

When a quantifier is used on the variable x, we say that this occurrence of the variable is **bound**. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**. All the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition. This can be done using a combination of universal quantifiers, existential quantifiers, and value assignments.

Definition 2.4. *The part of a logical expression to which a quantifier is applied is called the* ***scope*** *of this quantifier. Consequently, a variable is* ***free*** *if it is outside the scope of all quantifiers in the formula that specify this variable.*

Example 2.8. In the statement $\exists x(x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable. This illustrates that in the statement $\exists x(x + y = 1), x$ is bound, but y is free.

2.3.2 Logical Equivalences Involving Quantifiers

Definition 2.5. *Statements involving predicates and quantifiers are* ***logically equivalent*** *if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.*
We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example 2.9. Show that $\neg\forall xP(x) \equiv \exists x\neg P(x)$ and $\neg\exists xQ(x) \equiv \forall x\neg Q(x)$.
Solution: To show that $\neg\forall xP(x)$ and $\exists xP(x)$ are logically equivalent no matter what the propositional function $P(x)$ is and what the domain is, first note that $\neg\forall xP(x)$ is true if and only if $\forall xP(x)$ is false. Next, note that $\forall xP(x)$ is false if and only if there is an element x in the domain for which $P(x)$ is false. This holds if and only if there is an element x in the domain for

which $\neg P(x)$ is true. Finally, note that there is an element x in the domain for which $\neg P(x)$ is true if and only if $\exists x \neg P(x)$ is true. Putting these steps together, we can conclude that $\neg \forall x P(x)$ is true if and only if $\exists x \neg P(x)$ is true. It follows that $\neg \forall x P(x)$ and $\exists x \neg P(x)$ are logically equivalent.
Similarly we can prove that $\neg \exists x Q(x)$ and $\forall x \neg Q(x)$ are logically equivalent.

The rules for negations in the above example for quantifiers are called **De Morgan's laws for quantifiers**.

Example 2.10. What are the negations of the statements "There is an honest politician" and "All Indians eat parathas"?
Solution: Let $H(x)$ denote "x is honest." Then the statement "There is an honest politician" is represented by $\exists x H(x)$, where the domain consists of all politicians. The negation of this statement is $\neg \exists x H(x)$, which is equivalent to $\forall x \neg H(x)$. This negation can be expressed as "Every politician is dishonest." Let $C(x)$ denote "x eats parathas." Then the statement "All Indians eat parathas" is represented by $\forall x C(x)$, where the domain consists of all Indians. The negation of this statement is $\neg \forall x C(x)$, which is equivalent to $\exists x \neg C(x)$. This negation can be expressed in several different ways, including "Some Indian does not eat parathas" and "There is an Indian who does not eat parathas."

Example 2.11. What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?
Solution: The negation of $\forall x(x^2 > x)$ is the statement $\neg \forall x(x^2 > x)$, which is equivalent to $\exists x \neg (x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg \exists x(x^2 = 2)$, which is equivalent to $\forall x \neg (x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$. The truth values of these statements depend on the domain.

Example 2.12. Show that $\neg \forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

Solution: By De Morgan's law for universal quantifiers, we know that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(\neg(P(x) \rightarrow Q(x)))$ are logically equivalent. By the logical equivalence, we know that $\neg(P(x) \rightarrow Q(x))$ and $P(x) \wedge \neg Q(x)$ are logically equivalent for every x. Because we can substitute one logically equivalent expression for another in a logical equivalence, it follows that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

Example 2.13. Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Solution: First, we rewrite the statement so that we can clearly identify the appropriate quantifiers to use. Doing so, we obtain: "For every student in this class, that student has studied calculus." Next, we introduce a variable x so that our statement becomes "For every student x in this class, x has studied calculus." Continuing, we introduce $C(x)$, which is the statement "x has studied calculus." Consequently, if the domain for x consists of the students in the class, we can translate our statement as $\forall x C(x)$.

However, there are other correct approaches; different domains of discourse and other predicates can be used. The approach we select depends on the subsequent reasoning we want to carry out. For example, we may be interested in a wider group of people than only those in this class. If we change the domain to consist of all people, we will need to express our statement as "For every person x, if person x is a student in this class then x has studied calculus." If $S(x)$ represents the statement that person x is in this class, we see that our statement can be expressed as $\forall x(S(x) \rightarrow C(x))$.

Finally, when we are interested in the background of people in subjects besides calculus, we may prefer to use the two-variable quantifier $Q(x, y)$ for the statement "student x has studied subject y." Then we would replace $C(x)$ by $Q(x, \text{calculus})$ in both approaches to obtain $\forall x Q(x, \text{calculus})$ or $\forall x(S(x) \rightarrow Q(x, \text{calculus}))$.

Thus for different domains of discourse we get different logical expressions. In mathematical expressions we may need more than one quantifier. In the next section we will discuss a useful concept in mathematics called the nested quantifiers.

Exercises

1. Let $P(x)$ be the statement "x spends more than four hours every weekday in college," where the domain for x consists of all students. Express each of these quantifications in English.
   a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x \neg P(x)$ d) $\forall x \neg P(x)$

2. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.
   a) $\forall x(C(x) \to F(x))$ b) $\forall x(C(x) \wedge F(x))$
   c) $\exists x(C(x) \to F(x))$ d) $\exists x(C(x) \wedge F(x))$

3. Let P(x) be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?
   a) $P(0)$ b) $P(1)$ c) $P(2)$ d) $P(1)$ e) $\exists x P(x)$ f) $\forall x P(x)$

4. Suppose that the domain of the propositional function $P(x)$ consists of the integers $1, 2, 3, 4$, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.
   a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\neg\exists x P(x)$ d) $\neg\forall x P(x)$
   e) $\forall x((x \neq 3) \to P(x)) \vee \exists x \neg P(x)$

5. Suppose that the domain of $Q(x, y, z)$ consists of triples x, y, z, where $x = 0, 1$, or $2, y = 0$ or 1, and $z = 0$ or 1. Write out these propositions using disjunctions and conjunctions.
   a) $\forall y Q(0, y, 0)$ b) $\exists x Q(x, 1, 1)$ c) $\exists z \neg Q(0, 0, z)$ d) $\exists x \neg Q(x, 0, 1)$

6. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
   a) $\forall x(x^2 > x)$ b) $\forall x(x > 0 \vee x < 0)$ c) $\forall x(x = 1)$

7. Determine whether $\forall x(P(x) \to Q(x))$ and $\forall x P(x) \to \forall x Q(x)$ are logically equivalent. Justify your answer.

8. Show that $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x(P(x) \wedge Q(x))$ are not logically equivalent.

Hints and Solutions of Selected Problems

1. a) There is a student who spends more than 4 hours every weekday in college. b) Every student spends more than 4 hours every weekday in college. c) There is a student who does not spend more than 4 hours every weekday in college. d) No student spends more than 4 hours every weekday in college.

2. a) Every comedian is funny. b) Every person is a funny comedian.
   c) There exists a person such that if she or he is a comedian, then she or he is funny. d) Some comedians are funny.

3. a) T b) T c) F d) F e) T f) F

4. a) $P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$
   b)$P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$
   c) $\neg(P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5))$

5. a) $Q(0, 0, 0) \wedge Q(0, 1, 0)$ b) $Q(0, 1, 1) \vee Q(1, 1, 1) \vee Q(2, 1, 1)$

6. a) $x = 0$

2.4 Nested Quantifiers

In this section we will introduce nested quantifiers. Two quantifiers are **nested** if one is within the scope of the other, such as $\forall x \exists y(x + y = 0)$. Note that everything within the scope of a quantifier can be thought of as a propositional function.

Nested quantifiers commonly occur in mathematics and computer science. To understand statements involving nested quantifiers see the following examples.

Example 2.14. Assume that the domain for the variables x and y consists of all real numbers. The statement $\forall x \forall y(x+y = y+x)$, says that $x+y = y+x$ for all real numbers x and y. This is the commutative law for addition of real numbers. Similarly, the statement

$$\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$$

is the associative law for addition of real numbers.

Example 2.15. Translate into English the statement

$$\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0)),$$

where the domain for both variables consists of all real numbers.
Solution: This statement says that for every real number x and for every real number y, if $x > 0$ and $y < 0$, then $xy < 0$. That is, this statement says that for real numbers x and y, if x is positive and y is negative, then xy is negative. This can be stated in few words as “The product of a positive real number and a negative real number is always a negative real number.”

Thinking of Quantification as Loops

In working with quantifications of more than one variable, it is sometimes helpful to think in terms of nested loops. For example, to see whether

$\forall x \forall y P(x, y)$ is true, we loop through the values x, and for each x we loop through the values for y. If we find that $P(x, y)$ is true for all values for x and y, we have determined that $\forall x \forall y P(x, y)$ is true. If we ever hit a value x for which we hit a value y for which $P(x, y)$ is false, we have shown that $\forall x \forall y P(x, y)$ is false.

Similarly, to determine whether $\forall x \exists y P(x, y)$ is true, we loop through the values for x. For each x we loop through the values for y until we find a y for which $P(x, y)$ is true. If for every x we hit such a y, then $\forall x \exists y P(x, y)$ is true; if for some x we never hit such a y, then $\forall x \exists y P(x, y)$ is false.

To see whether $\exists x \forall y P(x, y)$ is true, we loop through the values for x until we find an x for which $P(x, y)$ is always true when we loop through all values for y. Once we find such an x, we know that $\exists x \forall y P(x, y)$ is true. If we never hit such an x, then we know that $\exists x \forall y P(x, y)$ is false.

Finally, to see whether $\exists x \exists y P(x, y)$ is true, we loop through the values for x, where for each x we loop through the values for y until we hit an x for which we hit a y for which $P(x, y)$ is true. The statement $\exists x \exists y P(x, y)$ is false only if we never hit an x for which we hit a y such that $P(x, y)$ is true.

2.4.1 The Order of Quantifiers

Many mathematical statements involve multiple quantifications of propositional functions involving more than one variable. It is important to note that the order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers. These remarks are illustrated in following examples.

Example 2.16. Let $P(x, y)$ be the statement "$x + y = y + x$". What are the truth values of the quantifications $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ where the domain for all variables consists of all real numbers?

Solution: The quantification $\forall x \forall y P(x, y)$ denotes the proposition "For all real numbers x, for all real numbers y, $x+y=y+x$." Because $P(x,y)$ is true for all real numbers x and y, the proposition $\forall x \forall y P(x,y)$ is true. Note that the statement $\forall y \forall x P(x,y)$ says "For all real numbers y, for all real numbers x, $x+y=y+x$." This has the same meaning as the statement "For all real numbers x, for all real numbers y, $x+y=y+x$." That is, $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$ have the same meaning, and both are true.

This illustrates the principle that the order of nested universal quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement.

Example 2.17. Let $Q(x,y)$ denote "$x+y=0$." What are the truth values of the quantifications $\exists y \forall x Q(x,y)$ and $\forall x \exists y Q(x,y)$, where the domain for all variables consists of all real numbers?

Solution: The quantification $\exists y \forall x Q(x,y)$ denotes the proposition "There is a real number y such that for every real number x, $Q(x,y)$." No matter what value of y is chosen, there is only one value of x for which $x+y=0$. Because there is no real number y such that $x+y=0$ for all real numbers x, the statement $\exists y \forall x Q(x,y)$ is false.

The quantification $\forall x \exists y Q(x,y)$ denotes the proposition "For every real number x there is a real number y such that $Q(x,y)$." Given a real number x, there is a real number y such that $x+y=0$; namely, $y=-x$. Hence, the statement $\forall x \exists y Q(x,y)$ is true.

The above example illustrates that the order in which quantifiers appear makes a difference. The statements $\exists y \forall x Q(x,y)$ and $\forall x \exists y Q(x,y)$ are not logically equivalent. The statement $\exists y \forall x Q(x,y)$ is true if and only if there is a y that makes $P(x,y)$ true for every x. So, for this statement to be true, there must be a particular value of y for which $P(x,y)$ is true regardless of the choice of x. On the other hand, $\forall x \exists y P(x,y)$ is true if and only if for every value of x there is a value of y for which $P(x,y)$ is true. So, for this statement to be true, no matter which x you choose, there must be a value of y (possibly depending on the x you choose) for which $P(x, y)$ is true. In other words, in the second case, y can depend on x, whereas in the first case, y is a constant independent of x. From these observations, it follows that if $\exists y \forall x P(x, y)$ is true, then $\forall x \exists y P(x, y)$ must also be true. However, if $\forall x \exists y P(x, y)$ is true, it is not necessary for $\exists y \forall x P(x, y)$ to be true.

Table 2.1 summarizes the meanings of the different possible quantifications involving two variables.

Statement	**When true?**	**When false?**
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Table 2.1: Quantification of two variables.

2.4.2 Translating Mathematical Statements

Mathematical statements expressed in English can be translated into logical expressions, as shown in following example.

Example 2.18. Translate the statement "The sum of two positive integers is always positive" into a logical expression.

Solution: To translate this statement into a logical expression, we first rewrite it so that the implied quantifiers and a domain are shown: "For every two integers, if these integers are both positive, then the sum of these integers is positive." Next, we introduce the variables x and y to obtain "For all positive integers x and y, $x+y$ is positive." Consequently, we can express this statement as

$$\forall x \forall y((x > 0) \wedge (y > 0) \to (x + y > 0))$$

where the domain for both variables consists of all integers. Note that we could also translate this using the positive integers as the domain. Then the statement "The sum of two positive integers is always positive" becomes "For every two positive integers, the sum of these integers is positive." We can express this as

$$\forall x \forall y(x + y > 0)$$

where the domain for both variables consists of all positive integers.

Example 2.19. Translate the statement "Every real number except zero has a multiplicative inverse." (A multiplicative inverse of a real number x is a real number y such that $xy = 1$.)
Solution: We first rewrite this as "For every real number x except zero, x has a multiplicative inverse." We can rewrite this as "For every real number x, if $x \neq 0$, then there exists a real number y such that $xy = 1$." This can be rewritten as

$$\forall x((x \neq 0) \to \exists y(xy = 1)).$$

One example that you may be familiar with is the concept of limit, which is important in calculus.

Example 2.20. Use quantifiers to express the definition of the limit of a real-valued function $f(x)$ of a real variable x at a point a in its domain.
Solution: Recall that the definition of the statement

$$\lim_{x \to a} f(x) = L$$

is: For every real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. This definition of a limit can be phrased in terms of quantifiers by

$$\forall\epsilon\exists\delta\forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon),$$

where the domain for the variables δ and ϵ consists of all positive real numbers and for x consists of all real numbers.
This definition can also be expressed as

$$\forall\epsilon > 0\exists\delta > 0\forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon),$$

when the domain for the variables ϵ and δ consists of all real numbers, rather than just the positive real numbers. [Here, restricted quantifiers have been used. Recall that $\forall x > 0 P(x)$ means that for all x with $x > 0, P(x)$ is true.]

2.4.3 Translating from Nested Quantifiers into English

Expressions with nested quantifiers expressing statements in English can be quite complicated. The first step in translating such an expression is to write out what the quantifiers and predicates in the expression mean. The next step is to express this meaning in a simpler sentence.

Example 2.21. Translate the statement

$$\forall x(C(x) \lor \exists y(C(y) \land F(x, y)))$$

into English, where $C(x)$ is "x has a computer", $F(x, y)$ is "x and y are friends," and the domain for both x and y consists of all students in your college.
Solution: The statement says that for every student x in your college, x has a computer or there is a student y such that y has a computer and x and y are friends. In other words, every student in your college has a computer or has a friend who has a computer.

Example 2.22. Translate the statement

$$\exists x \forall y \forall z((F(x,y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$

into English, where $F(a,b)$ means a and b are friends and the domain for x, y, and z consists of all students in your college.

Solution: We first examine the expression $(F(x,y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z)$. This expression says that if students x and y are friends, and students x and z are friends, and furthermore, if y and z are not the same student, then y and z are not friends. It follows that the original statement, which is triply quantified, says that there is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, then y and z are not friends. In other words, there is a student none of whose friends are also friends with each other.

2.4.4 Translating English Sentences into Logical Expressions

In this section we see translation of sentences into logical expressions using nested quantifiers.

Example 2.23. Express the statement "If a person is a woman and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution: The statement "If a person is a woman and is a parent, then this person is someone's mother" can be expressed as "For every person x, if person x is a woman and person x is a parent, then there exists a person y such that person x is the mother of person y." We introduce the propositional functions $F(x)$ to represent "x is a woman," $P(x)$ to represent "x is a parent," and $M(x,y)$ to represent "x is the mother of y." The original statement can be represented as

$$\forall x((F(x) \land P(x)) \to \exists y M(x, y)).$$

Example 2.24. Express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution: The statement "Everyone has exactly one best friend" can be expressed as "For every person x, person x has exactly one best friend." Introducing the universal quantifier, we see that this statement is the same as "$\forall x$ (person x has exactly one best friend)," where the domain consists of all people. To say that x has exactly one best friend means that there is a person y who is the best friend of x, and furthermore, that for every person z, if person z is not person y, then z is not the best friend of x. When we introduce the predicate $B(x, y)$ to be the statement "y is the best friend of x," the statement that x has exactly one best friend can be represented as

$$\exists y(B(x, y) \land \forall z((z \neq y) \to \neg B(x, z))).$$

Consequently, our original statement can be expressed as

$$\forall x \exists y(B(x, y) \land \forall z((z \neq y) \to \neg B(x, z))).$$

Example 2.25. Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution: Let $P(w, f)$ be "w has taken f" and $Q(f, a)$ be "f is a flight on a." We can express the statement as

$$\exists w \forall a \exists f(P(w, f) \land Q(f, a)),$$

where the domains of discourse for w, f , and a consist of all the women in the world, all airplane flights, and all airlines, respectively.

2.4.5 Negating Nested Quantifiers

Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier.

Example 2.26. Express the negation of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.
Solution: By successively applying De Morgan's laws for quantifiers, we can move the negation in $\neg \forall x \exists y(xy = 1)$ inside all the quantifiers. We find that $\neg \forall x \exists y(xy = 1)$ is equivalent to $\exists x \neg \exists y(xy = 1)$, which is equivalent to $\exists x \forall y \neg(xy = 1)$. Because $\neg(xy = 1)$ can be expressed more simply as $xy \neq 1$, we conclude that our negated statement can be expressed as $\exists x \forall y(xy \neq 1)$.

Example 2.27. Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."
Solution: This statement is the negation of the statement "There is a woman who has taken a flight on every airline in the world." Our statement can be expressed as $\neg \exists w \forall a \exists f(P(w, f) \wedge Q(f, a))$, where $P(w, f)$ is "w has taken f" and $Q(f, a)$ is "f is a flight on a." By successively applying De Morgan's laws for quantifiers to move the negation inside successive quantifiers and by applying De Morgan's law for negating a conjunction in the last step, we find that our statement is equivalent to each of this sequence of statements:

$$\begin{aligned} \forall w \neg \forall a \exists f(P(w, f) \wedge Q(f, a)) &\equiv \forall w \exists a \neg \exists f(P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f \neg(P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f(\neg P(w, f) \vee \neg Q(f, a)) \end{aligned}$$

This last statement states "For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline."

Example 2.28. Use quantifiers and predicates to express the fact that $\lim_{x \to a} f(x)$ does not exist where $f(x)$ is a real-valued function of a real variable

x and a belongs to the domain of f.

Solution: To say that $\lim_{x\to a} f(x)$ does not exist means that for all real numbers L, $\lim_{x\to a} f(x) \neq L$. The statement $\lim_{x\to a} f(x) \neq L$ can be expressed as

$$\neg\forall\epsilon > 0\exists\delta > 0\forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon).$$

Successively applying the rules for negating quantified expressions, we construct this sequence of equivalent statements

$$\begin{aligned}
&\neg\forall\epsilon > 0\exists\delta > 0\forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon).\\
&\equiv \exists\epsilon > 0\neg\exists\delta > 0\forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon)\\
&\equiv \exists\epsilon > 0\forall\delta > 0\neg\forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon)\\
&\equiv \exists\epsilon > 0\forall\delta > 0\exists x\neg(0 < |x - a| < \delta \to |f(x) - L| < \epsilon)\\
&\equiv \exists\epsilon > 0\forall\delta > 0\exists x(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon)
\end{aligned}$$

In the last step we used the equivalence $\neg(p \to q) \equiv p\wedge\neg q$. Because the statement "$\lim_{x\to a} f(x)$ does not exist" means for all real numbers L, $\lim_{x\to a} f(x) \neq L$ this can be expressed as

$$\forall L\exists\epsilon > 0\forall\delta > 0\exists x(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon)$$

This last statement says that for every real number L there is a real number $\epsilon > 0$ such that for every real number $\delta > 0$, there exists a real number x such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.

Exercises

1. Translate these statements into English, where the domain for each variable consists of all real numbers.
   a)$\forall x\exists y(x < y)$ b) $\forall x\forall y(((x \geq 0) \wedge (y \geq 0)) \to (xy \geq 0))$
   c)$\forall x\forall y\exists z(xy = z)$

2. Let $T(x, y)$ mean that "student x likes cuisine(method of cooking) y," where the domain for x consists of all students at your college and the

domain for y consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\neg T$ (Hussein, Japanese)

b) $\exists x T(x, \text{Korean}) \wedge \forall x T(x, \text{Mexican})$

c) $\exists y(T(\text{Monika}, y) \vee T(\text{Jay}, y))$

d) $\forall x \forall z \exists y((x \neq z) \rightarrow \neg(T(x, y) \wedge T(z, y)))$

e) $\exists x \exists z \forall y(T(x, y) \leftrightarrow T(z, y))$

3. Let $L(x, y)$ be the statement "x loves y," where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

   a) Everybody loves Jerry.

   b) Everybody loves somebody.

   c) There is somebody whom everybody loves.

   d) Nobody loves everybody.

   e) There is somebody whom Lisa does not love.

   f) There is somebody whom no one loves.

4. Use quantifiers and predicates with more than one variable to express these statements.

   a) Every computer science student needs a course in discrete mathematics.

   b) There is a student in this class who owns a personal computer.

   c) Every student in this class has taken at least one computer science course.

   d) There is a student in this class who has taken at least one course in computer science.

   e) Every student in this class has been in every building on campus.

5. Express each of these statements using mathematical and logical operators, predicates, and quantifiers, where the domain consists of all

integers.

a) The sum of two negative integers is negative.

b) The difference of two positive integers is not necessarily positive.

c) The sum of the squares of two integers is less than or equal to the square of their sum.

d) The absolute value of the product of two integers is the product of their absolute values.

6. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

   a)$\forall n \exists m (n^2 < m)$ b) $\exists n \forall m (n < m^2)$

   c) $\forall n \exists m (n + m = 0)$ d) $\exists n \forall m (nm = m)$

   e) $\exists n \exists m (n^2 + m^2 = 5)$ f) $\exists n \exists m (n^2 + m^2 = 6)$

   g) $\forall n \forall m \exists p \left(p = \dfrac{(m+n)}{2} \right)$ h) $\exists n \exists m (n + m = 4 \wedge n - m = 2)$

   i) $\exists n \exists m (n + m = 4 \wedge n - m = 1)$

7. Express the negations of each of these statements so that all negation symbols immediately precede predicates.

   a) $\forall x \exists y \forall z T(x, y, z)$ b) $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$

   c) $\forall x \exists y (P(x, y) \wedge \exists z R(x, y, z))$ d) $\forall x \exists y (P(x, y) \rightarrow Q(x, y))$

Hints and Solutions:

1. a) For every real number x there exists a real number y such that x is less than y. b) For every real number x and real number y, if x and y are both nonnegative, then their product is nonnegative. c) For every real number x and real number y, there exists a real number z such that $xy = z$.

2. a) Hussein does not like Japanese cuisine. b) Some student at your college likes Korean cuisine, and everyone at your college likes Mexican

cuisine. c) There is some cuisine that either Monika or Jay likes. d) For every pair of distinct students at your college, there is some cuisine that at least one them does not like. e) There are two students at your college who like exactly the same set of cuisines.

3. a) $\forall x L(x, \text{Jerry})$ b)$\forall x \exists y L(x, y)$ c) $\exists y \forall x L(x, y)$ d) $\forall x \exists y \neg L(x, y)$ e) $\exists x \neg L(\text{Lisa}, x)$ f) $\exists x \forall y \neg L(y, x)$

4. a) $\forall x P(x)$, where $P(x)$ is "x needs a course in discrete mathematics" and the domain consists of all computer science students b) $\exists x P(x)$, where $P(x)$ is "x owns a personal computer" and the domain consists of all students in this class c) $\forall x \exists y P(x, y)$, where $P(x, y)$ is "x has taken y," the domain for x consists of all students in this class, and the domain for y consists of all computer science classes d) $\exists x \exists y P(x, y)$, where $P(x, y)$ and domains are the same as in part (c) e) $\forall x \forall y P(x, y)$, where $P(x, y)$ is "x has been in y," the domain for x consists of all students in this class, and the domain for y consists of all buildings on campus.

5. a) $\forall x \forall y((x < 0) \wedge (y < 0) \rightarrow (x + y < 0))$
   b)$\neg \forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x - y > 0))$
   c) $\forall x \forall y(x^2 + y^2 \leq (x + y)^2)$
   d) $\forall x \forall y(|xy| = |x||y|)$

6. a) True b) True c) True d) True e) True f) False g) False h) True i) False

7. a)$\exists x \forall y \exists z \neg T(x, y, z)$
   b)$\exists x \forall y \neg P(x, y) \wedge \exists x \forall y \neg Q(x, y)$
   c) $\exists x \forall y(\neg P(x, y) \vee \forall z \neg R(x, y, z))$
   d) $\exists x \forall y(P(x, y) \wedge \neg Q(xy))$

Chapter 3

Methods of Proofs

Introduction

Proofs in mathematics are valid arguments that establish the truth of mathematical statements. To deduce new statements from statements we already have, we use rules of inference. Rules of inference are our basic tools for establishing the truth of statements.

Before we study mathematical proofs, we will look at arguments that involve only compound propositions. We define what it means for an argument involving compound propositions to be valid. Then we will introduce a collection of rules of inference in propositional logic. These rules of inference are among the most important ingredients in producing valid arguments. Later we illustrate how rules of inference are used to produce valid arguments.

After studying rules of inference in propositional logic, we will introduce rules of inference for quantified statements. We will describe how these rules of inference can be used to produce valid arguments. These rules of inference for statements involving existential and universal quantifiers play an important role in proofs in computer science and mathematics, although they are often used without being explicitly mentioned.

Finally, we will show how rules of inference for propositions and for quantified statements can be combined. These combinations of rule of inference are often used together in complicated arguments.

Later in this chapter we develop different proof methods that will enable us to prove many different types of results. After introducing many different methods of proof, we will introduce several strategies for constructing proofs.

3.1 Rules of Inference

Definition 3.1. *An* **argument** *in propositional logic is a sequence of propositions. All but the final proposition in the argument are called* **premises** *and the final proposition is called the* **conclusion***.*

An argument is **valid** if the truth of all its premises implies that the conclusion is true. An **argument form** in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is valid no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

"If you have a current password, then you can log onto the network."

"You have a current password."

Therefore,

"You can log onto the network."

We would like to determine whether this is a valid argument. That is, we would like to determine whether the conclusion "You can log onto the network" must be true when the premises "If you have a current password, then you can log onto the network" and "You have a current password" are both true.

Before we discuss the validity of this particular argument, we will look at its form. Use p to represent

"You have a current password" and q to represent "You can log onto the network." Then, the argument has the form

$$\begin{array}{l} p \to q \\ \underline{\quad\; p} \\ \;\therefore q \end{array}$$

where $\therefore$ is the symbol that denotes "therefore".

We know that when p and q are propositional variables, the statement $((p \to q) \wedge p) \to q$ is a tautology. In particular, when both $p \to q$ and p are true, we know that q must also be true. We say this form of argument is valid because whenever all its premises (all statements in the argument other than the final one, the conclusion) are true, the conclusion must also be true. Now suppose that both "If you have a current password, then you can log onto the network" and "You have a current password" are true statements. When we replace p by "You have a current password" and q by "You can log onto the network," it necessarily follows that the conclusion "You can log onto the network" is true. This argument is valid because its form is valid. Note that whenever we replace p and q by propositions where $p \to q$ and p are both true, then q must also be true.

In our discussion, to analyze an argument, we replaced propositions by propositional variables. This changed an argument to an argument form. We saw that the validity of an argument follows from the validity of the form of the argument.

From the definition of a valid argument form we see that the argument form with premises $p_1, p_2, ..., p_n$ and conclusion q is valid, when $(p_1 \wedge p_2 \wedge ... \wedge p_n) \to q$ is a tautology. The key to showing that an argument in propositional logic is valid is to show that its argument form is valid. Consequently, we would like techniques to show that argument forms are valid. We will now develop methods for accomplishing this task.

3.1.1 Rules of Inference for Propositional Logic

We can always use a truth table to show that an argument form is valid. We do this by showing that whenever the premises are true, the conclusion must also be true. However, this can be a tedious approach. Instead, we can first establish the validity of some relatively simple argument forms, called **rules of inference**. These rules of inference can be used as building blocks to construct more complicated valid argument forms. We will now introduce the most important rules of inference in propositional logic.
The tautology $(p \wedge (p \to q)) \to q$ is the basis of the rule of inference called **modus ponens**, or the **law of detachment**. This tautology leads to the following valid argument form, which we have already seen in our initial discussion about arguments:

$$\begin{array}{r} p \to q \\ \underline{\quad p} \\ \therefore q \end{array}$$

In particular, modus ponens tells us that if a conditional statement and the hypothesis of this conditional statement are both true, then the conclusion must also be true. Following example illustrates the use of modus ponens.

Example 3.1. Suppose that the conditional statement "If it snows today, then we will go skiing" and its hypothesis, "It is snowing today," are true. Then, by modus ponens, it follows that the conclusion of the conditional statement, "We will go skiing," is true.

As we mentioned earlier, a valid argument can lead to an incorrect conclusion if one or more of its premises is false. We illustrate this in next example.

Example 3.2. Determine whether the argument given here is valid and determine whether its conclusion must be true because of the validity of the argument.
If $\sqrt{2} > \frac{3}{2}$, then $(\sqrt{2})^2 > \left(\frac{3}{2}\right)^2$. We know that $\sqrt{2} > \frac{3}{2}$.
Consequently, $\left(\sqrt{2}\right)^2 = 2 > \left(\frac{3}{2}\right)^2 = \left(\frac{9}{4}\right)$.
Solution: Let p be the proposition "$\sqrt{2} > \frac{3}{2}$" and q the proposition "$2 > \left(\frac{3}{2}\right)^2$". The premises of the argument are $p \to q$ and p, and q is its conclusion. This argument is valid because it is constructed by using modus ponens, a valid argument form. However, one of its premise, $\sqrt{2} > \frac{3}{2}$, is false. Consequently, we cannot conclude that the conclusion is true. Furthermore, note that the conclusion of this argument is false, because $2 < \frac{9}{4}$.

There are many useful rules of inference for propositional logic. Perhaps the most widely used of these are listed in Table 3.1.

Example 3.3. State which rule of inference is the basis of the following argument: "It is below freezing now. Therefore, it is either below freezing or raining now."
Solution: Let p be the proposition "It is below freezing now" and q the proposition "It is raining now." Then this argument is of the form

$$\frac{p}{\therefore p \vee q}$$

This is an argument that uses the addition rule(refer to Table 3.1).

Example 3.4. State which rule of inference is the basis of the following argument:
"It is below freezing and raining now. Therefore, it is below freezing now."
Solution: Let p be the proposition "It is below freezing now," and let q be the proposition "It is raining now." This argument is of the form

Rule of Inference	**Tautology**	**Name**
p $\underline{p \to q}$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $\underline{p \to q}$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $\underline{q \to r}$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\underline{\neg p}$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q))$	Disjunctive syllogism
$\underline{p \quad}$ $\therefore p \vee q$	$p \to (p \to q)$	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p $\underline{q \quad}$ $\therefore p \wedge q$	$(p \wedge q) \to (p \to q)$	Conjunction
$p \vee q$ $\underline{\neg p \vee r}$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Table 3.1: Rules of Inference.

$$\begin{array}{rl} & \underline{p \wedge q} \\ \therefore & p \end{array}$$

This argument uses the simplification rule.

Example 3.5. State which rule of inference is used in the argument: "If it rains today, then we will not have a barbecue(outdoor meal) today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow."

Solution: Let p be the proposition "It is raining today," let q be the proposition "We will not have a barbecue today," and let r be the proposition "We will have a barbecue tomorrow." Then this argument is of the form

$$\begin{array}{rl} & p \rightarrow q \\ & \underline{q \rightarrow r} \\ \therefore & p \rightarrow r \end{array}$$

Hence, this argument is a hypothetical syllogism.

Using Rules of Inference to Build Arguments

When there are many premises, several rules of inference are often needed to show that an argument is valid. This is illustrated in following examples, where the steps of arguments are displayed on separate lines, with the reason for each step explicitly stated. These examples also show how arguments in English can be analyzed using rules of inference.

Example 3.6. Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a boat trip," and "If we take a boat trip, then we will be home by sunset lead to the conclusion." We will be home by sunset."

Solution: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a boat trip," and t the proposition "We will be home by sunset." Then the premises become $\neg p \wedge q, r \to p, \neg r \to s$, and $s \to t$. The conclusion is simply t. We need to give a valid argument with premises $\neg p \wedge q, r \to p, \neg r \to s, s \to t$ and conclusion t.

We construct an argument to show that our premises lead to the desired conclusion as follows:

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \to p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \to s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \to t$	Premise
8. t	Modus ponens using (6) and (7)

3.1.2 Rules of Inference for Quantified Statements

We have discussed rules of inference for propositions. We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x P(x)$. Universal instantiation is used when we conclude from the statement "All women are wise" that "Lisa is wise," where Lisa is a member of the domain of all women.

Universal generalization is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. Universal generalization is used when we show that $\forall x P(x)$ is true by taking an arbitrary element c from the domain and showing that $P(c)$ is true. The element c that we select must be an arbitrary, and not a specific, element of the domain. That is, when we assert from $\forall x P(x)$ the existence of an element c in the domain, we have no control over c and cannot make any other assumptions about c other than it comes from the domain.

Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\forall x P(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true. Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument.

Existential generalization is the rule of inference that is used to conclude that $\forall x P(x)$ is true when a particular element c with $P(c)$ true is known. That is, if we know one element c in the domain for which $P(c)$ is true, then we know that $\forall x P(x)$ is true.
We summarize these rules of inference in Table 3.2. We will illustrate how some of these rules of inference for quantified statements are used in following examples.

Rule of Inference	**Name**
$\underline{\forall x P(x)}$ $\therefore P(c)$	Universal instantiation
$\underline{P(c) \text{ for an ordinary c}}$ $\therefore \forall x P(x)$	Universal generalization
$\underline{\exists x P(x)}$ $\therefore P(c)$ for some element c	Existential instantiation
$\underline{P(c) \text{ for some element c}}$ $\therefore \exists x P(x)$	Existential generalization

Table 3.2: Rules of Inference for quantified statements.

Example 3.7. Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Maria is a student in this class" imply the conclusion "Maria has taken a course in computer science."

Solution: Let $D(x)$ denote "x is in this discrete mathematics class," and let $C(x)$ denote "x has taken a course in computer science." Then the premises are $\forall x(D(x) \to C(x))$ and $D(\text{Maria})$. The conclusion is $C(\text{Maria})$.

The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(D(x) \to C(x))$	Premise
2. $D(\text{Maria}) \to C(\text{Maria})$	Universal instantiation from (1)
3. $D(\text{Maria})$	Premise
4. $C(\text{Maria})$	Modus ponens from (2) and (3)

Exercises

1. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

   If Socrates is human, then Socrates is mortal.

   Socrates is human.

   $\therefore$ Socrates is mortal.

2. What rules of inference are used in this famous argument? "All men are mortal. Socrates is a man. Therefore, Socrates is mortal."

3. For each of these collections of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

   a) "If I take the day off, it either rains or snows." "I took Tuesday off or I took Thursday off." "It was sunny on Tuesday." "It did not snow

on Thursday."
b) "If I eat spicy foods, then I have strange dreams." "I have strange dreams if there is thunder while I sleep." "I did not have strange dreams."

4. What is wrong with this argument? Let $H(x)$ be "x is happy." Given the premise $\exists x H(x)$, we conclude that H(Lola). Therefore, Lola is happy.

5. Use rules of inference to show that if $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$ and $\forall x(P(x) \wedge R(x))$ are true, then $\forall x(R(x) \wedge S(x))$ is true.

Hints and Solutions to Selected Problems:

1. Modus ponens; valid; the conclusion is true, because the hypotheses are true.

2. Universal instantiation is used to conclude that "If Socrates is a man, then Socrates is mortal." Modus ponens is then used to conclude that Socrates is mortal.

3. a) Valid conclusions are "I did not take Tuesday off," "I took Thursday off," "It rained on Thursday." b) "I did not eat spicy foods and it did not thunder" is a valid conclusion.

4. We know that some x exists that makes $H(x)$ true, but we cannot conclude that Lola is one such x.

3.2 Introduction to Proofs

In this section we introduce the notion of a proof and describe methods for constructing proofs. A **proof** is a valid argument that establishes the truth of a mathematical statement. A proof can use the hypotheses of the theorem, if any, axioms assumed to be true, and previously proven theorems. Using these ingredients and rules of inference, the final step of the proof establishes the truth of the statement being proved.

The methods of proof discussed in this chapter are important not only because they are used to prove mathematical theorems, but also for their many applications to computer science. These applications include verifying that computer programs are correct, establishing that operating systems are secure, making inferences in artificial intelligence, showing that system specifications are consistent, and so on.

Formally, a **theorem** is a statement that can be shown to be true. In mathematical writing, the term theorem is usually reserved for a statement that is considered at least somewhat important. Less important theorems sometimes are called **propositions**. A mathematical system consists of axioms, definitions, and theorems. We demonstrate that a theorem is true with a **proof**. A proof is a valid argument that established the truth of a theorem. An **axiom** is a statement that is assumed to be true. A less important theorem that is helpful in the proof of other results is called a **lemma**. A **corollary** is a theorem that can be established directly from a theorem that has been proved. A **conjecture** is a statement that is being proposed to be a true statement.

3.2.1 Methods of Proving Theorems

Proving mathematical theorems can be difficult. To construct proofs we need all available ammunition, including a powerful battery of different proof methods. These methods provide the overall approach and strategy of proofs. Understanding these methods is a key component of learning how to read and construct mathematical proofs. Once we have chosen a method of proof, we use axioms, definitions of terms, previously proved results, and rules of inference to complete the proof. We will also assume the usual axioms whenever we prove a result about geometry.

Direct Proofs

A direct proof of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true. A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.

We will provide examples of several different direct proofs.

Example 3.8. Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution: Note that this theorem states $\forall n(P(n) \to Q(n))$, where $P(n)$ is "n is an odd integer" and $Q(n)$ is "n^2 is odd." As we have said, we will follow the usual convention in mathematical proofs by showing that $P(n)$ implies $Q(n)$. To begin a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that n is odd. By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer. We want to show that n^2 is also odd. We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses n^2. When we do this, we find that $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. By the definition of an odd integer, we can conclude that n^2 is an odd integer. Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer.

Example 3.9. Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer a is a perfect square if there is an integer b such that $a = b^2$.)

Solution: To produce a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that m and n are both perfect squares. By the definition of a perfect square, it follows that there are integers s and t such that $m = s^2$ and $n = t^2$. The goal of the proof is to show that mn must also be a perfect square when m and n are; looking ahead we see how we can show this by substituting s^2 for m and t^2 for n into mn. This tells us that $mn = s^2t^2$. Hence, $s^2t^2 = (ss)(tt) = (st)(st) = (st)^2$, using commutativity and associativity of multiplication. By the definition of perfect square, it follows that mn is also a perfect square, because it is the square of st, which is an integer. We have proved that if m and n are both perfect squares, then mn is also a perfect square.

Proof by Contraposition

Direct proofs lead from the premises of a theorem to the conclusion. They begin with the premises, continue with a sequence of deductions, and end with the conclusion. However, we will see that attempts at direct proofs often reach dead ends. We need other methods of proving theorems of the form $\forall x(P(x) \to Q(x))$. Proofs of theorems of this type that are not direct proofs, that is, that do not start with the premises and end with the conclusion, are called indirect proofs. An extremely useful type of indirect proof is known as proof by contraposition. Proofs by contraposition make use of the fact that the conditional statement $p \to q$ is equivalent to its contraposition, $\neg q \to \neg p$. This means that the conditional statement $p \to q$ can be proved by showing that its contraposition, $\neg q \to \neg p$, is true.

Example 3.10. Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution: We first attempt a direct proof. To construct a direct proof, we first assume that $3n + 2$ is an odd integer. This means that $3n + 2 = 2k + 1$

for some integer k. Can we use this fact to show that n is odd? We see that $3n + 1 = 2k$, but there does not seem to be any direct way to conclude that n is odd. Because our attempt at a direct proof failed, we next try a proof by contraposition.

The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $3n+2$ is odd, then n is odd" is false; namely, assume that n is even. Then, by the definition of an even integer, $n = 2k$ for some integer k. Substituting $2k$ for n, we find that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. This tells us that $3n + 2$ is even. This is the negation of the premise of the theorem. Thus we have proved the theorem "If $3n + 2$ is odd, then n is odd."

Example 3.11. If n is an integer such that n^2 is odd then n is also odd.

Solution: Suppose that n is an integer that is even. Then there exists an integer k such that $n = 2k$. But then $n^2 = 2(2k^2)$ which is even. This is negation of the hypothesis.

Vacuous and Trivial Proofs

A vacuous proof is a proof of an implication $p \to q$ in which it is shown that p is false. Consequently, if we can show that p is false, then we have a proof, called a vacuous proof, of the conditional statement $p \to q$.

Example 3.12. Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Solution: Note that $P(0)$ is "If $0 > 1$, then $0^2 > 0$." We can show $P(0)$ using a vacuous proof. Indeed, the hypothesis $0 > 1$ is false. This tells us that $P(0)$ is automatically true.

Example 3.13. Use the method of vacuous proof to show that if $x \in \phi$; then David is playing pool.

Solution: Since the proposition $x \in \phi$; is always false, the given proposition is vacuously true.

Proofs by Contradiction

Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \rightarrow q$ is true. Because q is false, but $\neg p \rightarrow q$ is true, we can conclude that $\neg p$ is false, which means that p is true. Because a proof by contradiction does not prove a result directly, it is another type of indirect proof. We provide three examples of proof by contradiction.

Example 3.14. Show that at least four of any 22 days must fall on the same day of the week.

Solution: Let p be the proposition "At least four of 22 chosen days fall on the same day of the week." Suppose that $\neg p$ is true. This means that at most three of the 22 days fall on the same day of the week. Because there are seven days of the week, this implies that at most 21 days could have been chosen, as for each of the days of the week, at most three of the chosen days could fall on that day. This contradicts the premise that we have 22 days under consideration. That is, if r is the statement that 22 days are chosen, then we have shown that $\neg p \rightarrow (r \wedge \neg r)$. Consequently, we know that p is true. We have proved that at least four of 22 chosen days fall on the same day of the week.

Example 3.15. If n^2 is an even integer so is n.

Solution: Suppose the contrary. That is suppose that n is odd. Then there is an integer k such that $n = 2k + 1$. In this case, $n^2 = 2(2k^2 + 2k) + 1$ is odd and this contradicts the assumption that n^2 is even. Hence, n must be even.

Proofs of Equivalence

To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we show that $p \rightarrow q$ and $q \rightarrow p$ are both true. The validity of this approach is based on the tautology $(p \leftrightarrow q) \leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$.

Example 3.16. Prove the theorem "If n is an integer, then n is odd if and only if n^2 is odd."

Solution: This theorem has the form "p if and only if q" where p is "n is odd" and q is "n^2 is odd." To prove this theorem, we need to show that $p \to q$ and $q \to p$ are true. We have already shown in Example 3.8 $p \to q$ is true and that $q \to p$ is true can be proved by direct method (left as an exercise). Hence, we have shown that the theorem is true.

Sometimes a theorem states that several propositions are equivalent. Such a theorem states that propositions $p_1, p_2, p_3, \ldots, p_n$ are equivalent. This can be written as $p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n$, which states that all n propositions have the same truth values, and consequently, that for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$, p_i and p_j are equivalent.

Example 3.17. Show that these statements about the integer n are equivalent:

$p_1 : n$ is even. $p_2 : n - 1$ is odd. $p_3 : n^2$ is even.

Solution: We will show that these three statements are equivalent by showing that the conditional statements $p_1 \to p_2, p_2 \to p_3$, and $p_3 \to p_1$ are true. We use a direct proof to show that $p_1 \to p_2$. Suppose that n is even. Then $n = 2k$ for some integer k. Consequently, $n - 1 = 2k - 1 = 2(k - 1) + 1$. This means that $n - 1$ is odd because it is of the form $2m + 1$, where m is the integer $k - 1$.

We also use a direct proof to show that $p_2 \to p_3$. Now suppose $n - 1$ is odd. Then $n - 1 = 2k + 1$ for some integer k. Hence, $n = 2k + 2$ so that $n^2 = (2k+2)^2 = 4k^2+8k+4 = 2(2k^2+4k+2)$. This means that n^2 is twice the integer $2k^2 + 4k + 2$, and hence is even.

To prove $p_3 \to p_1$, we use a proof by contraposition. That is, we prove that if n is not even, then n^2 is not even. This is the same as proving that if n is odd, then n^2 is odd, which we have already done. This completes the proof.

Counterexamples

In chapter 2, we stated that to show that a statement of the form $\forall x P(x)$ is false, we need only find a counterexample, that is, an example x for which $P(x)$ is false. We illustrate the use of counterexamples.

Example 3.18. Show that the statement "Every positive integer is the sum of the squares of two integers" is false.

Solution: To show that this statement is false, we look for a counterexample, which is a particular integer that is not the sum of the squares of two integers. It does not take long to find a counterexample, because 3 cannot be written as the sum of the squares of two integers. To show this is the case, note that the only perfect squares not exceeding 3 are $0^2 = 0$ and $1^2 = 1$. Furthermore, there is no way to get 3 as the sum of two terms each of which is 0 or 1. Consequently, we have shown that "Every positive integer is the sum of the squares of two integers" is false.

Exercises

1. Use a direct proof to show that the sum of two odd integers is even.

2. Show that the square of an even number is an even number using a direct proof.

3. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

4. Prove the proposition $P(0)$, where $P(n)$ is the proposition If n is a positive integer greater than 1, then $n^2 > n$. What kind of proof did you use?

5. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a + b)^n = a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

6. Show that at least ten of any 64 days chosen must fall on the same day of the week.

7. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$. [Hint: Assume that $r = \frac{a}{b}$ is a root, where a and b are integers and $\frac{a}{b}$ is in lowest terms. Obtain an equation involving integers by multiplying by b^3. Then look at whether a and b are each odd or even.]

8. Show that these statements about the integer x are equivalent: (i) $3x + 2$ is even, (ii) $x + 5$ is odd, (iii) x^2 is even.

9. Show that these statements about the real number x are equivalent: (i) x is irrational, (ii) $3x + 2$ is irrational, (iii) $\frac{x}{2}$ is irrational.

Chapter 4

Basics of Counting

Introduction

The origin of combinatorics goes far back in history. Magic squares (arrays where columns, rows and diagonals all sum to the same number) were popular subjects of mathematical study in medieval times. Jewish and Arab mathematicians in the early middle ages focused on combinatorial problems that counted the number of possibilities in a situation and evaluated their probability. This subject was studied in the seventeenth century, when combinatorial questions arose in the study of gambling games. Combinatorial approach to problem solving appears in the works of leading mathematicians such as Fibonacci, Pascal, Fermat and Euler. In modern times, the works of J. J. Sylvester (late 19^{th} century) and Percy MacMahon (early 20^{th} century) laid the foundation for enumerative and algebraic combinatorics. In the second half of 20^{th} century, combinatorics enjoyed a rapid growth. The growth was spurred by new connections and applications to other fields, ranging from algebra to probability, from functional analysis to number theory, etc. These connections shed the boundaries between combinatorics and parts of mathematics and theoretical computer science, but at the same time led to a partial fragmentation of the field.

Combinatorics is that part of mathematics which deals with counting and enumeration of specified objects, patterns or designs. Counting is also required to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand. Recently, it has played a key role in mathematical biology, especially in sequencing DNA.

4.1 Product and Sum Rule

Suppose that a password on a computer system consists of six, seven or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit. How many such passwords are there?

The techniques needed to answer this question and a wide variety of other counting problems will be introduced in this section. Here we study two basic counting principles, the product rule and the sum rule.

4.1.1 The Product Rule

Product Rule : Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \times n_2$ ways to do the procedure.
Note: The way to perform the second task does not depend on the way in which the first task is performed.

Example 4.1. Chairs of an auditorium are to be labelled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labelled differently?
Solution: The procedure of labelling a chair consists of two tasks, namely,

assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \times 100 = 2600$ different ways that a chair can be labelled. Therefore, the largest number of chairs that can be labelled differently is 2600.

Example 4.2. There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

Solution: The procedure of choosing a port consists of two tasks, first picking a microcomputer and then picking a port on this microcomputer. Because there are 32 ways to choose the microcomputer and 24 ways to choose the port no matter which microcomputer has been selected, the product rule shows that there are $32 \times 24 = 768$ ports.

Example 4.3. A new company with just two employees, Anil and Neel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution: The procedure of assigning offices to these two employees consists of assigning an office to Anil, which can be done in 12 ways, then assigning an office to Neel different from the office assigned to Anil and which can be done in 11 ways. By the product rule, there are $12 \times 11 = 132$ ways to assign offices to these two employees.

An extended version of the product rule is often useful.

Generalized Product Rule : Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_m$ in sequence. If each task T_i, $i = 1, 2, \ldots, m$, can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \times n_2 \times \cdots \times n_m$ ways to carry out the procedure.

Example 4.4. A certain type of car can be purchased in any of five colours, with a manual or automatic transmission, and with any of three engine sizes.

How many different car packages are available?
Solution: We can select colour in 5 ways, we can select transmission type in 2 ways, we can select engine type in 3 ways. Therefore by generalized product rule there are $5 \times 2 \times 3 = 30$ car packages available.

Example 4.5. Let L be the set of Washington state license plates, three numbers followed by three letters. How many license plates are in the set?
Solution: Each letter on license plate can be selected in 26 ways, each digit on license plate can be selected in 10 ways. Therefore by multiplication principle there are $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$ Washington state license plates.

Example 4.6. In the above example if letters and digits on license plate can not be repeated, then find the number of possible license plates.
Solution: First letter on license plate can be selected in 26 ways. Since there is no repetition, second letter on license plate can be selected in 25 ways, third letter on license plate can be selected in 24 ways. First digit on license plate can be selected in 10 ways, second digit on license plate can be selected in 9 ways, third digit on license plate can be selected in 8 ways. Therefore by multiplication principle there are $26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,000$ required license plates.

Example 4.7. How many different 4-letter radio station call letters (upper case) can be made
a) if the first letter must be a K or W and no letter may be repeated?
b) if repeats are allowed (but the first letter is a K or W).
c) How many of the 4-letter call letters (starting with K or W) with no repeats end in R.
Solution: a) Since first letter is K or W, there are 2 ways to select first letter. Since there is no repetition, there are 25 ways to select second letter, 24 ways to select third letter, 23 ways to select fourth letter. By multiplication

principle, there are $2 \times 25 \times 24 \times 23 = 27,600$ radio station call letters.
b) Since first letter is K or W, there are 2 ways to select first letter. Since repetition is allowed, there are 26 ways to select second letter, 26 ways to select third letter, 26 ways to select fourth letter. By multiplication principle, there are $2 \times 26 \times 26 \times 26 = 35,152$ radio station call letters.

$$\underline{\ 2\ } \times \underline{\ 24\ } \times \underline{\ 23\ } \times \underline{\ 1\ }$$

c) The last place can be filled in 1 way(with R). Since the first letter is K or W, there are 2 ways to select first letter. Since repetition is not allowed, there are 24 ways to select the second letter and 23 ways to select the third letter. By multiplication principle, there are $2 \times 24 \times 23 \times 1$ radio station call letters that can be made.

Example 4.8. How many different bit (each bit is either 0 or 1) strings of length seven are there?
Solution: Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven.

Theorem 4.1. (Counting Functions) *The number of functions from a set with r elements to a set with n elements is n^r.*

Proof. A function corresponds to a choice of one of the n elements in the codomain for each of the r elements in the domain. Hence, by the product rule there are $\underbrace{n \times n \times \ldots \times n}_{r\ times} = n^r$ functions from a set with r elements to one with n elements. □

Theorem 4.2. (Counting One-to-One Functions) *The number of one-to-one functions from a set with r elements to a set with n elements is $n \times (n-1) \times (n-2) \times ... \times (n-r+1)$*

Proof. First note that when $r > n$ there are no one-to-one functions from a set with r elements to a set with n elements.
Now let $r \leq n$. Suppose the elements in the domain are $a_1, a_2, ..., a_r$. There are n ways to choose the value of the function at a_1. Because the function is one-to-one, the value of the function at a_2 can be chosen in $n - 1$ ways (because the value used for a_1 cannot be used again). In general, the value of the function at a_k can be chosen in $n-(k-1)$ ways. By the product rule, there are $n \times (n-1) \times (n-2) \times ... \times (n-r+1)$ one-to-one functions from a set with r elements to one with n elements. □

Theorem 4.3. (Counting Subsets of a Finite Set) *The number of different subsets of a finite set X with n elements is 2^n.*

Proof. Let $X = \{a_1, a_2, ..., a_n\}$ be a finite set. For any subset A of X we define bit-string $S_A = b_1 b_2 ... b_n$, where $b_i = 0$ if $a_i \notin A$ and $b_i = 1$ if $a_i \in A$. define function ϕ from power set of X to set of all bit strings of length n as below.
$\phi(A) = S_A$ for all $A \subseteq X$. Note that ϕ is one-one and onto function. Therefore number of subsets of X is number of bit strings of length n. By the product rule, there are 2^n bit strings of length n. Hence total number of subsets of $X = 2^n$. □

Note: The product rule is often phrased in terms of sets in the following way.
If $A_1, A_2, ..., A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set. To relate this to the product rule, note that the task of choosing an element in the Cartesian product $A_1 \times A_2 \times \ldots \times A_m$ is done by choosing an element in A_1, an element in $A_2, \ldots,$ and an element in A_m. By the product rule it follows that $|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \times |A_2| \times \ldots \times |A_m|$.

4.1.2 The Sum Rule

The Sum Rule : If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example 4.9. Suppose there are 5 different types of burgers and 8 different types of pizzas. How many selections does a customer have ?
Solution: There are 5 choices for the burgers and 8 choices for the pizzas. We have to select one burger or one pizza. By addition principle there are $5 + 8 = 13$ possible selections.

We can extend the sum rule to more than two tasks.
Generalized Sum Rule :
Suppose that a task can be done in one of n_1 ways, in one of n_2 ways,. . . , or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m$. Then the number of ways to do the task is $n_1 + n_2 + \ldots + n_m$.

Example 4.10. Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?
Solution: There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of mathematics faculty is never same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick this representative.

Example 4.11. A student can choose a project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is in more than one list. How many possible projects are there to choose from?

Solution: The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is in more than one list, by the sum rule there are $23+15+19 = 57$ ways to choose a project.

Example 4.12. Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length $6, 7$, and 8, respectively. By the sum rule, $P = P_6 + P_7 + P_8$. We will now find P_6, P_7, and P_8. Finding P_6 directly is difficult. To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits. By the product rule, the number of strings of six characters is 36^6, and the number of strings with no digits is 26^6. Hence,

$P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560$. Similarly, we have $P_7 = 36^7 - 26^7 = 78,364,164,096 - 8,031,810,176 = 70,332,353,920$ and $P_8 = 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576 = 2,612,282,842,880$. Consequently, $P = P_6 + P_7 + P_8 = 2,684,483,063,360$.

4.2 The Division Rule

We have introduced the product and sum rules for counting. You may wonder whether there is also a division rule for counting. In fact, there is such a rule, which can be useful when solving certain types of enumeration problems.

The Division Rule : There are $\frac{n}{d}$ ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly

d of the n ways correspond to way w.

We can restate the division rule in terms of sets: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = \frac{|A|}{d}$.

We can also formulate the division rule in terms of functions: If f is a function from A to B where A and B are finite sets, and that for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$ (in which case, we say that f is d -to-one), then $|B| = \frac{|A|}{d}$.

Example 4.13. How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

Solution: We arbitrarily select a seat at the table and label it seat 1. We number the rest of the seats in numerical order, proceeding clockwise around the table. Note that there are four ways to select the person for seat 1, three ways to select the person for seat 2, two ways to select the person for seat 3, and one way to select the person for seat 4. Thus, there are $4! = 24$ ways to order the given four people for these seats. However, each of the four choices for seat 1 leads to the same arrangement, as we distinguish two arrangements only when one of the people has a different immediate left or immediate right neighbor. Because there are four ways to choose the person for seat 1, by the division rule there are $\frac{24}{4} = 6$ different seating arrangements of four people around the circular table.

Exercises

1. There are 18 mathematics majors and 325 computer science majors at a college.

   a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

   b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

2. A multiple-choice test contains 10 questions. There are four possible answers for each question.
   a) In how many ways can a student answer the questions on the test if the student answers every question?
   b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

3. Six different airlines fly from Chennai to Mumbai and seven fly from Mumbai to Delhi. How many different pairs of airlines can you choose on which to book a trip from Chennai to Delhi via Mumbai, when you pick an airline for the flight to Mumbai and an airline for the continuation flight to Delhi?

4. How many different three-letter initials can people have?

5. How many different three-letter initials are there that begin with an A?

6. How many bit strings with length not exceeding n, where n is a positive integer, consist entirely of $1's$, not counting the empty string?

7. How many strings of five $ASCII$ characters contain the character 'a' at least once? [Note: There are 128 different $ASCII$ characters.]

8. How many 6-element RNA sequences(A,C,G,U sequences)
   a) do not contain U?
   b) end with GU?
   c) start with C?
   d) contain only A or U?

9. How many positive integers between 100 and 999 inclusive

   a) are divisible by 7?

   b) are odd?

   c) have the same three decimal digits?

   d) are not divisible by 4?

10. How many strings of three decimal digits

    a) do not contain the same digit three times?

    b) begin with an odd digit?

    c) have exactly two digits that are $4's$?

11. A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

12. How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

13. How many license plates can be made using either two or three uppercase English letters followed by either two or three digits?

14. How many strings of eight English letters are there

    a) that contain no vowels, if letters can be repeated?

    b) that contain no vowels, if letters cannot be repeated?

    c) that start with vowel, if letters can be repeated?

    d) that start with a vowel, if letters cannot be repeated?

15. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

    a) 4 b) 5 c) 6 d) 7

Hints and Solutions

1. a) 5850 b) 343

2. a) 410 b) 510

3. 42

4. 263

5. 676

6. $n+1$ (counting the empty string)

7. 1,321,368,961

8. a) 729 b) 256 c) 1024 d) 64

9. a) 128 b) 450 c) 9 d) 675

10. a) 990 b) 500 c) 27

11. 350

12. 52,457,600

13. 20,077,200

14. a) 37,822,859,361 b) 8,204,716,800
    c) 40,159,050, 880 d) 12,113,640,000

15. a) 0 b) 120 c) 720 d) 2520

4.3 Permutations and Combinations of Distinguishable Objects

In this section we shall extend the sum and product rule studied in the previous section. Given a set of objects the problems involving arrangement of a subset according to some specification or selecting a subset as per given specification shall be discussed in this section.
Consider the following problems:

1. In how many ways can we arrange 4 out of 7 students for a photograph?

2. In how many ways can we form a committee of 4 out of 5 students?

In the above two problems one may observe that in the first problem we have to find the number of ways to arrange a specified number of distinguishable elements(in this case students to stand in a row for a photograph) of a set of a particular size (4 out of 7), where the order of these elements matters. We can solve the second problem by finding the number of ways to select a particular number of elements(in this case students to form a committee) from a set of a particular size (4 out of 5), where the order of the elements selected does not matter. Many counting problems involve these two basic ideas i.e. the idea of arrangement/selection.

4.3.1 Permutations without Repetition

Let us begin by solving the first problem.

Example 4.14. In how many ways can we arrange 4 out of the 7 students for a photograph?
Solution: We note that the order in which we select the students matters.

Let us begin by placing the students from left to right in a line for a photograph.

$$\underline{\ 7\ } \times \underline{\ 6\ } \times \underline{\ 5\ } \times \underline{\ 4\ }$$

There are seven ways to select the first student to stand at the leftmost position of the line. Once this student has been selected, then there are six ways to select the second student in the line. After the first and second student have been selected, there are five ways to select the third student in the line and then 4 ways to select the fourth student who will be in the rightmost position. By the product rule, there are $7 \times 6 \times 5 \times 4 = 840$ ways to select four students from a group of seven students to stand in a line for a photograph.

In case we had to arrange all the seven students in a line for a photograph the leftmost position would have been filled in 7 ways, the second in 6 ways, the third in 5 ways, the fourth in 4 ways, the fifth in 3 ways, the sixth in 2 ways and the last in 1 way. So by the product rule the number of ways for 7 students to stand in a line for a photograph would be $7\times6\times5\times4\times3\times2\times1 = 7!$ ways.
The above example illustrates how ordered arrangements of distinct objects can be counted. This leads to the idea of permutation.

Definition 4.1. *A **permutation** of a set of distinct objects is an ordered arrangement of these objects.*

We also are interested in ordered arrangements of some of the elements of a set.

Definition 4.2. *An ordered arrangement of r elements of a set is called an **r-permutation**.*

Remark 4.1. *Unless specifically mentioned, a permutation will always mean a linear permutation throughout this book.*

Example 4.15. Let $S = \{1, 2, 3\}$. The ordered arrangement $(2, 1, 3)$ is a permutation of S. The ordered arrangement $(2, 3)$ is a 2-permutation of S.

Remark 4.2. *An n-permutation of set S with n elements is simply referred to as a permutation of set S.*

The number of r-permutations of a set with n elements is denoted by $P(n, r)$.

Example 4.16. Let $S = \{a, b, c\}$. Find all the 2-permutations of S.
Solution: The 2-permutations of S are the ordered arrangements (a, b); (b, a); (a, c); (c, a); (b, c) and (c, b). There are exactly six 2-permutations of the set S. In fact there are always exactly six 2-permutations of a set with three elements since there are three ways to choose the first element of the arrangement and there are two ways to choose the second element of the arrangement, because it must be different from the first element. Hence, by the product rule, we see that $P(3, 2) = 3 \times 2 = 6$.

We now use the product rule to find a formula for $P(n, r)$ whenever n and r are positive integers with $1 \leq r \leq n$.

Theorem 4.4. *If n is a positive integer and r is an integer with $1 \leq r \leq n$, then the number of r-permutations of a set with n distinct elements is*

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1).$$

Proof: We will use the product rule to prove the formula. The first element of the permutation can be chosen in n ways since there are n distinct elements in the set. There are $n-1$ ways to choose the second element of the permutation, because there are $n-1$ elements left in the set after using the element picked for the first position. Similarly, there are $n-2$ ways to choose the third element, and so on, until there are exactly $n-(r-1) = n-r+1$ ways to choose the r^{th} element. Consequently, by the product rule, there are $n(n-1)(n-2)\cdots(n-r+1)$ r- permutations of the set.

Note that $P(n, 0) = 1$ whenever n is a nonnegative integer because there is exactly one way to order zero elements. That is, there is exactly one list with no elements in it, namely the empty list.

We now state a useful corollary of the above theorem.

Corollary 4.4.1. *If n and r are integers with $0 \leq r \leq n$, then*

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Proof: When n and r are integers with $1 \leq r \leq n$, then by Theorem 4.4 we have

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1). \tag{4.1}$$

By multiplying and dividing RHS of equation (4.1) by $(n-r)!$ we get,

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Also since

$$P(n, 0) = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1.$$

whenever n is a nonnegative integer, we see that the formula

$$P(n, r) = \frac{n!}{(n-r)!}$$

also holds when $r = 0$.

Note that if n is a positive integer then $P(n, n) = n!$

Let us now look at a few more examples:

Example 4.17. How many ways are there to select a captain and a vice captain from 15 members of a cricket team?

Solution: Because it matters who becomes the captain and who becomes the vice-captain, the number of ways to pick the captain and vice-captain is the number of ordered selections of two elements from a set of 15 elements, that is, the number of 2-permutations of a set of 15 elements. Consequently, the answer is $P(15, 2) = 15 \times 14 = 210$.

Example 4.18. Suppose that there are eight runners in a race. How many ways are there to award a gold medal(to the runner finishing the race first), a silver and a bronze medal(to the runners finishing second and third respectively) if all possible outcomes of the race can occur and there are no ties?
Solution: The number of different ways to award the medals is the number of 3-permutations of a set with eight elements. Hence, there are $P(8,3) = 8 \times 7 \times 6 = 336$ possible ways to award the medals.

Example 4.19. Suppose that Sachin has to visit eight different cities. He must begin his trip in a specified city, but later he can visit the other seven cities in any order he wishes. How many possible ways are there for Sachin to visit these cities?
Solution: The number of ways for Sachin to visit eight different cities starting from a specified city is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered in any way. Consequently, there are $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$ ways for Sachin to visit the eight cities.

Example 4.20. How many permutations of the letters ABCDEFG contain the string ABC(the order of ABC should not be changed)?
Solution: Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of the five objects, namely, the **block ABC** and the individual letters D, E, F and G. Because these five objects can occur in any order, there are $5! = 120$ permutations of the letters ABCDEFG in which ABC occurs as a block.

A slight variation of the above problem is:

Example 4.21. How many permutations of the letters ABCDEFG contain the letters ABC together in some order?
Solution: $5! \times 3!$.

4.3.2 Combinations without Repetition

We now turn our attention to counting unordered selections of r objects from n distinct objects where repetition is not allowed. We begin by solving a question posed at the beginning of this section.

Example 4.22. In how many ways can we form a committee of 4 out of 5 students?
Solution: Let us name the five students as A, B, C, D and E. Choosing four students is the same as not choosing one of the five students, as the remaining four students form the committee. The committee $\{B, C, D, E\}$ corresponds to not choosing student A. Thus there are five ways to choose the four students to form the committee, where the order in which these students are chosen does not matter.
You may observe that we have found the number of subsets with four elements(in this case students) from the set containing the five elements(students).

We shall quickly discuss one more illustration before we take up the general case of the number of ways to select r objects out of n objects.

Example 4.23. How many ways are there to select two out of five tennis players for a demonstration match?
Solution: Let the five players be named A, B, C, D and E. If two players are to be selected and the order mattered, then the answer would be $P(5, 2) = 5 \times 4 = 20$. But in this problem the order does not matter. Each selection of a pair of players, say, A and B, would have been counted in the 20 above, once as AB and once again as BA, that is, two times. The number 2 is actually the number of permutations of the two letters(or players) among themselves. So the number of 2 player selections would be $20/2 = 10$.

We shall see that many counting problems can be solved by finding the number of subsets of a particular size of a set with n-elements, where n is a non-negative integer.

Definition 4.3. *An* ***r-combination*** *of elements of a set is an unordered selection of r elements from the set.*

Remark 4.3. *Thus, an r-combination is simply a r-subset of the given n-set ($k-$set is a set containing k elements).*
This refers to an r-combination of n elements where repetition of elements is not allowed.

Example 4.24. Let S be the set $\{1, 2, 3\}$. Then $\{1, 3\}$ is a 2-combination from S. (Note that $\{3, 1\}$ is the same 2-combination as $\{1, 3\}$, because the order in which the elements of a set are listed does not matter.) In fact $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ are the only possible 2-combinations of the set S.

The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.
Note that $C(n, r)$ is also denoted by $\binom{n}{r}$ and is called a binomial coefficient.

Example 4.25. The 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}$ and $\{c, d\}$. From this we see that $C(4, 2) = 6$.

We can determine the number of r-combinations of a set with n elements using the formula for the number of r-permutations of a set.
Note that the r-permutations of a set can be obtained by first forming r-combinations and then ordering the elements in these combinations. The proof of Theorem 4.5, which gives the value of $C(n, r)$, is based on this observation.

Theorem 4.5. *The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals*

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

Proof: The different r-permutations of a set with n elements can be obtained by forming the $C(n, r)$ r-combinations of the set, and then ordering the elements in each r-combination, which can be done in $P(r, r)$ ways. Consequently, by the product rule, $P(n, r) = C(n, r) \times P(r, r)$. This implies that

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!\ (n-r)!}$$

as $0! = 1$. For computing $C(n, r)$, we first cancel out $(n - r)!$ from the numerator and denominator of the expression for $C(n, r)$ in Theorem 4.5. Thus we obtain,

$$C(n, r) = \frac{n!}{r!\ (n-r)!} = \frac{n(n-1)\cdots(n-(r-1))}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

We now obtain a very useful combinatorial identity as a consequence of Theorem 4.5.

Corollary 4.5.1. *Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.*

Proof: From Theorem 4.5 we get

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

and

$$C(n, n-r) = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}.$$

Thus we get, $C(n, r) = C(n, n - r)$.

Instead of algebraic manipulation, a combinatorial proof of Corollary 4.5.1 can also be given.

What is a Combinatorial Proof?

A combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity. These two types of proofs are called double counting proofs and bijective proofs, respectively.

A Double Counting proof:

By definition, the number of subsets of S with r elements equals $C(n, r)$. But each subset A of S is also determined by specifying which elements are not in A, and so are in $\overline{A}$. Because the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements. It follows that $C(n, r) = C(n, n - r)$.

A bijective proof:

Suppose that S is a set with n elements. The function that maps a subset A of S to $\overline{A}$ is a bijection between subsets of S with r elements and subsets with $n-r$ elements. The identity $C(n, r) = C(n, n-r)$ follows because when there is a bijection between two finite sets, the two sets must have the same number of elements.

Example 4.26. How many ways are there to select five players for an inter college basket ball match from a 10-member team?

Solution: It is the number of $5-$combinations of a set with 10 elements. By Theorem 4.5, the number of such combinations is

$$C(10, 5) = \frac{10 \times 9 \times \cdots \times (10 - 5 + 1)}{5!} = \frac{10 \times 9 \times 8 \times 7 \times 6}{120} = 252.$$

Example 4.27. A group of 30 astronauts has been trained to go on the first mission to Mars. In how many ways can a crew of six people be selected to go on this mission (assuming that all crew members have equal chance)?

Solution: The number of ways to select a crew of six from the 30 trained astronauts is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter. By Theorem 4.5, the number of such combinations is

$$C(30, 6) = \frac{30 \times 29 \times 28 \times 27 \times 26 \times 25}{6!} = 593,775.$$

Example 4.28. Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department of a college. How many ways are there to select a committee to develop a discrete mathematics course at the college if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?
Solution: By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4−combinations of a set with 11 elements. By Theorem 4.5, the number of ways to select the committee is

$$C(9,3)\cdot C(11,4) = \frac{9!}{3!6!}\times\frac{11!}{4!7!}.$$

4.3.3 Permutations with Repetition Allowed

Consider the following problem:

Example 4.29. How many 3 digit numbers can be formed using the digits 1, 3, 5, 7, 9, where we are allowed to repeat the digits?
Solution: To solve this problem, 3 places have to be filled:

$$\_\_\quad \_\_\quad \_\_$$

There are 5 digits which can be used to fill the first place. Because repetition is allowed, the same 5 digits can be used to fill the second place and also the last place. Thus

$$\underline{5}\times\underline{5}\times\underline{5} = 125$$

different numbers can be formed.

Example 4.30. Consider the problem of finding the number of strings of length 7 that can be formed using the lowercase letters of the English alphabet?

Solution: Since there are 26 lowercase letters in the English alphabet, and because each letter can be used repeatedly, we see that there are 26^7 strings of lowercase English letters of length 7.

Let us now obtain the general result.

Theorem 4.6. *The number of r-permutations of a set of n objects with repetition allowed is n^r.*

Proof: There are n ways to select an element from the set for each of the r positions in the r-permutation when repetition is allowed, because for each choice all n objects are available. Thus, using product rule there are n^r r-permutations when repetition is allowed.

Order?	Repetition?	Formula
Yes(Permutation)	No	$P(n,r)$
No(Combination)	No	$C(n,r)$
Yes(Permutation)	Yes	n^r

Note: We can also have an r-combination of n items with repetition allowed. But this concept is more involved than that of permutations with repetition allowed and we shall deal with it in the next chapter.
Let us summarize the results studied in this section in a table:

Exercises

1. List all the permutations of $\{a,b,c\}$.

2. How many permutations of $\{a,b,c,d,e,f\}$ end with a?

3. Find the number of $5-$permutations of a set with nine elements.

4. In how many different orders can five runners finish a race if no ties are allowed?

5. A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?

6. In how many ways can a set of five letters be selected from the English alphabet?

7. How many subsets with an odd number of elements does a set with 10 elements have?

8. How many subsets with more than two elements does a set with 100 elements have?

9. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes
   a) are there in total?
   b) contain exactly two heads?
   c) contain at most three tails?
   d) contain the same number of heads and tails?

10. How many permutations of the letters ABCDEFG contain
    a) the string BCD?
    b) the string CFGA?
    c) the strings BA and GF?
    d) the strings ABC and DE?
    e) the strings ABC and CDE?
    f) the strings CBA and BED?

11. How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [Hint: First position the men and then consider possible positions for the women.]

12. The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain
    a) exactly one vowel?
    b) exactly two vowels?
    c) at least one vowel?
    d) at least two vowels?

13. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

Hints and Solutions

1. $\{a, b, c\}, \{a, c, b\}, \{b, a, c\}, \{b, c, a\}, \{c, a, b\}, \{c, b, a\}$

2. Ans: 120.
   The first five letters can be in any order and the last letter is necessarily a.

3. $P(9, 5)$.

4. 5!=120

5. First place the men in $n!$ ways. The n women can then be placed either starting from left of the first man or from the right of the first man. Also consider the arrangements of the women among themselves.
   Ans: $2 \times n! \times n!$

6. $C(26, 5) = 65, 780$.

7. $C(10, 1) + C(10, 3) + C(10, 5) + C(10, 7) + C(10, 9) = 2^9$

8. $C(100, 3)+C(100, 4)+\ldots+C(100, 100) = 2^{100}-[C(100, 0)+C(100, 1)+C(100, 2)] = 2^{100} - 5051$.

9. a) 2^{10}= 1024
   b) C(10,2)= 45
   c) C(10,0)+C(10,1)+C(10,2)+C(10,3)=176
   d) C(10,5)=252.

10. a) 120 b) 24 c) 120 d) 24 e) 6 f) 0

11. $8! \times 9 \times 8 \times 7 \times 6 \times 5$

12. a) 122,523,030
    b) 72,930,375
    c) 223,149,655
    d) 100,626,625

13. 54,600

Chapter 5

Generalized Permutations and Combinations

Introduction

In the previous chapter, in case of distinguishable objects, we saw how to count combinations when repetition of objects was not allowed. To get the formula for $C(n, r)$, we used the notion of permutation where repetition of objects was not allowed ($P(n, r)$). We also saw how to count permutations when repetition of objects is allowed. The objects considered were distinguishable.

In section 5.2 of this chapter, we shall consider the case of combinations, but we will allow repetitions to occur (combinations with repetition). This will complete the matrix of counting formulae, indexed by order and repetition (No-order, Yes-Repetition) (Refer to Table 4.1). In this section we shall also deal with the number of permutations when some of the objects are indistinguishable(identical).

In section 5.3 we shall deal with distribution problems where we have to determine the number of ways in which the given objects can be distributed in the given boxes. The objects as well as the boxes may be distinguishable(distinct) or indistinguishable(identical). We find the number of ways to distribute the given objects in given boxes in 4 different situations.

Remark 5.1. *The number of* 2*-combinations of* $\{a, b, c\}$ *when repetition of letters is allowed are* $\{a, a\}; \{b, b\}; \{c, c\}; \{a, b\}; \{a, c\}; \{b, c\}$.

5.1 Generalized Permutations and Combinations

In this section we shall consider the problems of selecting objects when there are a sufficiently large number of indistinguishable objects of different types available for selection. For e.g., selecting 4 packets of biscuits in a shopping mall where large number of identical packets of Parle G, Britannia Marie Gold and Krackjack are available. A similar problem is:

Example 5.1. Suppose 3 different flavors of ice cream are available. Find the number of ways to choose 5 ice creams where each variety can be chosen repeatedly.

This contrasts with the counting problems discussed in the earlier chapter where we considered combinations in which each item could be used at most once.

Also, some counting problems involve finding the number of ways to arrange the elements of a multi set. A multi set is an extension of the concept of a set. While a set can contain only one occurrence of any given element, a multi set may contain multiple occurrences of the same element. For e.g. $S = \{a, a, b, b, b, c, d, e, e\}$ is a multi set. Consider the following problem:

Example 5.2. Find the number of ways in which the letters of the word MISSISSIPPI can be rearranged.

The methods to solve both these problems shall be discussed in this section.

5.1.1 Permutations of Multi sets.

Let us consider the problem mentioned above(Example 5.2). We have to find the number of ways to arrange the letters of the word MISSISSIPPI in a row. There are 4 identical I's, 4 identical S's, 2 identical P's and one M. Here we have to fill the 11 places shown below.

$$\_\_,\_\_,\_\_,\_\_,\_\_,\_\_,\_\_,\_\_,\_\_,\_\_,\_\_$$

For placing the I's, 4 of the places can be chosen in C(11,4) ways and the S's can then be placed in 4 out of the remaining 7 places in C(7,4) ways, the P's can be placed in C(3,2) ways and finally M in the remaining position. Thus the number of arrangements of the word MISSISSIPPI is

$$C(11,4) \times C(7,4) \times C(3,2) \times C(1,1) = \frac{11!}{4!7!} \times \frac{7!}{4!3!} \times \frac{3!}{2!1!} \times \frac{1!}{1!0!} = \frac{11!}{4!4!2!1!}.$$

In general we have,

Theorem 5.1. *The number of different permutations of n objects, where there are n_1 identical objects of type 1, n_2 identical objects of type 2,…, and n_k identical objects of type k, is*

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Proof: To determine the number of permutations, first note that the n_1 objects of type 1 can be placed among the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions free. Then the n_2 objects of type 2 can be placed in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions free. Continue placing the n_3 objects of type 3,… ,n_{k-1} objects of type $k - 1$, until at the last stage,

n_k objects of type k can be placed in $C(n - n_1 - n_2 - \ldots - n_{k-1}, n_k)$ ways. Hence, by the product rule, the total number of different permutations is

$$\begin{aligned} & C(n, n_1) \cdot C(n - n_1, n_2) \cdots C(n - n_1 - \ldots - n_{k-1}, n_k) \\ = & \frac{n!}{n_1!(n-n_1)!} \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{(n-n_1-n_2 \cdots - n_{k-1})!}{n_k!0!} \\ = & \frac{n!}{n_1!n_2! \ldots n_k!} \end{aligned}$$

5.1.2 Combinations with Repetition

Consider these examples of combinations with repetition of elements allowed.

Example 5.3. How many ways are there to select four fruits from a bag containing apples, oranges, and mangoes if the order in which the fruits are selected does not matter, only the type of fruit and not the individual fruit matters, and there are at least four fruits of each type in the bag? We also assume all the fruits of the same type are identical.

Solution: To solve this problem we list all the ways possible to select the fruit. There are 15 ways: 4 apples; 4 oranges; 4 mangoes; 3 apples and 1 orange; 3 apples and 1 mango; 3 oranges and 1 apple; 3 oranges and 1 mango; 3 mangoes and 1 apple; 3 mangoes and 1 orange; 2 apples and 2 oranges; 2 apples and 2 mangoes; 2 oranges and 2 mangoes; 2 apples, 1 orange and 1 mango; 2 oranges, 1 apple and 1 mango; 2 mangoes, 1 apple and 1 orange. The solution is the number of 4-combinations with repetition allowed from a three-element set, apple, orange, mango.

To solve more complex counting problems of this type, we need a general method for counting the r-combinations of an n-element set where repetition is allowed. Let us now illustrate a technique for the above problem, to count the combinations(selections) with repetition allowed. Suppose that a bowl has 3 compartments, the leftmost for placing apples, the middle one for oranges and the rightmost for mangoes. We require two separators to make three compartments in the box. The stars represent the fruits. If we select four apples it may be represented as shown below.

$$\star\star\star\star||$$

3 apples and 1 orange shall be represented as

$$\star\star\star|\star|$$

Note that:

$$\star|\star\star|\star$$

represents the selection 1 apple, 2 oranges and 1 mango.
Every selection listed above corresponds to a unique representation of 4 stars and 2 separators.
Thus counting the different combinations is equivalent to counting all possible arrangements of 4 identical stars and 2 identical separators.
We have to find the number of ways to fill the 6 places shown below

$$\text{—},\text{—},\text{—},\text{—},\text{—},\text{—}$$

with 4 identical stars and 2 identical separators. This can be done by either selecting the places for 2 separators or 4 stars. The selection of places for the separators can be done in $C(6,2) = 15$ ways. Alternatively we could have also selected the 4 places to place the stars. This can be done in $C(6,4) = 15$ ways.

The general theorem is given below:

Theorem 5.2. *There are $C(n+r-1,r) = C(n+r-1,n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.*

Proof: Each r-combination of a set with n elements when repetition is allowed can be represented by a list of $n-1$ bars(separators) and r stars. The $n-1$ bars are used to mark off n different cells, with the i^{th} cell containing a star for each time the i^{th} element of the set occurs in the combination. As we have seen in the above example, each different list containing $n-1$ bars and r stars corresponds to an r-combination of the set with n elements, when repetition is allowed. The number of such lists is $C(n-1+r, r)$, because each list corresponds to a choice of the r positions to place the r stars from the $n-1+r$ positions that contain r stars and $n-1$ bars. The number of such lists is also equal to $C(n-1+r, n-1)$, because each list can also be thought to correspond to a choice of the $n-1$ positions to place the $n-1$ bars.
Alternatively we may use theorem 5.1 to get that there are exactly

$$\frac{(n-1+r)!}{r!(n-1)!} = C(n-1+r, r) = C(n-1+r, n-1)$$

different arrangements of r identical stars and $n-1$ identical bars.

Example 5.4. Suppose that a cookie shop has four different kinds of cookies. In how many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements(repetition allowed). From Theorem 5.2 this equals $C(4+6-1, 6) = C(9, 6)$. Because $C(9, 6) = C(9, 3) = 84$, there are 84 different ways to choose the six cookies.

Theorem 5.2 can also be used to find the number of solutions of certain linear equations where the variables are integers subject to constraints. This is illustrated by the next example.

Example 5.5. How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, and x_3 are nonnegative integers?

Solution: To count the number of solutions, we note that a solution corresponds to a way of selecting 11 items from a set with three elements so that x_1 items of type one, x_2 items of type two, and x_3 items of type three are chosen. Hence, the number of solutions is equal to the number of 11-combinations with repetition allowed from a set with three elements. From Theorem 5.2 it follows that there are $C(3+11-1, 11) = C(13, 11) = C(13, 2) = 78$ solutions.

The number of solutions of the above equation can also be found when the variables are subject to constraints. For instance, we can find the number of solutions where the variables are integers with $x_1 > 1, x_2 > 2$, and $x_3 > 3$. A solution to the equation subject to these constraints corresponds to a selection of 11 items with x_1 items of type one, x_2 items of type two, and x_3 items of type three, where, there are at least two items of type one ($x_1 > 1$), three items of type two($x_2 > 2$), and four items of type three($x_3 > 3$). Since $x_1 > 1 \Rightarrow x_1 = 2 + y_1$, $x_2 > 2 \Rightarrow x_2 = 3 + y_2$ and $x_3 > 3 \Rightarrow x_3 = 4 + y_3$ where y_1, y_2, y_3 are non negative integers. Thus equivalently we now have to find the number of solutions of $y_1 + y_2 + y_3 = 11 - (2 + 3 + 4) = 2$ where y_1, y_2, y_3 are non negative integers.
In this case $n = 3$ and $r = 2$. From Theorem 5.2 it follows that there are $C(3 + 2 - 1, 2) = C(4, 2) = 6$ solutions. Thus, there are 6 solutions of the equation subject to the given constraints.

Theorem 5.3. *The number of non-negative integers solutions to $x_1 + x_2 + x_3 + \ldots + x_n = r$ is equal to $C(n + r - 1, r)$.*

Proof: Generalization of the above problems.

Exercises

1. In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed?

2. How many strings of six letters are there?

3. How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop?

4. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$, where x_1, x_2, x_3, and x_4 are nonnegative integers?

5. How many solutions are there to the equation
   $x_1 + x_2 + x_3 + x_4 + x_5 = 21$, where $x_i, i = 1, 2, 3, 4, 5$, is a nonnegative integer such that
   a) $x_1 \geq 1$?
   b) $x_i \geq 2$ for i = 1, 2, 3, 4, 5?

6. How many strings of 10 ternary digits (0, 1, or 2) are there that contain exactly two 0's, three 1's, and five 2's?

7. How many positive integers less than 1,000,000 have the sum of their digits equal to 19?

8. How many different strings can be made from the letters in AARDVARK, using all the letters, if all three As must be consecutive?

Hints and Solutions

1. 243

2. 26^6

3. $C(32, 12) = C(32, 20)$.

4. $C(20, 3) = C(20, 17)$.

5. a) $C(20, 4) = C(20, 16)$.
   b) $C(15, 4) = C(15, 11)$.

6. 2520

7. 30,492

8. Hint: Since the 3 A's are always together, let us call them as group G. We have to find number of ways to arrange G,R,R,D,V,K.
   Solution: $\frac{6!}{2}$

5.2 Distributing Objects into Boxes

Many counting problems can be solved by enumerating the ways objects can be placed into boxes (where the order these objects are placed into the boxes does not matter). The objects can be either distinguishable(distinct), or indistinguishable(identical). Distinguishable objects/boxes are sometimes said to be labelled, whereas indistinguishable objects/boxes are said to be unlabelled.

When you solve a counting problem using the model of distributing objects into boxes, you need to determine whether the objects are distinguishable or indistinguishable and whether the boxes are distinguishable of indistinguishable. If the objects are distinguishable, it is a good idea to label them. Similarly for the boxes. Although the context of the counting problem makes these two decisions clear, if the counting problems are sometimes ambiguous, then in such a case it is best to state whatever assumptions you are making and explain why the particular model you choose conforms to your assumptions.

We will see that there are closed formulae for counting the ways to distribute objects, distinguishable or indistinguishable, into distinguishable boxes.

We are not so lucky when we count the ways to distribute objects, distinguishable or indistinguishable, into indistinguishable boxes; there are no closed formulae to use in these cases.

5.2.1 Distinguishable Objects into Distinguishable Boxes

Example 5.6. In how many ways can 10 distinguishable objects be placed in 5 distinguishable boxes with no condition on the number of objects in any box?

Solution: Since the objects as well as boxes are distinguishable, let us label the objects as $O_1, O_2, \ldots, O_{10}$ and the boxes as $B_1, B_2, \ldots, B_5$. The first object O_1 can be placed in any of the 5 boxes and the second object can be placed in 5 ways and so on. Thus there are 5^{10} ways to distribute 10 distinguishable objects in 5 distinguishable boxes.

Theorem 5.4. *The number of ways to distribute n distinguishable objects into k distinguishable boxes with no condition on the number of objects in any box is k^n.*

Proof: Use product rule

Now consider another problem

Example 5.7. How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Solution: We will use the product rule to solve this problem. To begin, note that the first player can be dealt 5 cards in $C(52, 5)$ ways. The second player can be dealt 5 cards in $C(47, 5)$ ways, because only 47 cards are left. The third player can be dealt 5 cards in $C(42, 5)$ ways. Finally, the fourth player can be dealt 5 cards in $C(37, 5)$ ways. Hence, the total number of ways to deal four players 5 cards each is $C(52, 5) \times C(47, 5) \times C(42, 5) \times C(37, 5)$.

Theorem 5.5. *The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \dots, k$, equals* $\dfrac{n!}{n_1!n_2!\cdots n_k!}$.

Proof: Generalization of the above problem.

5.2.2 Indistinguishable Objects into Distinguishable Boxes

Counting the number of ways of placing n indistinguishable objects into k distinguishable boxes turns out to be the same as counting the number of n-combinations for a set with k elements when repetitions are allowed. The reason behind this is that there is a one-to-one correspondence between n-combinations from a set with k elements when repetition is allowed and the ways to place n indistinguishable balls into k distinguishable boxes. To set up this correspondence, we put a ball in the i^{th} bin each time the i^{th} element of the set is included in the n-combination.

Example 5.8. How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?

Solution: The number of ways to place 10 indistinguishable balls into eight bins equals the number of 10-combinations from a set with eight elements when repetition is allowed. Consequently, there are $C(8 + 10 - 1, 10) = C(17, 10) = \dfrac{17!}{10!7!} = 19,448$.

Theorem 5.6. *There are $C(k+n-1, k-1)$ ways to place n indistinguishable objects into k distinguishable boxes.*

5.2.3 Distinguishable Objects into Indistinguishable Boxes

Counting the ways to place n distinguishable objects into k indistinguishable boxes is more difficult than counting the ways to place objects, distinguishable or indistinguishable objects, into distinguishable boxes. We illustrate this with an example.

Example 5.9. How many ways are there to put four distinguishable employees into three indistinguishable offices, when each office can contain any number of employees?

Solution: We will solve this problem by enumerating all the ways these employees can be placed into the offices.We represent the four employees by A, B, C, and D. First, we note that we can distribute employees so that all four are put into one office, three are put into one office and a fourth is put into a second office, two employees are put into one office and two put into a second office, and finally, two are put into one office, and one each put into the other two offices.

Each way to distribute these employees to these offices can be represented by a way to partition the elements A, B, C, and D into disjoint subsets.

We can put all four employees into one office in exactly one way, represented by $\{\{A, B, C, D\}\}$. We can put three employees into one office and the fourth employee into a different office in exactly four ways, represented by $\{\{A, B, C\}, \{D\}\}, \{\{A, B, D\}, \{C\}\}, \{\{A, C, D\}, \{B\}\}$, and $\{\{B, C, D\}, \{A\}\}$. We can put two employees into one office and two into a second office in exactly three ways, represented by $\{\{A, B\}, \{C, D\}\}, \{\{A, C\}, \{B, D\}\}$, and $\{\{A, D\}, \{B, C\}\}$. Finally, we can put two employees into one office, and one each into each of the remaining two offices in six ways, represented by $\{\{A, B\}, \{C\}, \{D\}\}, \{\{A, C\}, \{B\}, \{D\}\}, \{\{A, D\}, \{B\}, \{C\}\}$,
$\{\{B, C\}, \{A\}, \{D\}\}, \{\{B, D\}\}, \{A\}, \{C\}\}$,and$\{\{C, D\}, \{A\}, \{B\}\}$. Counting all the possibilities, we find that there are 14 ways to put four different employees into three indistinguishable offices. Another way to look at this problem is to look at the number of offices into which we put employees. Note that there are six ways to put four different employees into three indistinguishable offices so that no office is empty, seven ways to put four different

employees into two indistinguishable offices so that no office is empty, and one way to put four employees into one office so that it is not empty.

There is no simple closed formula for the number of ways to distribute n distinguishable objects into j indistinguishable boxes. However, there is a formula involving a summation, which we will now describe. Let $S(n, j)$ denote the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty. The numbers $S(n, j)$ are called Stirling numbers of the second kind. For instance, Example 5.9 shows that $S(4, 3) = 6, S(4, 2) = 7$, and $S(4, 1) = 1$. We see that the number of ways to distribute n distinguishable objects into k indistinguishable boxes (where the number of boxes that are nonempty equals $k, k-1, \ldots, 2,$ or 1) equals $\sum_{j=1}^{k} S(n, j)$. For instance, following the reasoning in Example 5.9, the number of ways to distribute four distinguishable objects into three indistinguishable boxes equals $S(4, 1) + S(4, 2) + S(4, 3) = 1 + 7 + 6 = 14$.
Using Inclusion Exclusion Principle it can be shown that

Theorem 5.7.

$$S(n, j) = \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j - i)^n$$

(Without Proof)

5.2.4 Indistinguishable Objects into Indistinguishable Boxes

Some counting problems can be solved by determining the number of ways to distribute indistinguishable objects into indistinguishable boxes. We illustrate this principle with an example.

Example 5.10. How many ways are there to pack six copies of the same book into four indistinguishable boxes, where a box can contain at most six books?

Solution: We will enumerate all ways to pack the books. For each way to pack the books, we will list the number of books in the box with the largest number of books, followed by the numbers of books in each box containing at least one book, in order of decreasing number of books in a box. The ways we can pack the books are

6

5, 1

4, 2

4, 1, 1

3, 3

3, 2, 1

3, 1, 1, 1

2, 2, 2

2, 2, 1, 1.

For example, $(4, 1, 1)$ indicates that one box contains four books, a second box contains a single book, and a third box contains a single book (and the fourth box is empty). We conclude that there are nine allowable ways to pack the books.

Observe that distributing n indistinguishable objects into k indistinguishable boxes is the same as writing n as the sum of at most k positive integers in non increasing order. If $a_1 + a_2 + \ldots + a_j = n$, where $a_1, a_2, \ldots, a_j$ are positive integers with $a_1 \geq a_2 \ldots \geq a_j$, we say that $a_1, a_2, \ldots, a_j$ is a partition of the positive integer n into j positive integers. We see that if $p_k(n)$ is the number of partitions of n into at most k positive integers, then there are $p_k(n)$ ways to distribute n indistinguishable objects into k indistinguishable boxes. No simple closed formula exists for this number.

Let us summarize the results of this section in the table given below:

Table 5.1: Number of Ways of Distribution of n objects into k boxes

Objects(n)	**Boxes (k)**	**Distribution without any condition**
Distinguishable	Distinguishable	k^n
Indistinguishable	Distinguishable	$C(k+n-1, k-1)$
Distinguishable	Indistinguishable	$\sum_{j=1}^{k} S(n, j)$
Indistinguishable	Indistinguishable	$p_k(n)$

Exercises

1. How many ways are there to distribute six different coloured caps to 10 children?

2. How many ways are there to distribute 10 different books among 5 children such that each child gets at least one book?

3. How many ways are there to distribute six indistinguishable balls into nine distinguishable bins?

4. How many ways are there to distribute 12 indistinguishable balls into six distinguishable boxes if each box contains at least one ball?

5. How many ways are there to distribute 12 distinguishable objects into six distinguishable boxes so that two objects are placed in each box?

6. A student has three mangoes, two papayas, and two kiwi fruits. If the student eats one piece of fruit each day, and only the type of fruit matters, in how many different ways can these fruits be consumed?

7. How many ways are there to distribute five distinguishable objects into two indistinguishable boxes?

8. How many ways are there to distribute five indistinguishable objects into 5 indistinguishable boxes?

9. How many ways are there to distribute seven indistinguishable objects into four bins such that each bin contains at least one object?

Hints and Solutions

1. 10^6

2. $P(10, 5)$

3. It is the number of solutions of $x_1 + x_2 + \ldots + x_9 = 6$ where $x_i's$ are non-negative integers. Its the number of ways to arrange 6 $\star$ and 8 $|'s$ in a row. Ans: $C(14, 6) = C(14, 8)$.

4. $C(11, 5) = C(11, 6)$

5. $\frac{12!}{2^6} = 7,484,400.$

6. 210

7. $S(5, 2) + S(5, 1)$

8. $p_5(5)$

9. i) $C(6, 3)$ if the bins are distinguishable
   ii) 3 ways if the bins are indistinguishable.

Chapter 6

The Inclusion and Exclusion Principle

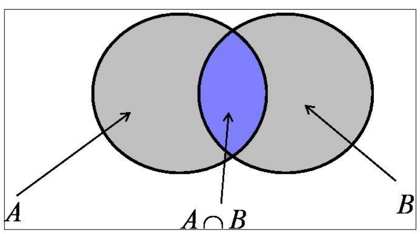
Introduction

The principle of Inclusion and Exclusion is doubtless very old; its origin is probably untraceable. The principle of Inclusion and Exclusion is sometimes referred to as "Poincare's Theorem". Sylvester and da Silva are the two mathematicians associated with the combinatorial form of the principle.
The principle of Inclusion and Exclusion is a way of thinking about combining sets with overlapping elements.

6.1 The Subtraction Rule

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
The subtraction rule is also known as the principle of inclusion-exclusion, especially when it is used to count the number of elements in the union of

two sets. Suppose that A_1 and A_2 are sets. Then, there are $|A_1|$ ways to select an element from A_1 and $|A_2|$ ways to select an element from A_2. The number of ways to select an element from A_1 or from A_2, that is, the number of ways to select an element from their union, is the sum of the number of ways to select an element from A_1 and the number of ways to select an element from A_2, minus the number of ways to select an element that is in both A_1 and A_2. Because there are $|A_1 \cup A_2|$ ways to select an element in either A_1 or in A_2, and $|A_1 \cap A_2|$ ways to select an element common to both sets, we have $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.



Example 6.1. How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

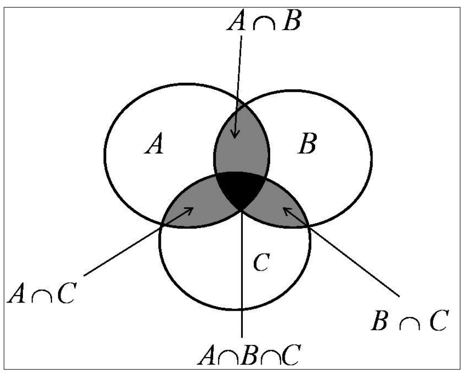
Solution: We can construct a bit string of length eight that either starts with a 1 bit or ends with the two bits 00, by constructing a bit string of length eight beginning with a 1 bit or by constructing a bit string of length eight that ends with the two bits 00. We can construct a bit string of length eight that begins with a 1 in $2^7 = 128$ ways. This follows by the product rule, because the first bit can be chosen in only one way and each of the other seven bits can be chosen in two ways. Similarly, we can construct a bit string of length eight ending with the two bits 00, in $2^6 = 64$ ways. This follows by the product rule, because each of the first six bits can be chosen in two ways and the last two bits can be chosen in only one way. Some of the ways to construct a bit string of length eight starting with a 1 are the same as the ways to construct a bit string of length eight that ends with the two bits 00. There are $2^5 = 32$ ways to construct such a string. This follows by the product rule, because the first bit can be chosen in only one way, each of the second through the sixth bits can be chosen in two ways, and the last two bits can be chosen in one way. Consequently, the number of bit strings of length eight that begin with a 1 or end with a 00, which equals the number of ways to construct a bit string of length eight that begins with a 1 or that ends with 00, equals $128 + 64 - 32 = 160$.

Example 6.2. A computer company receives 350 applications from graduates for a job. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Solution: To find the number of these applicants who majored neither in computer science nor in business, we can subtract the number of students who majored either in computer science or in business (or both) from the total number of applicants. Let A_1 be the set of students who majored in computer science and A_2 the set of students who majored in business. Then $A_1 \cup A_2$ is the set of students who majored in computer science or business (or both), and $A_1 \cap A_2$ is the set of students who majored both in computer science and in business. By the subtraction rule the number of students who majored either in computer science or in business (or both) equals $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316$. We conclude that $350 - 316 = 34$ of the applicants majored neither in computer science nor in business.

6.2 The Principle of Inclusion and Exclusion

The principle of Inclusion and Exclusion, hereafter called PIE, gives a formula for the size of the union of n finite sets. Usually the universe is finite too. It is a generalization of the familiar formulas $|A \cup B| = |A| + |B| - |A \cap B|$ and $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.



Theorem 6.1. *If $P_1, P_2, \ldots, P_n$ be finite sets, then*
$|P_1 \cup P_2 \cup \ldots \cup P_n| = |P_1| + |P_2| + \ldots + |P_n| - |P_1 \cap P_2| - |P_1 \cap P_3| - \ldots |P_{n-1} \cap P_n| + |P_1 \cap P_2 \cap P_3| + |P_1 \cap P_2 \cap P_4| + \ldots + |P_{n-2} \cap P_{n-1} \cap P_n| - \ldots + (-1)^{n+1}|P_1 \cap P_2 \cap \ldots \cap P_n|$
That is

$$|P_1 \cup P_2 \cup \ldots \cup P_n| = \sum_{1 \leq i \leq n} |P_i| - \sum_{1 \leq i < j \leq n} |P_i \cap P_j|$$

$$+ \sum_{1 \leq i < j < k \leq n} |P_i \cap P_j \cap P_k| - \ldots + (-1)^{n+1}|P_1 \cap P_2 \cap ... \cap P_n|$$

.

That is, the cardinality of the union $P_1 \cup P_2 \cup ... \cup P_n$ can be calculated by including (adding) the sizes of all of the sets together, then excluding (subtracting) the sizes of the intersections of all pairs of sets, then including the sizes of the intersections of all triples, excluding the sizes of the intersections of all quadruples, and so on until, finally, the size of the intersection of all of the sets has been included or excluded, as appropriate. If n is odd it is included, and if n is even it is excluded.

It is important to remember that all sets involved must be finite.

We should try to use PIE when we are trying to count something described by a bunch of conditions, any number of which might hold at the same time. Often PIE is used in conjunction with counting the complement. That is, you use it to count the number of objects in the universe that you do not want, and subtract this from the size of the universe (which needs to be finite). In applying PIE, the setup is of great importance. You need to be clear about what the sets are (what it means to belong to one or more of them), what the universe is, and how the principle gives you what you want. Once you have done this, things often reduce to more or less straightforward counting problems.

Example 6.3. In a discrete mathematics class every student is a major in computer science or mathematics, or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in this class?

Solution: Let A be the set of students in the class majoring in computer science and B be the set of students in the class majoring in mathematics. Then $A \cap B$ is the set of students in the class who are joint mathematics and computer science majors. Because every student in the class is majoring in either computer science or mathematics (or both), it follows that the number of students in the class is $|A \cup B|$. Therefore, $|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$. Therefore, there are 30 students in the class.

Example 6.4. How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution: Let A be the set of positive integers not exceeding 1000 that are divisible by 7, and let B be the set of positive integers not exceeding 1000 that are divisible by 11. Then $A \cup B$ is the set of integers not exceeding 1000 that are divisible by either 7 or 11, and $A \cap B$ is the set of integers not exceeding 1000 that are divisible by both 7 and 11. We know that among the positive integers not exceeding 1000 there are $\left\lfloor \frac{1000}{7} \right\rfloor$ integers divisible by 7 and $\left\lfloor \frac{1000}{11} \right\rfloor$ divisible by 11. Because 7 and 11 are relatively prime, the integers divisible by both 7 and 11 are those divisible by 7×11. Consequently, there are $\left\lfloor \frac{1000}{11 \times 7} \right\rfloor$ positive integers not exceeding 1000 that are divisible by both 7 and 11. It follows that there are
$|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220$ positive integers not exceeding 1000 that are divisible by either 7 or 11.

Example 6.5. Suppose that there are 1807 students in first year at your college. Of these, 453 are taking a course in computer science, 567 are taking a course in mathematics, and 299 are taking courses in both computer science and mathematics. How many are not taking a course either in computer science or in mathematics?

Solution: To find the number of first year students who are not taking a course in either mathematics or computer science, subtract the number that are taking a course in either of these subjects from the total number of first year students. Let A be the set of all first year students taking a course in computer science, and let B be the set of all first year students taking a course in mathematics. It follows that $|A| = 453$, $|B| = 567$, and $|A \cap B| = 299$. The number of first year students taking a course in either computer science or mathematics is $|A \cup B| = |A| + |B| - |A \cap B| = 453 + 567 - 299 = 721$. Consequently, there are $1807 - 721 = 1086$ first year students who are not taking a course in computer science or mathematics.

Example 6.6. A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then $|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 2092$. When we insert these quantities into the equation $|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$ we obtain $2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$. We now solve for $|S \cap F \cap R|$. We find that $|S \cap F \cap R| = 7$. Therefore, there are seven students who have taken courses in Spanish, French, and Russian.

Exercises

1. How many elements are in $A_1 \cup A_2$ if there are 12 elements in A_1, 18 elements in A_2, and a) $A_1 \cap A_2 = \phi$? b) $|A_1 \cap A_2| = 1$? c) $|A_1 \cap A_2| = 6$? d) $A_1 \subseteq A_2$?

2. A survey of households in the United States reveals that 96 percent have at least one television set, 98 percent have telephone service, and

95 percent have telephone service and at least one television set. What percentage of households in the United States have neither telephone service nor a television set?

3. Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and if

   a) the sets are pairwise disjoint.

   b) there are 50 common elements in each pair of sets and no elements in all three sets.

   c) there are 50 common elements in each pair of sets and 25 elements in all three sets.

   d) the sets are equal.

4. There are 2504 computer science students at a college. Of these, 1876 have taken a course in Java, 999 have taken a course in Linux, and 345 have taken a course in C. Further, 876 have taken courses in both Java and Linux, 231 have taken courses in both Linux and C, and 290 have taken courses in both Java and C. If 189 of these students have taken courses in Linux, Java, and C, how many of these 2504 students have not taken a course in any of these three programming languages?

5. How many students are enrolled in a course either in calculus, discrete mathematics, data structures, or programming languages at a college if there are $507, 292, 312$, and 344 students in these courses, respectively; 14 in both calculus and data structures; 213 in both calculus and programming languages; 211 in both discrete mathematics and data structures; 43 in both discrete mathematics and programming languages; and no student may take calculus and discrete mathematics, or data structures and programming languages, concurrently?

Hints and Solutions:

1. a) 30 b) 29 c) 24 d) 18

2. 1 percent

3. a) 300 b) 150 c) 175 d) 100

4. 492

5. 974

Suggested Problems for Practicals

Propositional Logic

1. Let p and q be the propositions "The election is decided" and "The votes have been counted," respectively. Express each of these compound propositions as an English sentence.
   a) $\neg p$ b) $p \vee q$ c) $\neg p \wedge q$ d) $\neg q \vee (\neg p \wedge q)$. [4]

2. Write each of these statements in the form "if p, then q" in English.
   a) That you get the job implies that you had the best credentials.
   b) It is necessary to have a valid password to log on to the server. [4]

3. Construct a truth table for each of these compound propositions.
   a) $\neg p \to (q \to r)$ b) $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$ [8]

4. Use De Morgan's laws to find the negation of each of the following statements.
   a) Kapil will take a job in industry or go to university.
   b) Yogesh knows algebra and calculus. [4]

5. Determine whether $(\neg p \wedge (p \to q)) \to \neg q$ is a tautology. [4]

6. Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent. [8]

Predicates and Quantifiers

1. Determine the truth value of each of these statements if the domain consists of all integers.
   a) $\forall n(n+1>n)$ b) $\exists n(2n=3n)$
   c) $\exists n(n=-n)$ d) $\forall n(3n \leq 4n)$. [4]

2. Express each of these statements using logical operators, predicates, and quantifiers.
   a) Some propositions are tautologies.
   b) The negation of a contradiction is a tautology.
   c) The disjunction of two contingencies can be a tautology.
   d) The conjunction of two tautologies is a tautology. [4]

3. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
   a) $\forall x(x^2 \geq x)$ b) $\forall x(x>0 \vee x<0)$
   c) $\forall x(x=1)$ d) $\forall x(x^2 \neq x)$. [4]

4. Let $Q(x,y)$ be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.
   a) $\exists x \exists y Q(x,y)$ b) $\exists x \forall y Q(x,y)$
   c) $\forall x \exists y Q(x,y)$ d) $\exists y \forall x Q(x,y)$ [4]

5. Let $S(x)$ be the predicate "x is a student," $F(x)$ the predicate "x is a faculty member," and $A(x,y)$ the predicate "x has asked y a question," where the domain consists of all people associated with your college. Use quantifiers to express each of these statements.

   a) Messi has asked Professor Maradona a question.
   b) Every student has asked Professor Muller a question.

c) Some student has not asked any faculty member a question.

d) There is a faculty member who has never been asked a question by a student. [8]

6. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

   a) $\neg\forall x\forall y P(x, y)$ b) $\neg\forall x(\exists y\forall z P(x, y, z) \wedge \exists z\forall y P(x, y, z))$

   c) $\neg\forall y\forall x(P(x, y) \vee Q(x, y))$ d) $\neg(\exists x\exists y\neg P(x, y) \wedge \forall x\forall y Q(x, y))$ [4]

Rules of Inference and Methods of Proof

1. What rule of inference is used in each of these arguments?

   a) Ankit is a mathematics major. Therefore, Ankit is either a mathematics major or a computer science major.

   b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

   c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

   d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today. [8]

2. Use rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job." [4]

3. Using direct proof, prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even. [4]

4. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

   a) a proof by contrapositive b) a proof by contradiction. [8]

5. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd. [4]

Basics of Counting

1. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$
   a) that are one-to-one?
   b) that assign 0 to both 1 and n?
   c) that assign 1 to exactly one of the positive integers less than n? [8]

2. A palindrome is a string whose reversal is identical to the string. How many bit strings of length n are palindromes? [4]

3. In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if
   a) the bride must be next to the groom?
   b) the bride is not next to the groom?
   c) the bride is positioned somewhere to the left of the groom? [8]

4. Let S = $\{1, 2, 3, 4\}$.
   a) List all the 2-permutations and 3-permutations of S.
   b) List all the 2-combinations and 3-combinations of S. [8]

5. The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain
   a) exactly one vowel? b) exactly two vowels?
   c) at least one vowel? d) at least two vowels? [4]

6. In how many ways can we form a committee of three from a set of 10 men and 8 women, such that our committee consists of at least one women? [4]

Generalized Permutations and Combinations

1. How many strings with seven or more characters can be formed from the letters in EVERGREEN? [4]

2. Suppose that a large family has 14 children, including 2 sets of identical triplets, 3 sets of identical twins, and 2 individual children. How many ways are there to seat these children in a row of chairs if the identical triplets or twins cannot be distinguished from one another? [4]

3. How many solutions are there to the equation
$x_1 + x_2 + x_3 + x_4 = 20$, where $x_i, i = 1, 2, 3, 4$ is a nonnegative integer such that
a) $x_1, x_2, x_3, x_4 \geq 0$? b) $x_1, x_2, x_3, x_4 \geq 1$? [8]

4. How many solutions are there to the equation
$x_1 + x_2 + x_3 = 15$, where $x_i, i = 1, 2, 3$ is a nonnegative integer such that $x_1 > 1, x_2 > 2$ and $x_3 > 3$? [4]

5. How many ways are there to distribute five balls into three boxes if each box must have at least one ball in it if
a) both the balls and boxes are labelled?
b) the balls are labelled, but the boxes are unlabelled?
c) the balls are unlabelled, but the boxes are labelled?
d) both the balls and boxes are unlabelled? [4]

The Inclusion Exclusion Principle

1. Find the number of positive integers not exceeding 100 that are either odd or square of an integer. [4]

2. How many bit strings of length eight do not contain six consecutive $0's$? [4]

3. How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123? [8]

4. How many elements are in the union of four sets if the sets have $50, 60, 70,$ and 80 elements, respectively, each pair of the sets has 5 elements in common, each triple of the sets has 1 common element, and no element is in all four sets? [8]

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