Pascal's Triangle

Pascal's Triangle is an infinite triangular array of numbers beginning with a 1 at the top. Pascal's Triangle can be constructed starting with just the 1 on the top by following one easy rule: suppose you are standing in the triangle and would like to know which number to put in the position you are standing on. Look up and to the left, then up and to the right, sum the numbers and you have the entry of Pascal's Triangle corresponding to your current location. Rows 0 thru 12 of Pascal's Triangle look like

$$
\begin{array}{ccccccccccccccccccccccccc}
&&&&&&&&&&&&1&&&&&&&&&&&&\\
&&&&&&&&&&&1&&1&&&&&&&&&&&\\
&&&&&&&&&&1&&2&&1&&&&&&&&&&\\
&&&&&&&&&1&&3&&3&&1&&&&&&&&&\\
&&&&&&&&1&&4&&6&&4&&1&&&&&&&&\\
&&&&&&&1&&5&&10&&10&&5&&1&&&&&&&\\
&&&&&&1&&6&&15&&20&&15&&6&&1&&&&&&\\
&&&&&1&&7&&21&&35&&35&&21&&7&&1&&&&&\\
&&&&1&&8&&28&&56&&70&&56&&28&&8&&1&&&&\\
&&&1&&9&&36&&84&&126&&126&&84&&36&&9&&1&&&\\
&&1&&10&&45&&120&&210&&252&&210&&120&&45&&10&&1&&\\
&1&&11&&55&&165&&330&&462&&462&&330&&165&&55&&11&&1&\\
1&&12&&66&&220&&495&&792&&924&&792&&495&&220&&66&&12&&1
\end{array}
$$

Notice if there is not a number either on the left or the right in the row above an entry then the missing number is replaced with a zero.

Patterns in Pascal's Triangle

Although it is quite easy to construct Pascal's Triangle, it contains many patterns, some surprising and some complex.

Counting

In the mathematical field of combinatorics, a subset of k elements from a larger set of n elements is called a *combination*. The number of combinations of size k denoted $C(n, k)$ and can be read as: how many different ways are there to choose k objects from a pool of n objects? Depending on how you solved the featured puzzle, you may have noticed the number of **rook paths** to each cell on the lower left triangle of the chess board gives rows 0 through 7 of Pascal's Triangle. This is because the entry in the k^{th} column of row n of Pascal's Triangle is $C(n, k)$.

The Fibonacci Numbers

Remember, the Fibonacci sequence is given by the recursive definition $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. This sequence can be found in Pascal's Triangle by drawing diagonal lines through the numbers of the triangle, starting with the 1's in the first column of each row, and adding the numbers the diagonal passes through.

Try it yourself:

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

Did you find the sum of the diagonals to be the same as the first six numbers in the Fibonacci sequence? If your numbers don't match those of the sequence, make sure your diagonal lines are really lines! For instance, the diagonal starting from row 1 passes through the empty space between the 1 of row 0 and the 1 in the right column of row 1. Or, you can use the array on the right, which is still Pascal's triangle, just shifted so all the rows start in the same column, by drawing diagonals starting from 1's on the left and adding the numbers each line crosses through.

Rectangular Sums

To see this next pattern it is best to redraw the triangle in yet another way. Write the diagonals of the triangle as columns. Now, pick any number in the triangle, the first 15, say, and draw a square around it. Starting from the upper left corner of the square, draw a line up and one to the left. It should look something like this:

1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210

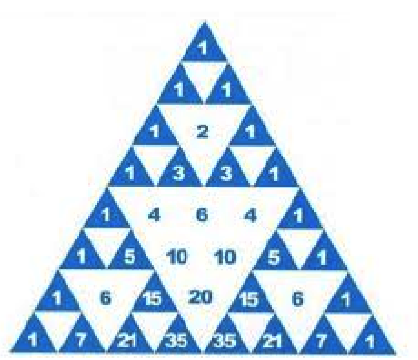
1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210

The sum of the numbers in the rectangular region is 14. Try this starting with a square around the 20, the 56 and any other number you like. **Notice anything?** What is the relationship between the sum and the number you drew a square around?

Congruent Numbers

If the integers n and m have the same remainder when divided by a, n and m are called congruent modulo a. For example, 7, 34 and 127 are all congruent to 1 modulo 3, and 6 is congruent to 0 modulo 3 since there is no remainder when 6 is divided by 3. The parity of a number can also be described in these terms: n is even if it is congruent to 0 modulo 2 and odd if it is congruent to 1 modulo 2.

Check this out! In the figure below all the numbers in Pascal's Triangle which are congruent to 1 modulo 2 have been shaded.

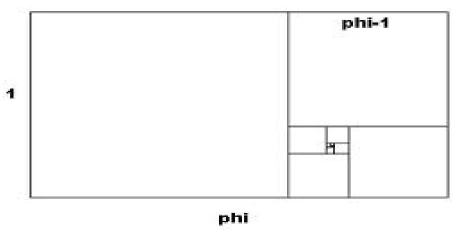


Does it look familiar? This is a fractal called Sierpinski's Triangle, which was featured in a puzzle last month.

You try. On the back page there is a triangle figure with each row drawn as a tessellation of equilateral triangles. Fill in the triangle by summing the two numbers above each location. Then, choose a new congruency class, fill in all of the triangles in that class and see what kind of pattern you get.

Fractals

Informally, a fractal is a set or geometric shape which posses self-similarity. You can see that the Sierpinski Triangle above has a self-similar pattern; if we zoom in, the pattern of smaller triangles appear the same as when we look at the entire triangle. The Cantor set, described below, and golden rectangle also have nice self-similarity patterns. All of these fractals can be defined iteratively. (Not all fractals are formed by an iterative process, but we will focus on those here.) Sierpinski's Triangle is constructed by beginning with a triangle and connecting the midpoints of each edge to make a new triangle. This triangle is then removed, and the same processes is carried out in each of the three remaining triangles and so on. The Cantors set, or comb, is similar to Sierpinski's triangle in that at each step a deletion occurs: a line of unit length is divided into thirds and the middle third is deleted. Each of the two remaining line segments is divided into thirds and the middle third of each is deleted and so on. The construction of the golden rectangle was described in Golden Ratio hand out (available online).

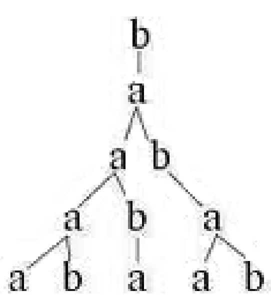


The ratio of the sides of each of the rectangles is the golden ratio.

The Cantor comb fractal.

L-systems

Iterative fractals can be described by L-systems, which consist of generators and rules of how to iterate these generators. For example, suppose we have generators a and b and rules $a \to ab,\ b \to a$, where an arrow means that whatever is on the left side of the arrow will be replaced with what is on the right side. If we specify that the patter will start with b, the first iteration gives an a, so the second iteration yields ab. A further iteration gives aba since, according to the rules, the a in the result of the second iteration is replaced with ab and the b is replaced by a. This can also be drawn in a tree form:

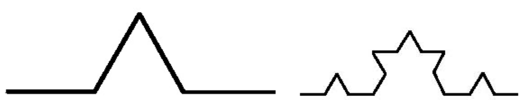


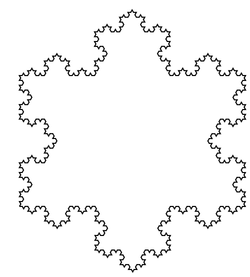
Carry out a few more iterations according to the replacement rules. Do you notice anything about the **number of letters in each row of the tree**?

The generators above are variables because they are replaced each iteration according to the rule. Constants can also be introduced. A generator is constant if it is just replaced with itself, so we can think of it as staying put while the variables around it change after an iteration. Consider a new system with variables F, constants $+$ and $-$, and one rule $F \to F+F-F-F+F$. Here, the variable means draw a line forward, $+$ means turn 60° counter-clockwise and $-$ means turn 60° clockwise. Starting with an F, one iteration yields $F+F-F-F+F$ and a second gives

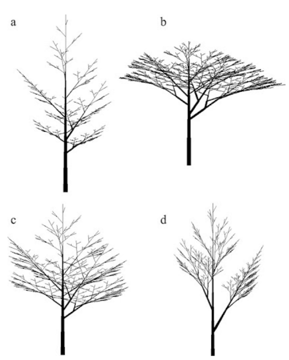
$$F+F-F-F+F+F+F-F-F+F-F+F-F-F+F-F+F-F-F+F+F+F-F-F+F.$$

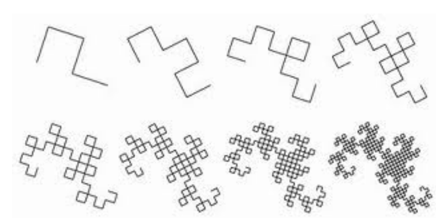
The first two iterations look like:





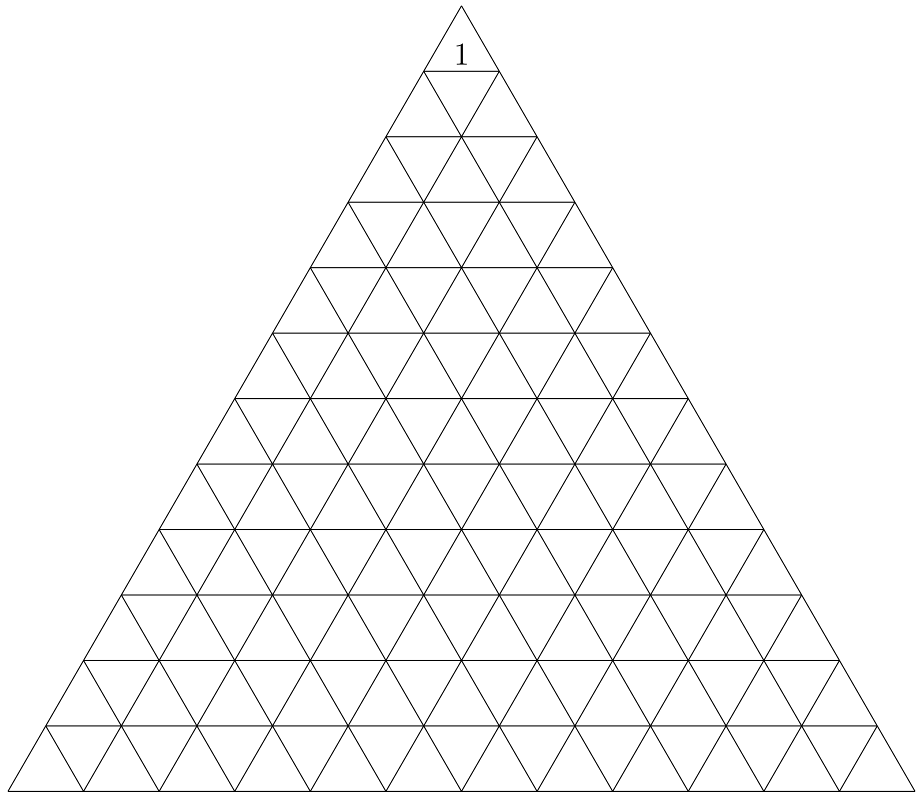
This is a fractal called the Koch curve. A Koch snowflake can be made by starting with a triangle and applying the rule to each edge at each iteration. The one shown on the right is the result of three iterations. Changing the angle assigned to $+$ and $-$ will produce variants of the above snowflake. Other fractals made by using an L-system are shown below.





By the way, the"L" in L-system is for Lindenmayer. Aristid Lindenmayer was a biologist who developed what are now called L-systems to model plant growth. (It should make sense now why you found the pattern you did in the number of letters in the rows of the first L-system example.)

Make your own! Many fractals can be generated this way. It is also fun to make up your own set of generators and rules and see what the result looks like! There are links to applets that allow you to do this on the last page. Have fun!



There is an applet at www.shodor.org/interactive/activities/ which shades in multiples of any number you choose.

Links to more information

(Also at: www.math.cornell.edu/ araymer/Puzzle/PuzzleNights.html)

Counting: http://betterexplained.com/articles/easy-permutations-and-combinations/
Combinations & Pascal's Triangle: http://www.passionatelycurious.com/files/combinations.html
Patterns in Pascal's Triangle: http://ptri1.tripod.com/
Modular Arithmetic: http://betterexplained.com/articles/fun-with-modular-arithmetic/
L-system generated fractals: http://ejad.best.vwh.net/java/fractals/process.shtml#lsystem
L-system Applets: http://nolandc.com/sandbox/fractals/#
http://www.kevs3d.co.uk/dev/lsystems/