Path Integrals in Quantum Field Theory – A Friendly Introduction

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1 Aims of this Talk

In this talk I hope to demystify (at least a little bit), why so much of modern physics is about defining and computing mysterious expressions of the form

$$\int d\phi \mathcal{O}(\phi) e^{iS(\phi)/\hbar}.$$

This question will lead me to introducing some of the basic ideas in *quantisation* and *quantum field theory*: huge fields leading to many beautiful ideas in both physics and mathematics.

As this is only a one hour talk, I'll only be able to begin the story. I hope to convince you that it's worth caring about these *Feynman path integrals*, but I'll only be able to touch on how one actually *defines* these heuristic expressions. Methods are known in many examples, most famously those involving *Feynman diagrams* and *renormalization*, but this will mostly go beyond the scope of this talk.

2 Classical Mechanics and Classical Field Theory

I'd like to start the story on relatively stable ground: the land of *classical* physics. More specifically, I'll introduce the idea of a classical *Lagrangian field theory*. The idea is that many physical systems take a similar form; there is a space of possible *configurations* of the system (which we usually refer to as *fields*), but those that arise in reality are those which minimise a certain *action* – a functional on the space of fields. In lieu of giving a (probably unenlightening) abstract definition, I'll give a few examples.

Examples 2.1. 1. **Classical Mechanics:** Consider a particle of mass m moving in a Riemannian manifold M under the influence of a forcefield with potential $V \in C^\infty(M)$. The physics of this particle can be described by a Lagrangian field theory where the fields are smooth maps $x : \mathbb{R} \to M$ (the possible *trajectories* the particle might move along), and the action functional is

$$S(x) = \int_{-\infty}^{\infty} \frac{1}{2} m|\dot{x}(t)|^2 - V(x(t))dt$$

where $\dot{x}(t)$ is the unit tangent vector to x at t, and $|\cdot|^2$ is the norm induced by the metric. The critical points of S can be computed to be those trajectories satisfying Newton's second law:

$$(dV)^\flat = m\nabla_t \dot{x}(t)$$

(which is a fancy differential-geometric way of writing $\boldsymbol{F} = m\boldsymbol{a}$). We might think of "minimising the action" as trying to balance the kinetic energy of the particle with its potential energy under an external force.

2. **Electromagnetism:** Now let M be a Riemannian 3-manifold, and consider pairs of vector fields E, B on $M \times \mathbb{R}$, which we think of as *electric* and *magnetic* fields varying over time. We can describe the classical physical relationship between these two fields as minimising an action functional, namely

$$S(E,B) = \int_{-\infty}^{\infty} |B(t)|^2 - |E(t)|^2 dt$$

again using the absolute value coming from the Riemannian metric on M. The critical points of S can be computed to be those fields satisfying Maxwell's equations in a vacuum. One can further generalise this story to include a fixed background charge distribution on M, and recover the more general form of Maxwell's equations.

3. **Gravity:** Let M be any manifold equipped with a (background) pseudo-Riemannian metric η of signature $(-,+,\ldots,+)$. The fields in this theory are pseudo-Riemannian metrics g of the same signature, and the action functional is the *Einstein-Hilbert action*

$$S(g) = \int_M R\sqrt{-\det g}\,\mathrm{dvol}_\eta$$

where dvol_η is the volume form induced by η, and where R is the Ricci scalar associated to g. The critical points of S can be described as the solutions to the Einstein field equation

$$\frac{1}{2}Rg = \mathrm{Ric},$$

where Ric is the Ricci tensor associated to g.

These examples illustrate a general phenomenon: given a classical field theory like this, the physical states – i.e. those fields extremising the action – arise as solutions to a system of differential equations: the *Euler-Lagrange equations*, or *equations of motion* of the system. We call these solutions *classical states* of the system.

In a classical field theory like these, we're interested in taking *measurements*, or making *observations*. That is, we investigate the state a physical system is in by evaluating a functional on the space of classical states.

Definition 2.2. A *classical observable* in a classical field theory is a functional on the space of classical states.

A key aspect of the rest of this talk is the question: what is the appropriate analogue of a classical observable in a quantum theory?

3 Why We Need Quantum Field Theory

Of course, the inadequacies of classical physics have been known for more than a hundred years by now. The classical theories we observe in nature actually arise as "approximations" or "limits" of *quantum theories*. I can't give a general definition of a quantum field theory (no-one can, at least not a satisfactory definition), but I can describe some properties these theories must have.

1. An important characteristic of quantum theory is the nature of measurement: what kind of thing *is* a quantum observation? Here's a sign that something genuinely *different* is going on to the classical theory: the observations we can make with "true" or "false" as possible answers *fail to form a Boolean algebra*. The famous counterexample is Young's two slit experiment. Suppose one has a screen with two slits at points $\boldsymbol{A}$ and $\boldsymbol{B}$, and a detector at a point $\boldsymbol{C}$ beyond it, and one fires a single photon at the screen. Then one can do two different experiments, measuring two different possible observables. One finds *different* results by performing the following two measurements:

$$(\boldsymbol{A} \text{ OR } \boldsymbol{B}) \text{ AND } \boldsymbol{C} \neq (\boldsymbol{A} \text{ AND } \boldsymbol{C}) \text{ OR } (\boldsymbol{B} \text{ AND } \boldsymbol{C}),$$

where by $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ I mean the *observables* "was a particle detected at this point?" There are two things to observe here. The first is the failure of the distributivity law (as satisfied by measurements in classical mechanics), the second is the *non-determinism* of the situation: one generally doesn't get the same result when one repeats the same experiment. Quantum measurements are inherently *probabilistic*.

As a result, while we cannot meaningfully talk about the *value* of an observable when the system is in some state, it does make sense to talk about the *expected* value of an observable.

2. Another famous characteristic that our model for quantum observables must possess is *failure of simultaneous measurability*. This is typified by Heisenberg's uncertainty principle: two observable quantities for a quantum particle are its position and its momentum. Suppose one tried to build an *algebra* of observables, where the product was "do both observables simultaneously". Measuring position and momentum simultaneously should certainly arise as a limit of "measure position, then measure momentum time ε later" as $\varepsilon \to 0$, or likewise of "measure momentum, then measure position time ε later". The uncertainty principle tells us that in fact these limits *necessarily* differ. While one can produce an algebra of observables, it is *necessarily* non-commutative in all non-trivial examples.

3. I should say something about the quantum notion of "states", and the wave-particle duality in quantum mechanics. One wants to represent our algebra of observables as *acting* on something. The principle of *superposition* says that any complex linear combination of two quantum states is also a state (as in the thought experiment of Schrödinger's cat, but in fact this is an experimentally verifiable phenomenon), so our space of states forms a *complex vector space*. One generally thinks of the space of states as a *separable Hilbert space*, with the observables acting by self-adjoint operators.

   For example, in the case of a quantum particle moving in $\mathbb{R}^n$, we have the position and momentum operators, which satisfy well-known commutation relations. The *Stone-von Neumann theorem* tells us that the representation of these operators is essentially unique, and can be described as multiplication and differentiation operators acting on the Hilbert space $L^2(\mathbb{R}^n)$.

4. I've mostly spoken just about quantum mechanics. In quantum field theory we really need to remember a piece of data we've been so far essentially forgetting: the underlying *spacetime* manifold. When we consider observables in this context we can remember the data of the *support* of a classical observable: does it only depend on a field in a certain neighbourhood? Quantisation should reflect this *locality* in a suitable way (I won't discuss this further, because it's somewhat orthogonal to the rest of the talk, but models for quantum field theory, both descriptions like TQFTs and descriptions like factorisation algebras have this locality built in as a hypothesis).

5. Finally, our system must behave well in the *classical limit*. That is, if we take a limit at low energies, or at long distances, we should recover the appropriate classical field theory. What does this mean in terms of observables? Well, broadly speaking, to any quantum observable there should correspond an underlying classical observable – a function on the classical state space – and the commutator of quantum observables should agree with the *Poisson bracket* of the classical observables up to a factor of $i\hbar$.

The example of *quantum mechanics* is the most well-understood: the quantisation of example 1 from the previous section. An important quantity to compute is the *propagator*, describing time evolution of quantum states. Let q_I and q_F be two points in spacetime: here I stands for 'initial' and F stands for 'final'. Associated to these points we associate quantum states (wavefunctions) $|q_I\rangle$ and $|q_F\rangle$ (eigenfunctions of the relevant position operators). The propagator describes the *transition probabilities* over a time interval from time 0 to time T: the probability density function for a particle to be observed in position q_F at time T having been observed in position q_I at time 0. This is written (and indeed computed) as

$$\langle q_F | e^{-iHT} | q_I \rangle$$

where e^{-iHT} is the *time evolution operator* in the theory. One can produce the *expectation value of an observable* for each T by integrating over q_I and q_F in some specified open sets.

Quantities of this form also appear in more general quantum field theories, where they are closely related to *scattering amplitudes* (or *S-matrix* elements): probability amplitudes for observing a particular ensemble of particles in specific positions at time T having observed some other ensemble of particles in specific positions at time 0. These amplitudes can be computed in terms of standard observables called *n-point functions*.

4 The Path Integral

Ok, so now I hope you're convinced that computing the expectation value of a quantum observable is a worthwhile thing to be doing. So how do we do it? Feynman's path integral gives an answer to that question (for the example of quantum mechanics) which is interesting for several reasons, for instance:

1. The path integral has a very interesting (if unintuitive) interpretation which links very neatly into the theory of the *classical* particle moving along critical points of the action.
2. Although the path integral initially makes sense only for quantum mechanics, it admits a natural generalisation to *any* quantum theory arising as a quantisation of a classical Lagrangian theory, with the same interpretation as the quantum particle.
3. Path integrals in quantum field theory are effectively computable in many examples, for instance via Feynman diagrams. What's more, the computations one does *themselves* have intriguing interpretations, as a sum over the ways particles might interact, split, merge and do all sorts of things on their journey through spacetime.

I'll give a sketch of Feynman's derivation of the path integral for a classical particle. Let's suppose first of all that we're considering a *free particle*, i.e. that there's no potential, then state at the end how to include a potential. We'll derive the propagator, as described in the previous section: the probability amplitude $\langle q_F|e^{-iHT/\hbar}|q_I\rangle$ where q_I and q_F are points in space (in $\mathbb{R}^n$ say).

To compute this time evolution, we'll split the time interval up into N pieces of equal length δt, then take the limit as $N \to \infty$. The scattering amplitude splits as an integral:

$$\langle q_F|e^{-iHT}|q_I\rangle = \int \langle q_F|e^{-iH\delta t}|q_1\rangle \cdots \int \langle q_N|e^{-iH\delta t}|q_I\rangle dq_1 \cdots dq_N.$$

Compute each factor by performing a Fourier transform (I'll be careless about units and omit $\hbar$). By "performing a Fourier transform", I mean we'll use Fourier inversion to write a function as an integral of its Fourier dual.

$$\begin{aligned}
\langle q_{n+1}|e^{-iH\delta t}|q_n\rangle &= \frac{1}{2\pi}\int \widehat{\langle q_{n+1}|e^{-i}}H\delta t|p\rangle e^{ipq_n}dp \\
&= \frac{1}{2\pi}\int e^{-\frac{i\delta t}{2m}p^2+ipq_n}\langle q_{n+1}|p\rangle dp \\
&= \frac{1}{2\pi}\int e^{-\frac{i\delta t}{2m}p^2-ip(q_{n+1}-q_n)}dp \\
&= \sqrt{\frac{-im}{2\pi\delta t}}e^{i\delta t m/2((q_{n+1}-q_n)/\delta t)^2}
\end{aligned}$$

Here we used a fact: that the Fourier transform of the functional $\langle q_{n+1}|e^{-iH\delta t}$ is $e^{-\frac{i\delta t}{2m}p^2}$ times pairing with $|q_{n+1}\rangle$, or as a physicist would put it, that the state $|p\rangle$ is an eigenvector for the time evolution operator, and we explicitly evaluated the Gaussian integral. One can, if one prefers, interpret the Fourier inversion step as decomposing the state $|q_n\rangle$ in terms of an eigenbasis of the time evolution operator. Now, we plug these pieces back into the product and take the limit. The integral over all the q_n, normalised by the constant terms, is interpreted as an integral over all *paths*, and the integrand in the limit becomes

$$e^{i\int_0^T \frac{1}{2}m\dot{q}^2 dt}.$$

If we did this calculation *including* a potential term $V(q)$ (and were careful to include $\hbar$), we would've found the integrand

$$e^{i/\hbar\int_0^T (\frac{1}{2}m\dot{q}^2 - V(q))dt}$$

which we recognise as $e^{i/\hbar S(q)}$, the complex exponential of the *action* for the classical particle.

Remark 4.1. This calculation can actually be made completely rigorous: this was done by Kac in the 40s using the *Wiener measure* on the space of continuous functions from an interval to $\mathbb{R}^n$.

Feynman interpreted this all in the following way: we compute the expectation value of an observable by considering *all* possible paths, i.e. all fields in the theory, weight them by the factor $e^{i/\hbar S(q)}$ and integrate. [1]

In a more general quantum theory quantising a classical theory with space of fields Φ and action functional S, we can try to generalise this idea. Motivated by the above calculation, we guess that the expectation value of an observable $\mathcal{O}$ can be computed by an integral over *all fields* where we weight the fields according to the action in the same way as above. That is, an integral of the form

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{\Phi} d\phi \mathcal{O}(\phi) e^{iS(\phi)/\hbar}$$

where Z is a normalisation. So far, this expression doesn't mean anything: we don't have a candidate for a measure $d\phi$ making this equality hold. In fact, generally no such measure can exist. However, don't panic! With some ingenuity we can still define the expression on the right-hand side. There's a great deal to say about this, and I don't have time to say more than a tiny bit, but there's some general techniques we might use:

- One can make sense of the integral fairly directly when S is a quadratic functional (we say the theory is *free* in this case), just because there's a good theory of infinite-dimensional Gaussian integrals. One can approximate the infinite-dimensional space Φ by finite-dimensional spaces and take a limit, which converges. This is called *regularisation*.
- If this is not the case then we can try to reduce to the free case, inspired by the analogous finite-dimensional calculations. One picks out the non-quadratic piece: $e^{iS/\hbar} = e^{iS_{\text{quad}}/\hbar} e^{igI/\hbar}$ and expands it as a power series in the variable g (a "coupling constant"). The terms can individually be computed combinatorially. Of course, the only works inside the radius of convergence of the power series.
- This actually still doesn't quite work: the individual terms diverge in the regularisation step. *Renormalization* is a method for "cancelling divergences" in these terms. It's not actually as arbitrary as it sounds: physically meaningful quantities can be proven independent of the method chosen, so the overall calculation gives a well-defined answer. Still, the details are beyond the scope of this talk.

Finally, let me note why the action *had* to show up in the path integral if we wanted an expression of this form to give the correct classical limit. In the classical limit $\hbar \to 0$ we expect the expectation value of an observable to be its expectation value as a function on the classical phase space. What does our path integral look like in this limit? Well, an answer in many contexts is given by the *principle of stationary phase*, which says (rather imprecisely) that an oscillating integral of the form

$$\int f(x) e^{i\eta g(x)} dx$$

converges, as $\eta \to \infty$, to (a constant times) the integral of $f(x)e^{i\eta g(x)}$ over the critical locus of g. A version of this holds, for instance, over finite-dimensional spaces, or for the path integrals in quantum mechanics. This is the sense in which the path integral closely mirrors and generalises the classical Euler-Lagrange story: where fields localise to the solutions to the equations of motion.

[1] Thirty-one years ago, Dick Feynman told me about his "sum over histories" version of quantum mechanics. "The electron does anything it likes," he said. "It just goes in any direction at any speed, forward or backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function." I said to him, "You're crazy." But he wasn't. — Freeman Dyson (in 1980)