Math 1C Section 10.1 Study Guide: Exploring Graphs of Parametric Equations

This study guide reviews graphing plane curves with parametric equations. There is no "calculus" in this section. Math 1C assumes that students know these concepts from prerequisite courses (covered in both Math 43 Precalculus III and Math 1A Calculus). Students who struggle with the concepts and skills in this review should get help in understanding the material. We will use some of these examples in our review of Section 10.1 in class. Students should review and complete this entire review guide.

You are expected to know the following skills and concepts:

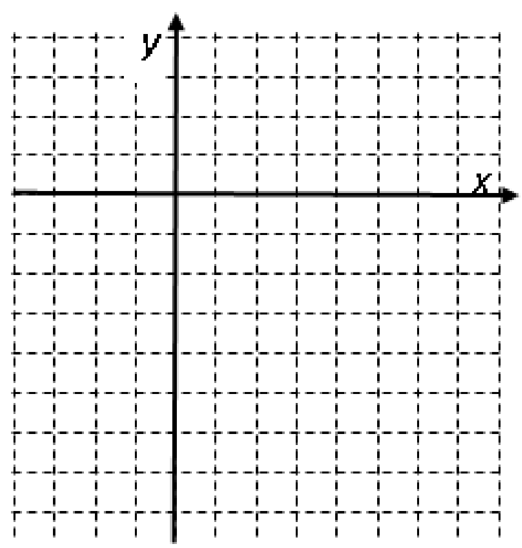
a. graph parametric equations by hand by plotting points
b. graph a curve by eliminating the parameter from the equations,
   (i) by substitution to eliminate t or (ii) by using a Pythagorean identity, as appropriate
c. use your graphing calculator (use both **PAR**ametric mode and **RADIAN** mode)
d. identify how a parametric curve is traced without depending on your calculator
e. have an excellent understanding of parametric equations for lines, circles, and ellipses in particular
f. find the value of the parameter when given the (x,y) coordinates of a point on the plane curve
g. parameterize an equation y = f(x) , or x = g(y) , to express both x and y as functions of a parameter t
h. identify the domain for t and the range for x and y when given a pair of parametric equations

A curve in the *xy* plane can be specified by a pair of parametric equations that express *x* and *y* as functions of a third variable, the parameter: $x = f(t)$, $y = g(t)$; *t* is the parameter. The parameter allows us to plot the points on the curve and indicates how the curve is traced.

1. $x = f(t) = 6 - t^2$ $y = g(t) = 2t - 4$

a. **Plotting a parametric curve**:

t	$x = f(t) = 6 - t^2$	$y = g(t) = 2t - 4$
-2		
-1		
0		
1		
2		
3		



Plot the points, label the (x,y) coordinates
Under each point(x,y), also write the value of t
Connect the points on the graph with a smooth curve.
Draw arrows on the graph to indicate the direction that the curve is traced between $t = -2$ and $t = 3$.

b. **Finding the value of the parameter *t*** (working backwards):

Find the value(s) of t at which the x coordinate is $x = 3.75$. Find the y coordinates at those times.

c. **Eliminate the parameter**: **Find a single equation for the curve, using *x* and *y* only,** eliminating t.

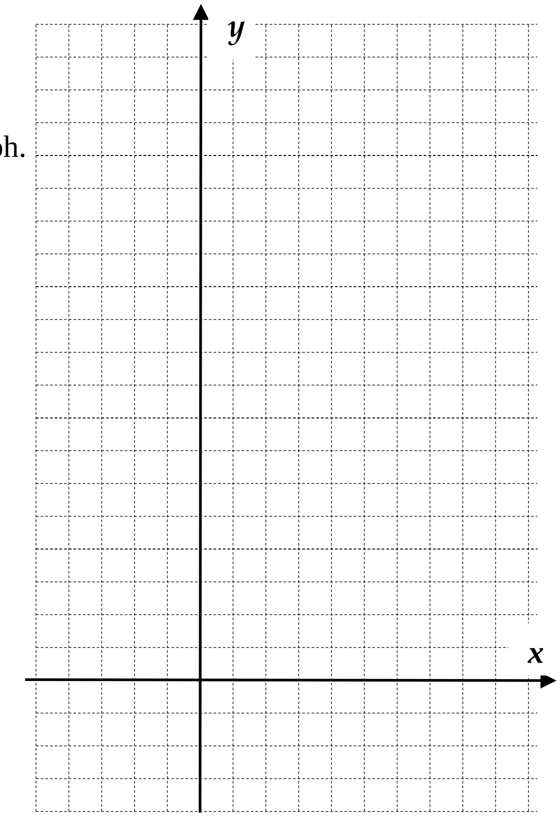
Steps to eliminate the parameter:

- Decide whether to solve for t in the x equation or the y equation.
  In this example it is better to solve $y = 2t - 4$ for t. (Why?)
- Substitute into the other equation, $x = 6 - t^2$, to eliminate t.
- Result will be an equation in x and y only, but not t

d. What information is missing from the implicit equation in x and y that is visible in parametric equations?

2. <u>Parametric Equations of Lines on a Plane</u>

$\boldsymbol{x = 4 - 2t} \qquad \boldsymbol{y = 5 + 3t}$



(a) Use a table of values with three values of t to plot the graph.

(b) Eliminate the parameter to find an EXPLICIT equation for y as a function of x

- Solve for t in terms of x.
- Substitute into the y equation to eliminate t.

(c) Explain how to find the slope of the line directly from the **parametric** equations, $\boldsymbol{x = 4 - 2t}$, $\boldsymbol{y = 5 + 3t}$.

(d) Write a set of parametric equations of a line that passes through the point (6, –2) and has slope 4/3
(There are many possible correct answers)

(e) Write a set of parametric equations of a line that has explicit equation $y = (-5/4)x - 3$
(There are many possible correct answers)

(f,g,h) **Parametric equations for a line are often linear equations of *t*, but not always**.
Each parametric equations below appear non-linear; however each pair of equations for x and y describe a line or a line segment. Eliminate the parameter to find a linear equation in y and x
(Hint: Identify a function $u(t)$ in both equations ; solve for $u(t)$in terms of x; do not solve for t itself)
For each, also explain how the graph differs from the line traced at the top of this page

(f) $x = 4 - 2t^3$, $y = 5 + 3t^3$	(g) $x = 4 - 2\sqrt{t}$, $y = 5 + 3\sqrt{t}$	(h) $x = 4 - 2\sin t$, $y = 5 + 3\sin t$

3. Parametric Equations for Circles:

$\boldsymbol{x = f(t) = \cos t \quad y = g(t) = \sin t \quad 0 \le t \le 2\pi}$ traces a curve with points $(x,y) = (\cos t, \sin t)$.

a. Remembering back to your trigonometry class, what curve does this represent?

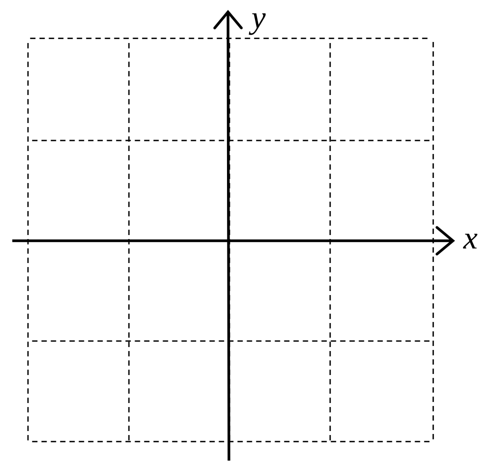
b **Eliminating the parameter**: (*A trick used more than once is a method*!)

$\boldsymbol{x = \cos t \quad y = \sin t}$

- Write the Pythagorean Trigonometric Identity that relates cos t and sin t
- Substitute x for cos t and y for sin t into the identity

c. We should be able to graph a circle by recognizing the parametric equations and then plotting several important points to understand how it is traced. (*Useful:* $\sqrt{2}/2 \approx 0.707$ $\sqrt{3}/2 \approx 0.866$)

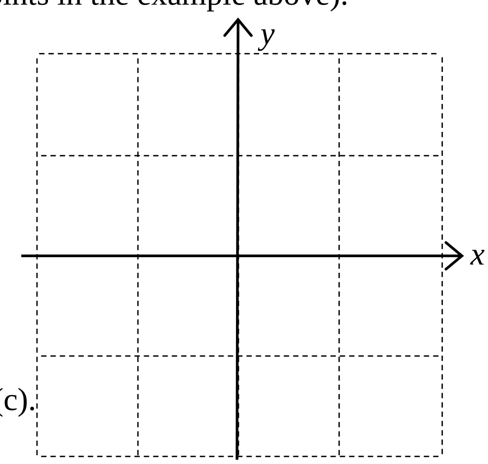
t	$x = f(t) = \cos t$	$y = g(t) = \sin t$
0		
$\pi/4$		
$\pi/2$		
$3\pi/4$		
π		
$5\pi/4$		
$3\pi/2$		
$7\pi/4$		
2π		



d. $x = f(t) = \sin t \quad y = g(t) = \cos t$ is also a circle.

If we can identify the type of curve from its equation, then to determine how it is traced, it is often sufficient to plot 5 points: t = 0, $\pi/2$, π, $3\pi/2$, 2π (instead of 9 points in the example above).

t	$x = f(t) = \sin t$	$y = g(t) = \cos t$
0		
$\pi/2$		
π		
$3\pi/2$		
2π		



Describe how this circle differs from the circle in the example 3(c).

4. **Circles and Ellipses:** We know that the unit circle with center (0,0) is $x = \cos t$, $y = \sin t$ which we can think of as $x = 0 + 1\cos t$, $y = 0 + 1\sin t$. In general, for parametric equations $\boldsymbol{x = h + a\cos t, y = k + b\sin t}$, we can use the Pythagorean Identity to eliminate the parameter:

For parametric curve: $x = h + a \cos t, y = k + b \sin t$, OR $x = h + a \sin t, y = k + b \cos t$,	
If $\|a\| = \|b\|$, the graph is:	If $\|a\| \neq \|b\|$, the graph is:
$\boldsymbol{x = 6 - 3 \sin t}$ and $\boldsymbol{y = -2 + 3 \cos t}$	$\boldsymbol{x = 1 + 2 \cos t\ ,\ y = -5 + 4 \sin t}$

5. **Hyperbola:** $\boldsymbol{x = -5 + 2 \sec t\ ,\ \ y = 2 + 3 \tan t}$
   - Solve for sec t in terms of x and solve for tan t in terms of y.
   - Write the Pythagorean Trigonometric Identity that relates tan t and sec t
   - Substitute the expressions for sec t and tan t above into the identity
   - Rearrange terms as necessary to get the righthand side equal to 1

CALCULATOR GRAPHING SKILLS:

Press MODE : On the line that reads Func Par Pol Seq , highlight Par

On the line that reads Radian Degree, highlight Radians ;then press 2nd Quit

Use Y= key to access the equation editor; use X,T,θ,n key to put T into the equations for X(T) and Y(T)

6. **Graphing paremetrically defined curves and finding appropriate graphing windows**

a. $\mathbf{x = 5 \cos t}$ and $\mathbf{y = 5 \sin t}$ represents a ________________________

Enter equations into the equation editor: Press ZOOM Arrow down to 6:ZStandard, press Enter

What does the shape of the graph look like on your calculator screen?__________________

Adjust to find an appropriate window: Press ZOOM. Arrow down to 5:ZSquare; press Enter.

What shape graph *now* appears on your calculator screen?___________

b. $\mathbf{x = 1 + 3 \cos t}$ and $\mathbf{y = -2 + 4 \sin t}$ represents an ________________________

Enter equations into the equation editor: Press ZOOM Arrow down to 6:ZStandard, press Enter

What does the shape of the graph look like on your calculator screen?__________________

Adjust to find an appropriate window: Press ZOOM. Arrow down to 5:ZSquare; press Enter.

What shape graph *now* appears on your calculator screen?___________

7 . **How different parameterizations affect the curve:** The same curve can be parameterized in different ways. The parameterization affects what part of the curve is shown and how it is traced.

In Window, adjust Tmin and Tmax as necessary. Calculator default uses Tmin =0 and Tmax = 2π.

In many cases we may want to extend that window to include negative values of t or larger positive values of t to view a larger portion of the graph.

Tstep needs to be small. The default Tstep = π/24 is good. Or use Tstep = 0.05 or 0.1.

(a) Graph $x = t^2$ $y = 3t$ on your calculator for the domain $-1 \le t \le 1$

What is the appropriate range for x? ___________

What is the appropriate range for y? __________

Sketch the curve

(b) Graph $x = t$, $y = 3\sqrt{t}$ on your calculator for the domain $0 \le t \le 1$

Why do we restrict t to be non-negative?

What is the appropriate range for x? __________

What is the appropriate range for y? __________

Sketch the curve

(c) $x = e^{-2t}$ $y = 3e^{-t}$ on your calculator for the domain for $-1 \le t \le 1$

What is the appropriate range for x? __________

What is the appropriate range for y? __________

Sketch the curve.

At what point does it start tracing?

At what point does it finish tracing?

Closely examine this graph on your calculator as t approaches 1 by adjusting the viewing window to $-1 \le x \le 2$, $-1 \le y \le 3$

CALCULATOR GRAPHING SKILLS:

Press MODE : On the line that reads Func Par Pol Seq , highlight Par

On the line that reads Radian Degree, highlight Radians ;then press 2nd Quit

8. Exploring how a curve is traced using your graphing calculator:

In Radian & Parametric Modes, set Window to have Tmin = 0 , Tmax = 2π ($\approx$6.283) , Tstep = $\pi/24$ ($\approx$ 0.131)

Graph each pair of parametric equations the equations on your calculator and complete the table:

Equations	At what point on the plane does the curve begin and end tracing?	In what direction does the circle trace on the plane?
$\boldsymbol{x = \cos t \quad y = \sin t}$		
$\boldsymbol{x = -\cos t \quad y = \sin t}$		
$\boldsymbol{x = -\cos t \quad y = -\sin t}$		
$\boldsymbol{x = \sin t \quad y = \cos t}$		
$\boldsymbol{x = \sin t \quad y = -\cos t}$		

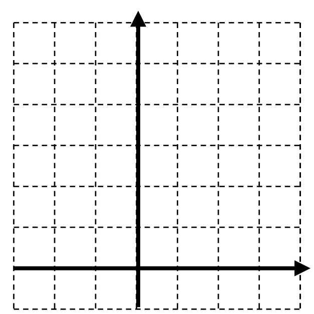
9. <u>Understanding how circles and ellipses are traced - without graphing calculator:</u>

We should recognize parametric equations for a circle or ellipse, and graph the curves by hand, without your calculator.

To figure out start and end points, and direction of tracing, use a table to calculate x and y when $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$.

- Complete the table without using a calculator.
- Knowing what the completed curve should look like, and using the information in the table, draw the graph.
- Indicate where the graph begins and ends tracing, and the direction it is traced.

t	$x = 1-2\sin t$	$y = 2+3\cos t$
0		
$\pi/2$		
π		
$3\pi/2$		
2π		



10. The domain for t affects how much of the curve is traced and how many times it is traced:

(a) Let $\boldsymbol{x = \cos t, y = \sin t}$ for $0 \le t \le \pi$. Describe the graph:

(b) Let $\boldsymbol{x = \cos t, y = \sin t}$ for $0 \le t \le 4\pi$. Describe the graph:

(c) Let $\boldsymbol{x = \cos .5t, y = \sin .5t}$ for $0 \le t \le \pi$. Describe the graph:

11. **Finding *t* at a point given on a curve:** If we are given the (x,y) coordinates of a point, we can solve the parametric equations $x = f(t)$ and $y = g(t)$ to find the value(s) of t that produce the given point. The value of t must satisfy BOTH parametric equations at the given point.

a. For the curve $x = \cos t$, $y = \sin^2 t$, find all values of t between $-\pi$ and π at the point $(0,1)$

b. For the curve $x = \cos t$, $y = 2\sin t$,find the value of t between 0 and 2π at the point $\left(\frac{\sqrt{2}}{2}, -2\right)$

c. For the curve $x = 2t^2 - 12$, $y = 3 + 4\sqrt{t}$, find the value of t at the point (20, 11)

12. **Parameterize the equations for a curve: Write a set of parametric equations for the given curve.**

Steps of parameterize equations: *Note: there are many correct answers*

- First, express one variable as a function of the parameter t. (Think carefully about which variable might be best to start with in the particular equation – start with the variable that is harder to solve for.)
- Then solve for the other variable in terms of t.

a. $\mathbf{x^2 + 2y = 60}$

b. $\mathbf{x - y^3 = y^5}$

If the equation appears to be a circle or ellipse, use sin and cos functions
If the equation appears to be a hyperbola, use sec and tan functions.
Use the appropriate Pythagorean identity to relate the variables.

c. $\mathbf{(x-20)^2 + (y+10)^2 = 144}$

d. $\mathbf{x^2 + 9y^2 = 36}$

e. $\mathbf{x^2 - 9y^2 = 36}$

13. For each ellipse write both a) an implicit equation in (x, y) and b) parametric equations (vertices are endpoints of the major axis; covertices are the endpoints of the minor axis).

i) ellipse with vertices (10, 80) and (90, 80) and covertices (50, 60) and (50, 100)

a) Implicit equation in (x, y) b) Parametric equations

ii) ellipse with vertices (60, 50) and (60, 100) and covertices (50, 75) and 70, 75)

a) Implicit equation in (x, y) b) Parametric equations

14. Practice: Domain and Range Find (i) domain for *t* , (ii) range for *x* , (iii) ranges for *y*

Don't graph it to observe the domain and range –think about the functions.
What values of t that can be input into the functions for x and y
What values can be output from the functions for x and y, based on the values of t in the domain

	Domain for t	Range for x	Range for y
a) $x = -1/t$, $y = \sqrt{t}$			
b) $x = -t^2$, $y = \sqrt{2t-1}$			
c) $x = \cos t$, $y = \sqrt[3]{t}$			
d) $x = \ln t$, $y = t^2$			
e) $x = e^t$, $y = \ln t$			
f) $x = 3 \sin 2t$, $y = 5+e^{-2t}$			

15. Exploration – Try this on your own:

Curves containing trig functions do not guarantee graphs of circles and ellipses.

It is tempting to think that whenever we have a pair of parametric equations that involve trigonometric functions, the result should be a circle or an ellipse, but that is not always true.

Circles and Ellipses:

$\boxed{x = h + a\cos u(t) \text{ AND } y = k + b\sin u(t)}$ OR $\boxed{x = h + a\sin u(t) \text{ AND } y = k + b\cos u(t)}$

- One equation contains cosine, the other equation contains sine.
- Both equations are "linear" functions of sin or cos (but may be compositions of functions)
- Sin and Cos in both equations have the same "argument" (the same input "inside" them): either both have *t* as input or both have the same function *u*(*t*) as input

Some examples that are NOT circles and ellipses: Graph them on the calculator to observe the result.

The following graphing window will work well to see most of these curves:

$0 \le t \le 2\pi$, Tstep = $\pi/24$, Xmin = – 3, Xmax = 3, Ymin = –2, Ymax = 3

$x = -1-2\sin\sqrt{t}$, $y = 1+\sin\sqrt{t}$: graph is ________________________________

$x = \cos^2 t$, $y = \cos t$: graph is ________________________________

$x = \cos 2t$, $y = \sin t$: graph is ________________________________

$x = \sin t$, $y = \sin^3 t$: graph is ________________________________

Graph the following on the calculator to see some members of the family called **Lissajous curves**

$x = \sin 2t, y = \cos t$ $x = \sin 4t, y = \cos t$

$x = \cos 2t, y = \sin 6t$ $x = \cos 3t, y = \sin t$