Passive Fault Tolerant Control

M. Benosman
*Mitsubishi Electric Research Laboratories 201 Broadway street, Cambridge, MA 02139,**
USA

1. Introduction

Today, as a result of increasing complexity of industrial automation technologies, fault handling of such automatic systems has become a challenging task. Indeed, although industrial systems are usually designed to perform optimally over time, performance degradation occurs inevitably. These are due, for example, to aging of system components, which have to be monitored to prevent system-wide failures. Fault handling is also necessary to allow redesign of the control in such a way to recover, as much as possible, an optimal performance. To this end, researchers in the systems control community have focused on a specific control design strategy, called Fault tolerant control (FTC). Indeed, FTC is aimed at achieving acceptable performance and stability for the safe, i.e. fault-free system as well as for the faulty system. Many methods have been proposed to deal with this problem. For survey papers on FTC, the reader may refer to (5; 38; 53). While the available schemes can be classified into two types, namely *passive* and *active* FTC (53), the work presented here falls into the first category of passive FTC. Indeed, active FTC is aimed at ensuring the stability and some performance, possibly degraded, for the post-fault model, and this by reconfiguring on-line the controller, by means of a fault detection and diagnosis (FDD) component that detects, isolates and estimates the current fault (53). Contrary to this active approach, the passive solution consists in using a unique robust controller that, will deal with *all the expected faults*. The passive FTC approach has the drawback of being reliable only for the class of faults expected and taken into account in the design. However, it has the advantage of avoiding the time delay required in active FTC for on-line fault diagnosis and control reconfiguration (42; 54), which is very important in practical situations where the time window during which the faulty system stay stabilizable is very short, e.g. the unstable double inverted pendulum example (37). In fact, in practical applications, passive FTC complement active FTC schemes. Indeed, passive FTC schemes are necessary during the fault detection and estimation phase (50), to ensure the stability of the faulty system, before switching to active FTC. Several passive FTC methods have been proposed, mainly based on robust theory, e.g. multi-objective linear optimization and LMIs techniques (25), QFT method (47; 48), H_∞ (36; 37), absolute stability theory (6), nonlinear regulation theory (10; 11), Lyapunov reconstruction (9) and passivity-based FTC (8). As for active FTC, many methods have been proposed for active linear FTC, e.g. (19; 29; 43; 46; 51; 52), as well as for nonlinear FTC, e.g. (4; 7; 13; 14; 20; 21; 28; 32–35; 39; 41; 49).
We consider in this work the problem of fault tolerant control for failures resulting from *loss of*

*E-mail: benosman@merl.com

actuator effectiveness. FTCs dealing with actuator faults are relevant in practical applications and have already been the subject of many publications. For instance, in (43), the case of uncertain linear time-invariant models was studied. The authors treated the problem of actuators stuck at unknown constant values at unknown time instants. The active FTC approach they proposed was based on an output feedback adaptive method. Another active FTC formulation was proposed in (46), where the authors studied the problem of loss of actuator effectiveness in linear discrete-time models. The loss of control effectiveness was estimated via an adaptive Kalman filter. The estimation was complemented by a fault reconfiguration based on the LQG method. In (30), the authors proposed a multiple-controller based FTC for linear uncertain models. They introduced an active FTC scheme that ensured the stability of the system regardless of the decision of FDD.

However, as mentioned earlier and as presented for example in (50), the aforementioned active schemes will incur a delay period during which the associate FDD component will have to converge to a best estimate of the fault. During this time period of FDD response delay, it is essential to control the system with a passive fault tolerant controller which is robust against actuator faults so as to ensure at least the stability of the system, before switching to another controller based on the estimated post-fault model, that ensures optimal post-fault performance. In this context, we propose here passive FTC schemes against actuator loss of effectiveness. The results presented here are based on the work of the author introduced in (6; 8). We first consider linear FTC and present some results on passive FTC for loss of effectiveness faults based on absolute stability theory. Next we present an extension of the linear results to some nonlinear models and use passivity theory to write nonlinear fault tolerant controllers. In this chapter several controllers are proposed for different problem settings: a) Linear time invariant (LTI) certain plants, b) uncertain LTI plants, c) LTI models with input saturations, d) nonlinear plants affine in the control with single input, e) general nonlinear models with constant as well as time-varying faults and with input saturation. We underline here that we focus in this chapter on the theoretical developments of the controllers, readers interested in numerical applications should refer to (6; 8).

2. Preliminaries

Throughout this chapter we will use the L_2 norm denoted $||.||$, i.e. for $x \in \mathbb{R}^n$ we define $||x|| = \sqrt{x^T x}$. The notation $L_f h$ denotes the standard Lie derivative of a scalar function $h(.)$ along a vector function $f(.)$. Let us introduce now some definitions from (40), that will be frequently used in the sequel.

Definition 1 ((40), p.45): The solution $x(t, x_0)$ of the system $\dot{x} = f(x),\ x \in \mathbb{R}^n$, f locally Lipschitz, is stable conditionally to Z, if $x_0 \in Z$ and for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$||\tilde{x}_0 - x_0|| < \delta \text{ and } \tilde{x}_0 \in Z \Rightarrow ||x(t, \tilde{x}_0) - x(t, x_0)|| < \epsilon,\ \forall t \geq 0.$$

If furthermore, there exist $r(x_0) > 0$, s.t. $||x(t, \tilde{x}_0) - x(t, x_0)|| \Rightarrow 0,\ \forall ||\tilde{x}_0 - x_0|| < r(x_0)$ *and* $\tilde{x}_0 \in Z$, the solution is asymptotically stable conditionally to Z. If $r(x_0) \rightarrow \infty$, the stability is global.

Definition 2 ((40), p.48): Consider the system $H : \dot{x} = f(x, u),\ y = h(x, u),\ x \in \mathbb{R}^n,\ u, y \in \mathbb{R}^m$, with zero inputs, i.e. $\dot{x} = f(x, 0),\ y = h(x, 0)$ and let $Z \subset \mathbb{R}^n$ be its largest positively invariant set contained in $\{x \in \mathbb{R}^n | y = h(x, 0) = 0\}$. We say that H is globally zero-state detectable (GZSD) if $x = 0$ is globally asymptotically stable conditionally to Z. If $Z = \{0\}$, the system H is zero-state observable (ZSO).

Definition 3 ((40), p.27): We say that H is dissipative in $X \subset \mathbb{R}^n$ containing $x = 0$, if there exists a function $S(x),\ S(0) = 0$ such that for all $x \in X$

$$S(x) \geq 0 \text{ and } S(x(T)) - S(x(0)) \leq \int_0^T \omega(u(t), y(t))dt,$$

for all $u \in U \subset \mathbb{R}^m$ and all $T > 0$ such that $x(t) \in X,\ \forall\ t \in [0, T]$. Where the function $\omega : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ called the supply rate, is locally integrable for every $u \in U$, i.e. $\int_{t_0}^{t_1} |\omega(u(t), y(t))|dt < \infty,\ \forall\ t_0 \leq t_1$. S is called the storage function. If the storage function is differentiable the previous conditions writes as

$$\dot{S}(x(t)) \leq \omega(u(t), y(t)).$$

The system H is said to be passive if it is dissipative with the supply rate $w(u, y) = u^T y$.
Definition 4 ((40), p.36): We say that H is output feedback passive (OFP(ρ)) if it is dissipative with respect to $\omega(u, y) = u^T y - \rho y^T y$ for some $\rho \in \mathbb{R}$.
We will also need the following definition to study the case of time-varying faults in Section 8.
Definition 5 (24): A function $\overline{x} : [0, \infty) \to \mathbb{R}^n$ is called a limiting solution of the system $\dot{x} = f(t, x)$, f a smooth vector function, with respect to an unbounded sequence t_n in $[0, \infty)$, if there exist a compact $\kappa \subset \mathbb{R}^n$ and a sequence $\{x_n : [t_n, \infty) \to \kappa\}$ of solutions of the system such that the associated sequence $\{\hat{x}_n :\to x_n(t + t_n)\}$ converges uniformly to $\overline{x}$ on every compact subset of $[0, \infty)$.
Also, throughout this paper it is said that a statement $P(t)$ holds almost everywhere (a.e.) if the Lebesgue measure of the set $\{t \in [0, \infty)\ |P(t)\ is\ false\}$ is zero. We denote by df the differential of the function $f : \mathbb{R}^n \to \mathbb{R}$. We also mean by semiglobal stability of the equilibrium point x^0 for the autonomous system $\dot{x} = f(x),\ x \in \mathbb{R}^n$ with f a smooth function, that for each compact set $K \subset \mathbb{R}^n$ containing x^0, there exist a locally Lipschitz state feedback, such that x^0 is asymptotically stable, with a basin of attraction containing K ((44), Definition 3, p. 1445).

3. FTC for known LTI plants

First, let us consider linear systems of the form

$$\dot{x} = Ax + B\alpha u, \tag{1}$$

where, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and input vector, respectively, and $\alpha \in \mathbb{R}^{m \times m}$ is a diagonal time variant fault matrix, with diagonal elements $\alpha_{ii}(t),\ i = 1, ..., m$ s.t., $0 < \epsilon_1 \leq \alpha_{ii}(t) \leq 1$. The matrices $A,\ B$ have appropriate dimensions and satisfy the following assumption.
Assumption(1): The pair (A, B) is controllable.

3.1 Problem statement

Find a state feedback controller $u(x)$ such that the closed-loop controlled system (1) admits $x = 0$ as a globally uniformly asymptotically (GUA) stable equilibrium point $\forall\alpha(t)$ (s.t. $0 < \epsilon_1 \leq \alpha_{ii}(t) \leq 1$).

3.2 Problem solution

Hereafter, we will re-write the problem of stabilizing (1), for $\forall\alpha(t)$ s.t., $0 < \epsilon_1 \leq \alpha_{ii}(t) \leq 1$, as an absolute stability problem or Lure's problem (2). Let us first recall the definition of sector nonlinearities.

Definition 6 ((22), p. 232): A static function $\psi: [0,\infty)\times\mathbb{R}^m \rightarrow \mathbb{R}^m$, s.t. $[\psi(t,y)-K_1y]^T[\psi(t,y)-K_2y] \leq 0,\ \forall(t,y)$, with $K = K_2 - K_1 = K^T > 0$, where $K_1 = diag(k1_1,...,k1_m),\ K_2 = diag(k2_1,...,k2_m)$, is said to belong to the sector $[K_1,\ K_2]$.
We can now recall the definition of absolute stability or Lure's problem.
Definition 7 (Absolute stability or Lure's problem (22), p. 264): We assume a linear system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ u &= -\psi(t,y), \end{aligned} \tag{2}$$

where, $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^m$, (A,B) controllable, (A,C) observable and $\psi: [0,\infty)\times\mathbb{R}^m \rightarrow \mathbb{R}^m$ is a static nonlinearity, piecewise continuous in t, locally Lipschitz in y and satisfies a sector condition as defined above. Then, the system (2) is absolutely stable if the origin is GUA stable for any nonlinearity in the given sector. It is absolutely stable within a finite domain if the origin is uniformly asymptotically (UA) stable within a finite domain.
We can now introduce the idea used here, which is as follows:
Let us associate with the faulty system (1) a virtual output vector $y \in \mathbb{R}^m$

$$\begin{aligned} \dot{x} &= Ax + B\alpha u \\ y &= Kx, \end{aligned} \tag{3}$$

and let us write the controller as an output feedback

$$u = -y. \tag{4}$$

From (3) and (4), we can write the closed-loop system as

$$\begin{aligned} \dot{x} &= Ax + Bv \\ y &= Kx \\ v &= -\alpha(t)y. \end{aligned} \tag{5}$$

We have thus transformed the problem of stabilizing (1), for all bounded matrices $\alpha(t)$, to the problem of stabilizing the system (5) for all $\alpha(t)$. It is clear that the problem of GUA stabilizing (5) is a Lure's problem in (2), with the linear time varying stationarity $\psi(t,y) = \alpha(t)y$, and where the 'nonlinearities' admit the sector bounds $K_1 = diag(\epsilon_1,...,\epsilon_1),\ K_2 = I_{m\times m}$.
Based on this formulation we can now solve the problem of passive fault tolerant control of (1) by applying the absolute stability theory (26).
We can first write the following result:
Proposition 1: Under Assumption 1, the closed-loop of (1) with the static state feedback

$$u = -Kx, \tag{6}$$

where K is solution of the optimal problem

$$\begin{aligned} &\min_{k_{ij}}\left(\textstyle\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} k_{ij}^2\right) \\ &\begin{bmatrix} P\hat{A}(K) + \hat{A}^T(K)P & (\hat{C}^T - P\hat{B})W^{-1} \\ ((\hat{C}^T - P\hat{B})W^{-1})^T & -I \end{bmatrix} < 0 \\ &P > 0 \\ &rank \begin{bmatrix} K \\ KA \\ \vdots \\ KA^{n-1} \end{bmatrix} = n, \end{aligned} \tag{7}$$

for $P = P^T > 0$, $W = (\hat{D} + \hat{D}^T)^{0.5}$ and $\{\hat{A}(K),\ \hat{B}(K),\ \hat{C}(K),\ \hat{D}(K)\}$ is a minimal realization of the transfer matrix

$$\hat{G} = [I + K(sI - A)^{-1}B][I + \epsilon_1 \times I_{m\times m}K(sI - A)^{-1}B]^{-1}, \tag{8}$$

admits the origin $x = 0$ as GUA stable equilibrium point.

Proof: We saw that the problem of stabilizing (1) with a static state feedback $u = -Kx$ is equivalent to the stabilization of (5). Studying the stability of (5) is a particular case of Lure's problem defined by (2), with the 'nonlinearity' function $\psi(t, y) = -\alpha(t)y$ associated with the sector bounds $K_1 = \epsilon_1 \times I_{m\times m}$, $K_2 = I_{m\times m}$ (introduced in Definition 1). Then based on Theorem 7.1, in ((22), p. 265), we can write that under Assumption1 and the constraint of observability of the pair (A, K), the origin $x = 0$ is GUA stable equilibrium point for (5), if the matrix transfer function

$$\hat{G} = [I + G(s)][I + \epsilon_1 \times I_{m\times m}G(s)]^{-1},$$

where $G(s) = K(sI - A)^{-1}B$, is strictly positive real (SPR). Now, using the KYP lemma as presented in (Lemma 6.3, (22), p. 240), we can write that a sufficient condition for the GUA stability of $x = 0$ along the solution of (1) with $u = -Kx$ is the existence of $P = P^T > 0$, L and W, s.t.

$$\begin{array}{l} P\hat{A}(K) + \hat{A}^T(K)P = -L^TL - \epsilon P,\ \epsilon > 0 \\ P\hat{B}(K) = \hat{C}^T(K) - L^TW \\ W^TW = \hat{D}(K) + \hat{D}^T(K), \end{array} \tag{9}$$

where, $\{\hat{A},\ \hat{B},\ \hat{C},\ \hat{D}\}$ is a minimal realization of $\hat{G}$. Finally, adding to equation (9), the observability condition of the pair (A, K), we arrive at the condition

$$\begin{array}{l} P\hat{A}(K) + \hat{A}^T(K)P = -L^TL - \epsilon P,\ \epsilon > 0 \\ P\hat{B}(K) = \hat{C}^T(K) - L^TW \\ W^TW = \hat{D}(K) + \hat{D}^T(K) \\ rank\begin{bmatrix} K \\ KA \\ \vdots \\ KA^{n-1} \end{bmatrix} = n. \end{array} \tag{10}$$

Next, if we choose $W = W^T$ we can write $W = (\hat{D} + \hat{D}^T)^{0.5}$. The second equation in (10) leads to $L^T = (\hat{C}^T - P\hat{B})W^{-1}$. Finally, from the first equation in (10), we arrive at the following condition on P

$$P\hat{A}(K) + \hat{A}^T(K)P + (\hat{C}^T - P\hat{B})W^{-1}((\hat{C}^T - P\hat{B})W^{-1})^T < 0,$$

which is in turn equivalent to the LMI

$$\begin{bmatrix} P\hat{A}(K) + \hat{A}^T(K)P & (\hat{C}^T - P\hat{B})W^{-1} \\ ((\hat{C}^T - P\hat{B})W^{-1})^T & -I \end{bmatrix} < 0. \tag{11}$$

Thus, to solve equation (10) we can solve the constrained optimal problem

$$\begin{array}{l}\min\limits_{k_{ij}}(\sum_{i=1}^{i=m}\sum_{j=1}^{j=n}k_{ij}^2)\\ \begin{bmatrix} P\hat{A}(K)+\hat{A}^T(K)P & (\hat{C}^T-P\hat{B})W^{-1} \\ ((\hat{C}^T-P\hat{B})W^{-1})^T & -I \end{bmatrix} < 0 \\ P>0 \\ rank\begin{bmatrix} K \\ KA \\ \vdots \\ KA^{n-1} \end{bmatrix} = n. \qquad \square \end{array} \tag{12}$$

Note that the inequality constraints in (7) can be easily solved by available LMI algorithms, e.g. *feasp* under Matlab. Furthermore, to solve equation (10), we can propose two other different formulations:

1. Through nonlinear algebraic equations: Choose $W = W^T$ which implies by the third equation in (10) that $W = (\hat{D}(K) + \hat{D}^T(K))^{0.5}$, for any K s.t.

$$\begin{array}{l} P\hat{A}(K)+\hat{A}^T(K)P = -L^TL-\epsilon P,\ \epsilon>0,\ P=P^T>0 \\ P\hat{B}(K) = \hat{C}^T(K) - L^TW \\ rank\begin{bmatrix} K \\ KA \\ \vdots \\ KA^{n-1} \end{bmatrix} = n. \end{array} \tag{13}$$

   To solve (13) we can choose $\epsilon = \tilde{\epsilon}^2$ and $P = \tilde{P}^T\tilde{P}$, which leads to the nonlinear algebraic equation

$$F(k_{ij}, \tilde{p}_{ij}, l_{ij}, \tilde{\epsilon}) = 0, \tag{14}$$

   where $k_{ij},\ i = 1,...,m,\ j = 1,...n,\ \tilde{p}_{ij},\ i = 1,...,\tilde{n}\ (\hat{A} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}),\ j = 1,...\tilde{n}$ and $l_{ij},\ i = 1,...,m,\ j = 1,...\tilde{n}$ are the elements of K, $\tilde{P}$ and L, respectively. Equation (14) can then be resolved by any nonlinear algebraic equations solver, e.g. *fsolve* under Matlab.

2. Through Algebraic Riccati Equations (ARE): It is well known that the positive real lemma equations, i.e. the first three equations in (10) can be transformed to the following ARE ((3), pp. 270-271):

$$P(\hat{\hat{A}} - \hat{B}R^{-1}\hat{C}) + (\hat{\hat{A}}^T - \hat{C}^TR^{-1}\hat{B}^T)P + P\hat{B}R^{-1}\hat{B}^TP + \hat{C}^TR^{-1}\hat{C} = 0, \tag{15}$$

   where $\hat{\hat{A}} = \hat{A} + 0.5\epsilon.I_{\tilde{n}\times\tilde{n}},\ R = \hat{D}(K) + \hat{D}^T(K) > 0$. Then, if a solution $P = P^T > 0$ is found for (15) it is also a solution for the first three equation in (10), together with

$$W = -VR^{1/2},\ L = (P\hat{B} - \hat{C}^T)R^{-1/2}V^T,\ VV^T = I.$$

   To solve equation (10), we can then solve the constrained optimal problem

$$\begin{array}{l} \min\limits_{k_{ij}}(\sum_{i=1}^{i=m}\sum_{j=1}^{j=n}k_{ij}^2) \\ P>0 \\ rank\begin{bmatrix} K \\ KA \\ \vdots \\ KA^{n-1} \end{bmatrix}=n, \end{array} \tag{16}$$

where P is the symmetric solution of the ARE (15), that can be directly computed by available solvers, e.g. *care* under Matlab.

There are other linear controllers for LPV system, that might solve the problem stated in Section 3.1, e.g. (1). However, the solution proposed here benefits from the simplicity of the formulation based on the absolute stability theory, and allows us to design FTCs for uncertain and saturated LTI plants, as well as nonlinear affine models, as we will see in the sequel. Furthermore, reformulating the FTC problem in the absolute stability theory framework may be applied to solve the FTC problem for several other systems, like infinite dimensional systems, i.e. PDEs models, stochastic systems and systems with delays (see (26) and the references therein). Furthermore, compared to optimal controllers, e.g. LQR, the proposed solution offers greater robustness, since it compensates for the loss of effectiveness over $[\epsilon_1,\ 1]$. Indeed, it is well known that *in the time invariant case*, optimal controllers like LQR compensates for a loss of effectiveness over $[1/2,\ 1]$ ((40), pp. 99-102). A larger loss of effectiveness can be covered but at the expense of higher control amplitude ((40), Proposition 3.32, p.100), which is not desirable in practical situations.
Let us consider now the more practical case of LTI plants with parameter uncertainties.

4. FTC for uncertain LTI plants

We consider here models with structured uncertainties of the form

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)\alpha u, \tag{17}$$

where $\Delta A \in \circ\mathcal{A} = \{\Delta A \in \mathbb{R}^{n\times n} | \Delta A_{min} \leq \Delta A \leq \Delta A_{max},\ \Delta A_{min}, \Delta A_{max} \in \mathbb{R}^{n\times n}\}$,
$\Delta B \in \circ\mathcal{B} = \{\Delta B \in \mathbb{R}^{n\times m} | \Delta B_{min} \leq \Delta B \leq \Delta B_{max},\ \Delta B_{min}, \Delta B_{max} \in \mathbb{R}^{n\times m}\}$,
$\alpha = diag(\alpha_{11}, ..., \alpha_{mm}),\ 0 < \epsilon_1 \leq \alpha_{ii} \leq 1\ \forall i \in \{1, ..., m\}$, and $A,\ B,\ x,\ u$ as defined before.

4.1 Problem statement

Find a state feedback controller $u(x)$ such that the closed-loop controlled system (17) admits $x = 0$ as a globally asymptotically (GA) stable equilibrium point $\forall\alpha(s.t.\ 0 < \epsilon_1 \leq \alpha_{ii} \leq 1),\ \forall\Delta A \in \circ\mathcal{A},\ \Delta B \in \circ\mathcal{B}$.

4.2 Problem solution

We first re-write the model (17) as follows:

$$\begin{array}{l} \dot{x} = (A + \Delta A)x + (B + \Delta B)v \\ y = Kx \\ v = -\alpha y. \end{array} \tag{18}$$

The formulation given by (18), is an uncertain Lure's problem (as defined in (15) for example). We can write the following result:

Proposition 2: Under Assumption 1, the system (17) admits $x = 0$ as GA stable equilibrium point, with the static state feedback $u = -\tilde{K}\tilde{H}^{-1}x$, where $\tilde{K},\ \tilde{H}$ are solutions of the LMIs

$$\begin{array}{l} \tilde{Q} + \tilde{H}A^T - \tilde{K}^T L^T B^T + A\tilde{H} - BL\tilde{K} \leq 0\ \forall L \in L^v,\ \tilde{Q} = \tilde{Q}^T > 0,\ \tilde{H} > 0 \\ -\tilde{Q} + \tilde{H}\Delta A^T - \tilde{K}^T L^T \Delta B^T + \Delta A\tilde{H} - \Delta BL\tilde{K} < 0,\ \forall(\Delta A, \Delta B, Ł) \in \circ\mathcal{A}^v \times \circ\mathcal{B}^v \times L^v, \end{array} \tag{19}$$

where, L^v is the set containing the vertices of $\{\epsilon_1 I_{m\times m},\ I_{m\times m}\}$, and $\circ\mathcal{A}^v,\ \circ\mathcal{B}^v$ are the set of vertices of $\circ\mathcal{A},\ \circ\mathcal{B}$ respectively.
Proof: Under Assumption 1, and using Theorem 5 in ((15), p. 330), we can write the stabilizing static state feedback $u = -Kx$, where K is such that, for a given $H > 0,\ Q = Q^T > 0$ we have

$$\begin{cases} Q + (A - BLK)^T H + H(A - BLK) \leq 0\ \forall L \in L^v \\ -Q + ((\Delta A - \Delta BLK)^T H + H(\Delta A - \Delta BLK)) < 0\ \forall(\Delta A, \Delta B, Ł) \in \circ\mathcal{A}^v \times \circ\mathcal{B}^v \times L^v, \end{cases} \tag{20}$$

where, L^v is the set containing the vertices of $\{\epsilon_1 I_{m\times m},\ I_{m\times m}\}$, and $\circ\mathcal{A}^v,\ \circ\mathcal{B}^v$ are the set of vertices of $\circ\mathcal{A},\ \circ\mathcal{B}$ respectively. Next, inequalities (20) can be transformed to LMIs by defining the new variables $\tilde{K} = KH^{-1},\ \tilde{H} = H^{-1},\ \tilde{Q} = H^{-1}QH^{-1}$ and multiplying both sides of the inequalities in (20) by H^{-1}, we can write finally (20) as

$$\begin{array}{l} \tilde{Q} + \tilde{H}A^T - \tilde{K}^T L^T B^T + A\tilde{H} - BL\tilde{K} \leq 0\ \forall L \in L^v,\ \tilde{Q} = \tilde{Q}^T > 0,\ \tilde{H} > 0 \\ -\tilde{Q} + \tilde{H}\Delta A^T - \tilde{K}^T L^T \Delta B^T + \Delta A\tilde{H} - \Delta BL\tilde{K} < 0\ \forall(\Delta A, \Delta B, Ł) \in \circ\mathcal{A}^v \times \circ\mathcal{B}^v \times L^v, \end{array} \tag{21}$$

the controller gain will be given by $K = \tilde{K}\tilde{H}^{-1}$.□
Let us consider now the practical problem of input saturation. Indeed, in practical applications the available actuators have limited maximum amplitudes. For this reason, it is more realistic to consider bounded control amplitudes in the design of the fault tolerant controller.

5. FTC for LTI plants with control saturation

We consider here the system (1) with input constraints $|u_i| \leq u_{max_i},\ i = 1, ..., m$, and study the following FTC problem.

5.1 Problem statement

Find a bounded feedback controller, i.e. $|u_i| \leq u_{max_i},\ i = 1, ..., m$, such that the closed-loop controlled system (1) admits $x = 0$ as a uniformly asymptotically (UA) stable equilibrium point $\forall\alpha(t)$ $(s.t.\ 0 < \epsilon_1 \leq \alpha_{ii}(t) \leq 1),\ i = 1, ..., m$, within an estimated domain of attraction.

5.2 Problem solution

Under the actuator constraint $|u_i| \leq u_{max_i},\ i = 1, ..., m$, the system (1) can be re-written as

$$\begin{array}{l} \dot{x} = Ax + BU_{max}v \\ y = Kx \\ v = -\alpha(t)sat(y), \end{array} \tag{22}$$

where $U_{max} = diag(u_{max_1}, ..., u_{max_m}),\ sat(y) = (sat(y_1), ..., sat(y_m))^T,\ sat(y_i) = sign(y_i)min\{1, |y_i|\}$.
Thus we have rewritten the system (1) as a MIMO Lure's problem with a generalized sector condition, which is a generalization of the SISO case presented in (16).
Next, we define the two functions $\psi_1 :\ \mathbb{R}^n \rightarrow \mathbb{R}^m,\ \psi_1(x) = -\epsilon_1 I_{m\times m} sat(Kx)$ and

$\psi_2 : \mathbb{R}^n \to \mathbb{R}^m,\ \psi_2(x) = -sat(Kx)$.
We can then write that v is spanned by the two functions $\psi_1,\ \psi_2$:

$$v(x,t) \in co\{\psi_1(x), \psi_2(x)\},\ \forall x \in \mathbb{R}^n, t \in \mathbb{R}, \tag{23}$$

where $co\{\psi_1(x), \psi_2(x)\}$ denotes the convex hull of $\psi_1,\ \psi_2$, i.e.

$$co\{\psi_1(x), \psi_2(x)\} := \{\sum_{i=1}^{i=2} \gamma_i(t)\psi_i(x),\ \sum_{i=1}^{i=2} \gamma_i(t) = 1,\ \gamma_i(t) \geq 0\ \forall t\}.$$

Note that in the SISO case, the problem of analyzing the stability of $x = 0$ for the system (22) under the constraint (23) is a Lure's problem with a generalized sector condition as defined in (16).
Let us recall now some material from (16; 17), that we will use to prove Proposition 4.
Definition 8 ((16), p.538): The ellipsoid level set $\varepsilon(P,\rho) := \{x \in \mathbb{R}^n :\ V(x) = x^T P x \leq \rho\},\ \rho > 0,\ P = P^T > 0$ is said to be contractive invariant for (22) if

$$\dot{V} = 2x^T P(Ax - BU_{max}\alpha sat(Kx)) < 0,$$

for all $x \in \varepsilon(P,\rho)\backslash\{0\},\ \forall t \in \mathbb{R}$.
Proposition 3 *((16), P. 539):* An ellipsoid $\varepsilon(P,\rho)$ is contractively invariant for

$$\dot{x} = Ax + Bsat(Fx),\ B \in \mathbb{R}^{n\times 1}$$

if and only if

$$(A + BF)^T P + P(A + BF) < 0,$$

and there exists an $H \in \mathbb{R}^{1\times n}$ such that

$$(A + BH)^T P + P(A + BH) < 0,$$

and $\varepsilon(P,\rho) \subset \{x \in \mathbb{R}^N :\ |Fx| \leq 1\}$.
Fact 1 ((16), p.539): Given a level set $L_V(\rho) = \{x \in \mathbb{R}^n /\ V(x) \leq \rho\}$ and a set of functions $\psi_i(u),\ i \in \{1,...,N\}$. Suppose that for each $i \in \{1,...,N\}$, $L_V(\rho)$ is contractively invariant for $\dot{x} = Ax + B\psi_i(u)$. Let $\psi(u,t) \in co\{\psi_i(u),\ i \in \{1,...,N\}\}$ for all $u,t \in \mathbb{R}$, then $L_V(\rho)$ is contractively invariant for $\dot{x} = Ax + B\psi(u,t)$.
Theorem 1((17), p. 353): Given an ellipsoid level set $\varepsilon(P,\rho)$, if there exists a matrix $H \in \mathbb{R}^{m\times n}$ such that

$$(A + BM(v,K,H))^T P + P(A + BM(v,K,H)) < 0,$$

for all[1] $v \in \mathcal{V} := \{v \in \mathbb{R}^n | v_i = 1\ or\ 0\}$, and $\varepsilon(P,\rho) \subset \mathcal{L}(H) := \{x \in \mathbb{R}^N :\ |h_i x| \leq 1,\ i = 1,...,m\}$, where

$$M(v,K,H) = \begin{bmatrix} v_1 k_1 + (1 - v_1)h_1 \\ \vdots \\ v_m k_m + (1 - v_m)h_m \end{bmatrix}, \tag{24}$$

then $\varepsilon(P,\rho)$ is a contractive domain for $\dot{x} = Ax + Bsat(Kx)$.
We can now write the following result:
Proposition 4: Under Assumption 1, the system (1) admits $x = 0$ as a UA stable equilibrium

[1] Hereafter, $h_i,\ k_i$ denote the ith line of $H,\ K$, respectively.

point, within the estimated domain of attraction $\varepsilon(P, \rho)$, with the static state feedback $u = Kx = YQ^{-1}x$, where $Y,\ Q$ solve the LMI problem

$$\begin{array}{l} inf_{Q>0,Y,G} J \\ \begin{bmatrix} JR & I \\ I & Q \end{bmatrix} \geq 0,\ J > 0 \\ QA^T + AQ + M(v,Y,G)^T (BU_{max}\alpha_\epsilon)^T + (BU_{max}\alpha_\epsilon) M(v,Y,G) < 0,\ \forall v \in \mathcal{V} \\ QA^T + AQ + M(v,Y,G)^T (BU_{max})^T + (BU_{max}) M(v,Y,G) < 0,\ \forall v \in \mathcal{V} \\ \begin{bmatrix} 1 & g_i \\ g_i^T & Q \end{bmatrix} \geq 0,\ i = 1, ..., m \end{array} \tag{25}$$

where $g_i \in \mathbb{R}^{1\times n}$ is the ith line of G, $\alpha_\epsilon = \epsilon_1 \times I_{m\times m}$, M given by (24), $P = \rho Q^{-1}$, and $R > 0,\ \rho > 0$ are chosen.

Proof: Based on Theorem 1 recalled above, the following inequalities

$$(A + BU_{max}\alpha_\epsilon M(v, -K, H))^T P + P(A + BU_{max}\alpha_\epsilon M(v, -K, H)) < 0, \tag{26}$$

together with the condition $\varepsilon(P, \rho) \subset \mathcal{L}(H)$ are sufficient to ensure that $\varepsilon(P, \rho)$ is contractive invariant for (1) with $\alpha = \epsilon_1 I_{m\times m},\ u = -u_{max} sat(Kx)$.

Again based on Theorem 1, the following inequalities

$$(A + BU_{max} M(v, -K, H))^T P + P(A + BU_{max} M(v, -K, H)) < 0, \tag{27}$$

together with $\varepsilon(P, \rho) \subset \mathcal{L}(H)$ are sufficient to ensure that $\varepsilon(P, \rho)$ is contractive invariant for (1) with $\alpha = I_{m\times m},\ u = -u_{max} sat(Kx)$. Now based on the direct extension to the MIMO case, of Fact 1 recalled above, we conclude that $\varepsilon(P, \rho)$ is contractive invariant for (1) with $u = -u_{max} sat(Kx),\ \forall \alpha_{ii}(t),\ i = 1, ..., m,\ s.t., 0 < \epsilon_1 \leq \alpha_{ii}(t) \leq 1$.

Next, the inequalities conditions (26), (27) under the constraint $\varepsilon(P, \rho) \subset \mathcal{L}(H)$ can be transformed to LMI conditions ((17), p. 355) as follows: To find the control gain K such that we have the bigger estimation of the attraction domain, we can solve the LMI problem

$$\begin{array}{l} inf_{Q>0,Y,G} J \\ \begin{bmatrix} JR & I \\ I & Q \end{bmatrix} \geq 0,\ J > 0 \\ QA^T + AQ + M(v,Y,G)^T (BU_{max}\alpha_\epsilon)^T + (BU_{max}\alpha_\epsilon) M(v,Y,G) < 0,\ \forall v \in \mathcal{V} \\ QA^T + AQ + M(v,Y,G)^T (BU_{max})^T + (BU_{max}) M(v,Y,G) < 0,\ \forall v \in \mathcal{V} \\ \begin{bmatrix} 1 & g_i \\ g_i^T & Q \end{bmatrix} \geq 0,\ i = 1, ..., m, \end{array} \tag{28}$$

where $Y = -KQ,\ Q = (P/\rho)^{-1},\ G = H(P/\rho)^{-1},\ M(v, Y, G) = M(v, -K, H)Q,\ g_i = h_i(P/\rho)^{-1}, h_i \in \mathbb{R}^{1\times n}$ is the ith line of H and $R > 0$ is chosen. □

Remark 1: To solve the problem (25) we have to deal with $2^{m+1} + m + 1$ LMIs, to reduce the number of LMIs we can force $Y = G$, which means $K = -H(P/\rho)^{-1}Q^{-1}$. Indeed, in this case the second and third conditions in (25) reduce to the two LMIs

$$\begin{array}{l} QA^T + AQ + G^T (BU_{max}\alpha_\epsilon)^T + (BU_{max}\alpha_\epsilon) G < 0 \\ QA^T + AQ + G^T (BU_{max})^T + (BU_{max}) G < 0, \end{array} \tag{29}$$

which reduces the total number of LMIs in (25) to $m + 3$. ⧫

In the next section, we report some results in the extension of the previous linear controllers to single input nonlinear affine plants.

6. FTC for nonlinear single input affine plants

Let us consider now the nonlinear affine system

$$\dot{x} = f(x) + g(x)\alpha u, \tag{30}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}$ represent, respectively, the state vector and the scalar input. The vector fields f, columns of g are supposed to satisfy the classical smoothness assumptions, with $f(0) = 0$. The fault coefficient is such that $0 < \epsilon_1 \leq \alpha \leq 1$.

6.1 Problem statement

Find a state feedback controller $u(x)$ such that the closed-loop controlled system (44) admits $x = 0$ as a local (global) asymptotically stable equilibrium point $\forall \alpha$ $(s.t.\ 0 < \epsilon_1 \leq \alpha \leq 1)$.

6.2 Problem solution

We follow here the same idea used above for the linear case, and associate with the faulty system (44) a virtual scalar output, the corresponding system writes as

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha u \\ y &= k(x), \end{aligned} \tag{31}$$

where $k : \mathbb{R} \to \mathbb{R}$ is a continuous function.
Let us chose now the controller as the simple output feedback

$$u = -k(x). \tag{32}$$

We can then write from (31) and (32) the closed-loop system as

$$\begin{aligned} \dot{x} &= f(x) + g(x)v \\ y &= k(x) \\ v &= -\alpha y. \end{aligned} \tag{33}$$

As before we have cast the problem of stabilizing (44), for all α as an absolute stability problem (33) as defined in ((40), p.55). We can then use the absolute stability theory to solve the problem.
Proposition 5: The closed-loop system (44) with the static state feedback

$$u = -k(x), \tag{34}$$

where k is such that there exist a C^1 function $S : \mathbb{R}^n \to \mathbb{R}$ positive semidefinite, radially unbounded, i.e. $S(x) \to +\infty$, $||x|| \to +\infty$, that satisfies the PDEs

$$\begin{aligned} L_f S(x) &= -0.5 q^T(x) q(x) + \left(\frac{\epsilon_1}{1-\epsilon_1}\right) k^2(x) \\ L_g S(x) &= \left(\frac{1+\epsilon_1}{1-\epsilon_1}\right) k(x) - q^T w, \end{aligned} \tag{35}$$

where the function $w : \mathbb{R}^n \to \mathbb{R}^l$ is s.t. $w^T w = \frac{2}{1-\epsilon_1}$, and $q : \mathbb{R}^n \to \mathbb{R}^l$, $l \in \mathbb{N}$, under the condition of local (global) detectability of the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)v \\ y &= k(x), \end{aligned} \tag{36}$$

admits the origin $x = 0$ as a local (global) asymptotically stable equilibrium point.
Proof: We saw the equivalence between the problem of stabilizing (44), and the absolute stability problem (33), with the 'nonlinearities' sector bounds ϵ_1 and 1. Based on this, we can use the sufficient condition provided in Proposition 2.38 in ((40), p. 55) to ensure the absolute stability of the origin $x = 0$ of (33), for all $\alpha \in [\epsilon_1,\ 1]$.
First we have to ensure that the parallel interconnection of the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)v \\ y &= k(x), \end{aligned} \tag{37}$$

with the trivial unitary gain system

$$y = v, \tag{38}$$

is OFP($-\tilde{k}$), with $\tilde{k} = \frac{\epsilon_1}{1-\epsilon_1}$ and with a C^1 radially unbounded storage function S.
Based on Definition 4, this is true if the parallel interconnection of (37) and (38) is dissipative with respect to the supply rate

$$\omega(v, \tilde{y}) = v^T \tilde{y} + \left(\frac{\epsilon_1}{1-\epsilon_1}\right) \tilde{y}^T \tilde{y}, \tag{39}$$

where $\tilde{y} = y + v$. This means, based on Definition 3, that it exists a C^1 function $S : \mathbb{R}^n \to \mathbb{R}$, with $S(0) = 0$ and $S(x) \geq 0,\ \forall x$, s.t.

$$\begin{aligned} \dot{S}(x(t)) &\leq \omega(v, \tilde{y}) \\ &\leq v^T y + ||v||^2 + \left(\tfrac{\epsilon_1}{1-\epsilon_1}\right) ||y + v||^2. \end{aligned} \tag{40}$$

Furthermore, S should be radially unbounded.
From the condition (40) and Theorem 2.39 in ((40), p. 56), we can write the following condition on $S,\ k$ for the dissipativity of the parallel interconnection of (37) and (38) with respect to the supply rate (39):

$$\begin{aligned} L_f S(x) &= -0.5 q^T(x) q(x) + \left(\tfrac{\epsilon_1}{1-\epsilon_1}\right) k^2(x) \\ L_g S(x) &= k(x) + 2\left(\tfrac{\epsilon_1}{1-\epsilon_1}\right) k(x) - q^T w, \end{aligned} \tag{41}$$

where the function $w : \mathbb{R}^n \to \mathbb{R}^l$ is s.t. $w^T w = \frac{2}{1-\epsilon_1}$, and $q : \mathbb{R}^n \to \mathbb{R}^l,\ l \in \mathbb{N}$. Finally, based on Proposition 2.38 in ((40), p. 55), to ensure the local (global) asymptotic stability of $x = 0$, the system (37) has to be locally (globally) ZSD, which is imposed by the local (global) detectability of (36). □
Solving the condition (41) might be computationally demanding, since it requires to solve a system of PDEs. We can simplify the static state feedback controller, by considering a lower bound of the condition (40). Indeed, condition (40) is true if the inequality

$$\dot{S} \leq v^T y, \tag{42}$$

is satisfied. Thus, it suffices to ensure that the system (37) is passive with the storage function S. Now, based again on the necessary and sufficient condition given in Theorem 2.39 ((40), p.56), the storage function and the feedback gain have to satisfy the condition

$$\begin{aligned} L_f S(x) &\leq 0 \\ L_g S(x) &= k(x). \end{aligned} \tag{43}$$

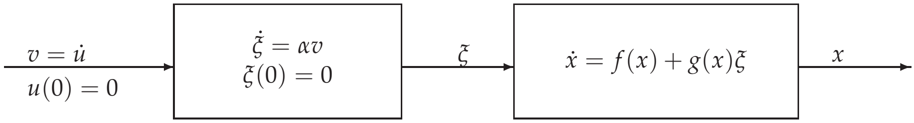

Fig. 1. The model (45) in cascade form

However, by considering a lower bound of (40), we are considering the extreme case where $\epsilon_1 \rightarrow 0$, which may result in a conservative feedback gain (refer to (6)). It is worth noting that in that case the controller given by (52), (41), reduces to the classical damping or Jurdjevic-Quinn control $u = -L_g S(x)$, e.g. ((40), p. 111), but based on a semidefinite function S.

7. FTC for nonlinear multi-input affine plants with constant loss of effectiveness actuator faults

We consider here affine nonlinear models of the form

$$\dot{x} = f(x) + g(x)u, \tag{44}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ represent respectively the state and the input vectors. The vector fields f, and the columns of g are assumed to be C^1, with $f(0) = 0$.
We study actuator's faults modelled by a multiplicative constant coefficient, i.e. a loss of effectiveness, which implies the following form for the faulty model[2]

$$\dot{x} = f(x) + g(x)\alpha u, \tag{45}$$

where $\alpha \in \mathbb{R}^{m \times m}$ is a diagonal constant matrix, with the diagonal elements $\alpha_{ii},\ i = 1, ..., m$ s.t., $0 < \epsilon_1 \leq \alpha_{ii} \leq 1$. We write then the FTC problem as follows.
Problem statement: *Find a feedback controller such that the closed-loop controlled system (45) admits $x = 0$ as a globally asymptotically stable (GAS) equilibrium point $\forall \alpha$ $(s.t.\ 0 < \epsilon_1 \leq \alpha_{ii} \leq 1)$.*

7.1 Problem solution

Let us first rewrite the faulty model (45) in the following cascade form (see figure 1)

$$\begin{aligned} \dot{x} &= f(x) + g(x)h(\xi) \\ \dot{\xi} &= \alpha v,\ \xi(0) = 0 \\ y &= h(\xi) = \xi, \end{aligned} \tag{46}$$

where we define the virtual input $v = \dot{u}$ with $u(0) = 0$. This is indeed, a cascade form where the controlling subsystem, i.e. ξ dynamics, is linear (40). Using this cascade form, it is possible to write a stabilizing controller for the faulty model (45), as follows.

[2] Hereafter, we will denote by x the states of the faulty system (45) to avoid cumbersome notations. However, we remind the reader that the solutions of the healthy system (44) and the faulty system (45) are different.

Theorem 2: Consider the closed-loop system that consists of the faulty system (45) and the dynamic state feedback

$$\begin{aligned} \dot{u} &= -L_g W(x)^T - k\xi,\ u(0) = 0 \\ \dot{\xi} &= \epsilon_1(-(L_g W(x))^T - k\xi),\ \xi(0) = 0, \end{aligned} \tag{47}$$

where W is a C^1 radially unbounded, positive semidefinite function, s.t. $L_f W \leq 0$, and $k > 0$. Consider the fictitious system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= \epsilon_1(-(L_g W)^T + \tilde{v}) \\ y &= h(\xi) = \xi. \end{aligned} \tag{48}$$

If the system (48) is (G)ZSD with the input $\tilde{v}$ and the output y, then the closed-loop system (45) with (47) admits the origin $(x, \xi) = (0, 0)$ as (G)AS equilibrium point.
Proof: We first prove that the cascade system (48) is passive from $\tilde{v}$ to $y = \xi$.
To do so, let us first consider the linear part of the cascade system

$$\begin{aligned} \dot{\xi} &= \epsilon_1 v,\ \xi(0) = 0 \\ y &= h(\xi) = \xi. \end{aligned} \tag{49}$$

The system (49) is passive, with the C^1 positive definite, radially unbounded, storage function $U(\xi) = \frac{1}{2}\xi^T\xi$. Indeed, we can easily see that $\forall\ T > 0$

$$\begin{aligned} U(\xi(T)) = \frac{1}{2}\xi^T(T)\xi(T) &\leq \int_0^T v^T y dt \\ &\leq \frac{1}{\epsilon_1}\int_0^T \dot{\xi}^T \xi dt \\ &\leq \frac{1}{\epsilon_1}\int_{\xi(0)}^{\xi(T)} \xi^T d\xi \leq \frac{1}{2}\frac{1}{\epsilon_1}\xi^T(T)\xi(T), \end{aligned}$$

which is true for $0 < \epsilon_1 \leq 1$
Next, we can verify that the nonlinear part of the cascade

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ y &= L_g W(x), \end{aligned} \tag{50}$$

is passive, with the C^1 radially unbounded, positive semidefinite storage function W. Since, $\dot{W} = L_f W + L_g W\xi \leq L_g W\xi$. Thus we have proved that both the linear and the nonlinear parts of the cascade are passive , we can then conclude that the feedback interconnection (48) of (49) and (50) (see figure 2) is passive from the new input $\tilde{v} = L_g W + v$ to the output ξ, with the storage function $S(x, \xi) = W(x) + U(\xi)$ (see Theorem 2.10 in (40), p. 33).
Finally, the passivity associated with the (G)ZSD implies that the control $\tilde{v} = -k\xi,\ k > 0$ achieves (G)AS (Theorem 2.28 in (40), p. 49).
Up to now, we proved that the negative feedback output feedback $\tilde{v} = -k\xi,\ k > 0$ achieves the desired AS for $\alpha = \epsilon_1 I_{m\times m}$. We have to prove now that the result holds for all α s.t. $0 < \epsilon_1 \leq \alpha_{ii} \leq 1$, even if ξ is fed back from the fault's model (46) with $\alpha = \epsilon_1 I_{m\times m}$, since we do not know the actual value of α. If we multiply the control law (47) by a constant gain matrix $\tilde{k} = diag(\tilde{k}_1, ..., \tilde{k}_m),\ 1 \leq \tilde{k}_i \leq \frac{1}{\epsilon_1}$, we can write the new control as

$$\begin{aligned} \dot{u} &= -\tilde{k}(L_g W(x)^T - k\xi),\ k > 0,\ u(0) = 0 \\ \dot{\xi} &= \epsilon_1\tilde{k}(-(L_g W(x))^T - k\xi),\ \xi(0) = 0. \end{aligned} \tag{51}$$

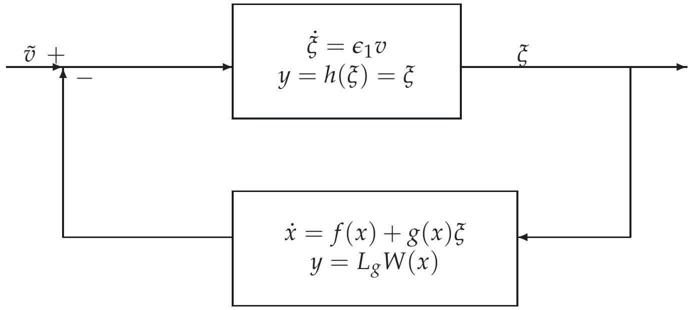

Fig. 2. Feedback interconnection of (49) and (50)

It is easy to see that this gain does not change the stability result, since we can define for the nonlinear cascade part (50) the new storage function $\tilde{W} = \tilde{k}W$ and the passivity is still satisfied from its input ξ to the new output $\tilde{k}L_gW(x)$. Next, since the ZSD property remains unchanged, we can chose the new stabilizing output feedback $\tilde{v} = -\tilde{k}k\xi$, which is still a stabilizing feedback with the new gain $\tilde{k}k > 0$, and thus the stability result holds for all $\alpha\ s.t.\ 0 < \epsilon_1 \leq \alpha_{ii} \leq 1,\ i = 1, ..., m.$ □

The stability result obtained in Theorem 2, depends on the ZSD property. Indeed, if the ZSD is global, the stability obtained is global otherwise only local stability is ensured. Furthermore, we note here that with the dynamic controller (47) we ensure that the initial control is zero, regardless of the initial value of the states. This might be important for practical applications, where an abrupt switch from zero to a non zero initial value of the control is not tolerated by the actuators.

In Theorem 2, one of the necessary conditions is the existence of $W \geq 0$, s.t. the uncontrolled part of (45) satisfies $L_fW \leq 0$. To avoid this condition that may not be satisfied for some practical systems, we propose the following Theorem.

Theorem 3: Consider the closed-loop system that consists of the faulty system (45) and the dynamic state feedback

$$\begin{aligned}\dot{u} &= \tfrac{1}{\epsilon_1}(-k(\xi - \beta K(x)) - \beta L_g W^T + \beta \tfrac{\partial K}{\partial x}(f + g\xi)),\ \beta = diag(\beta_{11}, ..., \beta_{mm}),\ 0 < \tfrac{\tilde{\epsilon}_1}{\epsilon_1} \leq \beta_{ii} \leq 1,\\ \dot{\xi} &= -k(\xi - \beta K(x)) - \beta L_g W^T + \beta \tfrac{\partial K}{\partial x}(f + g\xi),\ \xi(0) = 0,\ u(0) = 0,\end{aligned} \tag{52}$$

where $k > 0$ and the C^1 function $K(x)$ is s.t. there exists a C^1 radially unbounded, positive semidefinite function W satisfying

$$\frac{\partial W}{\partial x}(f(x) + g(x)\beta K(x)) \leq 0,\ \forall x \in \mathbb{R}^n,\ \forall \beta = diag(\beta_{11}, ..., \beta_{mm}),\ 0 < \tilde{\epsilon}_1 \leq \beta_{ii} \leq 1. \tag{53}$$

Consider the fictitious system

$$\begin{aligned}\dot{x} &= f(x) + g(x)\xi\\ \dot{\xi} &= \beta \tfrac{\partial K}{\partial x}(f + g\xi) - \beta L_g W^T + \tilde{\tilde{v}}\\ \tilde{y} &= \xi - \beta K(x).\end{aligned} \tag{54}$$

If (54) is (G)ZSD with the input $\tilde{\tilde{v}}$ and the output $\tilde{y}$, for for all β s.t. $\beta_{ii},\ i = 1, ..., m, 0 < \tilde{\epsilon}_1 \leq \beta_{ii} \leq 1$. Then, the closed-loop system (45) with (52) admits the origin $(x, \xi) = (0, 0)$ as (G)AS

equilibrium point.
Proof: We will first prove that the controller (52) achieves the stability results for a faulty model with $\alpha = \epsilon_1 I_{m\times m}$ and then we will prove that the stability result holds the same for all α s.t. $\alpha_{ii},\ i = 1, ..., m, 0 < \epsilon_1 \leq \alpha_{ii} \leq 1$.
First, let us define the virtual output $\tilde{y} = \xi - \beta K(x)$, we can write the model (46) with $\alpha = \epsilon_1 I_{m\times m}$ as

$$\begin{aligned} \dot{x} &= f(x) + g(x)(\tilde{y} + \beta K(x)) \\ \dot{\xi} &= \epsilon_1 I_{m\times m} v \\ \tilde{y} &= \xi - \beta K(x), \end{aligned} \tag{55}$$

we can then write

$$\dot{\tilde{y}} = \epsilon_1 I_{m\times m} v - \beta \frac{\partial K}{\partial x}(f + g(\tilde{y} + \beta K(x))) = \tilde{v}.$$

To study the passivity of (55), we define the positive semidefinite storage function

$$V = \beta W(x) + \frac{1}{2}\tilde{y}^T\tilde{y},$$

and write

$$\dot{V} = \beta L_{f+g\beta K} W + \beta L_g W \tilde{y} + \tilde{y}^T \tilde{v},$$

and using the condition (53), we can write

$$\dot{V} \leq \tilde{y}^T(\beta L_g W^T + \tilde{v}),$$

which establishes the passivity of (55) from the new input $\tilde{\tilde{v}} = \tilde{v} + \beta L_g W^T$ to the output $\tilde{y}$.
Finally, using the (G)ZSD condition for $\alpha = \epsilon_1 I_{m\times m}$, we conclude about the (G)AS of (46) for $\alpha = \epsilon_1 I_{m\times m}$, with the controller (52) (Theorem 2.28 in (40), p. 49). Now, remains to prove that the same result holds for all α s.t. $\alpha_{ii},\ i = 1, ..., m,\ 0 < \epsilon_1 \leq \alpha_{ii} \leq 1$, i.e. the controller (52) has the appropriate gain margin. In our particular case, it is straightforward to analyse the gain margin of (52), since if we multiply the controller in (52) by a matrix α, s.t. $\alpha_{ii},\ i = 1, ..., m$, $0 < \epsilon_1 \leq \alpha_{ii} \leq 1$, the new control writes as

$$\begin{aligned} \dot{u} &= \tfrac{1}{\epsilon_1} I_{m\times m}(-\alpha k(\xi - \beta K(x)) - \alpha\beta L_g W^T + \alpha\beta \tfrac{\partial K}{\partial x}(f + g\xi)), \\ & k > 0,\ \beta = diag(\beta_1, ..., \beta_m),\ 0 < \beta_i \leq 1, \\ \dot{\xi} &= \alpha k(\xi - \beta K(x)) - \alpha\beta L_g W^T + \alpha\beta \tfrac{\partial K}{\partial x}(f + g\xi),\ \xi(0) = 0,\ u(0) = 0. \end{aligned} \tag{56}$$

We can see that this factor will not change the structure of the initial control (52), since it will be directly absorbed by the gains, i.e. we can write $\tilde{k} = \alpha k$, with all the elements of diagonal matrix $\tilde{k}$ positive, we can also define $\tilde{\beta} = \alpha\beta$ which is still a diagonal matrix with bounded elements in $[\tilde{\epsilon}_1, 1]$, s.t. (53) and (54) are is still satisfied. Thus the stability result remains unchanged. □
The previous theorems may guaranty *global* AS. However, the conditions required may be difficult to satisfy for some systems. We present below a control law ensuring, under less demanding conditions, semiglobal stability instead of global stability.
Theorem 4: Consider the closed-loop system that consists of the faulty system (45) and the dynamic state feedback

$$\begin{aligned} \dot{u} &= -k(\xi - u_{nom}(x)),\ k > 0, \\ \dot{\xi} &= -k\epsilon_1(\xi - u_{nom}(x)),\ \xi(0) = 0,\ u(0) = 0, \end{aligned} \tag{57}$$

where the nominal controller $u_{nom}(x)$ achieves semiglobal asymptotic and local exponential stability of $x = 0$ for the safe system (44). Then, the closed-loop (45) with (57) admits the origin $(x, \xi) = (0, 0)$ as semiglobal AS equilibrium point.

Proof: The prove is a direct application of the Proposition 6.5 in ((40), p. 244), to the system (46), with $\alpha = \epsilon_1 I_{m\times m}$. Any positive gain α, s.t. $1 \leq \alpha_{ii} \leq \frac{1}{\epsilon_1},\ i = 1, ..., m$, will be absorbed by $k > 0$, keeping the stability results unchanged. Thus the control law (57) stabilize (46) and equivalently (45) for all α s.t. $\alpha_{ii},\ i = 1, ..., m, 0 < \epsilon_1 \leq \alpha_{ii} \leq 1$. □

Let us consider now the practical problem of input saturation. Indeed, in practical systems the actuator powers are limited, and thus the control amplitude bounds should be taken into account in the controller design. To solve this problem, we consider a more general model than the affine model (44). In the following we first study the problem of FTC with input saturation, on the general model

$$\dot{x} = f(x) + g(x, u)u, \tag{58}$$

where, $x,\ u,\ f$ are defined as before, g is now function of both the states and the inputs, and is assumed to be C^1 w.t.r. to x, u.

The actuator faut model, writes as

$$\dot{x} = f(x) + g(x, \alpha u)\alpha u, \tag{59}$$

with the loss of effectiveness matrix α defined as before. This problem is treated in the following Theorem, for the scalar case where $\alpha \in [\epsilon_1,\ 1]$, i.e. when the same fault occurs on all the actuators.

Theorem 5: Consider the closed-loop system that consists of the faulty system (59), for $\alpha \in [\epsilon_1,\ 1]$, and the static state feedback

$$\begin{array}{c} u(x) = -\lambda(x)G(x,0)^T \\ G(x,0) = \frac{\partial W(x)}{\partial x}\epsilon_1 g(x,0) \\ \lambda(x) = \frac{2\overline{u}}{(1+\gamma_1(|x|^2+4\overline{u}^2|G(x,0)|^2))(1+|G(x,0)|^2)} > 0 \\ \gamma_1 = \int_0^{2s} \frac{\overline{\gamma}_1(s)}{1+\overline{\gamma}_1(1)} ds \\ \overline{\gamma}_1(s) = \frac{1}{s}\int_s^{2s}(\tilde{\gamma}_1(t) - 1)dt + s \\ \tilde{\gamma}_1(s) = max_{\{(x,u)||x|^2+|u|^2\leq s\}}\{1 + \int_0^1 \frac{\partial W(x)}{\partial x}\frac{\partial g(x,\tau\epsilon_1 u)}{\partial u}d\tau\}, \end{array} \tag{60}$$

where W is a C^2 radially unbounded, positive semidefinite function, s.t. $L_f W \leq 0$. Consider the fictitious system

$$\begin{array}{c} \dot{x} = f(x) + g(x, \epsilon_1 u)\epsilon_1 u \\ y = \frac{\partial W(x)}{\partial x}\epsilon_1 g(x, \epsilon_1 u). \end{array} \tag{61}$$

If (61) is (G)ZSD, then the closed-loop system (59) with (60) admits the origin as (G)AS equilibrium point. Furthermore $|u(x)| \leq \overline{u},\ \forall x$.

Proof: Let us first consider the faulty model (59) with $\alpha = \epsilon_1$. For this model, we can compute the derivative of W as

$$\dot{W}(x) = L_f W + \frac{\partial W(x)}{\partial x}\epsilon_1 g(x, \epsilon_1 u)u$$

$$\dot{W}(x) \leq \frac{\partial W(x)}{\partial x}\epsilon_1 g(x, \epsilon_1 u)u.$$

Now, using Lemma II.4 (p.1562 in (31)), we can directly write the controller (60), s.t.

$$\dot{W} \leq -\frac{1}{2}\lambda(x)|G(x,0)|^2.$$

Furthermore $|u(x)| \leq \overline{u}, \ \forall x$.
We conclude then that the trajectories of the closed-loop equations converge to the invariant set $\{x| \ \lambda(x)|G(x,0)|^2 = 0\}$ which is equivalent to the set $\{x| \ G(x,0) = 0\}$. Based on Theorem 2.21 (p. 43, (40)), and the assumption of (G)ZSD for (61), we conclude about the (G)AS of the origin of (59), (60), with $\alpha = \epsilon_1$. Now multiplying u by any positive coefficient α, s.t. $0 < \epsilon_1 \leq \alpha \leq 1$ does not change the stability result. Furthermore, if $|u(x)| \leq \overline{u}, \ \forall x$, then $|\alpha u(x)| \leq \overline{u}, \ \forall x$, which completes the proof. □
Remark 2: In Theorem 5, we consider only the case of scalar fault $\alpha \in [\epsilon_1, \ 1]$, i.e. the case of uniform fault, since we need this assumption to be able to apply the result of Lemma II.4 in (31). However, this assumption can be satisfied in practice by a class of actuators, namely pneumatically driven diaphragm-type actuators (23), for which the failure of the pressure supply system might lead to a uniform fault of all the actuators. Furthermore, in Proposition 6 below we treat for the case of systems affine in the control, i.e. $g(x,u) = g(x)$, the general case of any diagonal matrix of loss of effectiveness coefficients.
Proposition 6: Consider the closed-loop system that consists of the faulty system (45), and the static state feedback

$$\begin{aligned} u(x) &= -\lambda(x)G(x)^T \\ G(x) &= \tfrac{\partial W(x)}{\partial x}\epsilon_1 g(x) \\ \lambda(x) &= \tfrac{2\overline{u}}{1+|G(x)|^2}. \end{aligned} \tag{62}$$

where W is a C^2 radially unbounded, positive semidefinite function, s.t. $L_f W \leq 0$. Consider the fictitious system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\epsilon_1 u \\ y &= \tfrac{\partial W(x)}{\partial x}\epsilon_1 g(x). \end{aligned} \tag{63}$$

If (63) is (G)ZSD, then the closed-loop system (45) with (62) admits the origin as (G)AS equilibrium point. Furthermore $|u(x)| \leq \overline{u}, \ \forall x$.
Proof: The proof follows the same steps as in the proof of Theorem 5, except that in this case the constraint of considering that the same fault occurs on all the actuators, i.e. for a scalar α, is relaxed. Indeed, in this case we can directly ensure the negativeness of $\dot{W}$, since if u is such that $\dot{W} \leq -\lambda(x)L_gW(x)\epsilon_1 L_gW(x)^T \leq 0$, then in the case of a diagonal fault matrix, the derivative writes as $\dot{W} \leq -\lambda(x)L_gW(x)\epsilon_1\alpha L_gW(x)^T \leq -\lambda(x)\epsilon_1^2 L_gW(x)L_gW(x)^T \leq -\epsilon_1^2\lambda(x)|G(x)|^2$. Thus, the stability result remains unchanged. □
Up to now we have considered the case of abrupt faults, modelled with constant loss of effectiveness matrices. However, in practical applications, the faults are usually time-varying or incipient, modelled with time-varying loss of effectiveness coefficients, e.g. (50). We consider in the following section this case of time-varying loss of effectiveness matrices.

8. FTC for nonlinear multi-input affine plants with time-varying loss of effectiveness actuator faults

We consider here faulty models of the form

$$\dot{x} = f(x) + g(x)\alpha(t)u, \tag{64}$$

where $\alpha(t)$ is a diagonal time-varying matrix, with C^1 diagonal elements $\alpha_{ii}(t), \ i = 1, ..., m$ s.t., $0 < \epsilon_1 \leq \alpha_{ii}(t) \leq 1, \ \forall t$. We write then the FTC problem as follows.
Problem statement: *Find a feedback controller such that the closed-loop controlled system (64) admits $x = 0$ as a uniformly asymptotically stable (UAS) equilibrium point $\forall \alpha(t)$ $(s.t. \ 0 < \epsilon_1 \leq \alpha_{ii}(t) \leq 1)$.*

8.1 Problem solution

To solve this problem we use some of the tools introduced in (24), where a generalization of Krasovskii-LaSalle theorem, has been proposed for nonlinear time-varying systems.
We can first write the following result.
Theorem 6: Consider the closed-loop system that consists of the faulty system (64) with the dynamic state feedback

$$\begin{aligned} \dot{u} &= -L_g W(x)^T - k\xi,\ k > 0,\ u(0) = 0 \\ \dot{\xi} &= \tilde{\alpha}(t)(-(L_g W(x))^T - k\xi),\ \xi(0) = 0, \end{aligned} \tag{65}$$

where $\tilde{\alpha}(t)$ is a C^1 function, s.t. $0 < \epsilon_1 \leq \tilde{\alpha}(t) \leq 1,\ \forall t$, and W is a C^1, positive semidefinite function, such that:
1- $L_f W \leq 0$,
2- The system $\dot{x} = f(x)$ is AS conditionally to the set $M = \{x \mid W(x) = 0\}$,
3- $\forall(\overline{x}, \overline{\xi})$ limiting solutions for the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= \alpha(t)(-(L_g W)^T - k\xi) \\ y &= h(x, \xi) = \xi, \end{aligned} \tag{66}$$

w.r.t. unbounded sequence $\{t_n\}$ in $[0, \infty)$, then if $h(\overline{x}, \overline{\xi}) = 0$, a.e., then either $(\overline{x}, \overline{\xi})(t_0) = (0, 0)$ for some $t_0 \geq 0$ or $(0, 0)$ is a ω-limit point of $(\overline{x}, \overline{\xi})$, i.e. $\lim_{t\to\infty}(\overline{x}, \overline{\xi})(t) \to (0, 0)$.
Then the closed-loop system (64) with (65) admits the origin $(x, \xi) = (0, 0)$ as UAS equilibrium point.
Proof: Let us first rewrite the system (64) for $\alpha(t) = \tilde{\alpha}(t)$, in the cascade form

$$\begin{aligned} &\dot{x} = f(x) + g(x)h(\xi) \\ &\dot{\xi} = \tilde{\alpha}(t)v,\ v = \dot{u},\ \xi(0) = 0,\ u(0) = 0 \\ &y = h(\xi) = \xi. \end{aligned} \tag{67}$$

Replacing $v = \dot{u}$ by its value in (65) gives the feedback system

$$\begin{aligned} &\dot{x} = f(x) + g(x)h(\xi) \\ &\dot{\xi} = \tilde{\alpha}(t)(-L_g W(x)^T + \tilde{v}),\ \xi(0) = 0,\ u(0) = 0 \\ &y = h(\xi) = \xi. \end{aligned} \tag{68}$$

We prove that (68) is passive from the input $\tilde{v}$ to the output ξ. We consider first the linear part of (67)

$$\begin{aligned} \dot{\xi} &= \tilde{\alpha}(t)v,\ \xi(0) = 0 \\ y &= h(\xi) = \xi, \end{aligned} \tag{69}$$

which is passive with the storage function $U(\xi) = \frac{1}{2}\xi^T\xi$, i.e. $\dot{U}(t, \xi) = \xi^T\dot{\xi} = \xi^T\tilde{\alpha}(t)v \leq \xi^T v = v^T\xi$.
Next, we consider the nonlinear part

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ y &= L_g W(x), \end{aligned} \tag{70}$$

which is passive with the storage function $W(x)$, s.t. $\dot{W} = L_f W + L_g W\xi \leq L_g W\xi$.
We conclude that the feedback interconnection (68) of (69) and (70) is passive from $\tilde{v}$ to ξ, with

the storage function $S(x, \xi) = W(x) + U(\xi)$ (see Theorem 2.10, p. 33 in (40)).
This implies that the derivative of S along (68) with $\tilde{v} = -k\xi,\ k > 0$, writes

$$\dot{S}(t, x, \xi) \leq \tilde{v}^T \xi \leq 0.$$

Now we define for (68) with $\tilde{v} = -k\xi,\ k > 0$, the positive invariant set

$$M = \{(x, \xi) | W(x) + U(\xi) = 0\}$$

$$M = \{(x, 0) | W(x) = 0\}.$$

We note that the restriction of (68) with $\tilde{v} = -k\xi,\ k > 0$ on M is $\dot{x} = f(x)$, then applying Theorem 5 in (18), we conclude that, under Condition 2 of Theorem 6, the origin $(x, \xi) = (0, 0)$ is US for the system (64) for $\alpha = \tilde{\alpha}$ and the dynamic controller (65). Now, multiplying u by any $\alpha(t)$, s.t. $0 < \epsilon_1 \leq \alpha_{ii}(t) \leq 1,\ \forall t$, does not change neither the passivity property, nor the AS condition of $\dot{x} = f(x)$ on M, which implies the US of $(x, \xi) = (0, 0)$ for (64), (65) $\forall \alpha(t)$, s.t. $0 < \epsilon_1 \leq \alpha_{ii}(t) \leq 1,\ \forall t$.
Now we first note the following fact: for any $\sigma > 0$ and any $t \geq t_0$ we can write

$$S(t, x(t), \xi(t)) - S(t_0, x(t_0), \xi(t_0)) \leq -\int_{t_0}^{t} \mu(h(\xi(\tau)))d\tau = -\int_{t_0}^{t} k|\xi(\tau)|^2 d\tau,$$

thus we have

$$\int_{t_0}^{t} (\mu(h(\xi(\tau))) - \sigma)d\tau \leq \int_{t_0}^{t} \mu(h(\xi(\tau)))d\tau \leq S(t_0, x(t_0), \xi(t_0)) < \tilde{M}; \tilde{M} > 0.$$

Finally, using Theorem 1 in (24), under Condition 3 of Theorem 6, we conclude that $(x, \xi) = (0, 0)$ is UAS for (64), (65). □

Remark 3: The function $\tilde{\alpha}$ in (65) has been chosen to be any C^1 time varying function, s.t. $0 < \epsilon_1 \leq \tilde{\alpha}(t) \leq 1,\ \forall t$. The general time-varying nature of the function was necessary in the proof to be able to use the results of Theorem 5 in (18) to prove the US of the faulty system's equilibrium point. However, in practice one can simply chose $\tilde{\alpha}(t) = 1,\ \forall t$ ♦.

Remark 4: Condition 3 in Theorem 6 is general and has been used to properly prove the stability results in the time-varying case. However, in practical application it can be further simplified, using the notion of reduced limiting system. Indeed, using Theorem 3 and Lemma 7 in (24), Condition 3 simplifies to:
$\forall (\overline{x}, \overline{\xi})$ solutions for the reduced limiting system

$$\begin{aligned} \dot{\overline{x}} &= f(\overline{x}) + g(\overline{x})\overline{\xi} \\ \dot{\overline{\xi}} &= \alpha_\gamma(t)(-(L_g W(\overline{x}))^T - k\overline{\xi}) \\ y &= h(\overline{x}, \overline{\xi}) = \overline{\xi}, \end{aligned} \tag{71}$$

where the limiting function $\alpha_\gamma(t)$ is defined us $\alpha_\gamma(t) \overset{\triangle}{=} \lim_{n \to \infty} \alpha(t + t_n)$ w.r.t. unbounded sequence $\{t_n\}$ in $[0, \infty)$. Then, if $h(\overline{x}, \overline{\xi}) = 0$, a.e., then either $(\overline{x}, \overline{\xi})(t_0) = (0, 0)$ for some $t_0 \geq 0$ or $(0, 0)$ is a ω-limit point of $(\overline{x}, \overline{\xi})$. Now, since in our case the diagonal matrix-valued function α is s.t. $0 < \epsilon_1 \leq \alpha_{ii}(t) \leq 1,\ \forall t$, then it obviously satisfies a permanent excitation (PE) condition of the form

$$\int_{t}^{t+T} \alpha(\tau)\alpha(\tau)^T d\tau \geq rI,\ T > 0,\ r > 0,\ \forall t,$$

which implies, based on Lemma 8 in (24), that to check Condition 3 we only need to check the classical ZSD condition:
$\forall \overline{x}$ solutions for the system

$$\begin{array}{c} \dot{\overline{x}} = f(\overline{x}) \\ L_g W(\overline{x}) = 0, \end{array} \tag{72}$$

either $\overline{x}(t_0) = 0$ for some $t_0 \geq 0$ or 0 is a ω-limit point of $\overline{x}$. ♦
Let us consider again the problem of input saturation. We consider here again the more general model (58), and study the problem of FTC with input saturation for the time-varying faulty model

$$\dot{x} = f(x) + g(x, \alpha(t)u)\alpha(t)u, \tag{73}$$

with the diagonal loss of effectiveness matrix $\alpha(t)$ defined as before. This problem is treated in the following Theorem, for the scalar case where $\alpha(t) \in [\epsilon_1,\ 1],\ \forall t$, i.e. when the same fault occurs on all the actuators.
Theorem 7: Consider the closed-loop system that consists of the faulty system (73) for $\alpha \in [\epsilon_1,\ 1],\ \forall t$, with the static state feedback

$$\begin{array}{c} u(x) = -\lambda(x)G(x,0)^T \\ G(x,0) = \frac{\partial W(x)}{\partial x} g(x,0) \\ \lambda(x) = \frac{2\overline{u}}{(1+\gamma_1(|x|^2+4\overline{u}^2|G(x,0)|^2))(1+|G(x,0)|^2)} > 0 \\ \gamma_1 = \int_0^{2s} \frac{\overline{\gamma}_1(s)}{1+\overline{\gamma}_1(1)} ds \\ \overline{\gamma}_1(s) = \frac{1}{s}\int_s^{2s} (\tilde{\gamma}_1(t) - 1)dt + s \\ \tilde{\gamma}_1(s) = max_{\{(x,u) | |x|^2+|u|^2 \leq s\}}\{1 + \int_0^1 \frac{\partial W(x)}{\partial x}\frac{\partial g(x,\tau\epsilon_1 u)}{\partial u} d\tau\}, \end{array} \tag{74}$$

where W is a C^2, positive semidefinite function, such that:
1- $L_f W \leq 0$,
2- The system $\dot{x} = f(x)$ is AS conditionally to the set $M = \{x \mid W(x) = 0\}$,
3- $\forall \overline{x}$ limiting solutions for the system

$$\begin{array}{c} \dot{x} = f(x) + g(x, \epsilon_1 u(x))(-\lambda(x)\alpha(t)\frac{\partial W}{\partial x}(x)g(x,0))^T \\ y = h(x) = \lambda(x)^{0.5}|\frac{\partial W}{\partial x}(x)g(x,0)|, \end{array} \tag{75}$$

w.r.t. unbounded sequence $\{t_n\}$ in $[0, \infty)$, then if $h(\overline{x}) = 0$, a.e., then either $\overline{x}(t_0) = 0$ for some $t_0 \geq 0$ or 0 is a ω-limit point of $\overline{x}$.
Then the closed-loop system (73) with (74) admits the origin $x = 0$ as UAS equilibrium point. Furthermore $|u(x)| \leq \overline{u},\ \forall x$.
Proof: We first can write, based on Condition 1 in Theorem 7

$$\dot{W} \leq \frac{\partial W}{\partial x} g(x, \alpha(t)u)\alpha(t)u,$$

using Lemma II.4 in (31), and considering the controller (74), we have

$$\dot{W} \leq -\frac{\epsilon_1}{2}\lambda(x)|G(x,0)|^2,\ |u(x)| \leq \overline{u}\ \forall x.$$

Next, we define for (73) and the controller (74) the positive invariant set $M = \{x |\ W(x) = 0\}$. Note that we can also write

$$M = \{x |\ \dot{W}(x) = 0\} \Leftrightarrow \{x |\ G(x,0) = 0\} \Leftrightarrow \{x |\ u(x) = 0\}.$$

Thus, the restriction of (73) on M is the system $\dot{x} = f(x)$. Finally, using Theorem 5 in (18), and under Condition 2 in Theorem 7, we conclude that $x = 0$ is US for (73) and the controller (74). Furthermore if $|u(x)| \leq \overline{u}$ then $|\alpha(t)u(x)| \leq \overline{u}\ \forall t, x$.
Now we note that for the virtual output $y = h(x) = \lambda(x)^{0.5}|\frac{\partial W}{\partial x}(x)g(x,0)|$, and $\sigma > 0$ we can write

$$W(t, x(t)) - W(t_0, x(t_0)) \leq -\frac{\epsilon_1}{2}\int_{t_0}^{t}|y(\tau)|^2 d\tau = -\int_{t_0}^{t}\mu(y(\tau))d\tau,$$

thus we have

$$\int_{t_0}^{t}(\mu(y(\tau)) - \sigma)d\tau \leq \int_{t_0}^{t}\mu(y(\tau))d\tau \leq W(t_0, x(t_0)) \leq \tilde{M},\ \tilde{M} > 0.$$

Finally, based on this last inequality and under Condition 3 in Theorem 7, using Theorem 1 in (24), we conclude that $x = 0$ is UAS equilibrium point for (73), (74). □

Remark 5: Here again we can simplify Condition 3 of Theorem 7, as follows. Based on Proposition 3 and Lemma 7 in (24), this condition is equivalent to: $\forall\overline{x}$ solutions for the reduced limiting system

$$\begin{array}{c} \dot{\overline{x}} = f(\overline{x}) + g(\overline{x}, \epsilon_1 u(\overline{x}))(-\lambda(\overline{x})\alpha_\gamma(t)\frac{\partial W}{\partial x}(\overline{x})g(\overline{x},0))^T \\ y = h(\overline{x}) = \lambda(\overline{x})^{0.5}|\frac{\partial W}{\partial x}(\overline{x})g(\overline{x},0)|, \end{array} \tag{76}$$

where the limiting function $\alpha_\gamma(t)$ is defined us $\alpha_\gamma(t) \overset{\triangle}{=} \lim_{n\to\infty}\alpha(t + t_n)$ w.r.t. unbounded sequence $\{t_n\}$ in $[0, \infty)$. Then, if $h(\overline{x}) = 0$, a.e., then either $\overline{x}(t_0) = 0$ for some $t_0 \geq 0$ or 0 is a ω-limit point of $\overline{x}$. Which writes directly as the ZSD condition:
$\forall\overline{x}$ solutions for the system

$$\begin{array}{c} \dot{\overline{x}} = f(\overline{x}) \\ \frac{\partial W}{\partial x}(\overline{x})g(\overline{x},0) = 0, \end{array} \tag{77}$$

either $\overline{x}(t_0) = 0$ for some $t_0 \geq 0$ or 0 is a ω-limit point of $\overline{x}$. ◆

Theorem 7 deals with the case of the general nonlinear model (73). For the particular case of affine nonlinear models, i.e. $g(x, u) = g(x)$, we can directly write the following Proposition.

Proposition 7: Consider the closed-loop system that consists of the faulty system (64) with the static state feedback

$$\begin{array}{c} u(x) = -\lambda(x)G(x)^T \\ G(x) = \frac{\partial W(x)}{\partial x}g(x) \\ \lambda(x) = \frac{2\overline{u}}{1+|G(x)|^2}. \end{array} \tag{78}$$

where W is a C^2, positive semidefinite function, such that:
1- $L_f W \leq 0$,
2- The system $\dot{x} = f(x)$ is AS conditionally to the set $M = \{x \mid W(x) = 0\}$,
3- $\forall\overline{x}$ limiting solutions for the system

$$\begin{array}{c} \dot{x} = f(x) + g(x)(-\lambda(x)\alpha(t)\frac{\partial W}{\partial x}(x)g(x))^T \\ y = h(x) = \lambda(x)^{0.5}|\frac{\partial W}{\partial x}(x)g(x)|, \end{array} \tag{79}$$

w.r.t. unbounded sequence $\{t_n\}$ in $[0, \infty)$, then if $h(\overline{x}) = 0$, a.e., then either $\overline{x}(t_0) = 0$ for some $t_0 \geq 0$ or 0 is a ω-limit point of $\overline{x}$.
Then the closed-loop system (64) with (78) admits the origin $x = 0$ as UAS equilibrium point. Furthermore $|u(x)| \leq \overline{u}$, $\forall x$.

Proof: The proof is a direct consequence of Theorem 7. However in this case the constraint of considering that the same fault occurs on all the actuators, is relaxed. Indeed, in this case we can directly write $\forall \alpha(t) \in \mathbb{R}^{m\times m},\ s.t.\ 0 < \epsilon_1 \leq \alpha_{ii}(t) \leq 1,\ \forall t$:

$$\dot{W} \leq -\lambda(x) L_g W(x) \alpha(t) L_g W(x)^T$$

$$\dot{W} \leq -\lambda(x) \epsilon_1 L_g W(x) L_g W(x)^T \leq -\epsilon_1 |G(x)|^2.$$

The rest of the proof remains unchanged. □

If we compare the dynamic controllers proposed in the Theorems 2, 3, 4, 6 and the static controllers of Theorems 5, 7, we can see that the dynamic controllers ensure that the control at the initialization time is zero, whereas this is not true for the static controllers. In the opposite, the static controllers have the advantage to ensure that the feedback control amplitude stays within the desired bound $\overline{u}$. We can also notice that, except for the controller in Theorem 3, all the remaining controllers proposed here do not involve the vector field f in there computation. This implies that these controllers are robust with respect to any uncertainty Δf as long as the conditions on f, required in the different theorems are still satisfied by the uncertain vector field $f + \Delta f$. Furthermore, the dynamic controller of Theorem 4 inherits the same robustness properties of the nominal controller u_{nom} used to write equation (57) (refer to Proposition 6.5, (40), p. 244).

9. Conclusion and future work

In this chapter we have presented different passive fault tolerant controllers for linear as well as for nonlinear models. Firstly, we have formulated the FTC problem in the context of the absolute stability theory, which has led to direct solutions to the passive FTC problem for LTI systems with uncertainties as well as input saturations. Open problems to which this formulation may be applied include infinite dimension models, stochastic models as well as time-delay models. Secondly, we have proposed several fault tolerant controllers for nonlinear models, by formulating the FTC problem as a cascade passivity-based control. Although, the proposed formulation has led to solutions for a large class of loss of actuator effectiveness faults for nonlinear systems, a more general result treating component faults entering the system through the vector field f plus additive faults on g, as well as the complete loss of some actuators is still missing and should be the subject of future work.

10. References

[1] F. Amato. *Robust control of linear systems subject to uncertain time-varying parameters*. Lecture Notes in Control and Information Sciences. Springer edition, 2006.

[2] M. Aizerman and F. Gantmacher. *Absolute stability of regulator systems*. Holden-Day, INC. 1964.

[3] B. Anderson and S. Vongpanitlerd. *Network analysis and synthesis*. Network series. Prentice-hall edition, 1973.

[4] M. Basin and M. Pinsky. Stability impulse control of faulted nonlinear systems. *IEEE, Transactions on Automatic Control*, 43(11):1604–1608, 1998.

[5] M. Benosman. A survey of some recent results on nonlinear fault tolerant control. *Mathematical Problems in Engineering*, 2010. Article ID 586169, 25 pages, doi:10.1155/2010/586169.

[6] M. Benosman and K.-Y. Lum. Application of absolute stability theory to robust control against loss of actuator effectiveness. *IET Control Theory and Applications*, 3(6):772–788, June 2009.

[7] M. Benosman and K.-Y. Lum. On-line references reshaping and control re-allocation for nonlinear fault tolerant control. *IEEE, Transactions on Control Systems Technology*, 17(2):366–379, March 2009.

[8] M. Benosman and K.-Y. Lum. Application of passivity and cascade structure to robust control against loss of actuator effectiveness. *Int. Journal of Robust and Nonlinear Control*, 20:673–693, 2010.

[9] M. Benosman and K.-Y. Lum. A passive actuators' fault tolerant control for affine nonlinear systems. *IEEE, Transactions on Control Systems Technology*, 18(1):152–163, 2010.

[10] C. Bonivento, L. Gentili, and A. Paoli. Internal model based fault tolerant control of a robot manipulator. In *IEEE, Conference on Decision and Control*, 2004.

[11] C. Bonivento, A. Isidori, L. Marconi, and A. Paoli. Implicit fault-tolerant control: application to induction motors. *Automatica*, 40(3):355–371, 2004.

[12] C. Byrnes, A. Isidori, and J.C.Willems. Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE, Transactions on Automatic Control*, 36(11):1228–1240, November 1991.

[13] M. Demetriou, K. Ito, and R. Smith. Adaptive monitoring and accommodtion of nonlinear actuator faults in positive real infinite dimensional systems. *IEEE, Transactions on Automatic Control*, 52(12):2332–2338, 2007.

[14] A. Fekih. Effective fault tolerant control design for nonlinear systems: application to a class of motor control system. *IET Control Theory and Applications*, 2(9):762–772, 2008.

[15] T. Gruijic and D. Petkovski. On robustness of Lurie systems with multiple non-linearities. *Automatica*, 23(3):327–334, 1987.

[16] T. Hu, B. Huang, and Z. Lin. Absolute stability with a generalized sector condition. *IEEE, Transactions on Automatic Control*, 49(4):535–548, 2004.

[17] T. Hu, Z. Lin, and B. Chen. An analysis and design method for linear systems subject to actuator saturation and disturbance. *Automatica*, 38:351–359, 2002.

[18] A. Iggidr and G. Sallet. On the stability of noautonomous systems. *Automatica*, 39:167–171, 2003.

[19] B. Jiang and F. Chowdhury. Fault estimation and accommodation for linear MIMO discret-time systems. *IEEE, Transactions on Control Systems Technology*, 13(3):493–499, 2005.

[20] B. Jiang and F. Chowdhury. Parameter fault detection and estimation of a class of nonlinear systems using observers. *Journal of Franklin Institute*, 342(7):725–736, 2005.

[21] B. Jiang, M. Staroswiecki, and V. Cocquempot. Fault accomodation for nonlinear dynamics systems. *IEEE, Transactions on Automatic Control*, 51(9):1578–1583, 2006.

[22] H. Khalil. *Nonlinear systems*. Prentice-Hall, third edition, 2002.

[23] M. Karpenkoa, N. Sepehri, and D. Scuseb. Diagnosis of process valve actuator faults using a multilayer neural network. *Control Engineering Practice*, 11(11):1289–1299, November 2003.

[24] T.-C. Lee and Z.-P. Jiang. A generalization of Krasovskii-LaSalle theorem for nonlinear time-varying systems: converse results and applications. *IEEE, Transactions on Automatic Control*, 50(8):1147–1163, August 2005.

[25] F. Liao, J. Wang, and G. Yang. Reliable robust flight tracking control: An LMI approach. *IEEE, Transactions on Control Systems Technology*, 10(1):76–89, 2002.

[26] M. Liberzon. Essays on the absolute stability theory. *Automation and remote control*, 67(10):1610–1644, 2006.

[27] X. Mao. Exponential stability of stochastic delay interval systems with markovian switching. *IEEE, Transactions on Automatic Control*, 47(10):1604–1612, 2002.
[28] H.-J. Ma and G.-H. Yang. FTC synthesis for nonlinear systems: Sum of squares optimization approach. In *IEEE, Conference on Decision and Control*, pages 2645–2650, New Orleans, USA,, 2007.
[29] M. Mahmoud, J. Jiang, and Y. Zhang. Stabilization of active fault tolerant control systems with imperfect fault detection and diagnosis. *Stochastic Analysis and Applications*, 21(3):673–701, 2003.
[30] M. Maki, J. Jiang, and K. Hagino. A stability guaranteed active fault-tolerant control system against actuator failures. *Int. Journal of Robust and Nonlinear Control*, 14:1061–1077, 2004.
[31] F. Mazenc and L. Praly. Adding integrations, saturated controls, and stabilization for feedforward systems. *IEEE, Transactions on Automatic Control*, 41(11):1559–1578, November 1996.
[32] P. Mhaskar. Robust model predictive control design for fault tolrant control of process systems. *Ind. Eng. Chem. Res.*, 45:8565–8574, 2006.
[33] P. Mhaskar, A. Gani, and P. Christofides. Fault-tolrant control of nonlinear processes: Performance-based reconfiguration and robustness. *Int. Journal of Robust and Nonlinear Control*, 16:91–111, 2006.
[34] P. Mhaskar, A. Gani, N. El-Farra, C. McFall, P. Christofides, and J. Davis. Integrated fault detection and fault-tolerant control of nonlinear process systems. *AIChE J.*, 52:2129–2148, 2006.
[35] P. Mhaskar, C. McFall, A. Gani, P. Christofides, and J. Davis. Isolation and handling of actuator faults in nonlinear systems. *Automatica*, 44(1):53–62, January 2008.
[36] H. Niemann and J. Stoustrup. Reliable control using the primary and dual Youla parametrization. In *IEEE, Conference on Decision and Control*, pages 4353–4358, 2002.
[37] H. Niemann and J. Stoustrup. Passive fault tolerant control of a double inverted pendulium-a case study. *Control Engineering Practice*, 13:1047–1059, 2005.
[38] R. Patton. Fault-tolerant control systems: The 1997 situation. In *IFAC Symposium Safe-Process' 97*, pages 1033–1055, U.K., 1997.
[39] P.G. de Lima and G.G. Yen. Accommodation controller malfunctions through fault tolerant control architecture. *IEEE Transactions on Aerospace and Electronic Systems*, 43(2):706–722, 2007.
[40] R. Sepulchre, M. Jankovic, and P. Kokotovic. *Constructive nonlinear control*. Springer edition, 1997.
[41] A. Shumsky. Algebraic approach to the problem of fault accommodation in nonlinear systems. In *IFAC 17th Word Congress*, pages 1884–1889, 2008.
[42] M. Staroswiecki, H. Yang, and B. Jiang. Progressive accommodation of aircraft actuator faults. In *6th IFAC Symposium of Fault Detection Supervision and Safety for Technical Processes*, pages 877–882, 2006.
[43] G. Tao, S. Chen, and S. Joshi. An adaptive actuator failure compensation controller using output feedback. *IEEE, Transactions on Automatic Control*, 47(3):506–511, 2002.
[44] A. Teel and L. Praly. Tools for semiglobal stabilization by partial state and output feedback. 33(5):1443–1488, 1995.
[45] M. Vidyasagar. *Nonlinear systems analysis*. Prentice-Hall, second edition, 1993.

[46] N. Wu, Y. Zhang, and K. Zhou. Detection, estimation, and accomodation of loss of control effectiveness. *Int. Journal of Adaptive Control and Signal Processing*, 14:775–795, 2000.

[47] S. Wu, M. Grimble, and W. Wei. QFT based robust/fault tolerant flight control design for a remote pilotless vehicle. In *IEEE International Conference on Control Applications*, China, August 1999.

[48] S. Wu, M. Grimble, and W. Wei. QFT based robust/fault tolerant flight control design for a remote pilotless vehicle. *IEEE, Transactions on Control Systems Technology*, 8(6):1010–1016, 2000.

[49] H. Yang, V. Cocquempot, and B. Jiang. Robust fault tolerant tracking control with application to hybrid nonlinear systems. *IET Control Theory and Applications*, 3(2):211–224, 2009.

[50] X. Zhang, T. Parisini, and M. Polycarpou. Adaptive fault-tolerant control of nonlinear uncertain systems: An information-based diagnostic approach. *IEEE, Transactions on Automatic Control*, 49(8):1259–1274, 2004.

[51] Y. Zhang and J. Jiang. Design of integrated fault detection, diagnosis and reconfigurable control systems. In *IEEE, Conference on Decision and Control*, pages 3587–3592, 1999.

[52] Y. Zhang and J. Jiang. Integrated design of reconfigurable fault-tolerant control systems. *Journal of Guidance, Control, and Dynamics*, 24(1):133–136, 2000.

[53] Y. Zhang and J. Jiang. Bibliographical review on reconfigurable fault-tolerant control systems. In *Proceeding of the 5th IFAC symposium on fault detection, supervision and safety for technical processes*, pages 265–276, Washington DC, 2003.

[54] Y. Zhang and J. Jiang. Issues on integration of fault diagnosis and reconfigurable control in active fault-tolerant control systems. In *6th IFAC Symposium on fault detection supervision and safety of technical processes*, pages 1513–1524, China, August 2006.

Robust Control, Theory and Applications

Edited by Prof. Andrzej Bartoszewicz

ISBN 978-953-307-229-6
Hard cover, 678 pages
Publisher InTech
Published online 11, April, 2011
Published in print edition April, 2011

The main objective of this monograph is to present a broad range of well worked out, recent theoretical and application studies in the field of robust control system analysis and design. The contributions presented here include but are not limited to robust PID, H-infinity, sliding mode, fault tolerant, fuzzy and QFT based control systems. They advance the current progress in the field, and motivate and encourage new ideas and solutions in the robust control area.

INTECH
open science | open minds

InTech Europe
University Campus STeP Ri
Slavka Krautzeka 83/A
51000 Rijeka, Croatia
Phone: +385 (51) 770 447
Fax: +385 (51) 686 166
www.intechopen.com

InTech China
Unit 405, Office Block, Hotel Equatorial Shanghai
No.65, Yan An Road (West), Shanghai, 200040, China
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元
Phone: +86-21-62489820
Fax: +86-21-62489821