Exercise 1. Verify that both $u = \log(x^2+y^2)$ and $u = \arctan(y/x)$ are solutions of Laplace's equation $u_{xx} + u_{yy} = 0$.

If $u = \log(x^2 + y^2)$, then by the chain rule

$$u_x = \frac{2x}{x^2+y^2} \quad \Rightarrow \quad u_{xx} = \frac{(x^2+y^2)(2) - (2x)(2x)}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2},$$

and by the symmetry of u in x and y,

$$u_{yy} = \frac{2x^2-2y^2}{(x^2+y^2)^2}.$$

Clearly then $u_{xx} + u_{yy} = 0$ in this case.

If $u = \arctan(y/x)$, then by the chain rule again

$$u_x = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} \quad \Rightarrow \quad u_{xx} = \frac{(x^2+y^2)(0)-(-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}.$$

Likewise

$$u_y = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \quad \Rightarrow \quad u_{yy} = \frac{(x^2+y^2)(0)-(x)(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

so that once again we have $u_{xx} + u_{yy} = 0$.

Exercise 2. Solve the boundary value problem.

a. $r\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = e^{3x}, \quad (x,y) \in \mathbb{R} \times (0,\infty), \;\; u(x,0) = f(x)$

Because the coefficients of the derivatives are constants (r and 1), we perform the linear change of variables

$$\alpha = ax + by, \tag{1}$$
$$\beta = cx + dy, \tag{2}$$
$$ad - bc \neq 0. \tag{3}$$

The usual application of the chain rule yields

$$\frac{\partial u}{\partial x} = a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta} \tag{4}$$
$$\frac{\partial u}{\partial y} = b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta} \tag{5}$$

so that the original PDE becomes

$$(ra+b)\frac{\partial u}{\partial \alpha}+(rc+d)\frac{\partial u}{\partial \beta}=e^{3x}.$$

Taking $a=0$, $b=1$, $c=-1$ and $d=r$, and noting that in this case (1) and (2) imply $r\alpha-\beta=x$, we obtain

$$\frac{\partial u}{\partial \alpha}=e^{3(r\alpha-\beta)}.$$

Integration with respect to α gives

$$u=\frac{1}{3r}e^{3(r\alpha-\beta)}+g(\beta)=\frac{1}{3r}e^{3x}+g(-x+ry). \tag{6}$$

We now impose the initial condition to solve for g. Setting $y=0$ we find that

$$f(x)=u(x,0)=\frac{1}{3r}e^{3x}+g(-x).$$

Solving for g and replacing x with $-x$ tells us that

$$g(x)=-\frac{1}{3r}e^{-3x}+f(-x).$$

Substituting this into the general solution (6) we finally arrive at

$$\boxed{u(x,y)=\frac{1}{3r}e^{3x}-\frac{1}{3r}e^{3(x-ry)}+f(x-ry).}$$

b. $\dfrac{\partial u}{\partial x}-3y\dfrac{\partial u}{\partial y}=0,\ \ (x,y)\in(0,\infty)\times\mathbb{R},\ \ u(0,y)=y^4-2$

Because this PDE has the form

$$\frac{\partial u}{\partial x}+p(x,y)\frac{\partial u}{\partial y}=0,$$

we may appeal to the naïve method of characteristics. The characteristic curves are given by

$$\frac{dy}{dx}=-3y \quad\Rightarrow\quad y=Ce^{-3x} \quad\Rightarrow\quad C=ye^{3x}.$$

The general solution therefore has the form

$$u(x,y)=f(ye^{3x}).$$

As for the initial condition, we simply set $y=0$:

$$y^4-2=u(0,y)=f(ye^0)=f(y).$$

Hence

$$\boxed{u(x,y)=y^4e^{12x}-2.}$$

c. $\dfrac{\partial u}{\partial x} - 2u\dfrac{\partial u}{\partial y} = 0, \ \ (x,y) \in (0,\infty)\times\mathbb{R}, \ \ u(0,y) = y$

This is a quasilinear PDE, but because of the coefficient $-2u$ multiplying the y derivative, the naïve method of characteristics is out. So we begin by parametrizing the initial curve, essentially taking y as the parameter:

$$x_0(a) = 0, \ \ y_0(a) = a, \ \ z_0(a) = a.$$

The characteristic ODEs are therefore

$$\frac{dx}{ds} = 1, \qquad \frac{dy}{ds} = -2z, \qquad \frac{dz}{ds} = 0,$$

$$x(0) = 0, \qquad y(0) = a, \qquad z(0) = a.$$

The first immediately yields $x = s$ and the last that $z = a$. The second the yields $y = -2as + a$. Since $x = s$ we can solve the equation for y to obtain a:

$$a = \frac{y}{1-2s} = \frac{y}{1-2x}.$$

Hence

$$\boxed{z = u(x,y) = a = \frac{y}{1-2x}.}$$

d. $4x\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 2y, \ \ (x,y) \in \mathbb{R}\times(0,\infty), \ \ u(x,0) = \log(8+x^2)$

This is a quasilinear PDE, and If we first divide through by $4x$ we can apply the naïve method of characteristics. However, we prefer to use the full strength method. The initial curve is given by

$$x_0(a) = a, \ \ y_0(a) = 0, \ \ z_0(a) = \log(8+a^2),$$

so that the characteristic ODEs are

$$\frac{dx}{ds} = 4x, \qquad \frac{dy}{ds} = 1, \qquad \frac{dz}{ds} = 2y,$$

$$x(0) = a, \quad y(0) = 0, \quad z(0) = \log(8+a^2).$$

The first equation is an exponential growth equation with solution $x = ae^{4s}$, and the second is readily integrated to yield $y = s$. This means the third becomes $\frac{dz}{ds} = 2s$ so that $z = s^2 + \log(8+a^2)$. To invert the $(x,y)-(a,s)$ system, simply note that $a = xe^{-4s} = xe^{-4y}$. Thus

$$\boxed{z = u(x,y) = s^2 + \log(8+a^2) = y^2 + \log\left(8 + x^2e^{-8y}\right).}$$

Exercise 3. Show that the general solution to $u_{xy} + u_x = 0$ has the form $u(x,y) = F(y) + e^{-y}G(x)$. [*Suggestion:* Notice that $u_{xy} + u_x = (u_y + u)_x$.]

Since $0 = u_{xy} + u_x = (u_y + u)_x$, we can integrate at once with respect to x to obtain $u_y + u = f(y)$. This is a first order linear "ODE" in the variable y. Introducing the integrating factor $\mu = \exp\left(\int 1\,dy\right) = e^y$, it becomes

$$\frac{\partial}{\partial y}(e^y u) = e^y f(y).$$

Integrating with respect to y this time yields

$$e^y u = \int e^y f(y)\,dy + G(x).$$

Finally, dividing by e^y gives

$$\boxed{u(x,y) = e^{-y}\int e^y f(y)\,dy + e^{-y}G(x) = F(y) + e^{-y}G(x),}$$

where we have replaced the arbitrary function $e^{-y}\int e^y f(y)\,dy$ with another we call F for convenience.

Exercise 4. Solve the wave equation subject to the initial conditions

$$u(x,0) = xe^{-x^2}, \quad u_t(x,0) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}.$$

According to Exercise of Assignment 2, the solution of the wave equation in this case is given by

$$u(x,t) = F(x+ct) + G(x-ct),$$

where

$$F = \frac{xe^{-x^2}}{2} + \frac{1}{2c}\int \frac{1}{1+x^2}\,dx = \frac{xe^{-x^2}}{2} + \frac{1}{2c}\arctan x,$$

$$G = \frac{xe^{-x^2}}{2} - \frac{1}{2c}\int \frac{1}{1+x^2}\,dx = \frac{xe^{-x^2}}{2} - \frac{1}{2c}\arctan x.$$

Hence

$$\boxed{u(x,t) = \frac{1}{2}\left((x+ct)e^{-(x+ct)^2} + (x-ct)e^{-(x-ct)^2}\right) + \frac{1}{2c}\left(\arctan(x+ct) - \arctan(x-ct)\right).}$$

Exercise 5. Suppose we want to find a solution of the (unbounded) wave equation that consists of a single traveling wave moving to the right with shape given by the graph of $f(x)$. What initial conditions are required to cause this to happen?

We want the solution to take the form $u(x,t) = f(x-ct)$. This requires $u_t(x,t) = -cf'(x-ct)$. To obtain the initial conditions we simply set $t = 0$:

$$\boxed{\begin{aligned} &u(x,0) = f(x), \\ &u_t(x,0) = -cf'(x). \end{aligned}}$$

Exercise 6. This problem concerns the partial differential equation

$$u_{xx} + 4u_{xy} + 3u_{yy} = 0. \tag{7}$$

a. If F and G are twice differentiable functions, show that

$$u(x, y) = F(3x - y) + G(x - y) \tag{8}$$

is a solution to (7).

We have

$$u_x = 3F'(3x - y) + G'(x - y) \quad \Rightarrow \quad \begin{cases} u_{xx} = 9F''(3x - y) + G''(x - y) \\ u_{xy} = -3F''(3x - y) - G''(x - y) \end{cases}$$

and

$$u_y = -F'(3x - y) - G'(x - y) \quad \Rightarrow \quad u_{yy} = F''(3x - y) + G''(x - y).$$

Hence

$$\begin{aligned} u_{xx} + 2u_{xy} + 3u_{yy} =& (9F''(3x - y) + G''(x - y)) + 4(-3F''(3x - y) - G''(x - y)) \\ &+ 3(F''(3x - y) + G''(x - y)) \\ =& (9 - 12 + 3)F''(3x - y) + (1 - 4 + 3)G''(x - y) \\ =& 0 + 0 = 0, \end{aligned}$$

as claimed.

b. Use a linear change of variables to show that every solution to (7) has the form (8).

Defining α and β as in (1) and (2), and applying the chain rule six times eventually leads us to

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= a^2 \frac{\partial^2 u}{\partial \alpha^2} + 2ac \frac{\partial^2 u}{\partial \alpha \partial \beta} + c^2 \frac{\partial^2 u}{\partial \beta^2}, \\ \frac{\partial^2 u}{\partial x \partial y} &= ab \frac{\partial^2 u}{\partial \alpha^2} + (ad + bc) \frac{\partial^2 u}{\partial \alpha \partial \beta} + cd \frac{\partial^2 u}{\partial \beta^2}, \\ \frac{\partial^2 u}{\partial y^2} &= b^2 \frac{\partial^2 u}{\partial \alpha^2} + 2bd \frac{\partial^2 u}{\partial \alpha \partial \beta} + d^2 \frac{\partial^2 u}{\partial \beta^2}. \end{aligned}$$

Substituting these into (7) and collecting common terms we arrive at the new PDE

$$(a^2 + 4ab + 3b^2) \frac{\partial^2 u}{\partial \alpha^2} + (2ac + 4ad + 4bc + 6bd) \frac{\partial^2 u}{\partial \alpha \partial \beta} + (c^2 + 4cd + 3d^2) \frac{\partial^2 u}{\partial \beta^2} = 0.$$

If we take $a = 3$, $b = -1$, $c = 1$ and $d = -1$ then $ad - bc = -3 + 1 = -2 \neq 0$ and the new PDE becomes

$$-4 \frac{\partial^2 u}{\partial \alpha \partial \beta} = 0 \iff \frac{\partial^2 u}{\partial \alpha \partial \beta} = 0.$$

Integration with respect to β gives

$$\frac{\partial u}{\partial \alpha} = f(\alpha)$$

for an arbitrary f and integration with respect to α then gives

$$u = F(\alpha) + G(\beta),$$

where F is an antiderivative of f. Since $\alpha = 3x - y$ and $\beta = x - y$, we finally find that

$$u(x, y) = F(3x - y) + G(x - y),$$

as desired.

c. Find the solution to (7) that satisfies the initial conditions

$$u(x, 0) = \frac{x}{x^2+1} \text{ and } u_y(x, 0) = 0 \text{ for all } x.$$

Using the general solution obtained in part **b**, we find that

$$u_y(x, y) = -F'(3x - y) - G'(x - y).$$

Hence the initial conditions require that

$$\begin{aligned} \frac{x}{x^2+1} &= u(x, 0) = F(3x) + G(x), \\ 0 &= u_y(x, 0) = -F'(3x) - G'(x). \end{aligned}$$

The second equation implies that $G'(x) = -F'(3x)$ so that $G(x) = -F(3x)/3 + C$. Substituting this into the first yields

$$\frac{2}{3}F(3x) + C = \frac{x}{x^2+1} \quad \Rightarrow \quad F(x) = \frac{9x}{2(x^2+9)} - \frac{3}{2}C.$$

Thus

$$G(x) = -\frac{1}{3}F(3x) + C = \frac{-x}{2(x^2+1)} + \frac{3}{2}C.$$

Hence we finally have

$$\boxed{u(x, y) = F(3x - y) + G(x - y) = \frac{9(3x - y)}{2((3x - y)^2 + 9)} + \frac{y - x}{2((x - y)^2 + 1)}.}$$

Exercise 7. Show that the functions

$$\cos x, \cos 3x, \cos 5x, \cos 7x, \ldots,$$

are pairwise orthogononal relative to the inner product $\langle f, g\rangle = \int_0^{\pi/2} f(x)g(x)\, dx$. [*Suggestion:* Use the identity $\cos(A + B) + \cos(A - B) = 2\cos A \cos B$.]

If $m, n \in \mathbb{N}$ are both odd (and distinct), then using the given identity we have

$$\begin{aligned}
\langle \cos mx, \cos nx \rangle &= \int_0^{\pi/2} \cos mx \cos nx \, dx \\
&= \frac{1}{2} \int_0^{\pi/2} \cos(m+n)x + \cos(m-n)x \, dx \\
&= \frac{1}{2} \left(\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \Big|_0^{\pi/2} \right) \\
&= \frac{1}{2} \left(\frac{\sin(m+n)\pi/2}{m+n} + \frac{\sin(m-n)\pi/2}{m-n} - \frac{\sin 0}{m+n} - \frac{\sin 0}{m-n} \right) \\
&= \frac{1}{2} \left(\frac{\sin(m+n)\pi/2}{m+n} + \frac{\sin(m-n)\pi/2}{m-n} \right)
\end{aligned}$$

since $\sin 0 = 0$. Because m and n are both odd, $m+n$ and $m-n$ are both even, so that $\frac{m+n}{2} = k$ and $\frac{m-n}{2} = \ell$ are both integers. Hence

$$\frac{\sin(m+n)\pi/2}{m+n} = \frac{\sin k\pi}{m+n} = 0$$

and

$$\frac{\sin(m-n)\pi/2}{m+n} = \frac{\sin \ell\pi}{m+n} = 0$$

as well, and the integral evaluates to zero, which is what we needed to show.

Exercise 8. Let

$$\begin{aligned}
f_1(x) &= 1, \\
f_2(x) &= 2x - 1, \\
f_3(x) &= 6x^2 - 6x + 1, \\
f_4(x) &= 20x^3 - 30x^2 + 12x - 1.
\end{aligned}$$

a. Verify that the polynomials f_1, f_2, f_3 and f_4 are pairwise orthogonal relative to the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx$.

We have

$$\langle f_1, f_2\rangle = \int_0^1 f_1(x) f_2(x)\, dx = x^2 - x\Big|_0^1 = 0,$$

$$\langle f_1, f_3\rangle = \int_0^1 f_1(x) f_3(x)\, dx = 2x^3 - 3x^2 + x\Big|_0^1 = 0,$$

$$\langle f_1, f_4\rangle = \int_0^1 f_1(x) f_4(x)\, dx = 5x^4 - 10x^3 + 6x^2 - x\Big|_0^1 = 0,$$

$$\langle f_2, f_3\rangle = \int_0^1 f_2(x) f_3(x)\, dx = 3x^4 - 6x^3 + 4x^2 - x\Big|_0^1 = 0,$$

$$\langle f_2, f_4\rangle = \int_0^1 f_2(x) f_4(x)\, dx = 8x^5 - 20x^4 + 18x^3 - 7x^2 + x\Big|_0^1 = 0,$$

$$\langle f_3, f_4\rangle = \int_0^1 f_3(x) f_4(x)\, dx = 20x^6 - 60x^5 + 68x^4 - 36x^3 + 9x^2 - x\Big|_0^1 = 0.$$

b. Let $p(x) = x^3 - 2$. Use part **a** to write p as a linear combination of f_1, f_2, f_3 and f_4. [*Suggestion:* Recall that since the f_i are orthogonal, if

$$p = a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4,$$

then $a_j = \langle p, f_j\rangle / \langle f_j, f_j\rangle$.]

According to the stated formulae

$$a_1 = \frac{\langle p, f_1\rangle}{\langle f_1, f_1\rangle} = \frac{\int_0^1 p(x) f_1(x)\, dx}{\int_0^1 f_1(x) f_1(x)\, dx} = -\frac{7}{4},$$

$$a_2 = \frac{\langle p, f_2\rangle}{\langle f_2, f_2\rangle} = \frac{\int_0^1 p(x) f_2(x)\, dx}{\int_0^1 f_2(x) f_2(x)\, dx} = \frac{9}{20},$$

$$a_3 = \frac{\langle p, f_3\rangle}{\langle f_3, f_3\rangle} = \frac{\int_0^1 p(x) f_3(x)\, dx}{\int_0^1 f_3(x) f_3(x)\, dx} = \frac{1}{4},$$

$$a_4 = \frac{\langle p, f_4\rangle}{\langle f_4, f_4\rangle} = \frac{\int_0^1 p(x) f_4(x)\, dx}{\int_0^1 f_4(x) f_4(x)\, dx} = \frac{1}{20},$$

so that

$$p = -\frac{7}{4} f_1 + \frac{9}{20} f_2 + \frac{1}{4} f_3 + \frac{1}{20} f_3,$$

as is easily verified.

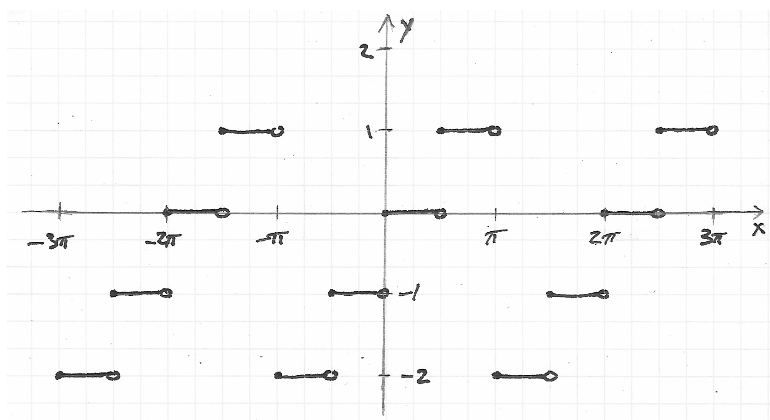
c. Explain why the procedure of part **b** *fails* if we take $p(x) = x^5 - 2x + 1$.

Although the coefficients a_1, a_2, a_3 and a_4 can still be computed, the linear combination $a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4$ will have degree at most 4 and so cannot equal p, which in this case has degree 5. The problem is that while the orthogonality of the set $\{f_1, f_2, f_3, f_4\}$ guarantees its linear independence, it is not a basis for the space of polynomials of degree at most 5 since it does not have enough vectors, namely 5.

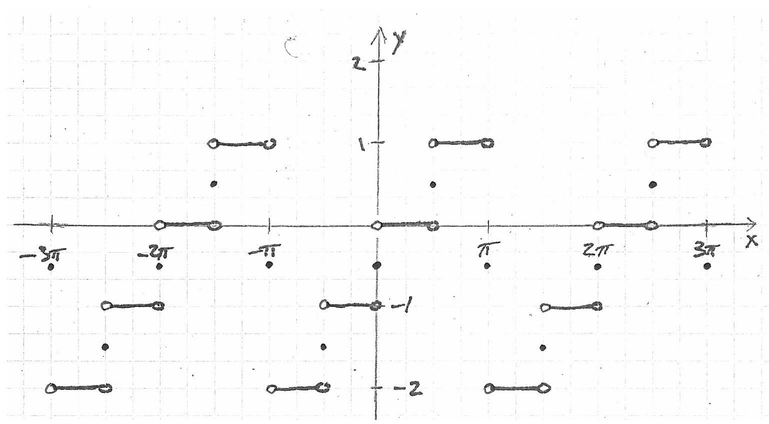
Exercise 9. For each 2π-periodic function f given: **i.** carefully sketch three periods of f and **ii.** carefully sketch three periods of the the Fourier series of f.

a. $f(x) = \begin{cases} \lfloor 2x/\pi \rfloor & \text{if } -\pi \le x < \pi, \\ f(x + 2\pi) & \text{otherwise.} \end{cases}$

Here's a sketch of f

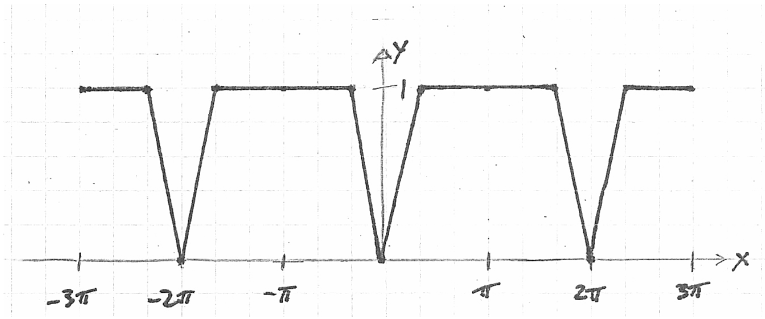


and a sketch of its Fourier series



b. $f(x) = \begin{cases} \min\{|3x/\pi|, 1\} & \text{if } -\pi \le x < \pi, \\ f(x + 2\pi) & \text{otherwise.} \end{cases}$

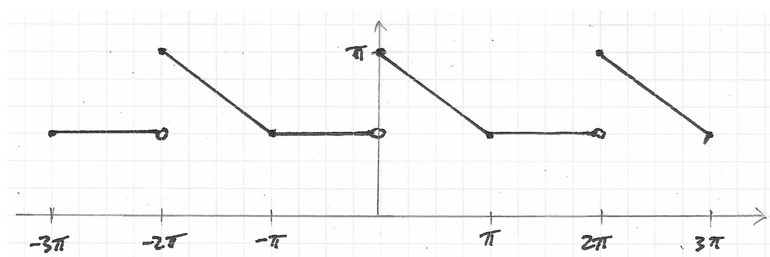
Here's a sketch of f



which is identical with its Fourier series since it is continuous everywhere.

c. $f(x) = \begin{cases} \pi/2 & \text{if } -\pi \le x < 0, \\ \pi - x/2 & \text{if } 0 \le x < \pi, \\ f(x+2\pi) & \text{otherwise.} \end{cases}$

Here's a sketch of f



and a sketch of its Fourier series