Parameter Selection and Model Calibration for an SIR Model

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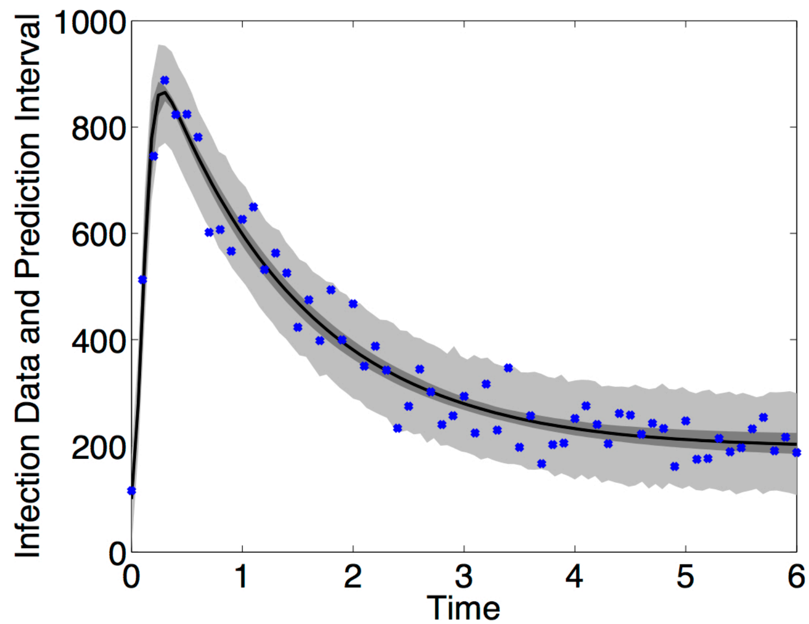
SIR Model

$$\frac{dS}{dt} = \delta N - \delta S - \underline{\gamma k} IS \quad , \; S(0) = S_0$$

$$\frac{dI}{dt} = \underline{\gamma k} IS - (r + \delta) I \quad , \; I(0) = I_0$$

$$\frac{dR}{dt} = rI - \delta R \quad , \; R(0) = R_0$$



SIR Disease Example

SIR Model:

$$\frac{dS}{dt} = \delta N - \delta S - \underline{\gamma k} IS \quad , \; S(0) = S_0 \qquad \text{Susceptible}$$

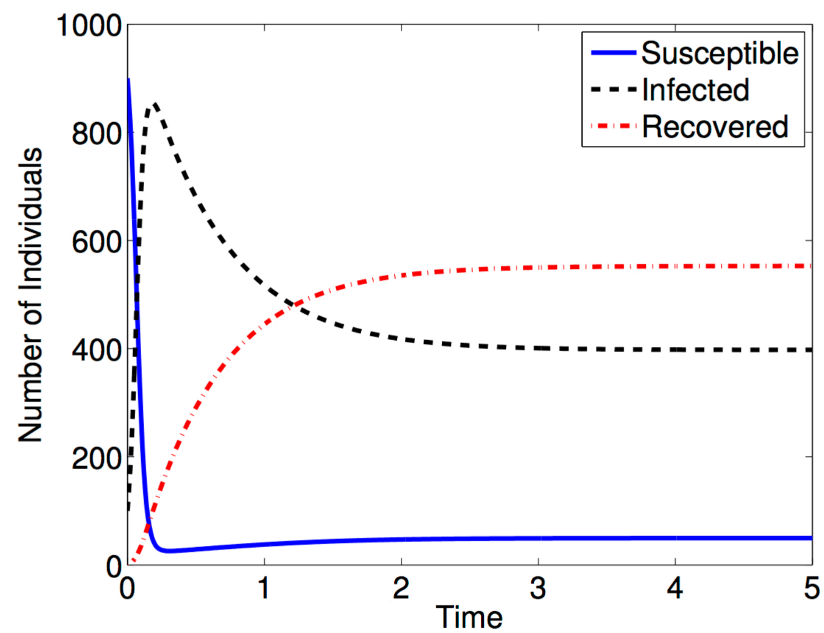
$$\frac{dI}{dt} = \underline{\gamma k} IS - (r + \delta) I \quad , \; I(0) = I_0 \qquad \text{Infectious}$$

$$\frac{dR}{dt} = rI - \delta R \quad , \; R(0) = R_0 \qquad \text{Recovered}$$

Note: Parameter set $q = [\gamma, k, r, \delta]$ is not identifiable

Objectives: Employ Bayesian analysis for

- Parameter selection
- Model calibration
- Uncertainty propagation



Delayed Rejection Adaptive Metropolis (DRAM)

Websites

- http://www4.ncsu.edu/~rsmith/UQ_TIA/CHAPTER8/index_chapter8.html
- http://helios.fmi.fi/~lainema/mcmc/

Examples

- Examples on using the toolbox for some statistical problems.

Delayed Rejection Adaptive Metropolis (DRAM)

We fit the Monod model

$$y = \theta_1 \frac{1}{\theta_2 + 1} + \epsilon \quad , \quad \epsilon \sim N(0, I\sigma^2)$$

to observations

x (mg / L COD): 28 55 83 110 138 225 375

y (1 / h): 0.053 0.060 0.112 0.105 0.099 0.122 0.125

First clear some variables from possible previous runs.

```
clear data model options
```

Next, create a data structure for the observations and control variables. Typically one could make a structure data that contains fields xdata and ydata.

```
data.xdata = [28   55   83   110   138   225   375]';   % x (mg / L COD)
data.ydata = [0.053 0.060 0.112 0.105 0.099 0.122 0.125]'; % y (1 / h)
```

Construct model

```
modelfun = @(x,theta) theta(1)*x./(theta(2)+x);
ssfun = @(theta,data) sum((data.ydata-modelfun(data.xdata,theta)).^2);
model.ssfun = ssfun;
model.sigma2 = 0.01^2;
```

Delayed Rejection Adaptive Metropolis (DRAM)

Input parameters

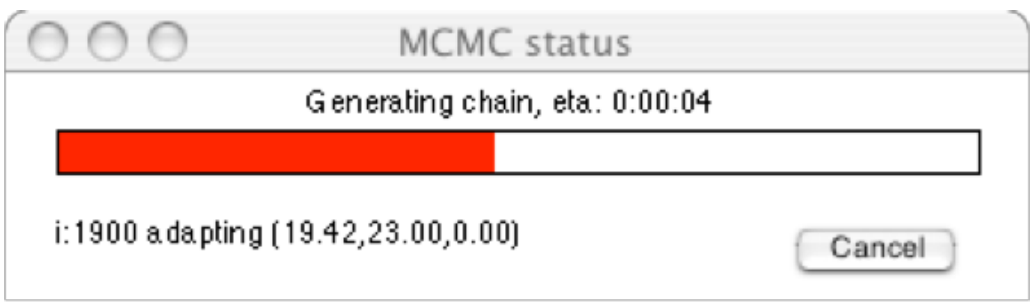
```
params = {
    {'theta1', tmin(1), 0}
    {'theta2', tmin(2), 0} };
```

and set options

```
options.nsimu = 4000;
options.updatesigma = 1;
options.qcov = tcov;
```

Run code

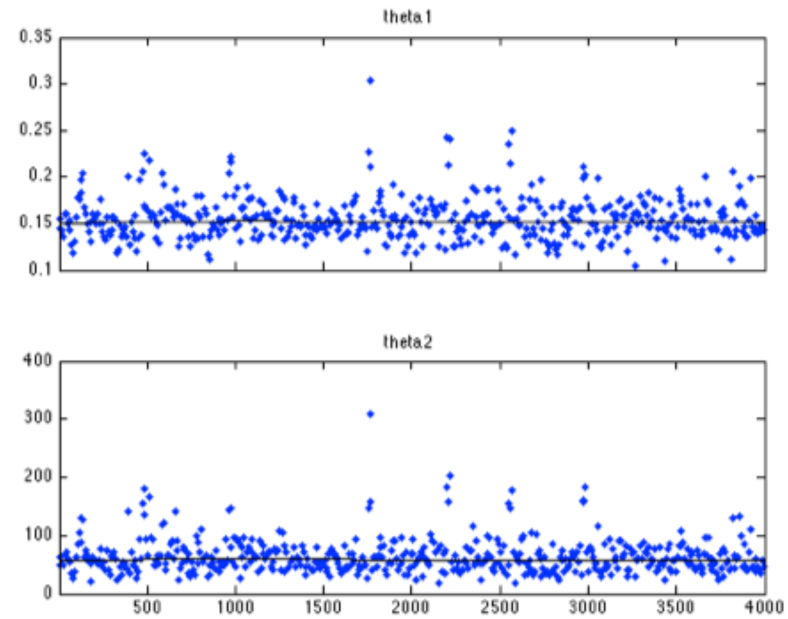
```
[res,chain,s2chain] = mcmcrun(model,data,params,options);
```

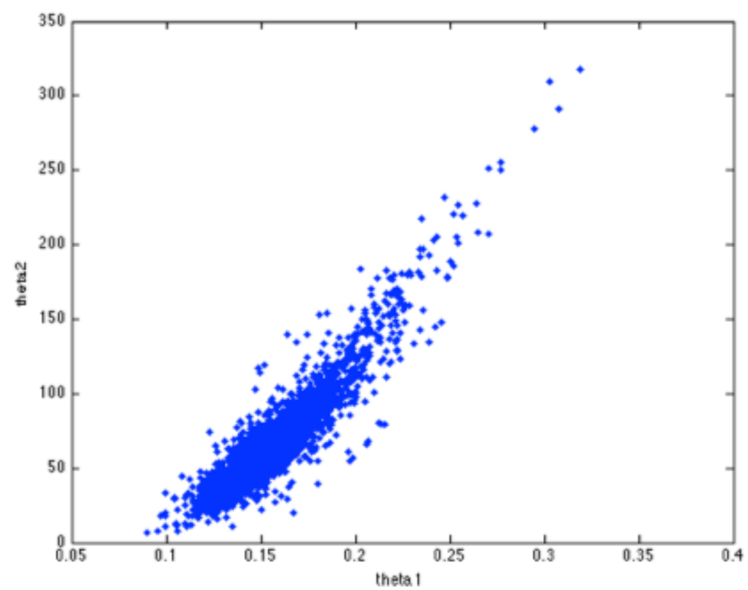


Delayed Rejection Adaptive Metropolis (DRAM)

Plot results

```
figure(2); clf
mcmcplot(chain,[],res,'chainpanel');
figure(3); clf
mcmcplot(chain,[],res,'pairs');
```





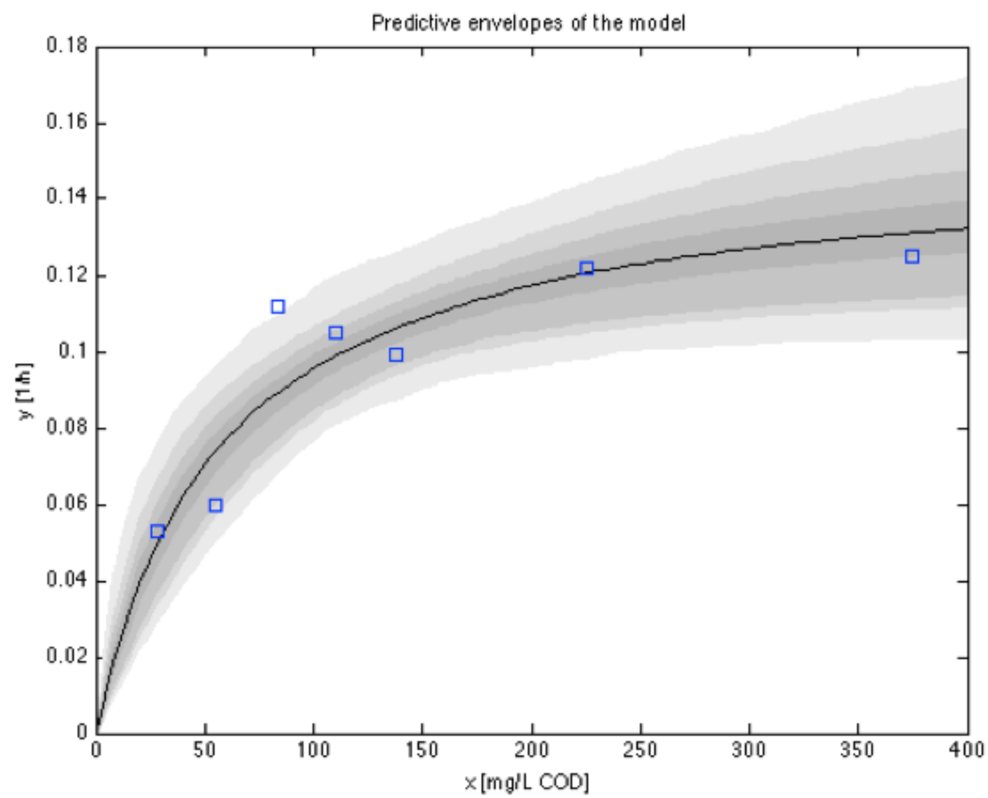
Examples:

- Several available in MCMC_EXAMPLES
- ODE solver illustrated in algae example

Delayed Rejection Adaptive Metropolis (DRAM)

Construct credible and prediction intervals

```
 figure(5); clf
 out = mcmcpred(res,chain,[],x,modelfun);
 mcmcpredplot(out);
 hold on
plot(data.xdata,data.ydata,'s'); % add data points to the plot
xlabel('x [mg/L COD]');
ylabel('y [1/h]');
hold off
title('Predictive envelopes of the model')
```



DRAM for SIR Example

SIR Model:

$$\frac{dS}{dt} = \delta N - \delta S - \gamma k I S \quad , \quad S(0) = S_0$$ Susceptible

$$\frac{dI}{dt} = \gamma k I S - (r + \delta) I \quad , \quad I(0) = I_0$$ Infectious

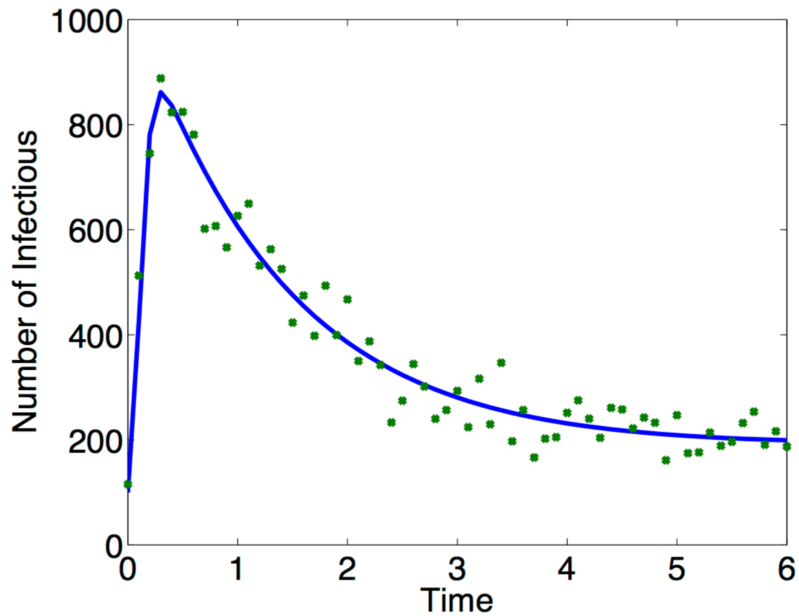
$$\frac{dR}{dt} = rI - \delta R \quad , \quad R(0) = R_0$$ Recovered

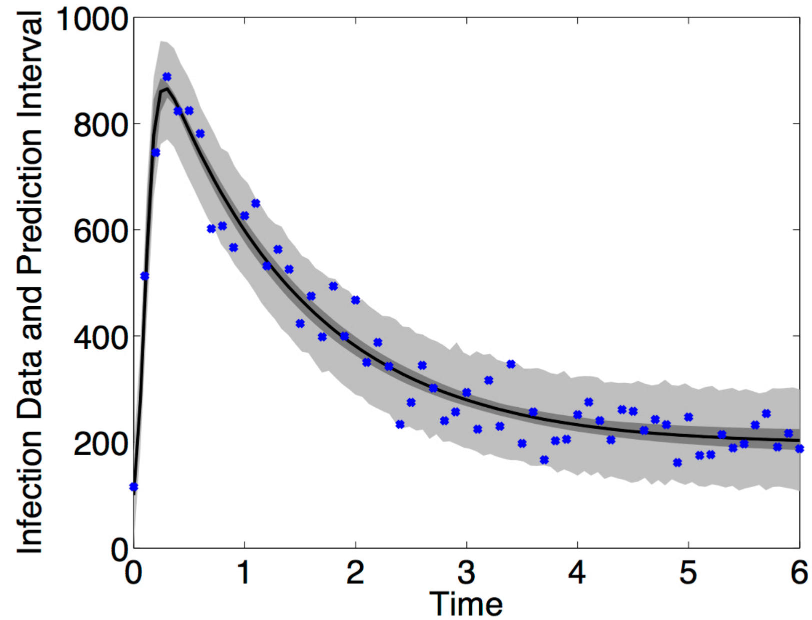
Note: Parameter set $q = [\gamma, k, r, \delta]$ is not identifiable

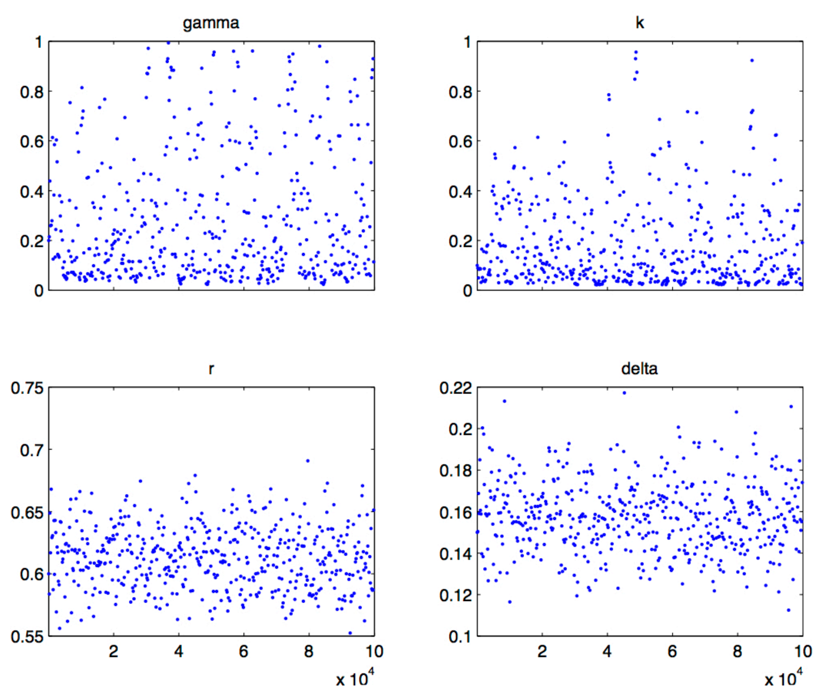
Website

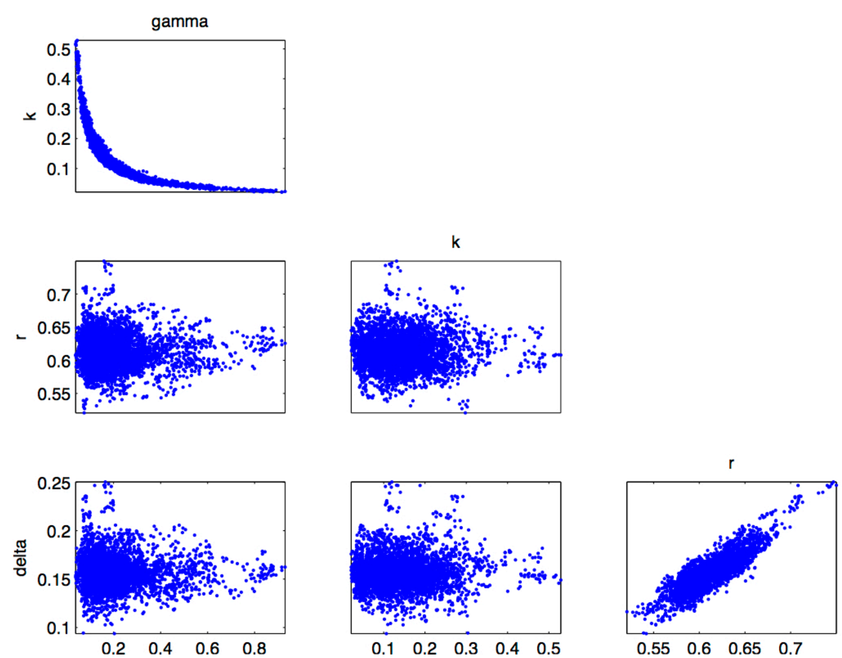
- http://helios.fmi.fi/~lainema/mcmc/
- http://www4.ncsu.edu/~rsmith/

DRAM for SIR Example: Results









SIR Project: Bayesian Inference

3 Parameter SIR Model:

$$\frac{dS}{dt} = \delta N - \delta S - \gamma I S \quad , \; S(0) = S_0 \qquad \text{Susceptible}$$

$$\frac{dI}{dt} = \gamma I S - (r + \delta) I \quad , \; I(0) = I_0 \qquad \text{Infectious}$$

$$\frac{dR}{dt} = rI - \delta R \quad , \; R(0) = R_0 \qquad \text{Recovered}$$

Note: Parameter set $q = [\gamma, r, \delta]$ is now identifiable

Exercise:

- Download MCMC_Stat and SIR_dram.m, SIR_rhs.m, SIR_fun.m, SIRss.m, SIR_data.mat and mcmcpredplot_custom.m from website

  https://rsmith.math.ncsu.edu/SAMSI_UNDERGRAD19/

- Modify the posted 4 parameter code for the 3 parameter model. How do your chains and results compare?
- You can set options.nsimu = 1000 when debugging to speed up the code.
- Consider various chain lengths to establish burn-in.

SIR Example: Sensitivity Analysis

SIR Model:

$$\frac{dS}{dt} = \delta N - \delta S - \gamma k I S \quad , \; S(0) = S_0 \qquad \text{Susceptible}$$

$$\frac{dI}{dt} = \gamma k I S - (r + \delta) I \quad , \; I(0) = I_0 \qquad \text{Infectious}$$

$$\frac{dR}{dt} = rI - \delta R \quad , \; R(0) = R_0 \qquad \text{Recovered}$$

Note: Parameter set $q = [\gamma, k, r, \delta]$ is not identifiable

Response:

$$y = \int_0^5 R(t, q) dt$$

Project Part 2: Here we are going to investigate various techniques to evaluate the sensitivity of y to the parameters q.

SIR Example: Sensitivity Analysis

1. Assume Parameter Distributions:

$$\gamma \sim \mathcal{U}(0,1)\ ,\ k \sim Beta(\alpha,\beta)\ ,\ r \sim \mathcal{U}(0,1)\ ,\ \delta \sim \mathcal{U}(0,1)$$

Infection Coefficient	Interaction Coefficient	Recovery Rate	Birth/death Rate

Results: Beta(2,7)

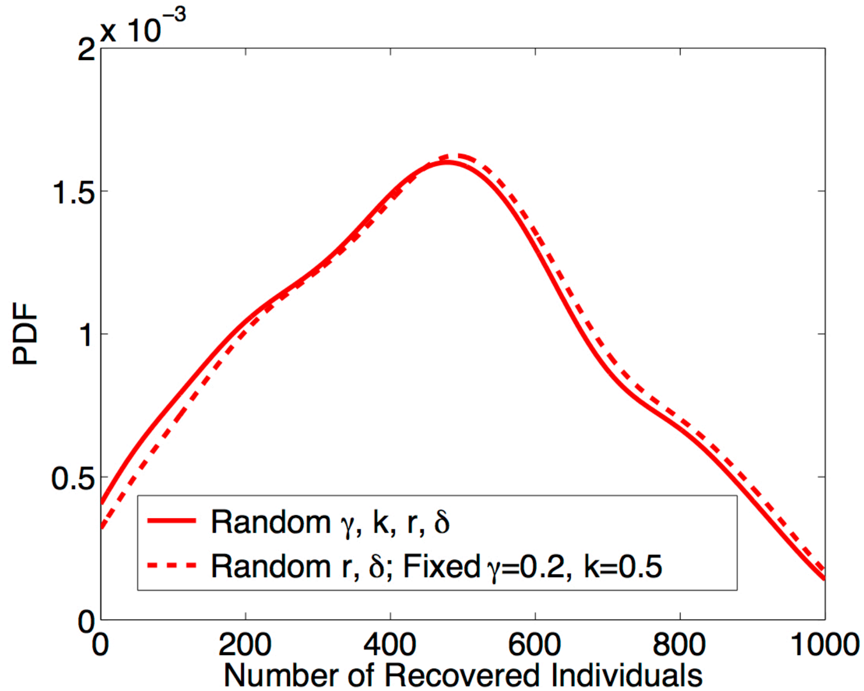
Global Sensitivity Measures:

		γ	k	r	δ
Sobol	S_i	0.0997	0.0312	0.7901	0.1750
	S_{T_i}	-0.0637	-0.0541	0.5634	0.2029
Morris	μ_i^* $(\times 10^3)$	0.2532	0.2812	2.0184	1.2328
	σ_i $(\times 10^3)$	0.9539	1.6245	6.6748	3.9886

Influential Parameters

SIR Example: Sensitivity Analysis

Result: Densities for $R(t_f)$ at $t_f = 5$



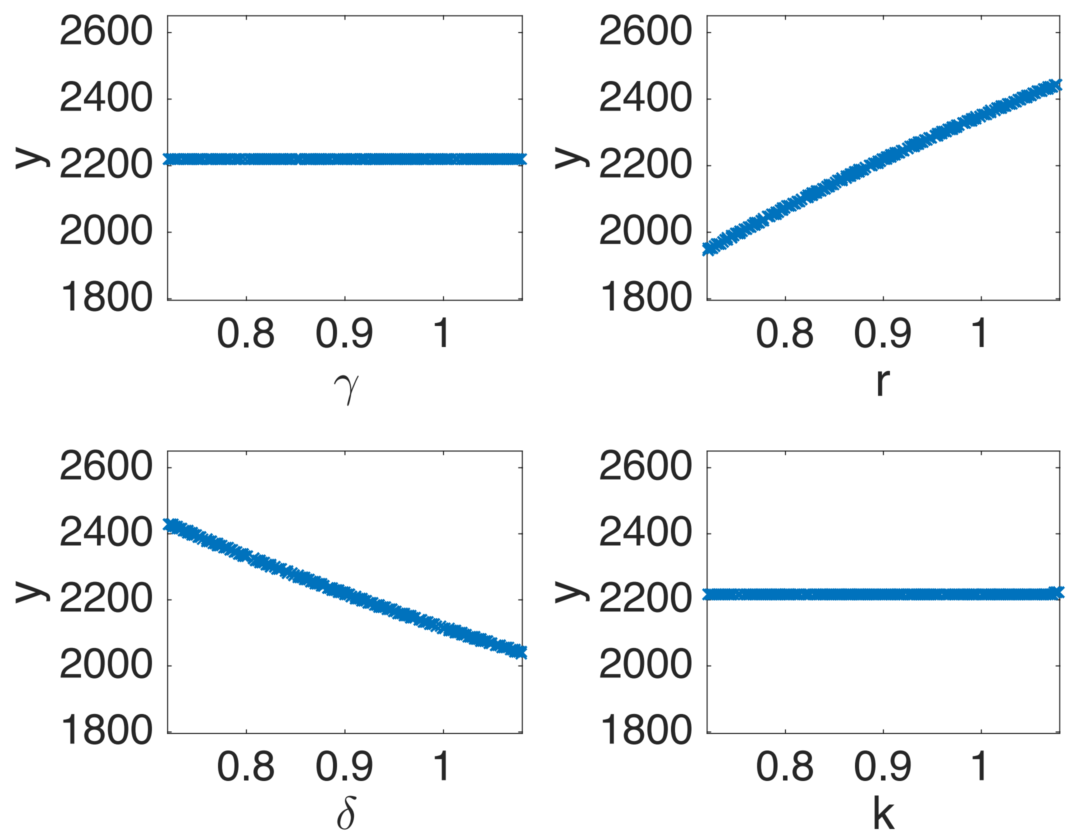
Note: Can fix non-influential parameters

Exercise 1: Run SIR_Saltelli.m with beta(2,7) and beta(0.2,15) and compare the relative sensitivities.

SIR Example: Sensitivity Analysis

2. Modify code SIR_OAT_1d.m to plot local sensitivities based on one-at-a-time sampling for

$$\frac{\partial y}{\partial \gamma}, \frac{\partial y}{\partial r}, \frac{\partial y}{\partial \delta}, \frac{\partial y}{\partial k}$$



Teams: Investigate the following nominal values

Team 1: nom = 0.01

Team 2: nom = 0.1

Team 3: nom = 0.3

Team 4: nom = 0.6

Team 5: nom = 1.0

Team 6: nom = 1.3

Team 7: nom = 1.5

Goals:

1. Plot responses and local variation of individual parameters with others fixed

2. Determine influential parameters

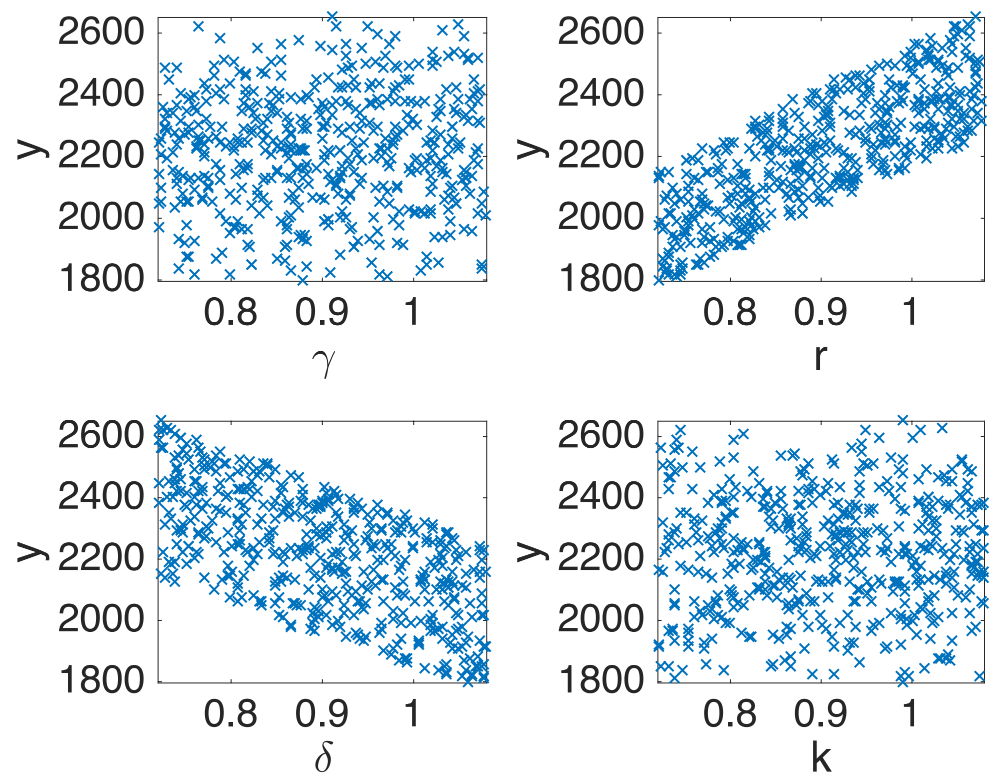
SIR Example: Sensitivity Analysis

3. Modify code SIR_SA_1d.m to sample from uniform distributions for gamma, r, k, delta; e.g.,

$$\gamma \sim \mathcal{U}(\gamma_\ell, \gamma_r)$$

$$\gamma_\ell = \gamma_{nom} - 0.2\gamma_{nom}$$

$$\gamma_r = \gamma_{nom} + 0.2\gamma_{nom}$$



Recall: MATLAB command to sample M samples from U(a,b)

$$>> q = a + (b - a) * \text{rand}(M, 1)$$

Teams: Investigate the following nominal values

Team 1: nom = 0.01

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Team 7: nom = 1.5

Goals:

1. Plot responses and scatter plots

2. Determine influential parameters