Parametrization

Dr. Allen Back

Nov. 17, 2014

Paraboloid $z = x^2 + 4y^2$

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Parameters u and v just different names for x and y resp.

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Use this idea if you can't think of something better.

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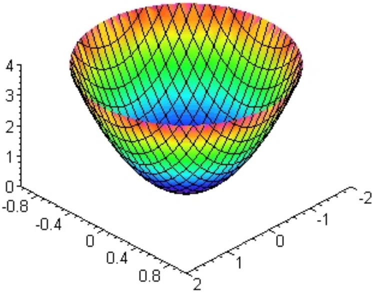

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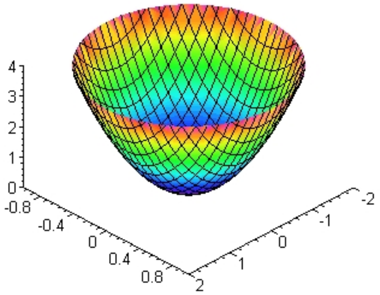

Note the curves where u and v are constant are visible in the wireframe.

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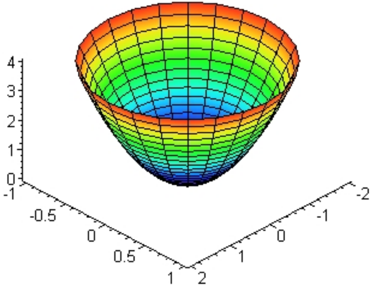

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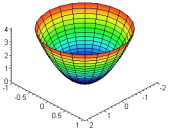

Algebraically, we are rescaling the algebra behind polar coordinates where

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

leads to $r^2 = x^2 + y^2$.

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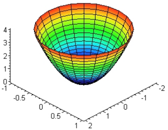

Here we want $x^2 + 4y^2$ to be simple. So

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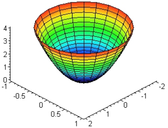

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will do better.

Plug x and y into $z = x^2 + 4y^2$ to get the z-component.

Parabolic Cylinder $z = x^2$

Graph parametrizations are often optimal for parabolic cylinders.

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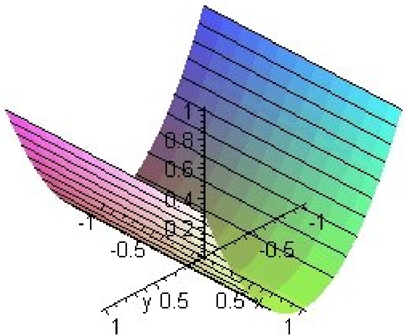

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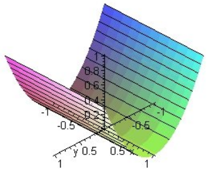

One of the parameters (v) is giving us the "extrusion" direction. The parameter u is just being used to describe the curve $z = x^2$ in the zx plane.

Elliptic Cylinder $x^2 + 2z^2 = 6$

The trigonometric trick is often good for elliptic cylinders

Elliptic Cylinder $x^2 + 2z^2 = 6$

$$\Phi(u, v) =< \sqrt{3} \cdot \sqrt{2} \cos v, u, \sqrt{3} \sin v >=< \sqrt{6} \cos v, u, \sqrt{3} \sin v >$$

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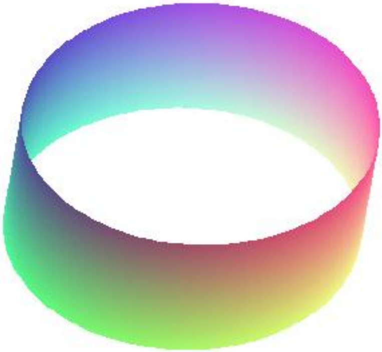

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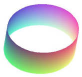

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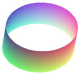

What happened here is we started with the polar coordinate idea

$$x = r \cos \theta$$
$$z = r \sin \theta$$

but noted that the algebra wasn't right for $x^2 + 2z^2$ so shifted to

$$x = \sqrt{2} r \cos \theta$$
$$z = r \sin \theta$$

Elliptic Cylinder $x^2 + 2z^2 = 6$

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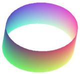

$$\begin{aligned} x &= \sqrt{2} r \cos\theta \\ z &= r \sin\theta \end{aligned}$$

makes the left hand side work out to $2r^2$ which will be 6 when $r = \sqrt{3}$.

Ellipsoid $x^2 + 2y^2 + 3z^2 = 4$

A similar trick occurs for using spherical coordinate ideas in parameterizing ellipsoids.

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$$\Phi(u, v) =< 2 \sin u \cos v, \sqrt{2} \sin u \sin v, \sqrt{\frac{4}{3}} \cos u >$$

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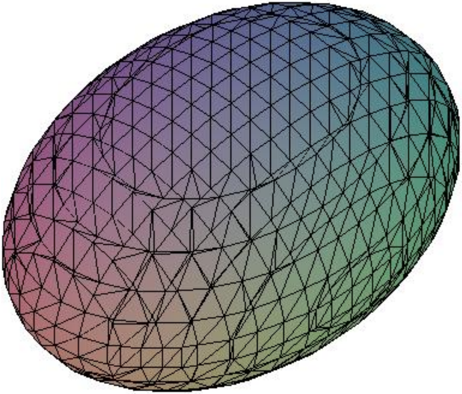

Hyperbolic Cylinder $x^2 - z^2 = -4$

You may have run into the hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
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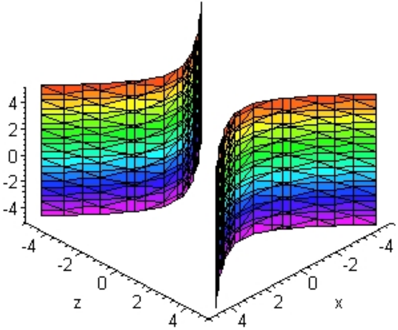

Saddle $z = x^2 - y^2$

The hyperbolic trick also works with saddles

Saddle $z = x^2 - y^2$

$$\Phi(u, v) =< u \cosh v, u \sinh v, u^2 >$$

Saddle $z = x^2 - y^2$

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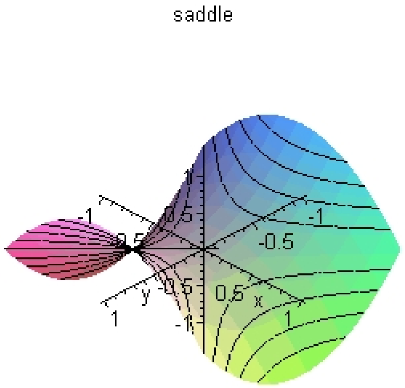

Hyperboloid of 1 Sheet $x^2 + y^2 - z^2 = 1$

The spherical coordinate idea for ellipsoids with $\sin \phi$ replaced by $\cosh u$ works well here.

Hyperboloid of 1 Sheet $x^2 + y^2 - z^2 = 1$

$$\Phi(u, v) = < \cosh u \cos v, \cosh u \sin v, \sinh u >$$

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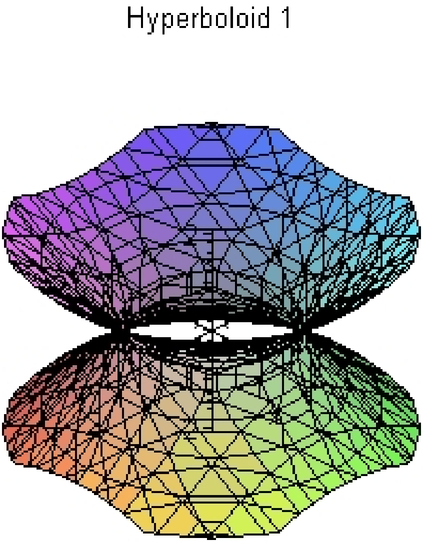

Hyperboloid of 2-Sheets $x^2 + y^2 - z^2 = -1$

$$\Phi(u, v) = < \sinh u \cos v, \sinh u \sin v, \cosh u >$$

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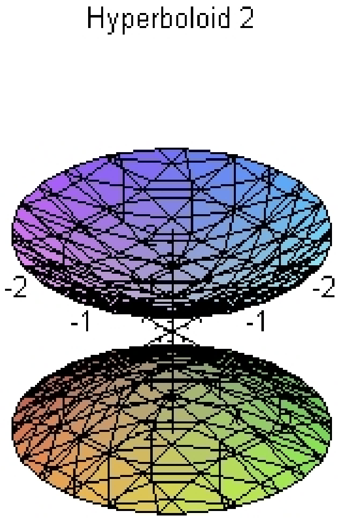

Top Part of Cone $z^2 = x^2 + y^2$

So $z = \sqrt{x^2 + y^2}$.

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The polar coordinate idea leads to

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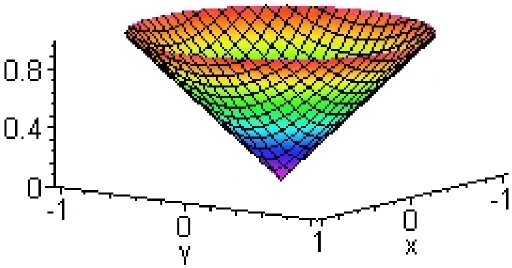
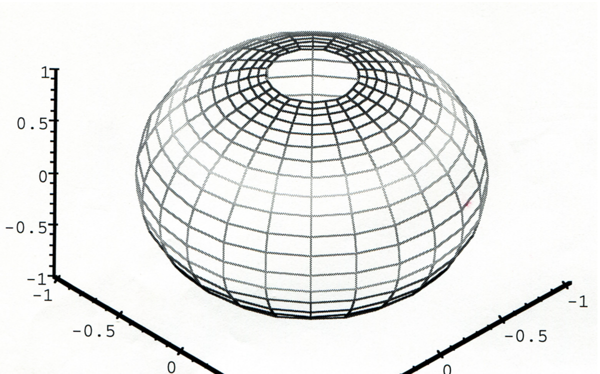


Mercator Parametrization of the Sphere

For $0 \leq v \leq \infty, \quad 0 \leq u \leq 2\pi$

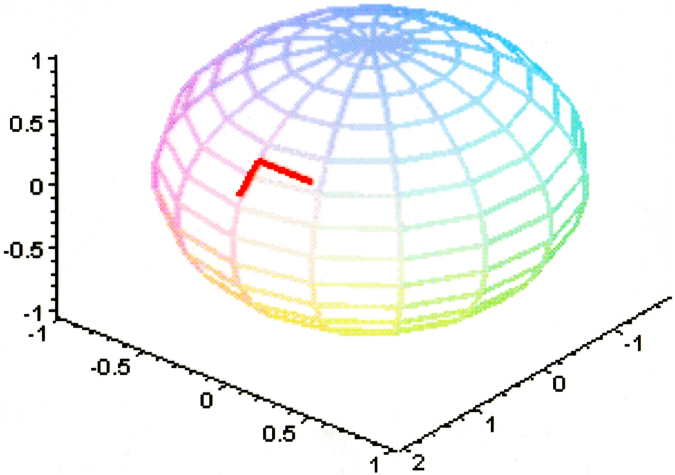
$$\Phi(u, v) = (sech(v) \cos u, sech(v) \sin u, tanh(v)).$$

$$(\text{Note } tanh^2(v) + sech^2(v) = 1)$$



Surface Integrals

Picture of $\vec{T}_u$, $\vec{T}_v$ for a Lat/Long Param. of the Sphere.



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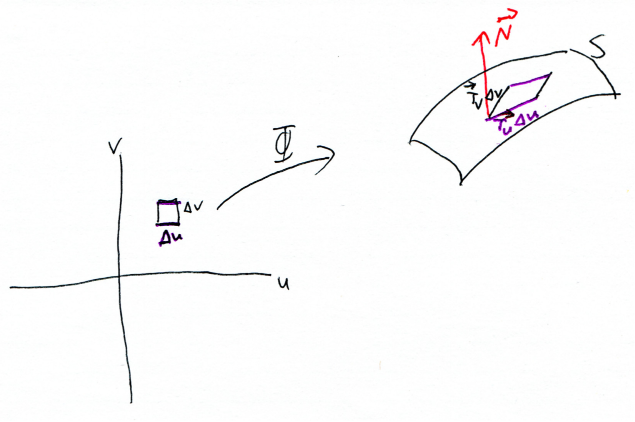
Basic Parametrization Picture



Surface Integrals

Parametrization $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$

Tangents $T_u = (x_u, y_u, z_u)$ $\qquad T_v = (x_v, y_v, z_v)$

Area Element $dS = \|\vec{T}_u \times \vec{T}_v\| \; du \; dv$

Normal $\vec{N} = \vec{T}_u \times \vec{T}_v$

Unit normal $\hat{n} = \pm \dfrac{\vec{T}_u \times \vec{T}_v|}{\|\vec{T}_u \times \vec{T}_v\|}$

(Choosing the $\pm$ sign corresponds to an *orientation* of the surface.)

Surface Integrals

Two Kinds of Surface Integrals

Surface Integral of a scalar function $f(x, y, z)$:

$$\iint_{\mathcal{S}} f(x, y, z)\ dS$$

Surface Integral of a vector field $\vec{F}(x, y, z)$:

$$\iint_{\mathcal{S}} \vec{F}(x, y, z) \cdot \hat{n}\ dS.$$

Surface Integrals

Surface Integral of a scalar function $f(x,y,z)$ calculated by

$$\iint_{\mathcal{S}} f(x,y,z)\, dS = \iint_{\mathcal{D}} f(\Phi(u,v))\, \|\vec{T}_u \times \vec{T}_v\|\, du\, dv$$

where $\mathcal{D}$ is the domain of the parametrization Φ.

Surface Integral of a vector field $\vec{F}(x,y,z)$ calculated by

$$\iint_{\mathcal{S}} \vec{F}(x,y,z) \cdot \hat{n}\, dS$$

$$= \pm \iint_{\mathcal{D}} \vec{F}(\Phi(u,v)) \cdot \left(\frac{\vec{T}_u \times \vec{T}_v|}{\|\vec{T}_u \times \vec{T}_v\|} \right) \|\vec{T}_u \times \vec{T}_v\|\, du\, dv$$

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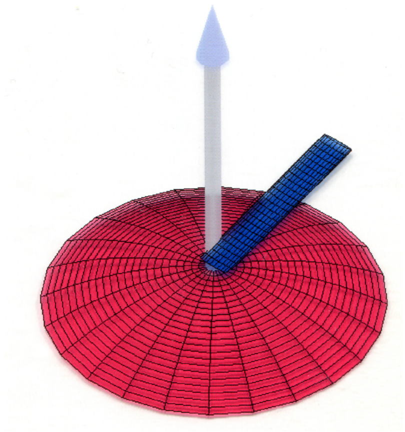
Parametrization

V1

Surface Parametrization

Surface Integrals

3d Flux Picture



Surface Integrals

The preceding picture can be used to argue that if $\vec{F}(x, y, z)$ is the velocity vector field, e.g. of a fluid of density $\rho(x, y, z)$, then the surface integral

$$\iint_{\mathcal{S}} \rho \vec{F} \cdot \hat{n} \; dS$$

(with associated Riemann Sum

$$\sum \rho(x_i^*, y_j^*, z_k^*) \vec{F}(x_i^*, y_j^*, z_k^*) \cdot \hat{n}(x_i^*, y_j^*, z_k^*) \; \Delta S_{ijk})$$

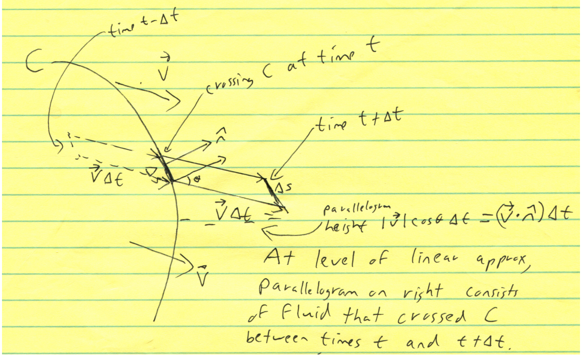
represents the rate at which material (e.g. grams per second) crosses the surface.

Surface Integrals

From this point of view the orientation of a surface simple tells us which side is accumulatiing mass, in the case where the value of the integral is positive.

Surface Integrals

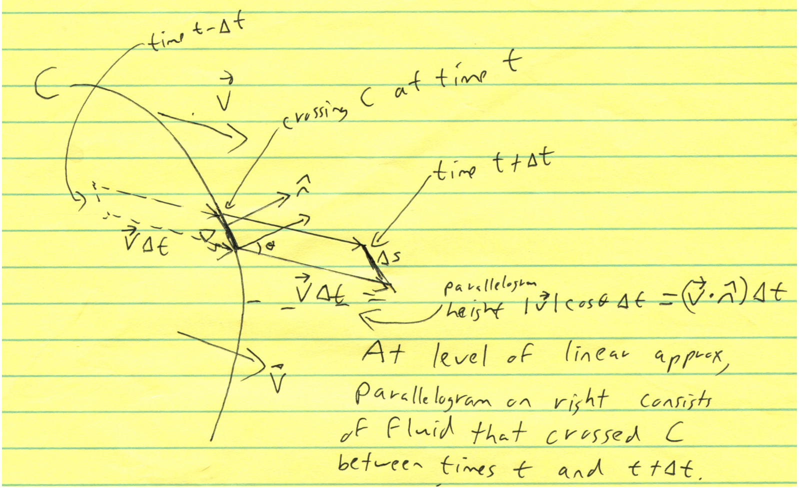
2d Flux Picture



There's an analagous 2d Riemann sum and interp of

$$\int_C \vec{F} \cdot \hat{n}\; ds.$$

Surface Integrals



Surface Integrals

Problem: Calculate

$$\iint_{\mathcal{S}} \vec{F}(x,y,z)\cdot\hat{n}\, dS$$

for the vector field $\vec{F}(x,y,z) = (x,y,z)$ and $\mathcal{S}$ the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane. Choose the positive orientation of the paraboloid to be the one with normal pointing downward.

Surface Integrals

Problem: Calculate the surface area of the above paraboloid.