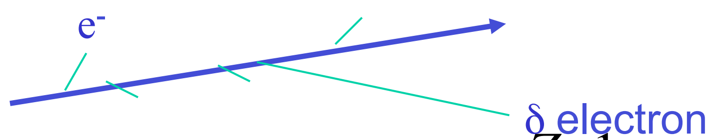
Particle Detectors

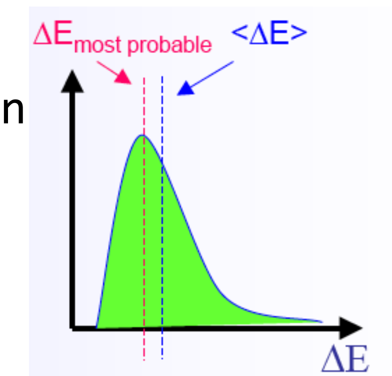
Lecture 5
16/03/16

a.a. 2015-2016
Emanuele Fiandrini

Energy loss probability (2)

- For thin layers or low density materials:
  - Few collisions, some with high energy transfer.
  - ⇒ Energy loss distributions show large fluctuation towards high losses
  - ⇒ Long Landau tails



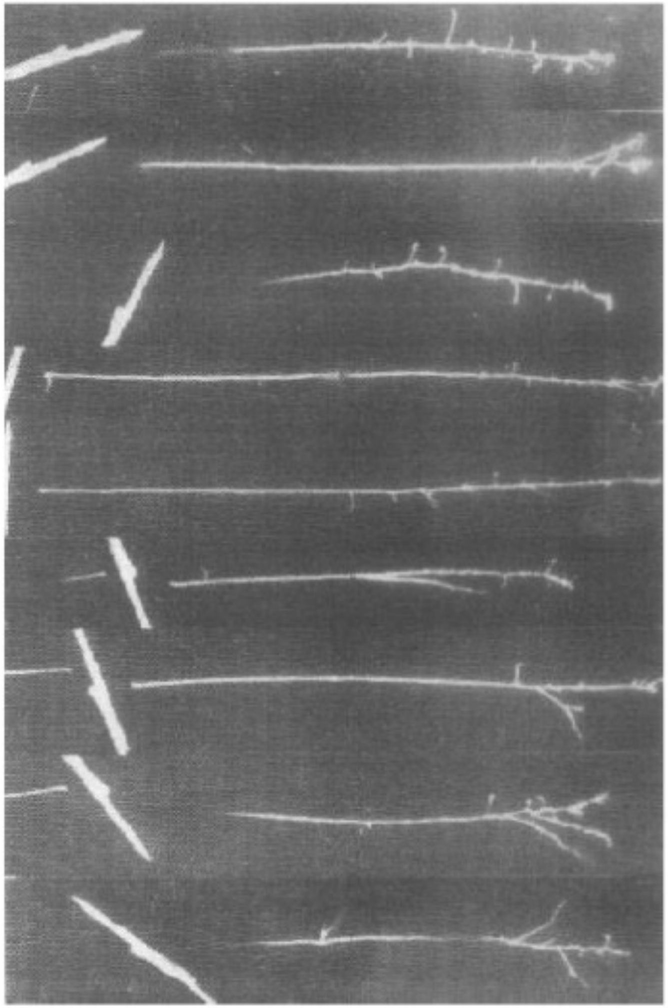


$$P(E)dE = k\frac{X}{E^2}dE \quad \text{con} \quad k = 2\pi N_A r_e^2 m_e c^2 z^2 \frac{Z}{A}\frac{1}{\beta^2}$$

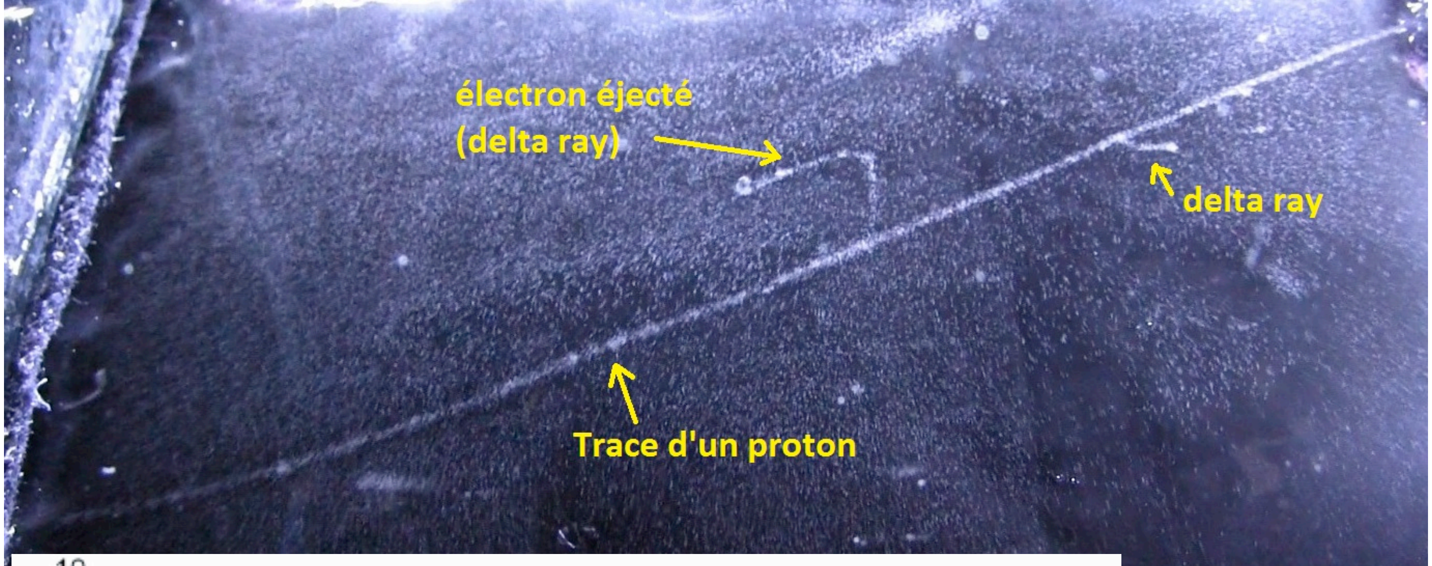
La probabilita' e' piccata a piccoli valori di E, cioe' per valori ~ Emin: la maggior parte delle collisioni comporta piccole perdite di E, ma c'e' una probabilita' finita che in una singola collisione avvenga una perdita "grande", vicina a Emax
Se N non e' sufficientemente elevato, singole collisioni con grande E non sono compensate da tante collisioni con piccola E: la distribuzione delle perdite non e' simmetrica intorno al valore medio. Le singole collisioni spostano il valore medio verso alti valori di E, mentre la maggior parte delle perdite e' con piccola E, cioe' il valore piu' probabile rimane piccolo

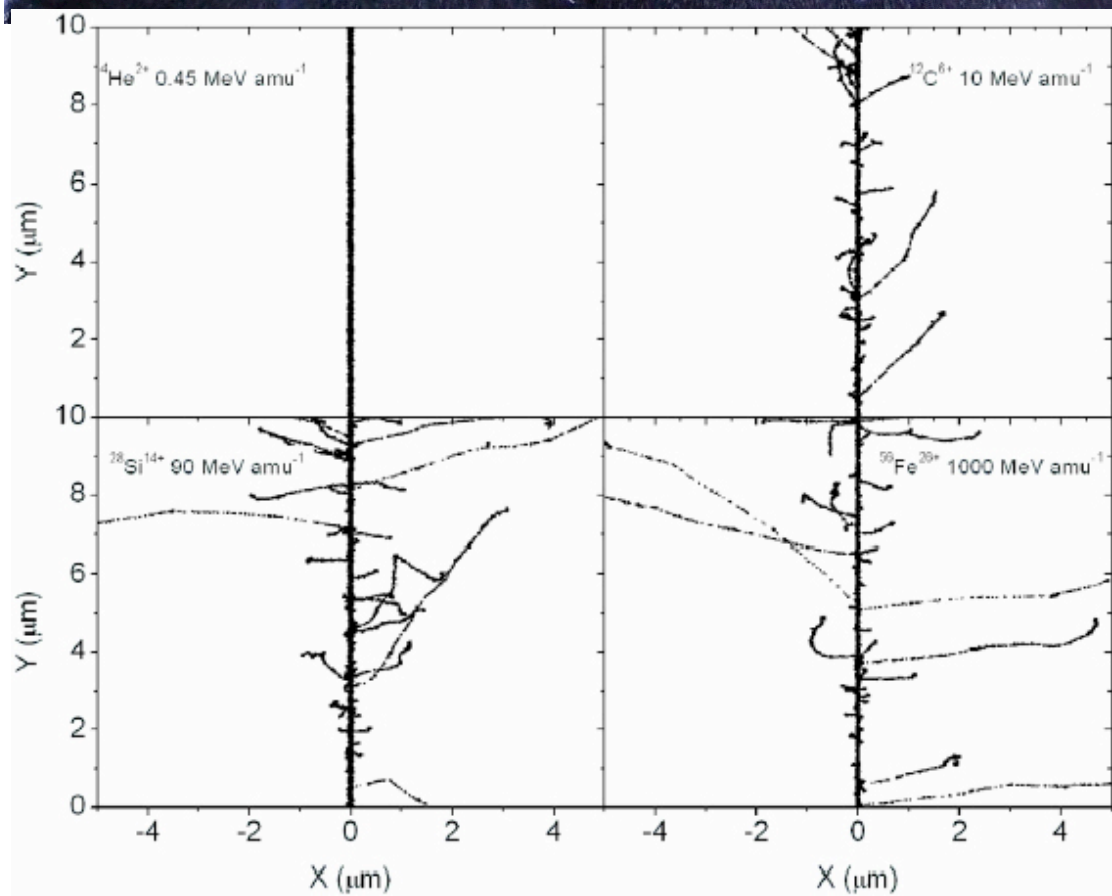
Delta Rays

Le perdite di energia appaiono come energia cinetica degli elettroni espulsi. I delta appaiono come sottili “rami” che dipartono dal tronco



From: http://www.lateralscience.co.uk/cloud/diff.html



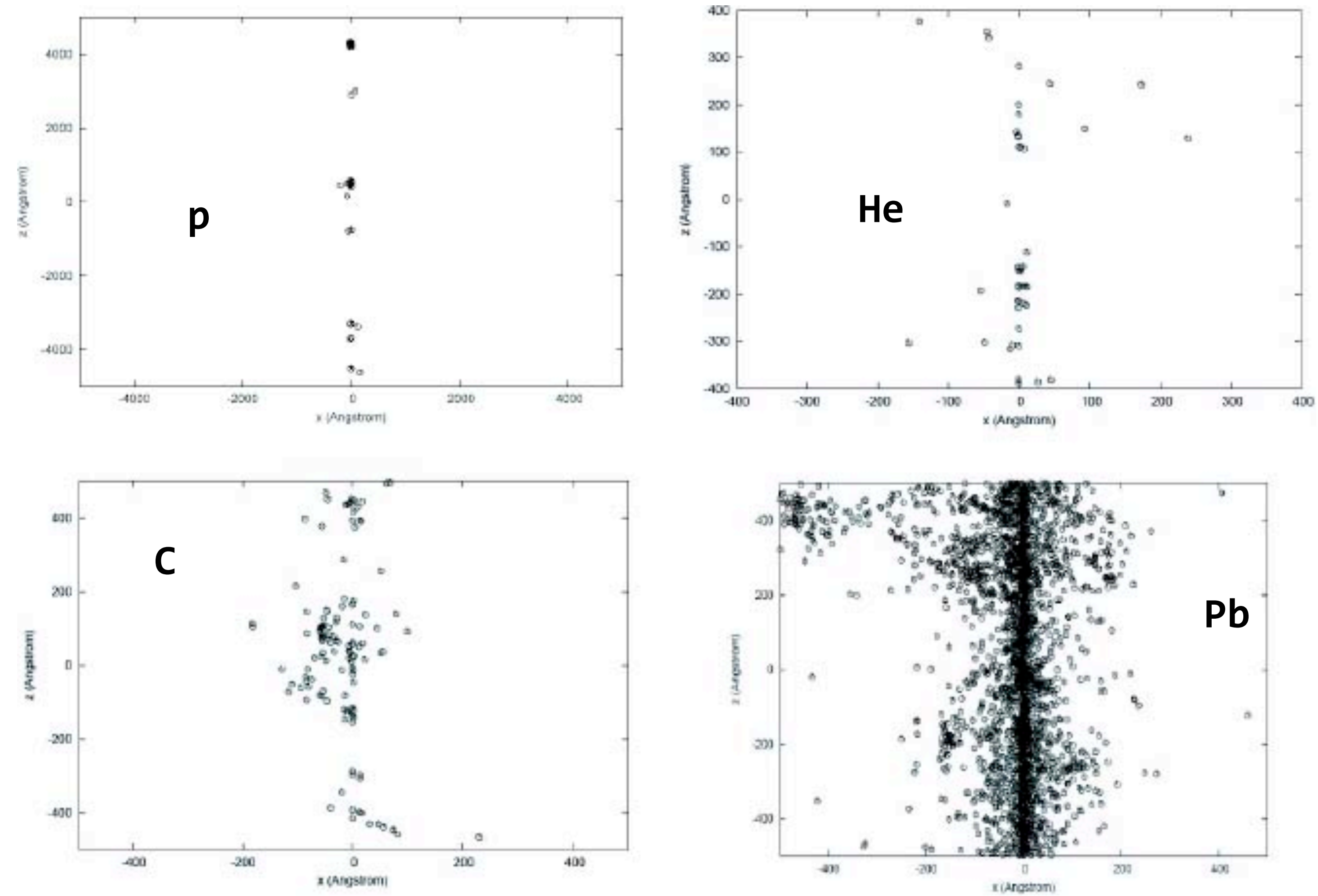


Delta rays can make additional secondary ionization trails that:

-may escape the active volume, so their energy can not be collected→potential bias to energy deposit

-confuse the main track, making difficult the pattern recognition, that is the task of assigning a energy deposit to a given track.

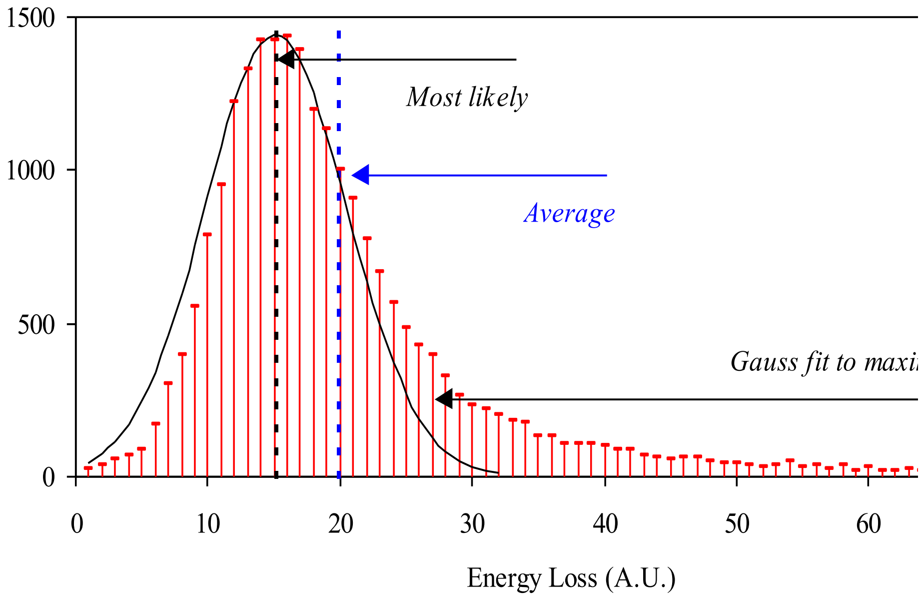
That's why we hate δ-rays



Delta rays grow quadratically with charge.
They worsen the capability to know the particle trajectory, i.e. position resolution and therefore

Energy loss probability

Energy loss distribution differs a lot from a gaussian.
Only around the peak the E loss distribution is approximated by a gaussian: the width is large and asymmetric, does not fall as 1/√X.
This has important consequences from a practical point of view: it makes the average E loss, <dE/dX>, useless for measuring the energy deposit in the medium: it will fluctuate in a non-gaussian way due high energy deposit fluctuations



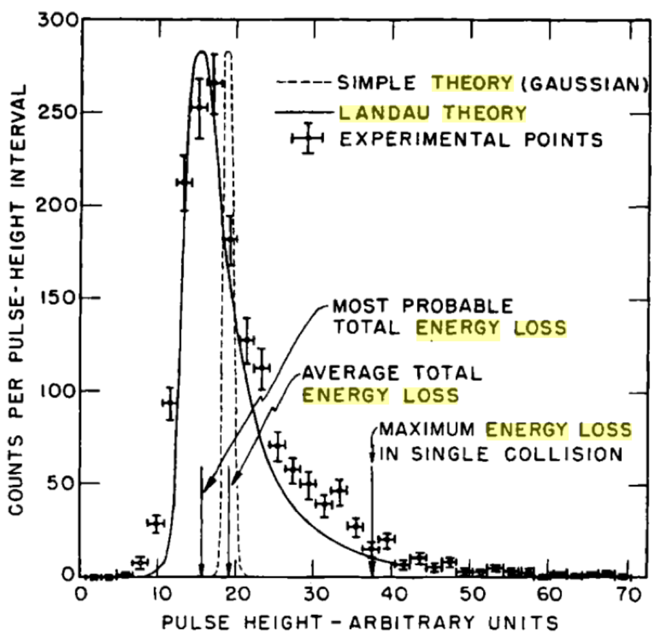
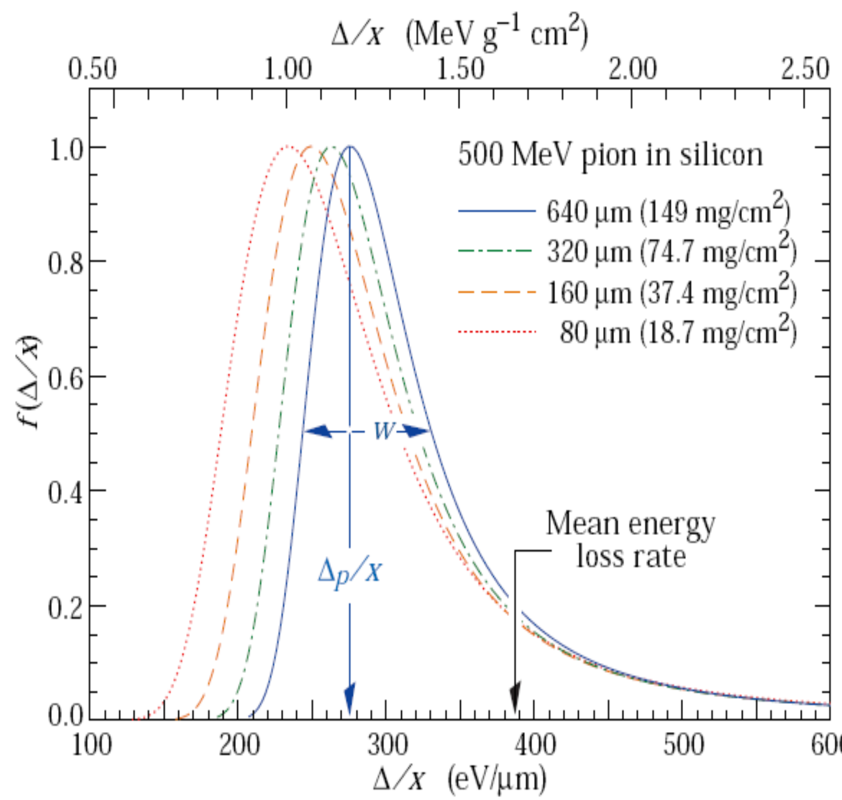


Fig. 2. Frequency distribution of energy losses of 31.5 MeV protons traversing a proportional counter filled with 96% Ar and 4% CO_2. The histogram of experimental points is compared to the theoretical Landau distribution and a gaussian distribution based on ion-electron pair statistics. From Igo *et al.* [3]

Delta rays, most probably energy loss

While <dE/dX> is affected by fluctuations, peak position (most probable energy loss) is not affected; therefore it is used when sampling dE/dx loss.

- It is important to keep in mind that the m.p. value and the average value do not coincide



Straggling functions in silicon for 500 MeV pions, normalized to unity at the most probable value Δ/x. The width w is the FWHM.

Bibliografia Fernow (Introduction to experimental particle physics)

http://pdg.lbl.gov

Landau Tails

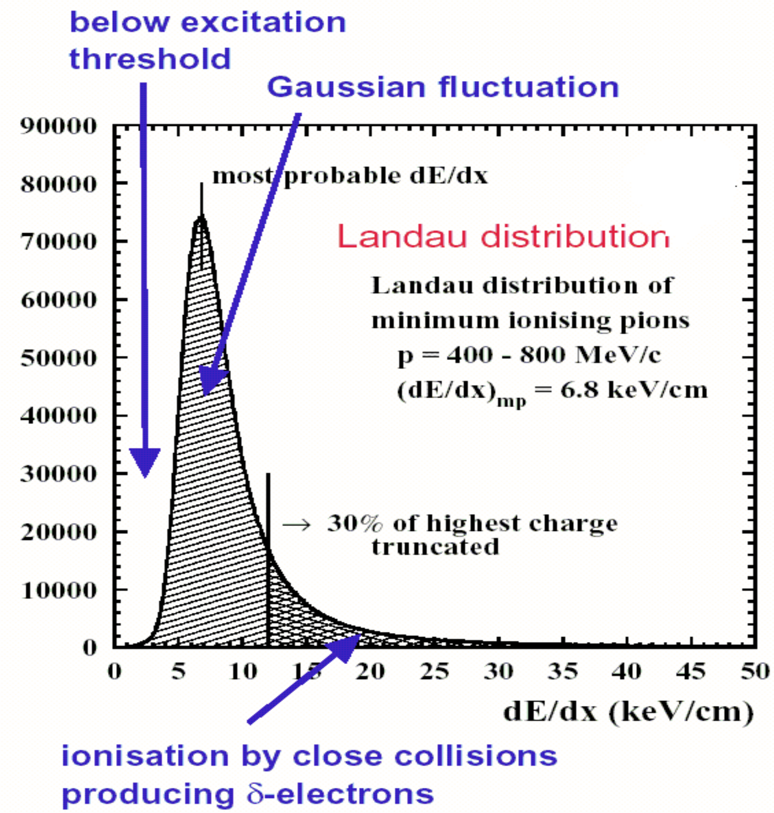
- Restricted dE/dX: mean of truncated distribution excluding energy transfers above some threshold T_{cut}
  - Useful for thin detectors in which δ-electrons can escape the detector (energy loss not measured)

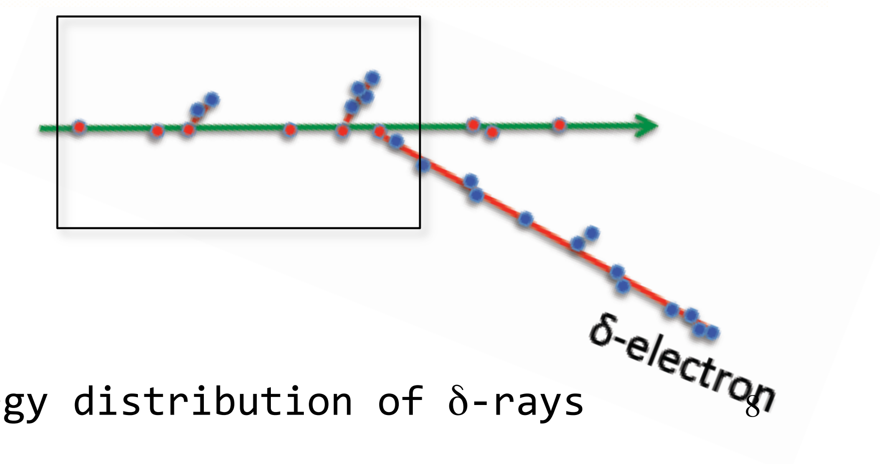
$$-\frac{dE}{dx}\bigg|_{T<T_{\rm cut}} = Kz^2\frac{Z}{A}\frac{1}{\beta^2}\left[\frac{1}{2}\ln\frac{2m_ec^2\beta^2\gamma^2T_{\rm cut}}{I^2} - \frac{\beta^2}{2}\left(1+\frac{T_{\rm cut}}{T_{\rm max}}\right) - \frac{\delta}{2}\right].$$

This form approaches the normal Bethe-Bloch function as Tcut→Tmax. It can be verified that the difference between restricted and full loss is equal to

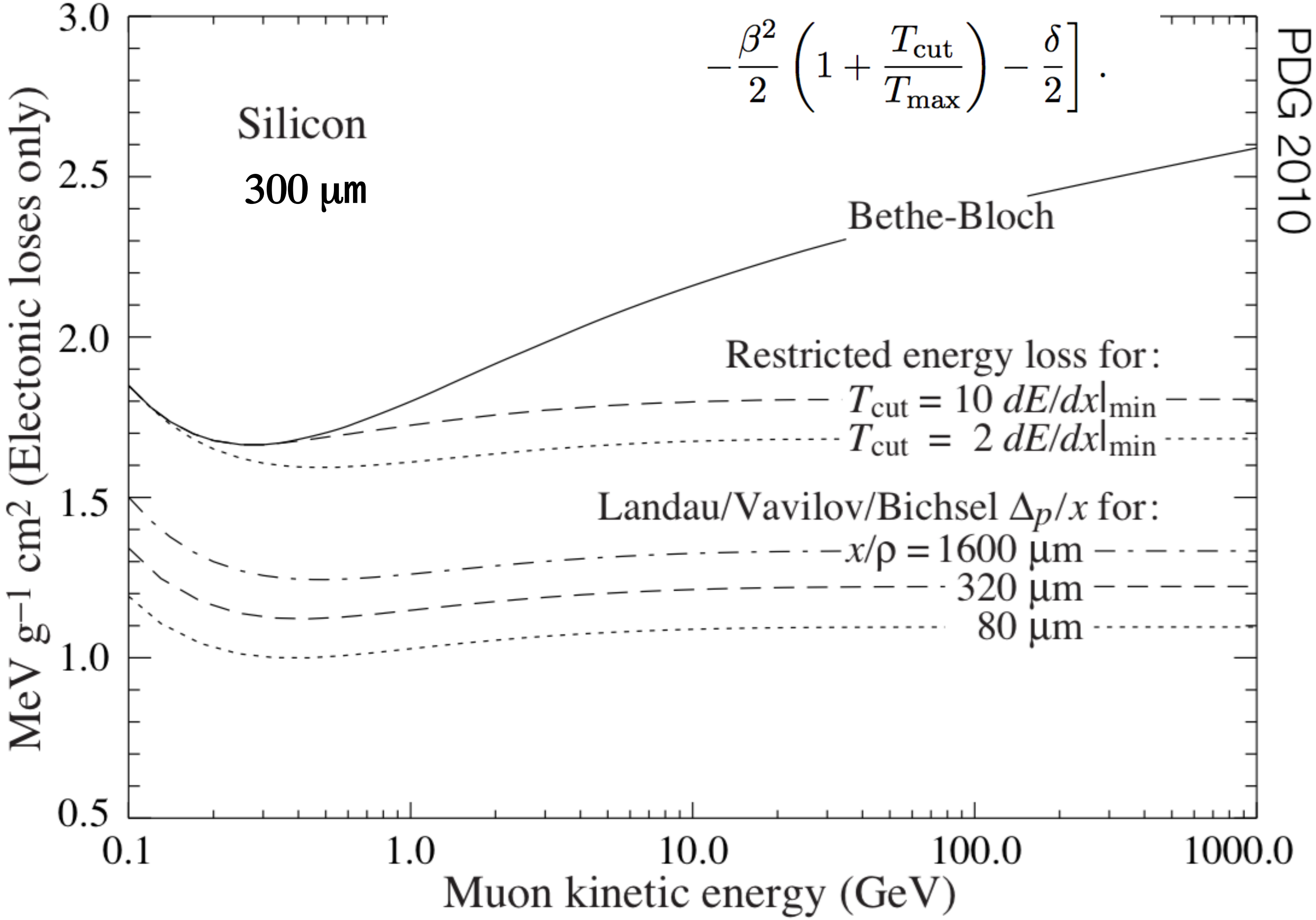
$$\int_{T_{cut}}^{T_{\max}} T\,\frac{dn}{dTdx}dT$$

Where dn/dTdX is energy distribution of δ-rays



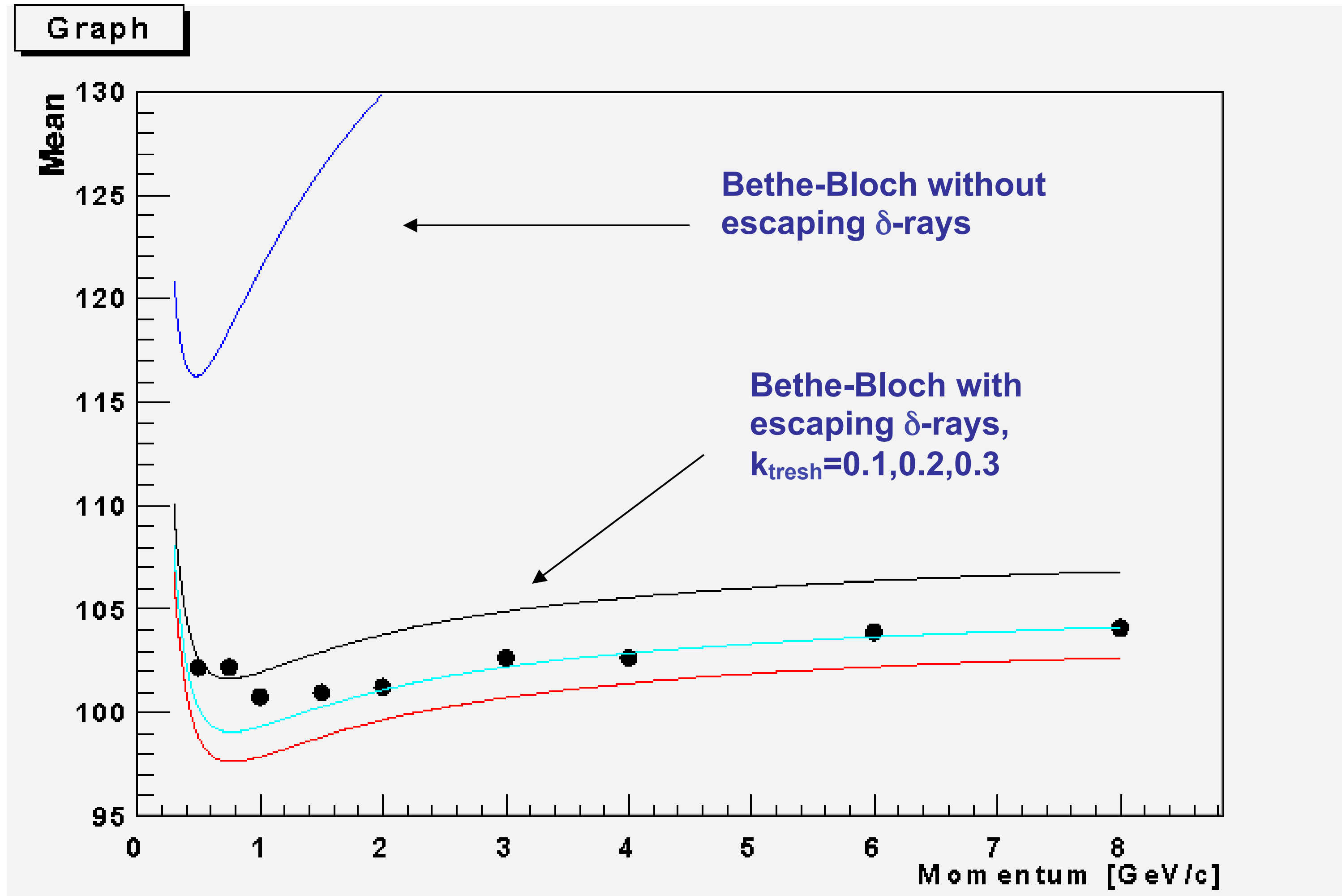


$$-\frac{dE}{dx}\bigg|_{T<T_{\rm cut}} = K z^2 \frac{Z}{A}\frac{1}{\beta^2}\left[\frac{1}{2}\ln\frac{2m_e c^2\beta^2\gamma^2 T_{\rm cut}}{I^2} - \frac{\beta^2}{2}\left(1+\frac{T_{\rm cut}}{T_{\rm max}}\right) - \frac{\delta}{2}\right].$$



Bethe-Bloch dE/dx, two examples of restricted energy loss, and the Landau most probable energy per unit thickness in silicon. The change of Δ_p/x with thickness x illustrates its a ln x + b dependence. Minimum ionization ($dE/dx|_{min}$) is 1.664 MeV g^{-1} cm^2. Radiative losses are excluded. The incident particles are muons.

Data + Bethe-Bloch - Silicon



Continuous curves are MonteCarlo simulations with different max δ-rays energy cuts

Assorbitori sottili infinitesimi

La probabilità che una particella incidente di energia E perda energia compresa fra W e W+dW attraversando un dx infinitesimo è:

$$\Phi(W)dWdx = n_a \frac{d\sigma(W)}{dW} dWdx$$

Dove $n_a = N_A \rho / A$= numero di atomi per unità di volume, $d\sigma/dW$= sezione d'urto differenziale per la particella incidente di perdere energia W in una <u>singola</u> collisione con un atomo.

La probabilità totale di una collisione di perdere qualunque W nell'infinitesimo dx sarà:

$$q dx = \left(n_a \int_{W\min}^{W\max} \frac{d\sigma}{dW} dW \right) dx$$

q si chiama rate di <u>ionizzazione primaria.</u>

Assorbitori sottili

Semplice se dx è infinitesimo, ma complicato per x finito.

Consideriamo un fascio di N particelle di energia E. Sia $\chi(W,x)dW$ la probabilità che una particella perda un'energia fra W e W+dW dopo avere attraversato uno spessore x.

La forma di χ può essere determinata considerando come varia il # di particelle che hanno perso W a x, quando le particelle attraversano un altro spessore dx:

variazione del # di part che subiscono perdite tra W, W+dW a x, x+dx=

part "in" (W, W+dW) a (x, x+dx) – part "out" da (W, W+dW) a (x, x+dx)

- Il numero di particelle con perdita di energia fra W e W+dW cresce perché qualcuna che ad x aveva perso meno energia di W colliderà e perderà un'energia fra W e W+dW in dx = N × [prob di perdere W - ε fino a x] × [Prob di perdere ε in dx], sommate su tutti i possibili valori di ε.
- Il numero di particelle con perdita fra W e W+dW diminuisce perché alcune particelle che avevano già perso l'energia giusta prima del tratto dx ne perderanno ancora e quindi ne perdono di più di W+dW = N × [prob di perdere W fino a x] × [Prob di perdere un'energia qualunque in dx]

Assorbitori sottili

Se assumiamo che le collisioni che avvengono successivamente siano statisticamente indipendenti, che il mezzo assorbitore sia omogeneo e che la perdita totale di energia sia piccola rispetto all'energia della particella incidente:

$$N\chi(W,x+dx)dW - N\chi(W,x)dW =$$

$$= N\int_{W\min}^{W\max}\chi(W-\varepsilon,x)\Phi(\varepsilon)\,d\varepsilon dW dx - N\chi(W,x)dWqdx$$

Cioè:

$$\frac{\partial\chi(W,x)}{\partial x} = \int_{W\min}^{W\max}\Phi(\varepsilon)\chi(W-\varepsilon,x)\,d\varepsilon - q\chi(W,x)$$

Equazione integro-differenziale molto difficile da risolvere. Le differenze nelle soluzioni derivano essenzialmente dalle assunzioni fatte sulla probabilità $\Phi(W)$ cioè dal trasferimento di energia per collisione singola.

Assorbitori sottili

- The description of ionisation fluctuations is characterised by the significance parameter k, which is proportional to the ratio of mean energy loss to the maximum allowed energy transfer in a single collision with an atomic electron, **$k = \xi/E_{max}$**
- ξ represents the energy above which at least a δ-ray will be emitted in the thickness X

$$\xi = \frac{2\pi z^2 e^4 N_{Av} Z\rho\delta x}{m_e \beta^2 c^2 A} = 153.4 \frac{z^2}{\beta^2}\frac{Z}{A}\rho\delta x \quad \text{keV},$$

NB: emission prob of δ with energy between E,E+dE is P(E)dE = (ξ/E^2)dE

κ measures the contribution of the collisions with energy transfer close to E_{max}. For a given absorber, κ tends towards large values if δx is large and/or if β is small. Likewise, κ tends towards zero if δx is small and/or if β approaches 1. The value of κ distinguishes two regimes which occur in the description of ionisation fluctuations :

Assorbitori sottili: teoria di Landau (cenni)

1. A large number of collisions involving the loss of all or most of the incident particle energy during the traversal of an absorber.

   As the total energy transfer is composed of a multitude of small energy losses, we can apply the central limit theorem and describe the fluctuations by a Gaussian distribution. This case is applicable to non-relativistic particles and is described by the inequality $\kappa > 10$ (i.e. when the mean energy loss in the absorber is greater than the maximum energy transfer in a single collision). **Already seen**

2. Particles traversing thin counters and incident electrons under any conditions.

   The relevant inequalities and distributions are $0.01 < \kappa < 10$, Vavilov distribution, and $\kappa < 0.01$, Landau distribution.

An additional regime is defined by the contribution of the collisions with low energy transfer which can be estimated with the relation ξ / I_0, where I_0 is the mean ionisation potential of the atom. Landau theory assumes that the number of these collisions is high, and consequently, it has a restriction $\xi / I_0 \gg 1$.

Assorbitori sottili: teoria di Landau (cenni)

Valida per **k=** $\xi/E_{max}<0.01$ $\xi=\rho k z^2 (Z/A)(1/\beta^2)x$

Assunzioni:

- Perdita di energia piccola rispetto al massimo possibile in una singola collisione ($\xi/E_{max} << 1$), in pratica si assume Emax $\rightarrow\infty$
- Perdita di energia grande se paragonata all'energia di legame degli elettroni (elettrone libero). Si trascurano quindi le piccole perdite di energia dovute alle collisioni lontane ($\xi/E_{bind} >> 1$).
- E sottinteso $\xi/E_{inc} << 1$, cosi' che $E_{inc} \approx$ cost.
- For gaseous detectors, typical energy losses are a few keV which is comparable to the binding energies of the inner electrons. In such cases a more sophisticated approach which accounts for atomic energy levels is necessary to accurately simulate data distributions.

Assorbitori sottili: teoria di Landau (cenni)

Con queste assunzioni χ può essere fattorizzata come segue:

$$\chi(W,x) = \frac{1}{\xi} f_L(\lambda)$$

$$\text{con} \quad \lambda = \frac{1}{\xi}\left[W - \bar{W} - \xi \ln\frac{\xi}{\varepsilon'} + 1 - c_E\right];$$

$$\ln\varepsilon' = \ln\frac{\left(1-\beta^2\right)I^2}{2mv^2} + \beta^2;$$

$$c_E = 0.577 \quad \text{(costante di Eulero)}$$

ε' è il taglio sulla minima energia trasferibile nel singolo urto, in accordo all'assunzione ξ/E_{bind} >> 1 (e in modo tale che la perdita media sia quella della bethe-bloch.

Assorbitori sottili: teoria di Landau (cenni)

La funzione universale $f_L(\lambda)$ può essere espressa come segue:

$$f_L(\lambda) = \frac{1}{\pi}\int_0^{\infty} e^{-u(\ln u+\lambda)} \sin(\pi u)\,du$$

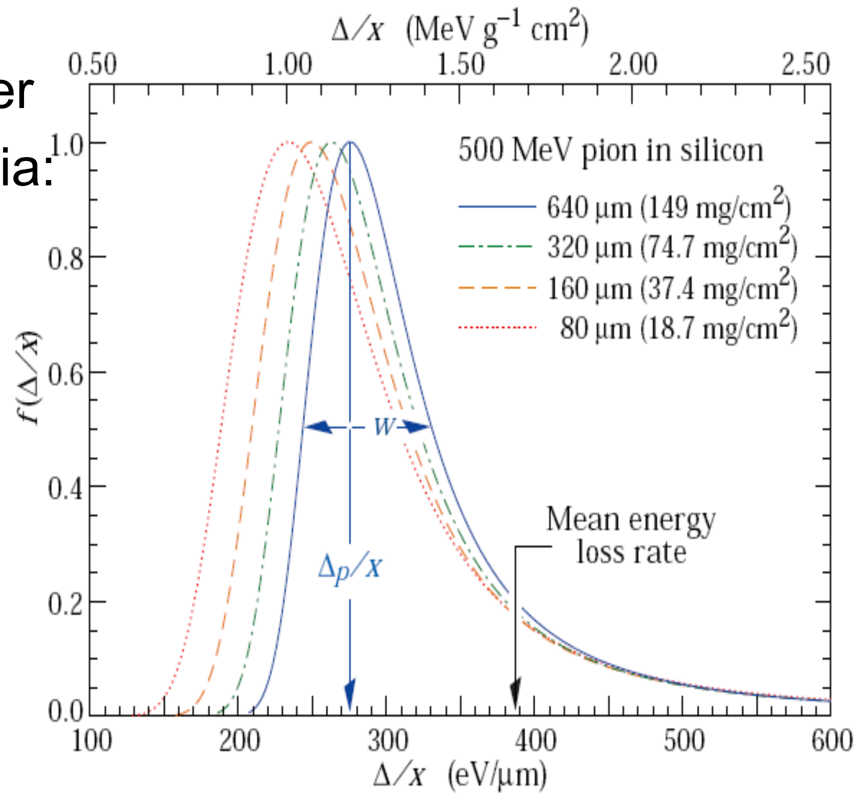
Valutando numericamente $f_L(\lambda)$ si ottiene per il valore più probabile per la perdita di energia:

$$W_{mp} = \xi\left[\ln \xi/_{\varepsilon'} + 0.198 - \delta\right]$$

δ= correzione per effetto densità

e FWHM=4.02ξ

$$\xi = \frac{2\pi z^2 e^4 N_{Av} Z \rho \delta x}{m_e \beta^2 c^2 A} = 153.4 \frac{z^2}{\beta^2}\frac{Z}{A}\rho\delta x \quad \text{keV},$$

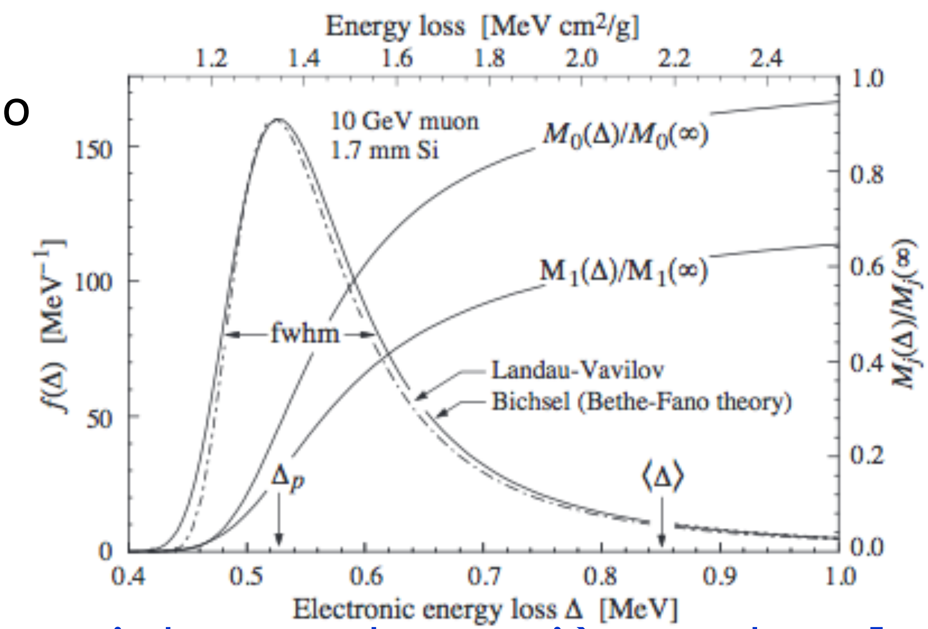


Assorbitori sottili: teoria di Vavilov (cenni)

Il modello di Landau non funziona bene in gas (ex. TPC) e silicio (tracciatori) dato che in questi materiali le perdite tipiche di energia sono paragonabili alle energia di legame, per cui l'assunzioni che $\xi/E_{bind} >> 1$ non e' piu' valida

Valida per k>0.01, tiene conto che Emax e' finito

Caratterizzata da code un po' meno asimmetriche.



Osserviamo:

Anche se il limite gaussiano si ha per k≥10 già per k≥1 la distribuzione assomiglia ad una gaussiana.

Vavilov → landau per k → 0 ed ad una gaussiana per k → ∞.

Assorbitori sottili: teoria di Vavilov (cenni)

Vavilov[5] derived a more accurate straggling distribution by introducing the kinematic limit on the maximum transferable energy in a single collision, rather than using $E_{\max} = \infty$. Now we can write[2]:

$$f(\epsilon, \delta s) = \frac{1}{\xi}\phi_v\left(\lambda_v, \kappa, \beta^2\right)$$

where

$$\phi_v\left(\lambda_v, \kappa, \beta^2\right) = \frac{1}{2\pi i}\int_{c+i\infty}^{c-i\infty}\phi(s)\, e^{\lambda s} ds \qquad c \geq 0$$

$$\phi(s) = \exp\left[\kappa(1+\beta^2\gamma)\right]\ \exp\left[\psi(s)\right],$$

$$\psi(s) = s\ln\kappa + (s+\beta^2\kappa)\left[\ln(s/\kappa) + E_1(s/\kappa)\right] - \kappa e^{-s/\kappa},$$

and

$$E_1(z) = \int_{\infty}^{z} t^{-1}e^{-t}dt \qquad \text{(the exponential integral)}$$

Assorbitori sottili: teoria di Vavilov (cenni)

$$\lambda_v = \kappa \left[\frac{\epsilon - \bar{\epsilon}}{\xi} - \gamma' - \beta^2 \right]$$

The Vavilov parameters are simply related to the Landau parameter by $\lambda_L = \lambda_v/\kappa - \ln\kappa$. It can be shown that as $\kappa \to 0$, the distribution of the variable λ_L approaches that of Landau. For $\kappa \leq 0.01$ the two distributions are already practically identical. Contrary to what many textbooks report, the Vavilov distribution *does not* approximate the Landau distribution for small κ, but rather the distribution of λ_L defined above tends to the distribution of the true λ from the Landau density function. Thus the routine `GVAVIV`

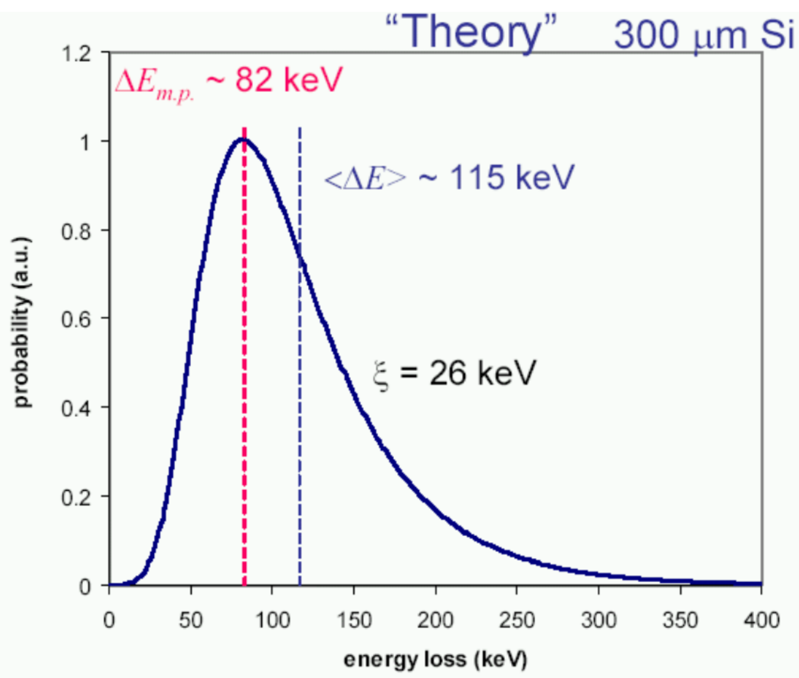
The Landau/Vavilov distribution depends on two parameters:

- the most probable energy loss W_{mp} and
- the FWHM of the distribution $\sim\xi$

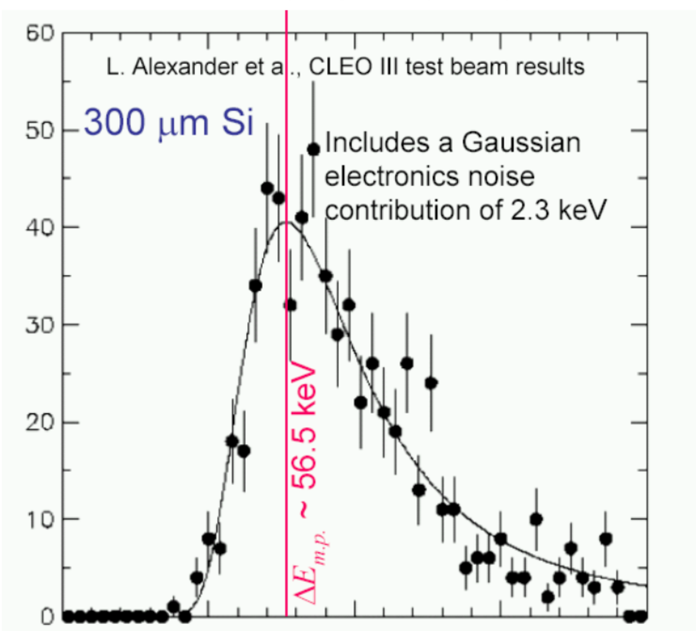
Typical application of the Landau distribution is the CHARGE Identification of the incident particle, since $W_{mp} = W_{mp}(Z_{inc},\beta)$ and $\langle W \rangle = W(Z_{inc},\beta)$

The particle crosses N active layers in which it deposits dEi, i=1,…,N according to the Landau distribution.

Then by a numerical fit, one finds the best parameters W_{mp},FWHM that describes the measured set of dEi. Knowing Wmp, and the width it is possible to get Z_{inc}



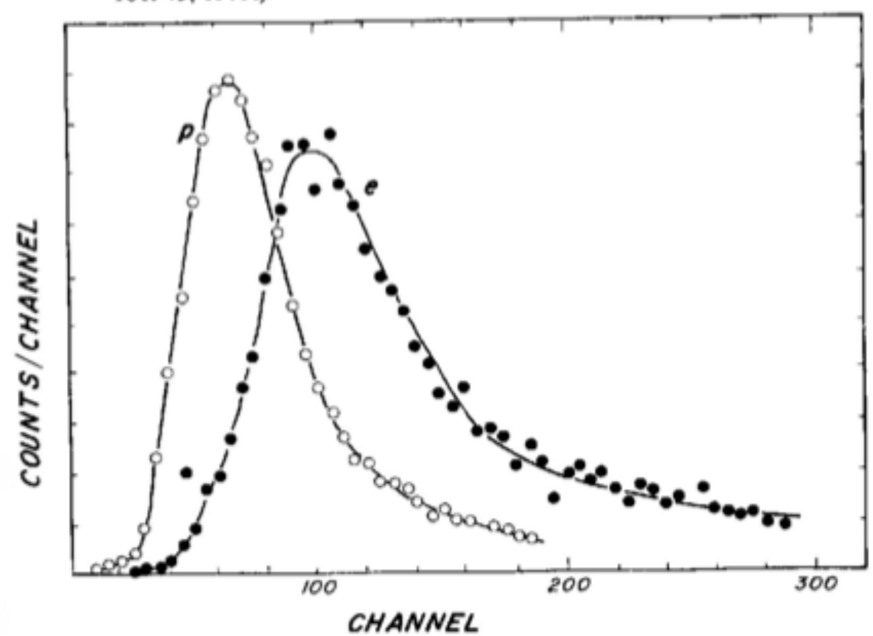
Measurement



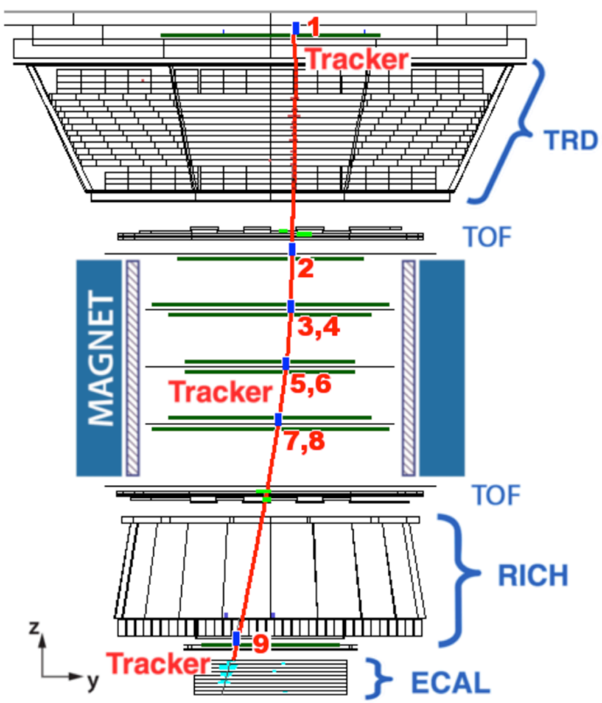
Charge ID

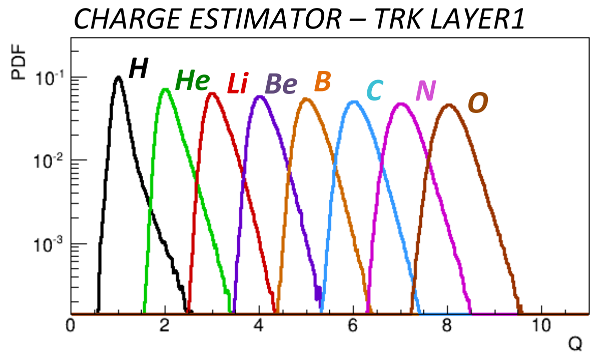
- Conseguenze sperimentali:
  - ID di carica fatta tramite la misura del deposito di energia (noto β e/o p): necessita' di avere piu' misure di dE/dX per campionare la distribuzione di Landau/Vavilov da cui ricavare Wmp e quindi il valore assoluto della carica dato che $W_{mp} \propto Z_{inc}^2$

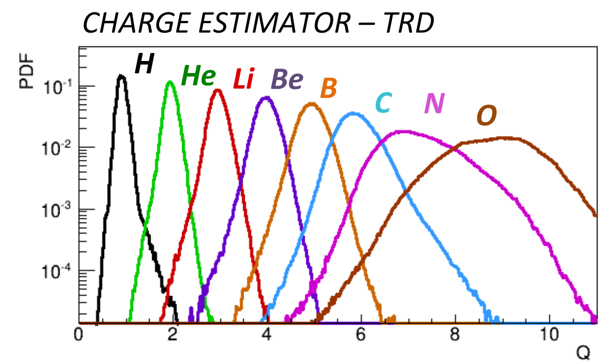
Figure 2.7 Measured pulse height distributions for 3-GeV/*c* protons and 2-GeV/*c* electrons in a 90% Ar + 10% CH_4 gas mixture. (After A. Walenta, J. Fischer, H. Okuno, and C. Wang, Nuc. Instr. Meth. 161: 45, 1979.)

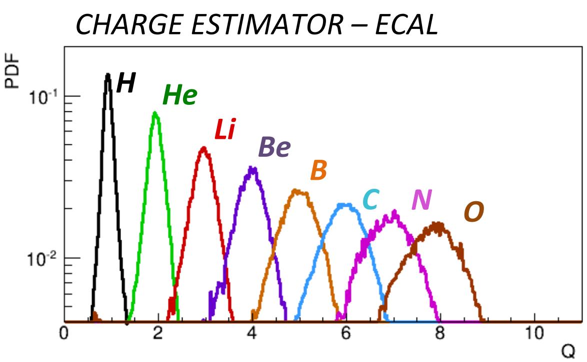


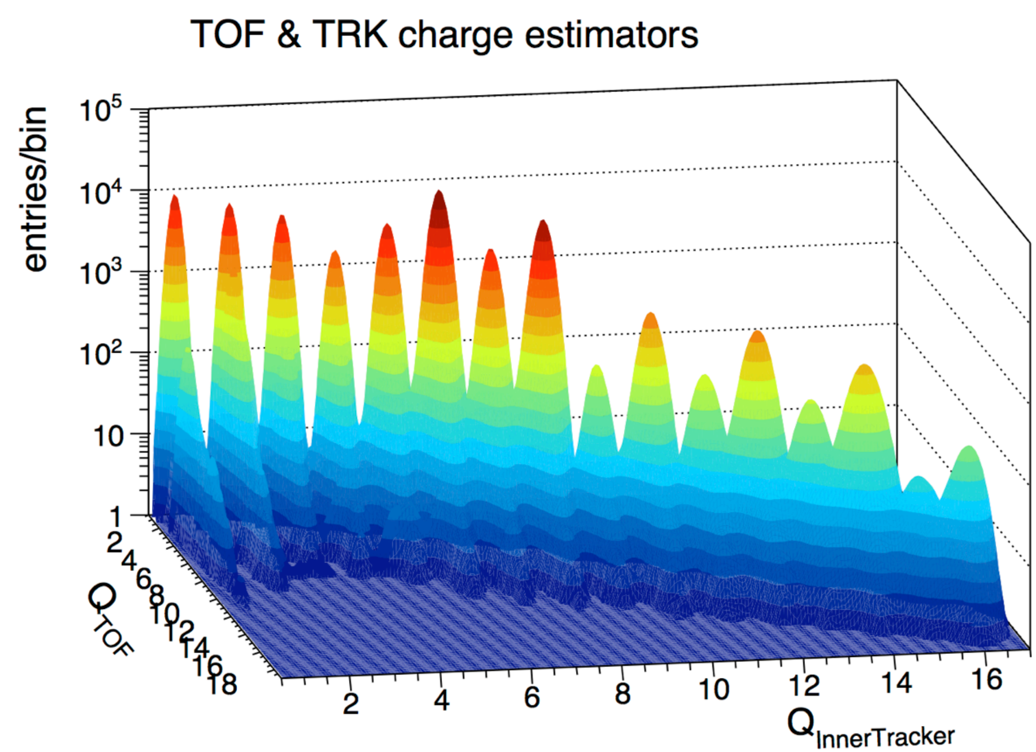
Charge ID: Multiple measurements of charge



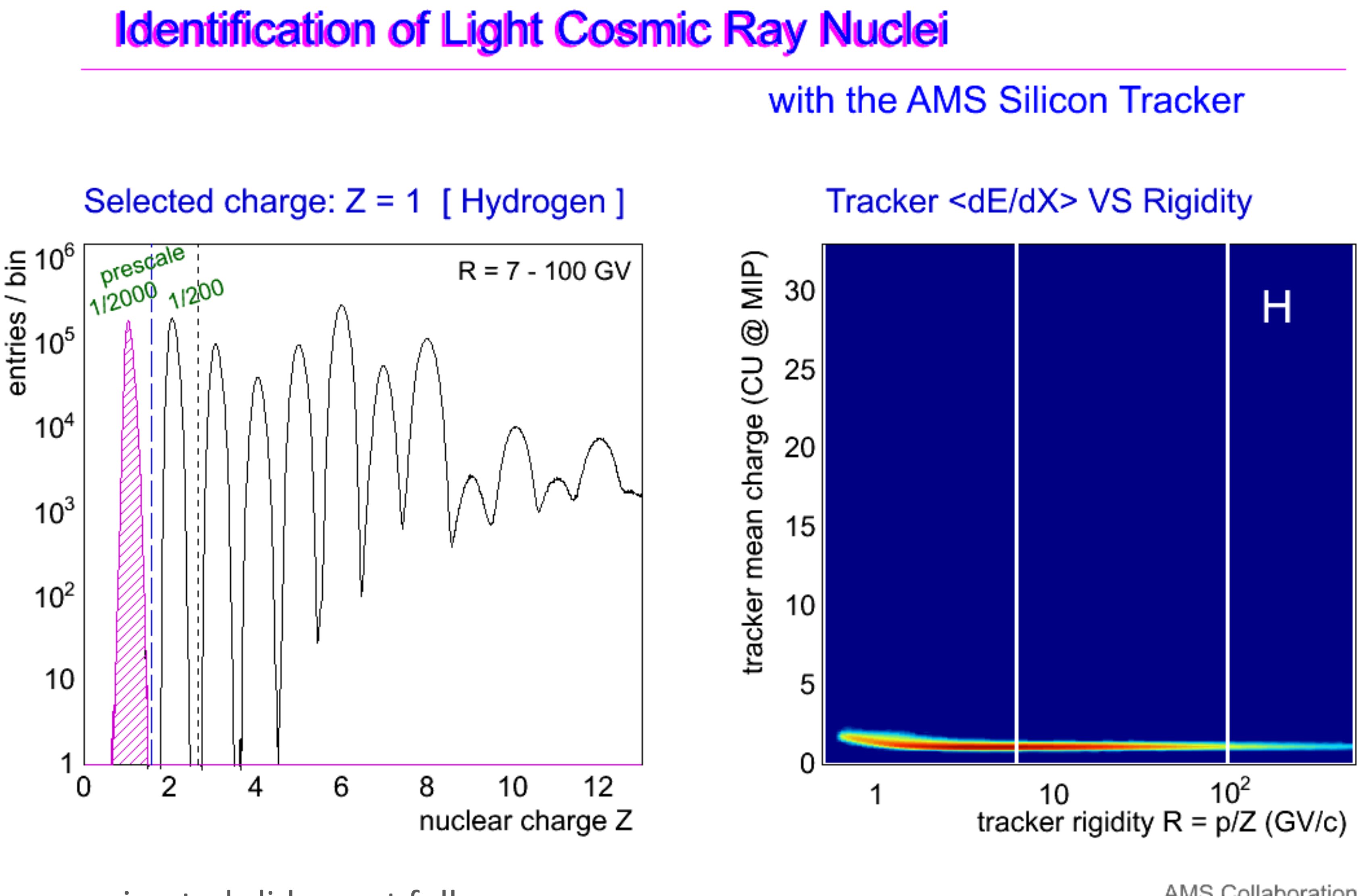






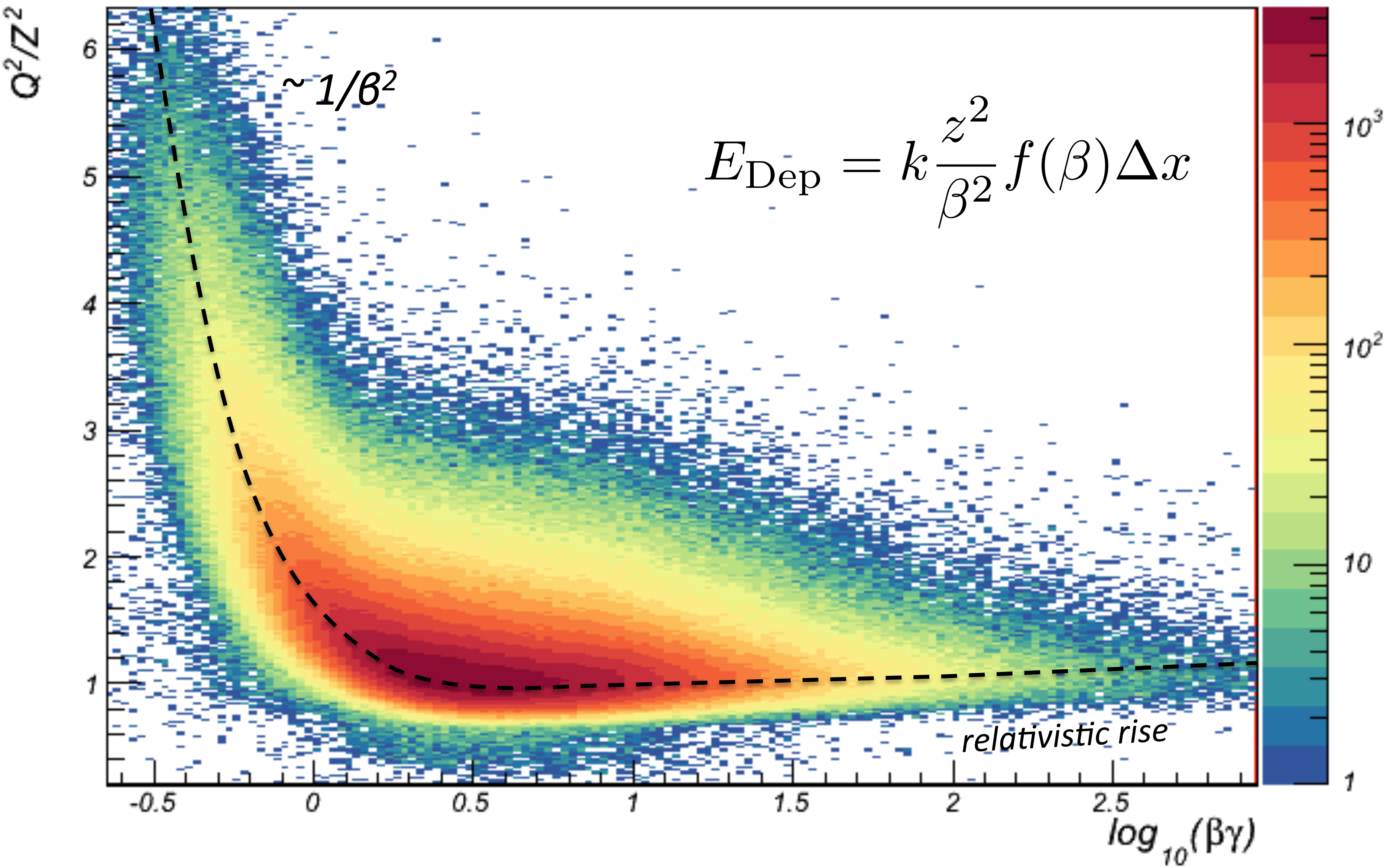


Charge measurement using Tracker



animated slide – set full screen

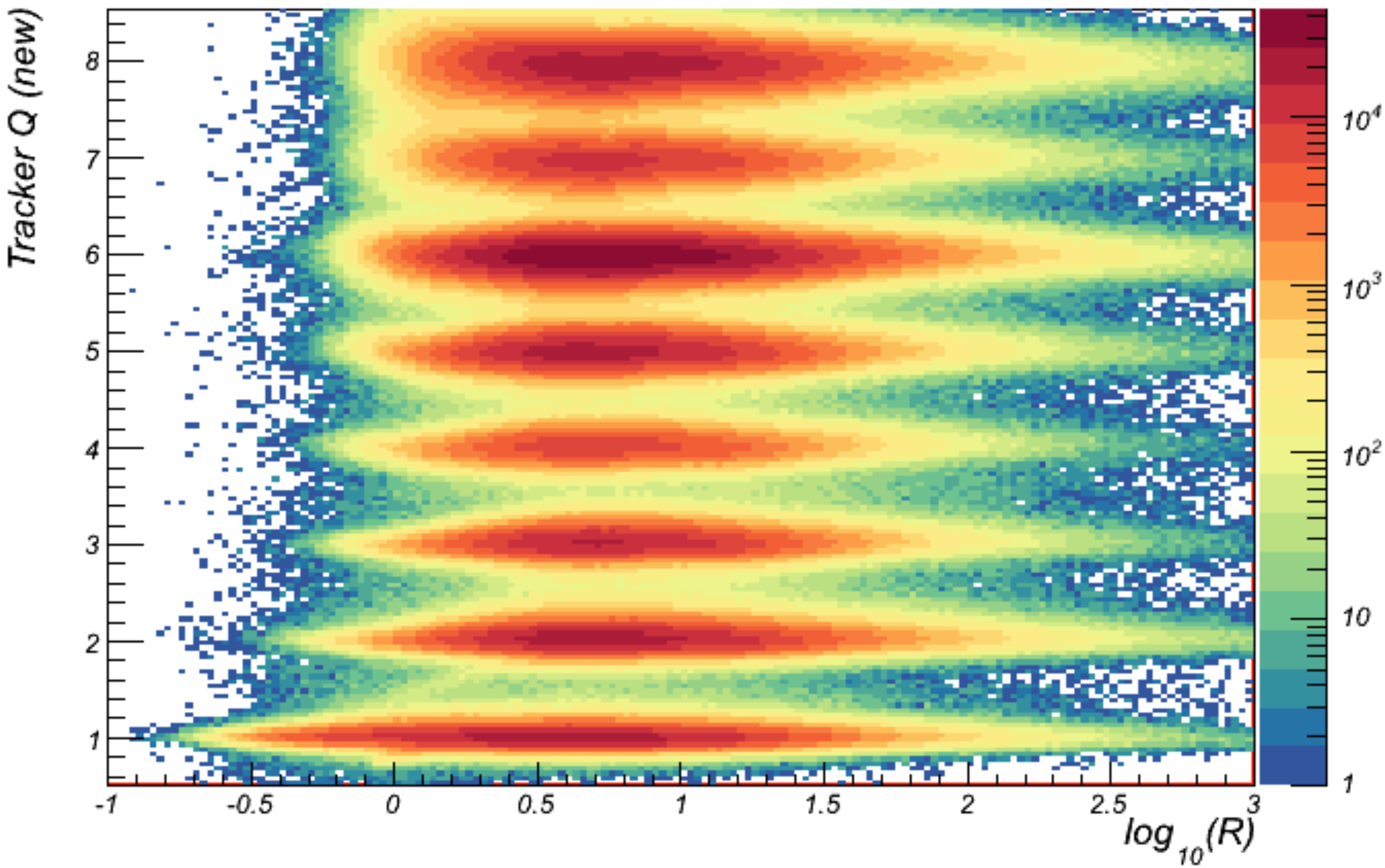
Bethe-Bloch for Protons



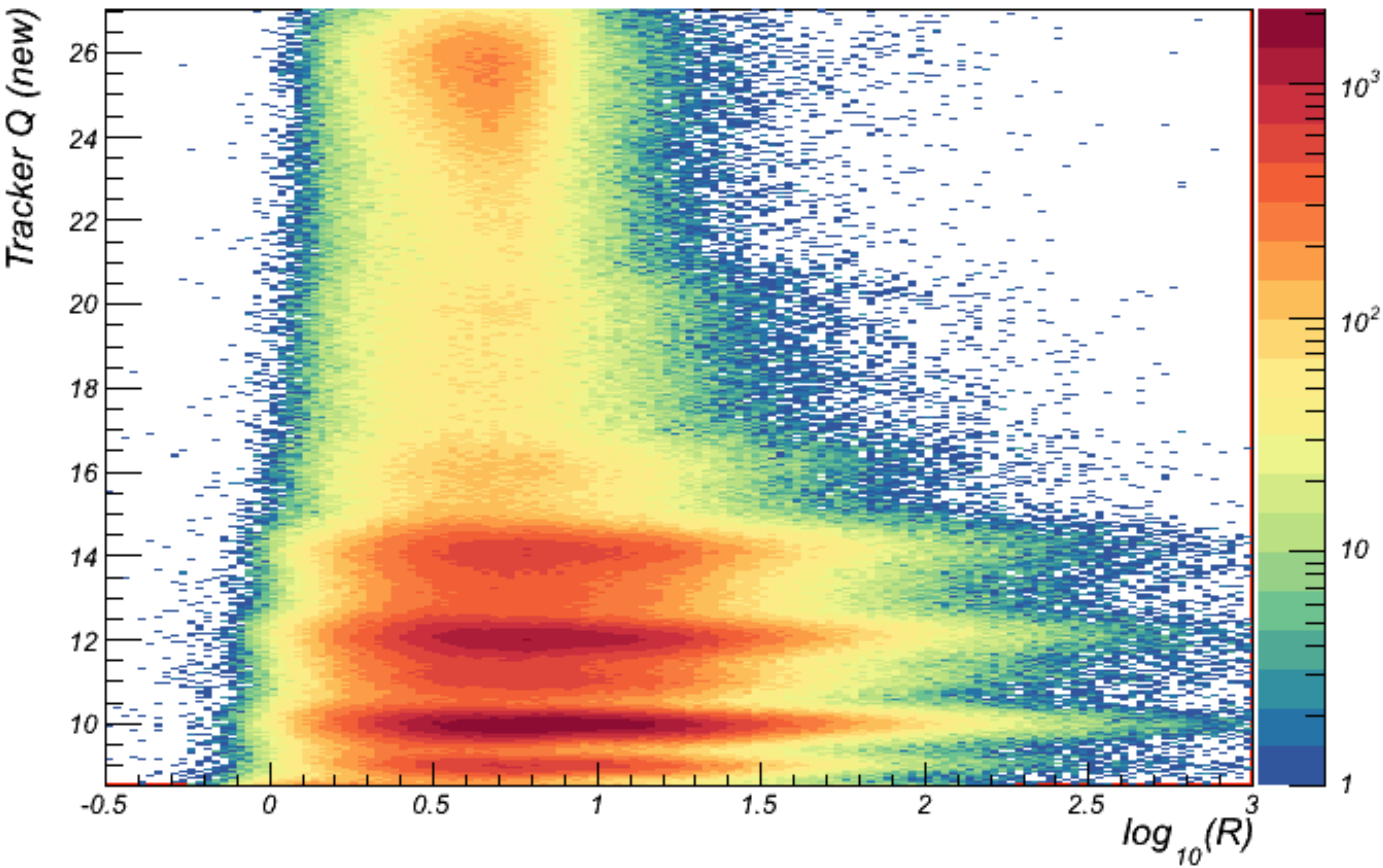
Velocity corrections to the measured dE/dX

To eliminate the velocity dependence of the charge, usually a velocity correction is applied to the measured energy depositions

$$\Delta E_{corr} = \Delta E_{meas} * \beta/F(\beta)$$



After Rigidity Correction (high-Z)

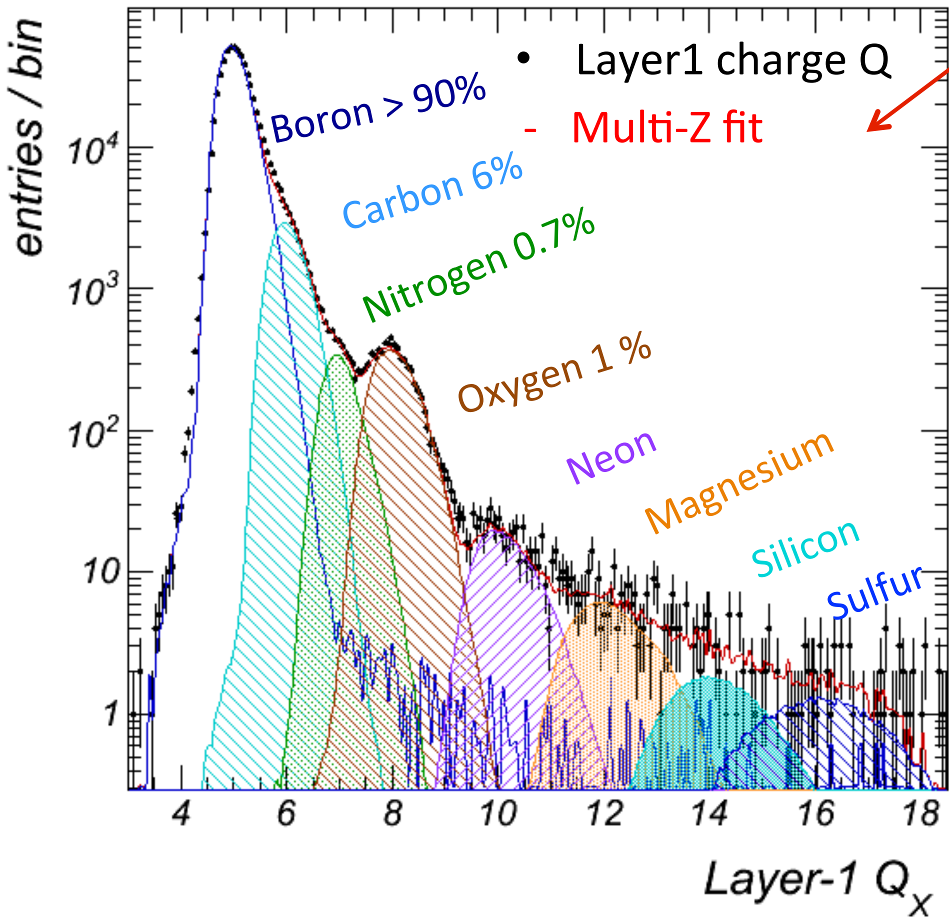


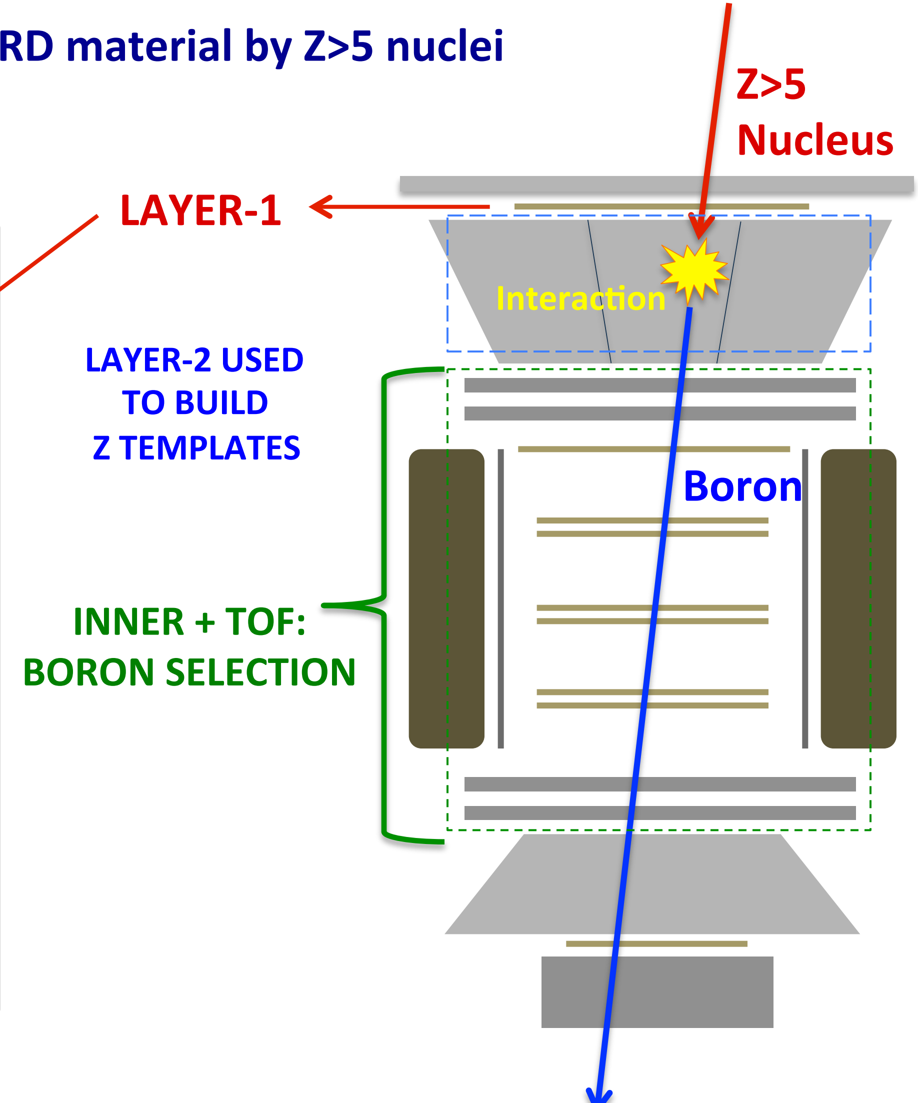
Fragmentation studies in the detector

Redundancy in Z-measurements allows us to study different fragmentation processes appearing at different levels in the detector.

Example: secondary production of Boron in TRD material by Z>5 nuclei

Layer-1 Q distribution

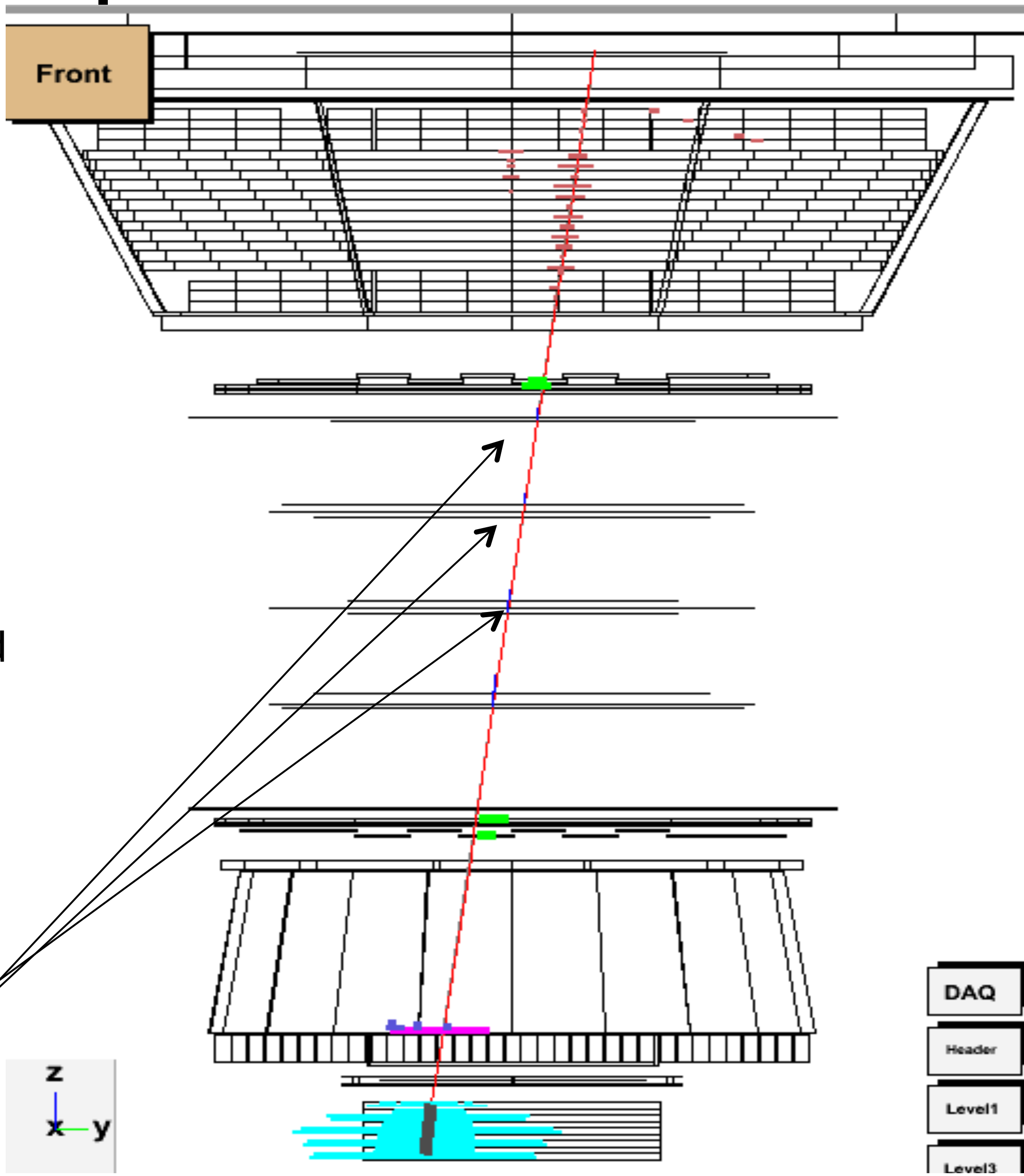




dE/dX and particle ID

dE/dX can (and it IS used) to signal the passage of a charged particle through an active medium, as the active planes (where the energy is deposited and converted in an electrical signal) of a tracking device (eg immersed in a magnetic field)

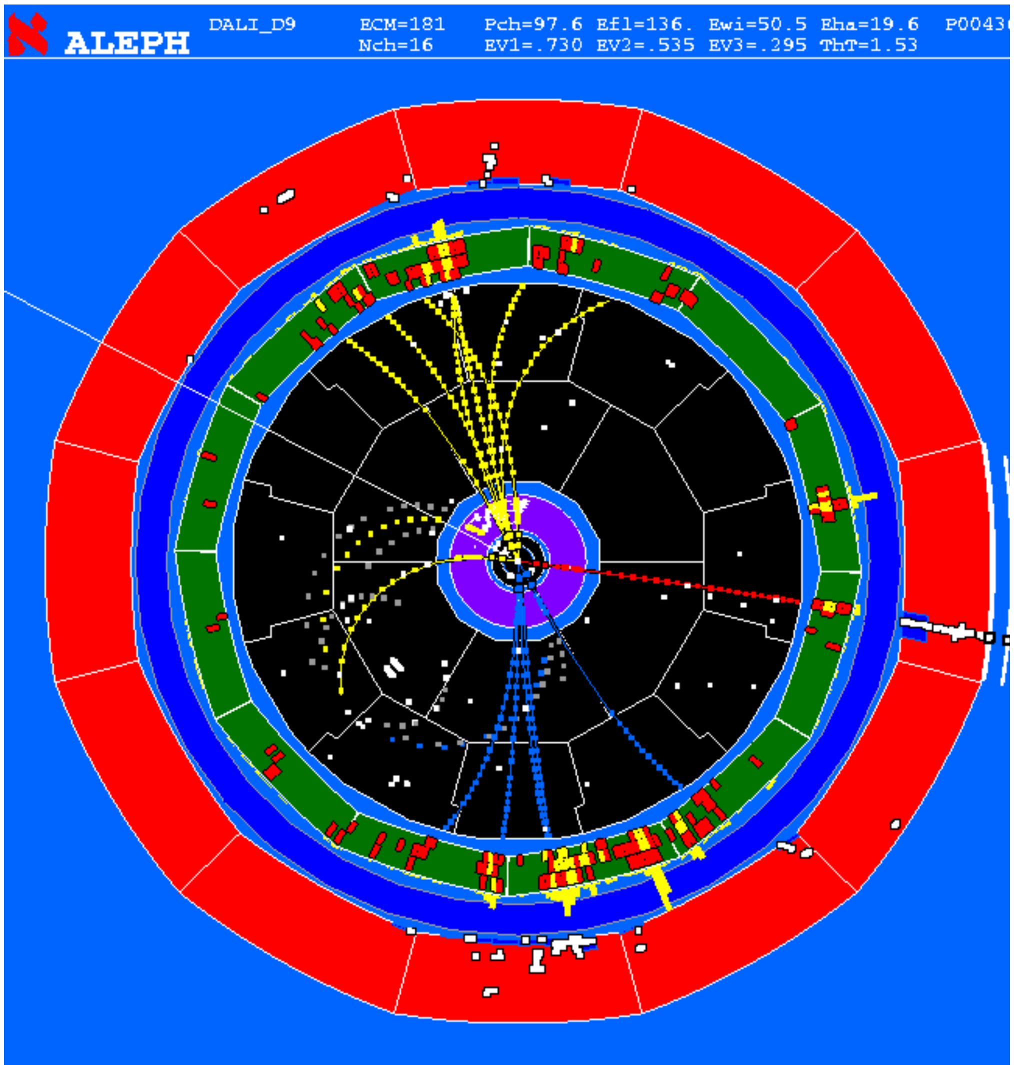
Each energy deposit is $\propto z^2$ and therefore they can be used to measure the particle $|z|$, provided that an indipendent measurement of β is available (since energy deposit $\propto z^2/\beta^2$)

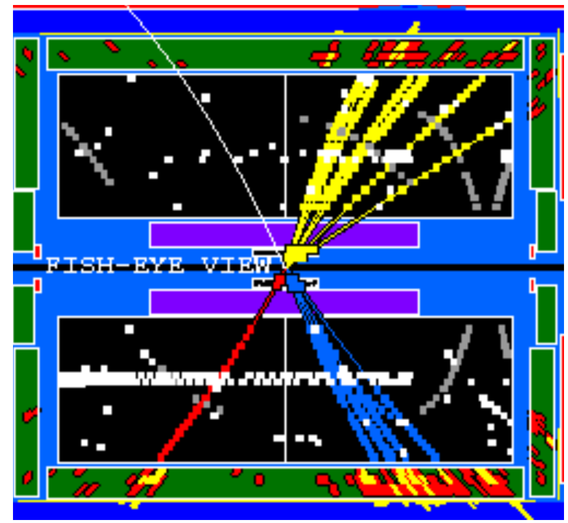


AMS Event display for a high energy electron: the passage at a given point is signalled by the energy deposits in the tracking planes

A W^+W^- decay in ALEPH (a LEP experiment)

e^+e^- ($\sqrt{s}$=181 GeV)
→ W^+W^- → $qq\mu\nu_\mu$
→ 2 hadronic jets + μ + missing momentum

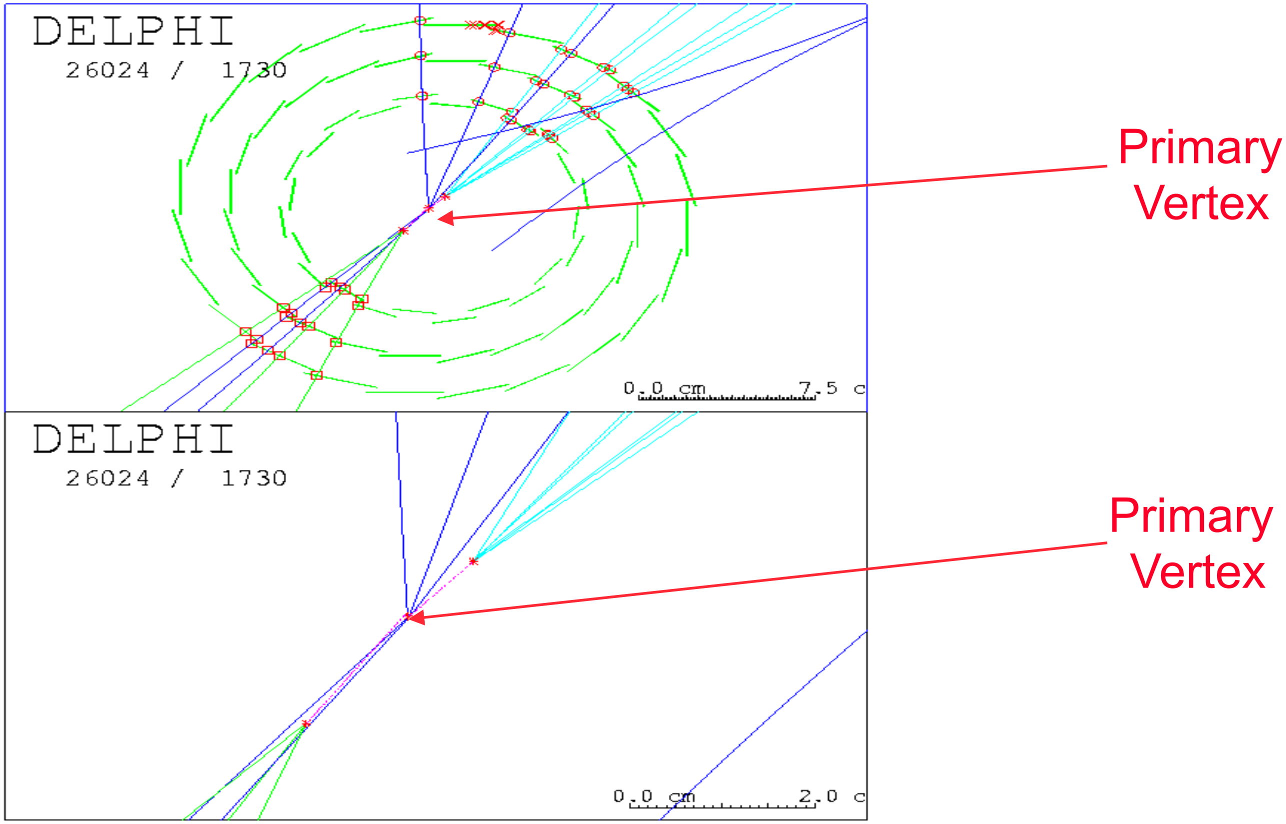


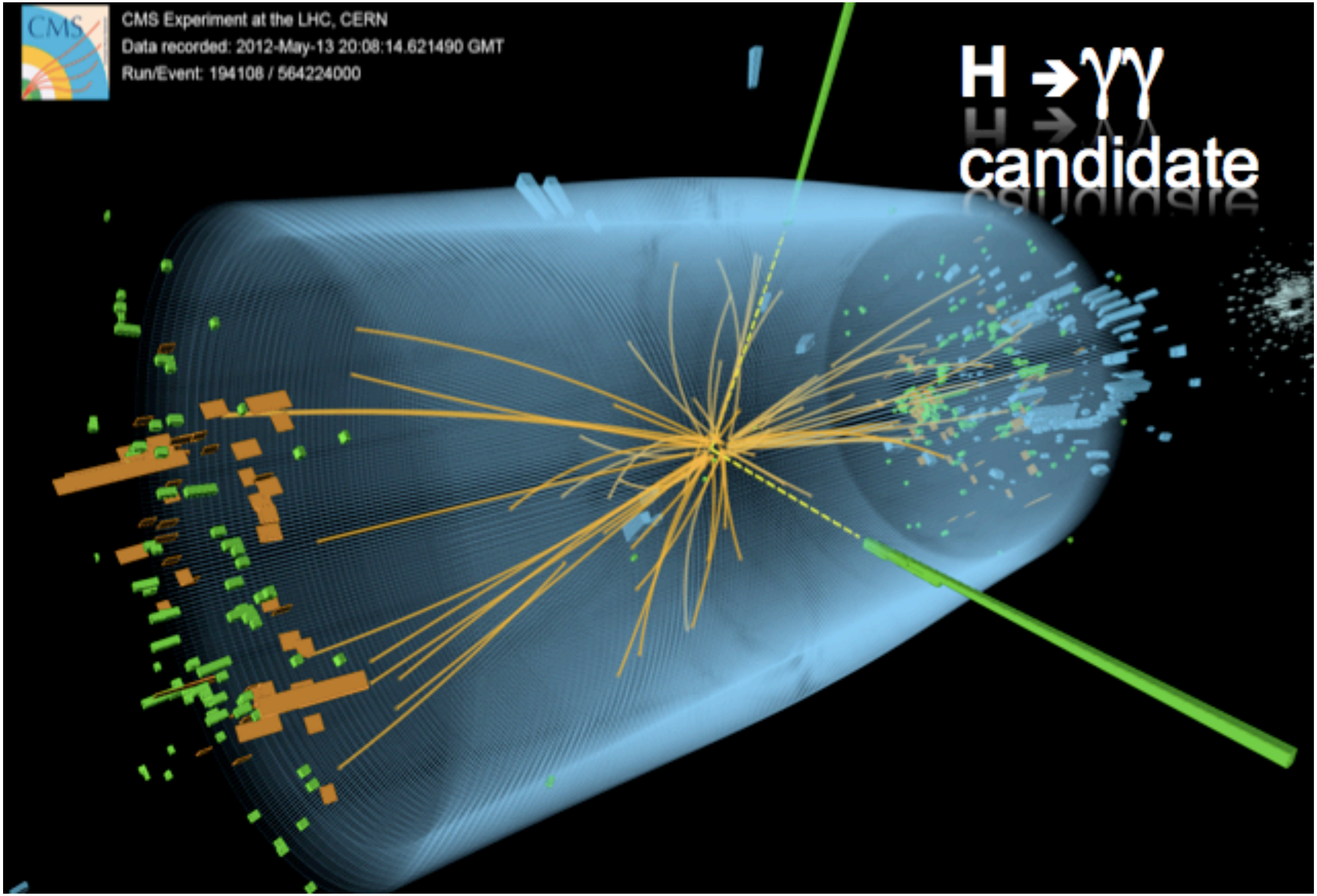


Most of position sensitive detectors are based on the energy deposit in the active medium by ionization, as tracking detectors used to reconstruct trajectories the particles

Reconstructed B-mesons in the DELPHI micro vertex detector

$\tau_B \approx 1.6$ ps $l = c\tau\gamma \approx 500$ μm·γ

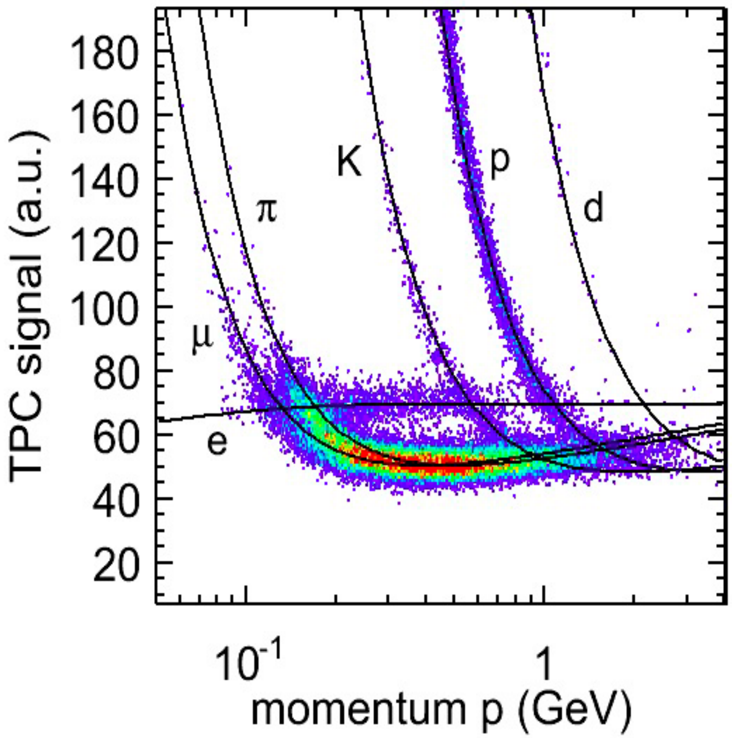


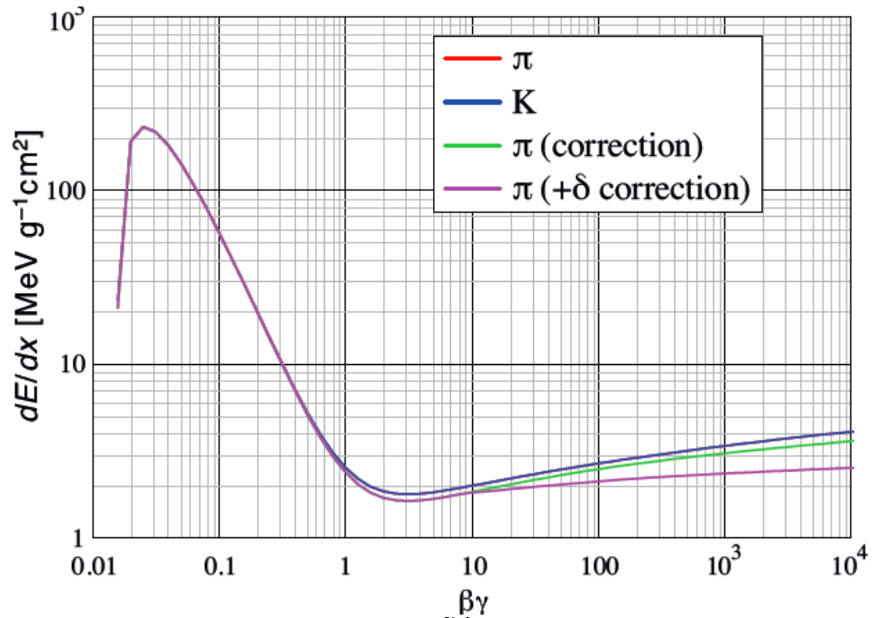

CMS Experiment at the LHC, CERN
Data recorded: 2012-May-13 20:08:14.621490 GMT
Run/Event: 194108 / 564224000
H → γγ
candidate

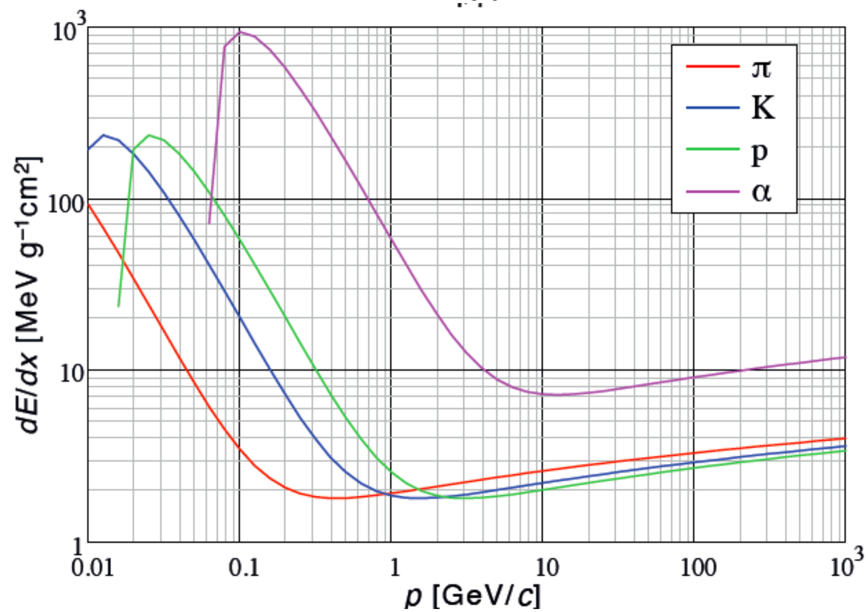
dE/dx and particle ID

For a given medium, dE/dX depends only on β

But for a given momentum, $\beta\gamma$ and hence dE/dX are different for particles with different masses:





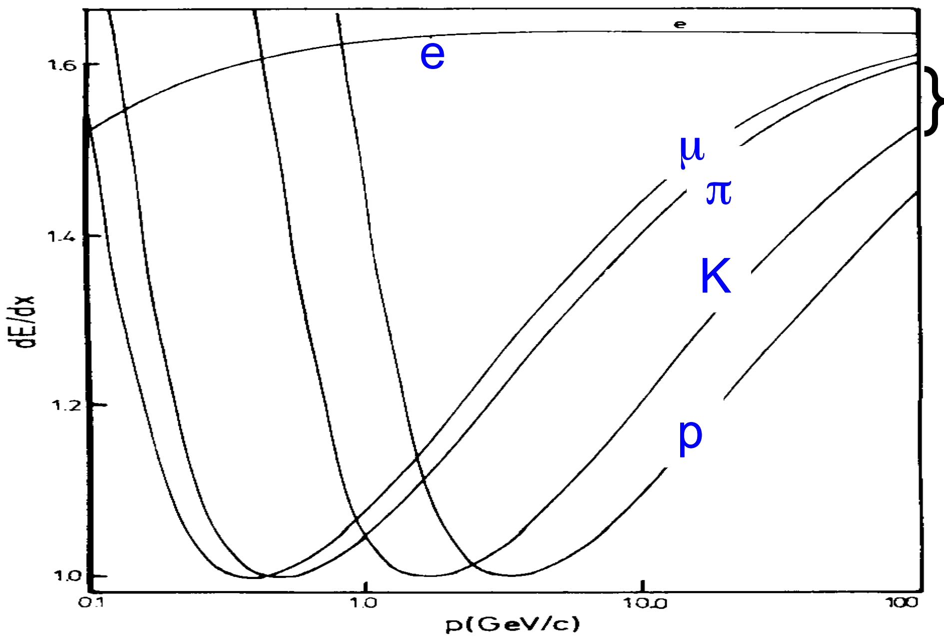


Particle ID using the specific energy loss dE/dx

$$p = m_0 \beta\gamma c$$

$$\frac{dE}{dx} \propto \frac{1}{\beta^2} \ln\left(\beta^2\gamma^2\right)$$

Simultaneous measurement of p and dE/dx defines mass m, hence the particle identity



π/K separation (2σ) requires a dE/dx resolution of < 5%

Average energy loss for e,μ,π,K,p in 80/20 Ar/CH_4 (NTP)

(J.N. Marx, Physics today, Oct.78)

But: Large fluctuations + Landau tails !

When $\beta \rightarrow 1$, there only a ln($\beta\gamma$) dependence, all the particles tend to have similar loss: difficult to separate particles from single dE/dX

→ very good resolution both in p and dE/dX measurements needed

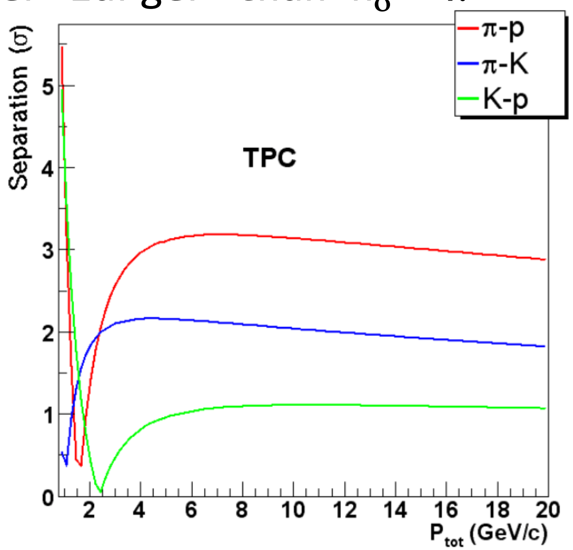
dE/dx and particle ID

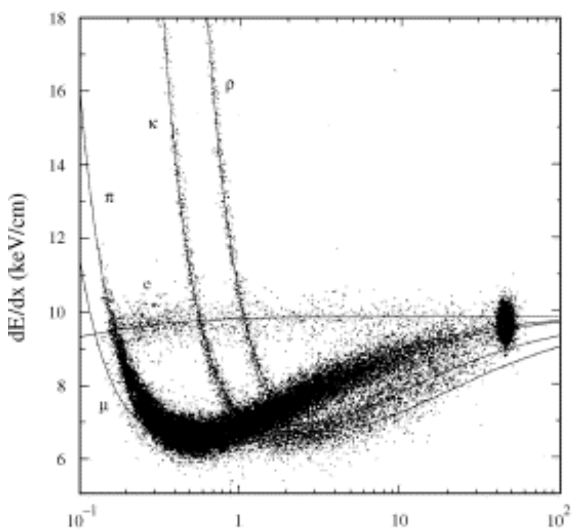
Any PID technique is based on the fact that the detector response is different “enough” to different types of particles. Therefore it is worthwhile to introduce the concept of “separation power” of a PID system, defined by the significance of the measurement of the detector response, herein generically indicated with S. The detector response could actually be, according to the method employed, either time (in the case of TOF detectors), the ionization energy loss or the Cherenkov angle.

If S_A and S_B are the mean values of such a quantity measured for particles of type A and B, respectively, and σ_{AB} is the average value between the standard deviations of the measured distributions, the separation power $\mathbf{n_\sigma}$ is given by:

$$\mathbf{n_\sigma = (S_A - S_B)/\sigma_{AB}}$$

By assuming that the measured quantities S_A and S_B are distributed according to a Gaussian, a particle contamination smaller than 1% requires a separation power larger than $n_\sigma = 4$.

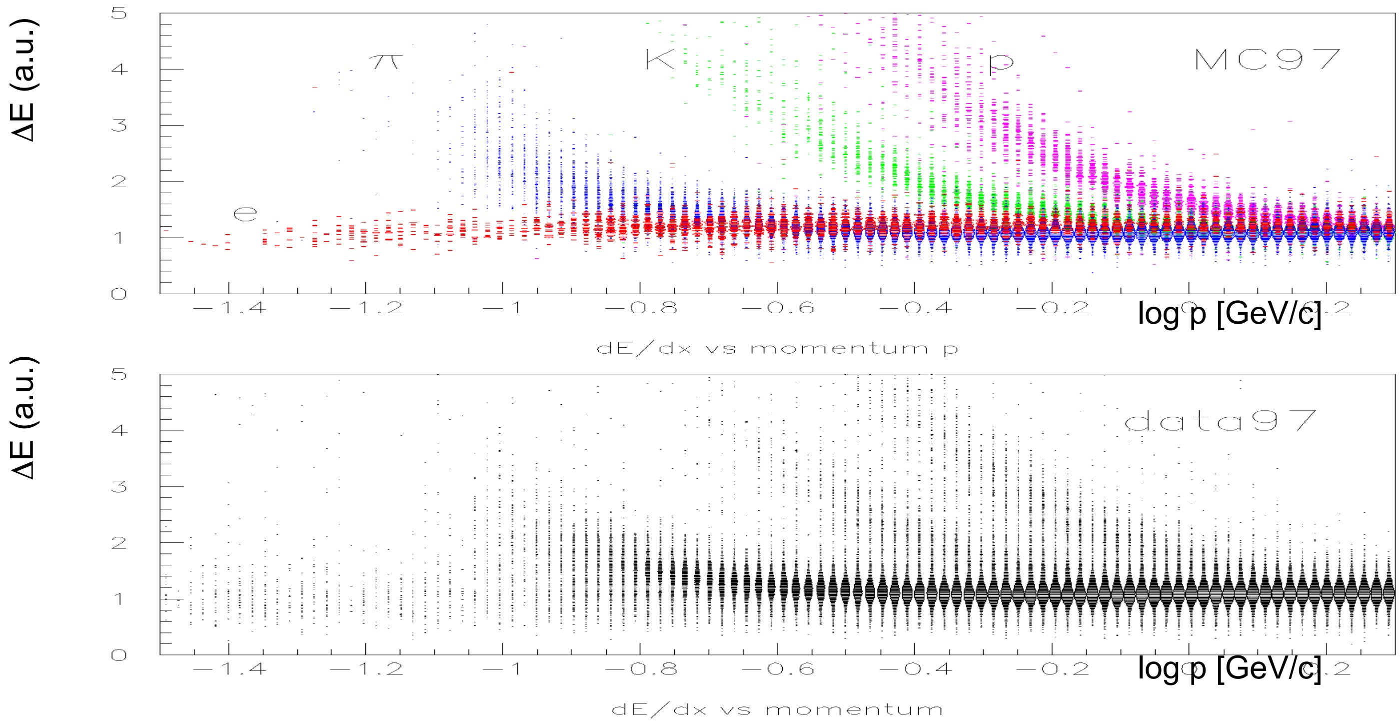




dE/dx and particle ID

dE/dx can also be used in
Silicon detectors

Example DELPHI microvertex detector (3 x 300 μm Silicon)



Time of flight

Particle ID using Time Of Flight (TOF)

$$t = \frac{L}{\beta c}$$

$$(p = m_0 \beta \gamma)$$

Combine TOF with momentum measurement

$$m = p\sqrt{\frac{c^2 t^2}{L^2} - 1}$$

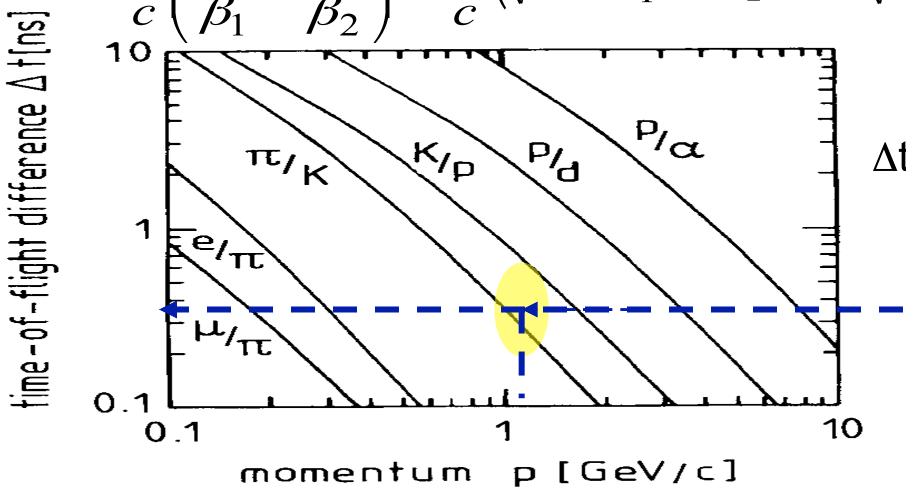
Mass resolution

$$\frac{dm}{m} = \frac{dp}{p} \oplus \gamma^2 \left(\frac{dt}{t} + \frac{dL}{L} \right)$$

Quadratic sum

TOF difference of 2 particles at a given momentum

$$\Delta t = \frac{L}{c}\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) = \frac{L}{c}\left(\sqrt{1 + m_1^2 c^2 / p^2} - \sqrt{1 + m_2^2 c^2 / p^2}\right) \approx \frac{Lc}{2p^2}\left(m_1^2 - m_2^2\right)$$



Δt for L = 1 m path length

σ_t = 300 ps
π/K separation up to 1 GeV/c

Collisioni fra particelle cariche: Multiple Scattering

Riprendiamo una slide gia' vista

Una particella di massa >> dell'elettrone in moto (veloce) in un materiale collide con:

- **Nuclei** → poca energia rilasciata al nucleo, ma angolo di scattering della particella incidente significativo.
- **Elettroni atomici** → gli elettroni (leggeri) si prendono energia dalla particella incidente, ma questa fa uno scattering trascurabile → Bethe-Bloch, Landau,…

Fatto

Collisioni fra particelle cariche: Multiple Scattering

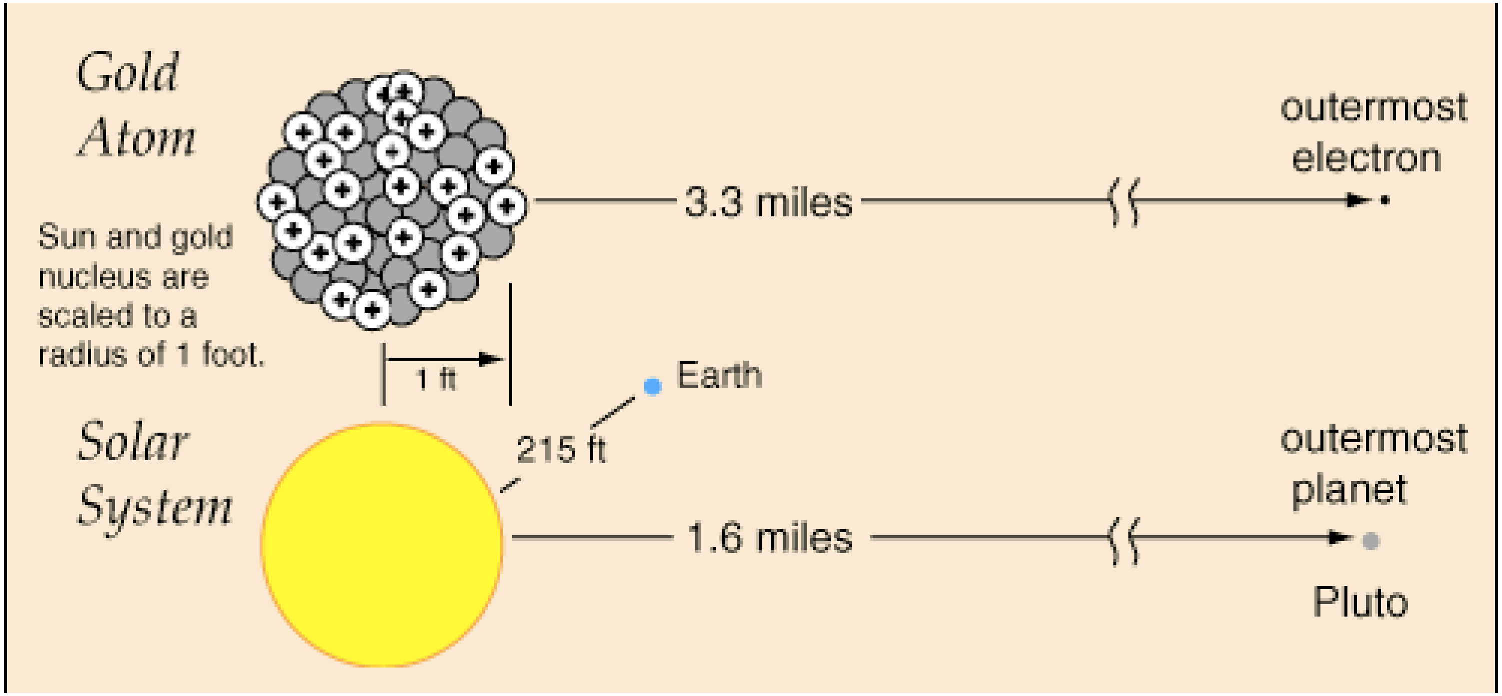
- Quindi quando una particella attraversa un mezzo continuo siamo interessati a conoscere:
  - quanta E deposita → Bethe-Bloch
  - **<u>che deviazione subisce</u>**
- Tratteremo i due processi in maniera indipendente (anche se perdite di energia e deviazione NON sono in realta' indipendenti)

Multiple Scattering

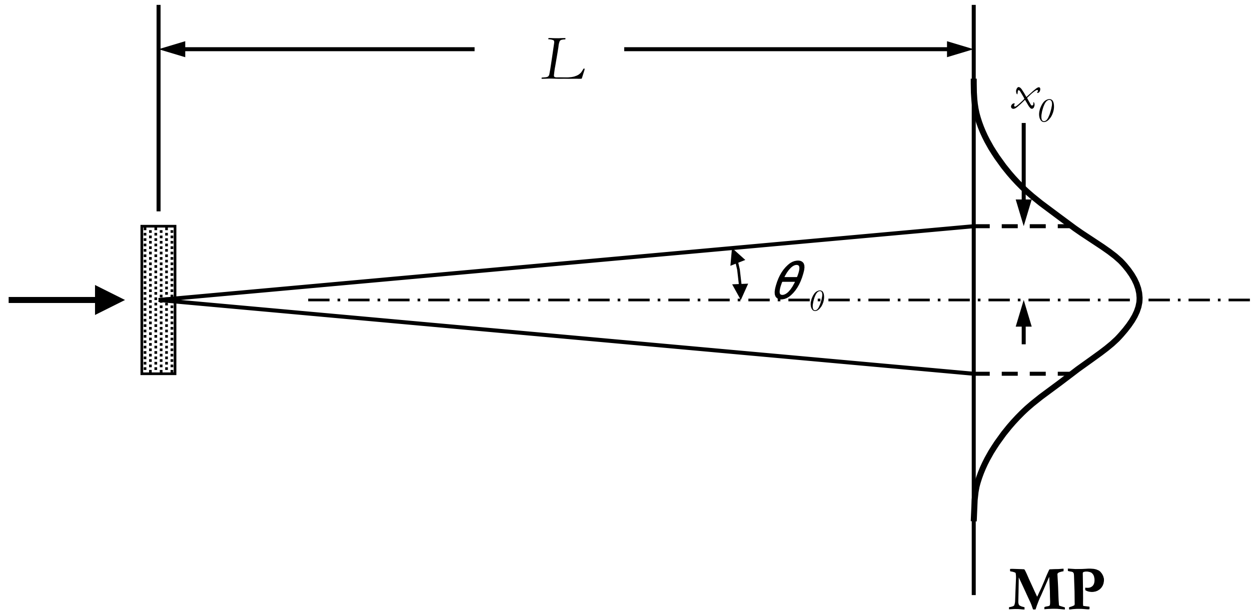
- Looking at dE/dx from ionization, ignore nuclei.
  - Energy transfer small compared to scattering from (lighter) electrons.
- However, scattering from nuclei does change the **direction** of the particles momentum, if not its **magnitude**.
  - Deflection of particle's path limits the accuracy with which the curvature in a magnetic field can be determined, and hence the momentum measured.

Size of Nuclei

Atomic radius of aluminum = 1.3 x 10^{-10} m
Nuclear radius aluminum = 3.6 x 10^{-15} m
The nucleus occupies a tiny fraction of atom volume.



Multiple Scattering



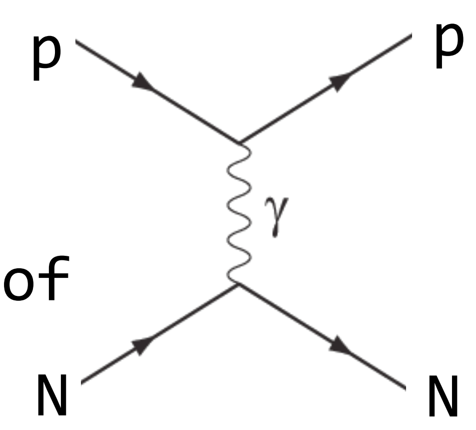
An initially monodirectional beam of particles shows a spread in directions after it crossed a slab of material. This is known as direction straggling or multiple scattering.

When particles, as protons, pass through a slab of material they suffer many collisions with atomic nuclei. The statistical outcome is a *multiple scattering angle* whose distribution is approximately Gaussian. The width parameter of the angular distribution is θ_0.

The task of multiple scattering theory is to predict θ_0 given the scattering material and thickness, and the incident particle energy.

Multiple Scattering

Basic process is the scattering p + N → p + N with a probability to scatter in $d\Omega$ around θ, ϕ as $dP = (d\sigma/d\Omega)d\Omega$



Each collision deviates the incident particle direction

We want to know the cumulative effect of many collisions, ie the amount of deviation after crossing a given thickness of material

As usual, two approaches are possible:

classical and quantistic

Classical Coulomb cross section

- Il primo passo e’ conoscere la probabilita' di diffusione ad un dato angolo in una collisione singola su un nucleo
- Quindi derivare l’effetto statistisco delle collisioni dopo un dato spessore x di materiale attraversato

- For **repulsive scattering** (what we mainly look at here) the situation is as shown in the figure:

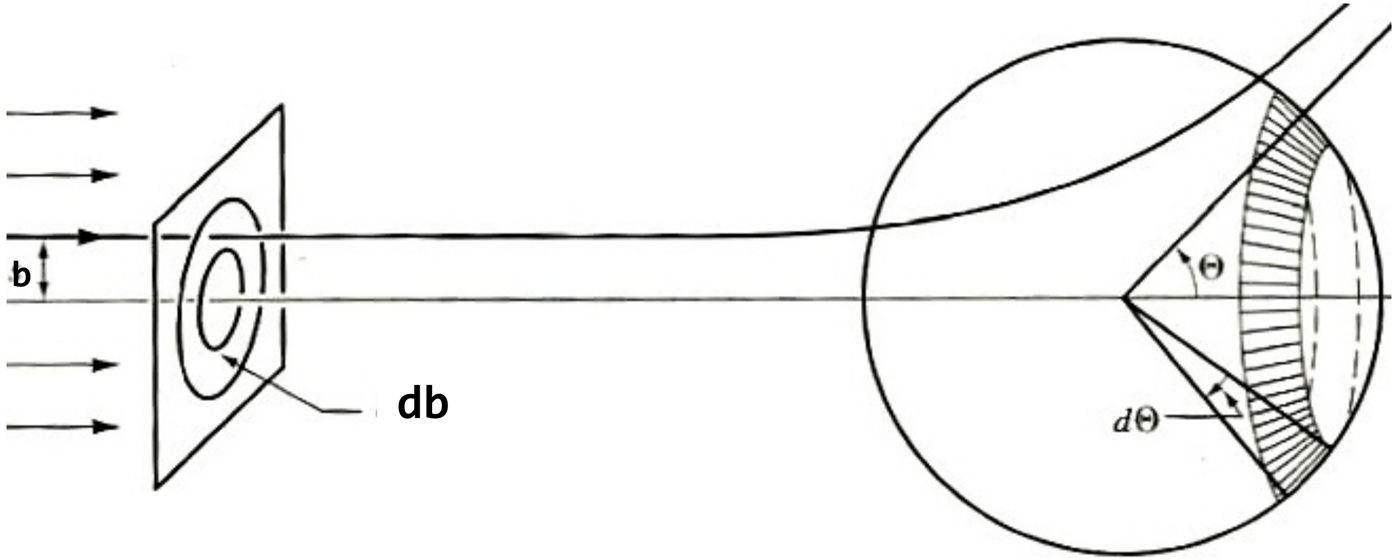


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- ***Define***: **Differential Cross Section for Scattering** in a given direction (into a given solid angle **dΩ**):

$\sigma(\Omega)d\Omega \equiv (N_s/I)$. With **I** = incident intensity

N_s = # particles/time scattered into solid angle **dΩ**

- **Scattering Cross Section:**

$$\sigma(\Omega)d\Omega \equiv (N_s/I)$$

I = incident intensity

N_s = # particles/time

scattered into angle $d\Omega$

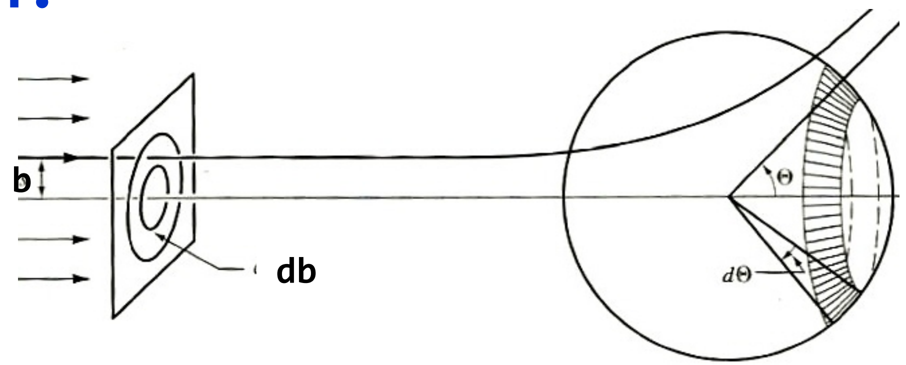


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- In general, the solid angle Ω depends on the spherical angles Θ, Φ. However, for central forces, there must be **symmetry** about the axis of the incident beam

$\Rightarrow \sigma(\Omega)\ (\equiv \sigma(\Theta))$ is independent of azimuth angle Φ

$$\Rightarrow d\Omega \equiv 2\pi \sin\Theta d\Theta\ ,\ \ \sigma(\Omega)d\Omega \equiv 2\pi\, \sigma(\Theta) \sin\Theta d\Theta,$$

$\Theta \equiv$ Angle between incident & scattered beams, as in the figure.

$\sigma \equiv$ "***cross section***". It has units of area

Also called the differential cross section.

- As in all Central Force problems, for a given particle, the orbit, & thus the amount of scattering,

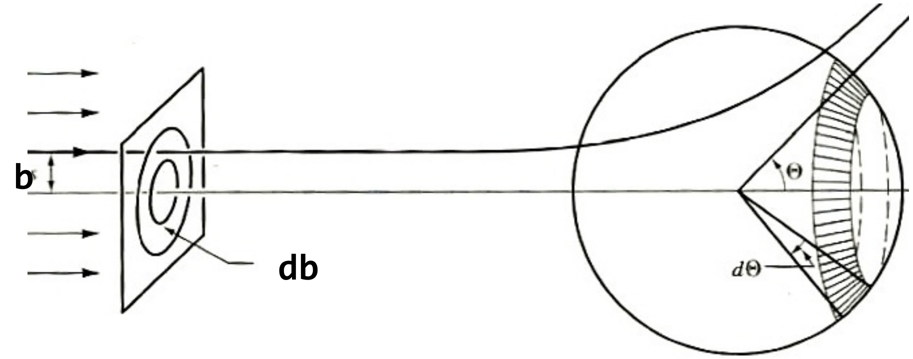


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

is determined by the energy **E** & the angular momentum ℓ

- ***Define*: Impact parameter**, **b**, & express the angular momentum ℓ in terms of **E** & **b**.
- Impact parameter **b** ≡ the ⊥ distance between the center of force & the incident beam velocity (fig).
- ***GOAL:*** **Given** the energy **E**, the impact parameter **b**, & the force **f(r)**, what is the cross section **σ(Θ)**?

- **Beam**, intensity **I**. Particles, mass **m**, incident speed (at **r** → ∞) = v_0.

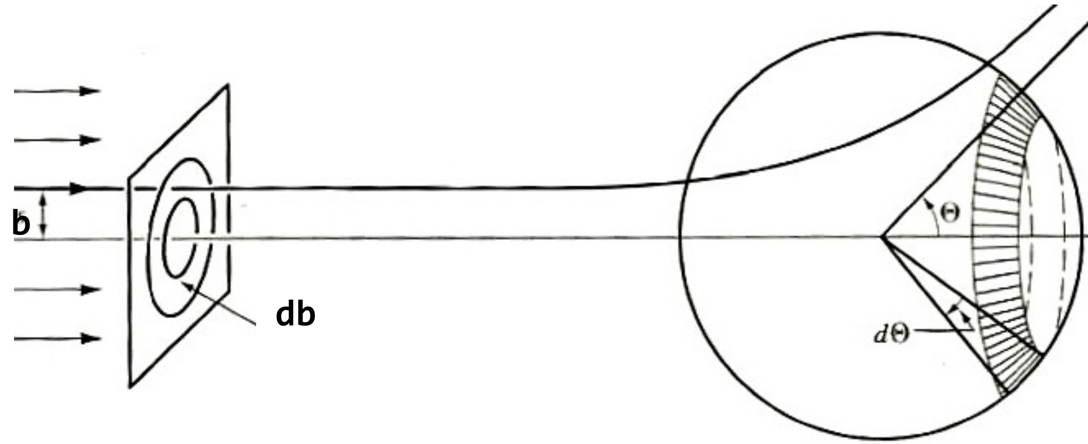


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- **Energy conservation:**

$$E = T + V = ½ mv^2 + V(r) = (½)mv_0^2 + V(r\to\infty)$$

- Assume $V(r\to\infty) = 0 \Rightarrow E = ½ m(v_0)^2$

$$\Rightarrow v_0 = (2E/m)^{½}$$

- **Angular momentum:** $\ell \equiv mv_0 b \equiv b(2mE)^{½}$

  ℓ (vector) is conserved for central forces

- **Angular momentum** $\ell \equiv mv_0b \equiv b(2mE)^{1/2}$
  Incident speed v_0.
- $N_s \equiv$ # particles scattered into solid angle $d\Omega$ between Θ & $\Theta + d\Theta$. **Cross section definition**

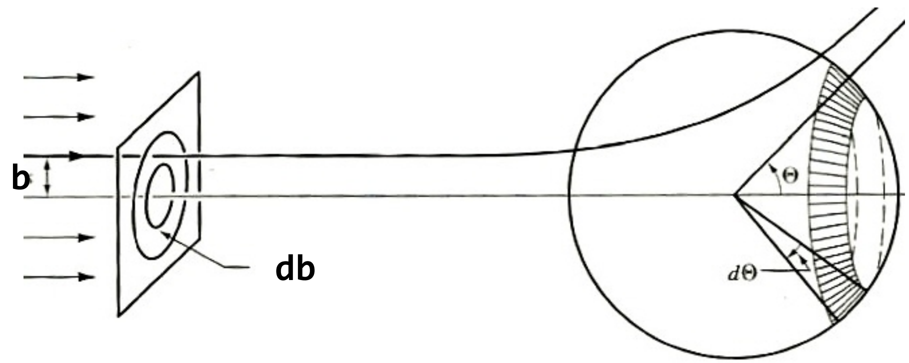


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

$$\Rightarrow \qquad N_s \equiv I\sigma(\Theta)d\Omega = 2\pi I\sigma(\Theta)\sin\Theta d\Theta$$

- $N_i \equiv$ # incident particles with impact parameter between b & $b + db$. $N_i = 2\pi Ibdb$
- **Conservation of particle number**

$$\Rightarrow N_s = N_i \quad \text{or:} \quad 2\pi I\sigma(\Theta)\sin\Theta|d\Theta| = 2\pi Ib|db|$$

$2\pi I$ cancels out! (Use absolute values because N's are always > 0, but db & $d\Theta$ can have any sign.)

$$\sigma(\Theta)\sin\Theta|d\Theta| = b|db| \quad (1)$$

- **b** = a function of energy **E**
- & scattering angle **Θ**:

$$b = b(\Theta, E)$$

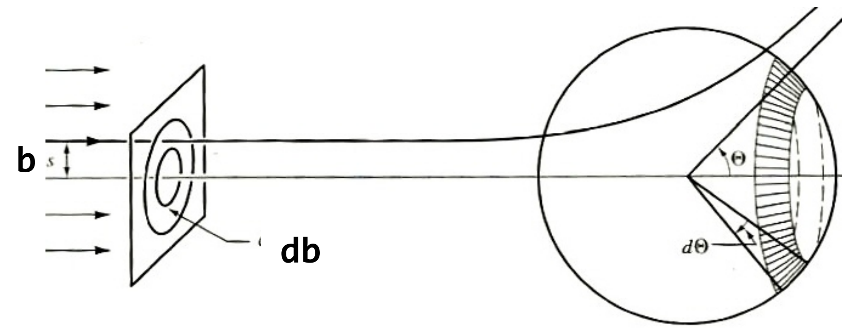


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

$$(1) \quad \Rightarrow \sigma(\Theta) = (b/\sin\Theta)\,(|db|/|d\Theta|) \qquad (2)$$

- To compute $\sigma(\Theta)$ we clearly need $b = b(\Theta,E)$
- Get $\Theta = \Theta(b,E)$ from the orbit eqtn. For general central force (θ is the angle which describes the orbit $r = r(\theta)$; $\theta \neq \Theta$)

$$\theta(r) = \int(\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}dr$$

- Orbit eqtn. **General central force:**

$$\theta(r) = \int(\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}dr \quad (3)$$

- Consider purely **repulsive scattering**. See figure:

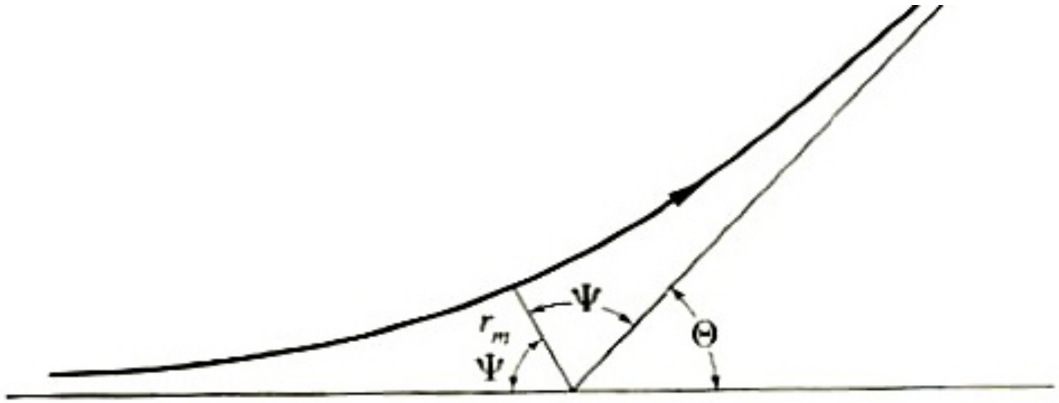


FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

- **Closest approach distance** $\equiv \mathbf{r_m}$. Orbit must be symmetric about $\mathbf{r_m} \Rightarrow$ Sufficient to look at angle Ψ (see figure): Scattering angle $\Theta \equiv \pi - 2\Psi$. Also, orbit angle

$$\theta = \pi - \Psi \textit{ in the special case } r = r_m$$

⇒ After some **manipulation** can write **(3)** as:

$$\Psi = \int (dr/r^2)[(2mE)/(\ell^2) - (2mV(r))/(\ell^2) - 1/(r^2)]^{-1/2} \quad (4)$$

- Integrate from r_m to $r \to \infty$
- Angular momentum in

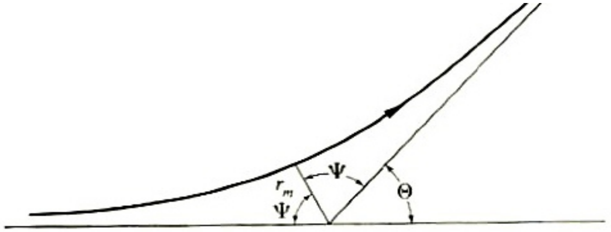


FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

terms of impact parameter **b** & energy **E**: $\ell \equiv mv_0 b \equiv b(2mE)^{1/2}$. Put this into **(4)**

& get for scattering angle **Θ** (after **manipulation**):

$$\Theta(b) = \pi - 2\int dr(b/r)[r^2\{1 - V(r)/E\} - b^2]^{-1/2} \qquad (4')$$

Changing integration variables to $u = 1/r$:

$$\Theta(b) = \pi - 2\int b\,du\,[1 - V(r)/E - b^2u^2]^{-1/2} \qquad (4'')$$

- Integrate from $u = 0$ to $u = u_m = 1/r_m$

- ***Summary:*** Scattering by a ***general central force***:
- **Scattering angle Θ** = Θ**(b,E)** (**b** = impact parameter, **E** = energy):

$$\Theta(b) = \pi - 2\int b\,du\,[1- V(r)/E - b^2u^2]^{-\frac{1}{2}} \quad (4'')$$

Integrate from $u = 0$ to $u = u_m = 1/r_m$

- **Scattering cross section:**

$$\sigma(\Theta) = (b/\sin\Theta)\,(|db|/|d\Theta|) \qquad (2)$$

- **"Recipe":** To solve a scattering problem:

  1. Given force **f(r),** compute potential **V(r).**

  2. Compute Θ**(b)** using **(4′′).**

  3. Compute $\sigma(\Theta)$ using **(2).**

Coulomb Scattering

- ***Special case:* Scattering by r^{-2} repulsive forces**:
  - **For this case**, as well as for others where the orbit eqtn

    $\mathbf{r = r(\theta)}$ is known analytically, instead of applying **(4´´)** directly to get $\mathbf{\Theta(b)}$ & then computing $\mathbf{\sigma(\Theta)}$ using **(2)**, we make use of the known expression for $\mathbf{r = r(\theta)}$ to get $\mathbf{\Theta(b)}$ & then use **(2)** to get $\mathbf{\sigma(\Theta)}$.
- Repulsive scattering by $\mathbf{r^{-2}}$ repulsive forces =

  Coulomb scattering by like charges
  - Charge **+Ze** scatters from the center of force, charge **Z´e**

$\Rightarrow$ $$\mathbf{f(r) \equiv (ZZ'e^2)/(r^2) \equiv - k/r^2}$$

  - Gaussian E&M units! Not SI! For SI, multiply by $\mathbf{(1/4\pi\varepsilon_0)}$!

  $\Rightarrow$ For $\mathbf{r(\theta)}$, in the previous formalism for $\mathbf{r^{-2}}$ attractive forces, make the replacement $\mathbf{k \rightarrow - ZZ'e^2}$

- **Like charges:** $f(r) \equiv (ZZ'e^2)/(r^2)$

  $\Rightarrow k \rightarrow - ZZ'e^2$ in orbit eqtn $r = r(\theta)$
- Orbit eqtn for r^{-2} force is a conic section:

$$[\alpha/r(\theta)] = 1 + \varepsilon\cos(\theta - \theta') \qquad (1)$$

With: **Eccentricity** $\equiv \varepsilon = [\, 1 + \{2E\ell^2/(mk^2)\}]^{1/2}$ &

$2\alpha = [2\ell^2/(mk)]$. Eccentricity = ε to avoid confusion with electric charge **e**. $E > 0 \Rightarrow \varepsilon > 1 \Rightarrow$ **Orbit is a hyperbola**, by previous discussion.

- Choose the integration const $\theta' = \pi$ so that r_{min} is at $\theta = 0$
- Make the changes in notation noted:

$$\Rightarrow \quad [1/r(\theta)] = [(mZZ'e^2)/(\ell^2)](\varepsilon\cos\theta - 1) \qquad (2)$$

$$f(r) = (ZZ'e^2)/(r^2)$$

$$[1/r(\theta)] = [(mZZ'e^2)/(\ell^2)](\varepsilon\cos\theta - 1) \qquad (2)$$

- **Hyperbolic orbit.**
- With change of notation, eccentricity is

$$\varepsilon = [\, 1 + \{2E\ell^2/(mZ^2Z'^2e^4)\}]^{1/2}$$

- Using the relation between angular momentum, energy, & impact parameter, $\ell^2 = 2mEb^2$ this is:

$$\varepsilon = [\, 1 + (2Eb)^2/(ZZ'e^2)^2\,]^{1/2}$$

$$[1/r(\theta,b)] = [(mZZ'e^2)/(\ell^2)](\varepsilon\cos\theta - 1) \qquad (2)$$

$$\varepsilon = [\, 1 + (2Eb)^2/(ZZ'e^2)^2\,]^{1/2} \qquad (3)$$

- From **(2)** get **θ(r,b)**. Then, use relations between orbit angle **θ** scattering angle **Θ**, & auxillary angle **Ψ** in the scattering problem, to get **Θ = Θ(b)** & thus **the scattering cross section.**

$$\Theta = \pi - 2\Psi$$

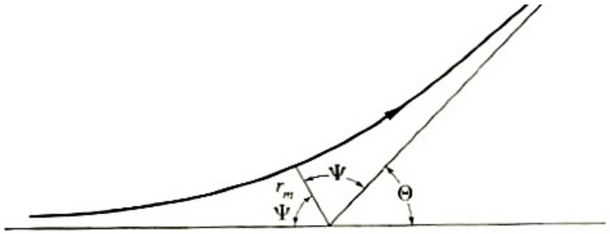


FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

- **Ψ** = direction of incoming asymptote. Determined by **r→ ∞** in **(2)** ⇒ **cosΨ = (1/ε)**.
- In terms of **Θ** this is: **sin(½Θ) = (1/ε). (4)**

$$\varepsilon = [\, 1 + (2Eb)^2/(ZZ'e^2)^2 \,]^{1/2} \qquad (3)$$

$$\sin(\tfrac{1}{2}\Theta) = (1/\varepsilon) \qquad (4)$$

- Put (4) into left side of (3) and solve for b:

$$\Rightarrow \quad b = b(\Theta,E) = (ZZ'e^2)/(2E)\cot(\tfrac{1}{2}\Theta) \qquad (7)$$

(7), the impact parameter as function of **Θ** & **E** for Coulomb scattering is an **important result!**

$$\mathbf{b = b(\Theta,E) = (ZZ'e^2)/(2E)\cot(\tfrac{1}{2}\Theta)} \qquad \mathbf{(7)}$$

- Now, use **(7)** to compute the **Differential Scattering Cross Section for Coulomb Scattering.**
- We had: $\boldsymbol{\sigma(\Theta) = (b/\sin\Theta)\,(|db|/|d\Theta|)}$ **(8)**

  (7) & **(8)** (after using trig identities):

$$\Rightarrow \boldsymbol{\sigma(\Theta) = (\tfrac{1}{4})[(ZZ'e^2)/(2E)]^2(1/\sin^4(\tfrac{1}{2}\Theta)} \qquad \mathbf{(9)}$$

(9) ≡ ***The Rutherford Scattering Cross Section***

- Get the same results in a (non-relativistic) QM derivation!
- Valid for spin 0 particles