PEARSON EDEXCEL INTERNATIONAL A LEVEL

MECHANICS 2

Student Book

Series Editors: Joe Skrakowski and Harry Smith

Authors: Greg Attwood, Dave Berry, Ian Bettison, Alan Clegg, Gill Dyer, Jane Dyer, Keith Gallick, Susan Hooker, Michael Jennings, Mohammed Ladak, Jean Littlewood, Bronwen Moran, James Nicholson, Su Nicholson, Laurence Pateman, Keith Pledger, Joe Skrakowski, Harry Smith, Jack Williams

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ABOUT THIS BOOK

The following three themes have been fully integrated throughout the Pearson Edexcel International Advanced Level in Mathematics series, so they can be applied alongside your learning.

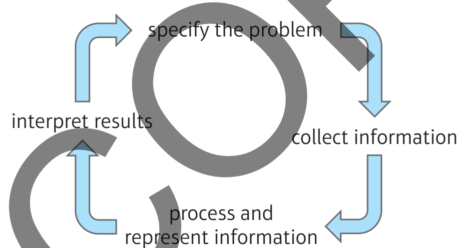
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols

2. Mathematical problem-solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Challenge questions provide extra stretch

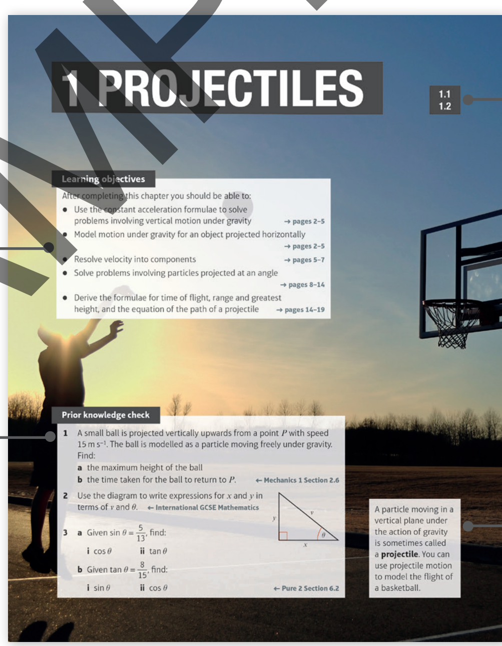
The Mathematical Problem-Solving Cycle



3. Transferable skills

- Transferable skills are embedded throughout this book, in the exercises and in some examples
- These skills are signposted to show students which skills they are using and developing

Finding your way around the book



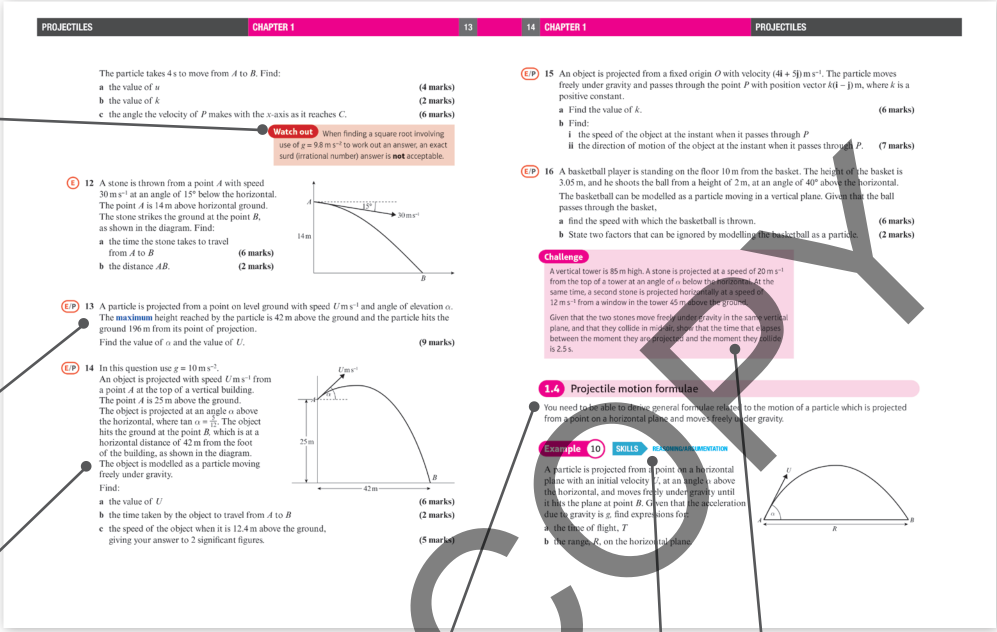
Each chapter is mapped to the specification content for easy reference

Each chapter starts with a list of *Learning objectives*

The *Prior knowledge check* helps make sure you are ready to start the chapter

The real world applications of the maths you are about to learn are highlighted at the start of the chapter.

Glossary terms will be identified by bold blue text on their first appearance.



Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

Step-by-step worked examples focus on the key types of questions you'll need to tackle

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard

Exercises are packed with exam-style questions to ensure you are ready for the exams

Each section begins with explanation and key learning points

Challenge boxes give you a chance to tackle some more difficult questions

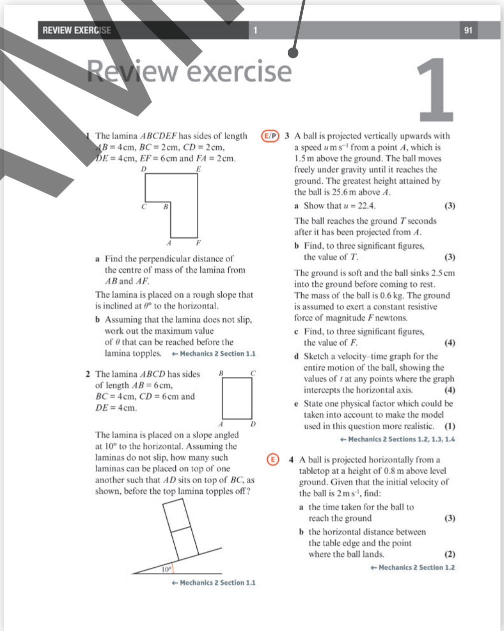
Exam-style questions are flagged with Ⓔ

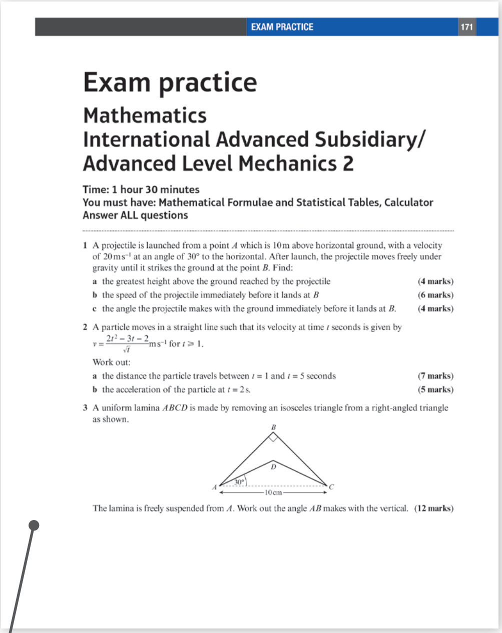
Problem-solving questions are flagged with Ⓟ

Transferable skills are signposted where they naturally occur in the exercises and examples

Each chapter ends with a *Chapter review* and a *Summary of key points*

After every few chapters, a *Review exercise* helps you consolidate your learning with lots of exam-style questions





A full practice paper at the back of the book helps you prepare for the real thing

QUALIFICATION AND ASSESSMENT OVERVIEW

Qualification and content overview

Mechanics 2 (M2) is a **optional** unit in the following qualifications:

International Advanced Subsidiary in Mathematics

International Advanced Subsidiary in Further Mathematics

International Advanced Level in Mathematics

International Advanced Level in Further Mathematics

Assessment overview

The following table gives an overview of the assessment for this unit.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

Unit	Percentage	Mark	Time	Availability
M2: Mechanics 2 Paper code WME02/01	$33\frac{1}{3}$ % of IAS $16\frac{2}{3}$ % of IAL	75	1 hour 30 mins	January, June and October First assessment June 2019

IAS: International Advanced Subsidiary, IAL: International Advanced A Level.

Assessment objectives and weightings

		Minimum weighting in IAS and IAL
AO1	Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of contexts.	30%
AO2	Construct rigorous mathematical arguments and proofs through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions, including the construction of extended arguments for handling substantial problems presented in unstructured form.	30%
AO3	Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinement of such models.	10%
AO4	Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.	5%
AO5	Use contemporary calculator technology and other permitted resources (such as formulae booklets or statistical tables) accurately and efficiently; understand when not to use such technology, and its limitations. Give answers to appropriate accuracy.	5%

Relationship of assessment objectives to units

P2	Assessment objective				
	AO1	AO2	AO3	AO4	AO5
Marks out of 75	20–25	20–25	10–15	7–12	5–10
%	$26\frac{2}{3}$–$33\frac{1}{3}$	$26\frac{2}{3}$–$33\frac{1}{3}$	$13\frac{1}{3}$–20	$9\frac{1}{3}$–16	$6\frac{2}{3}$–$13\frac{1}{3}$

Calculators

Students may use a calculator in assessments for these qualifications. Centres are responsible for making sure that calculators used by their students meet the requirements given in the table below.

Students are expected to have available a calculator with at least the following keys: +, –, ×, ÷, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, ln x, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory.

Prohibitions

Calculators with any of the following facilities are prohibited in all examinations:

- databanks
- retrieval of text or formulae
- built-in symbolic algebra manipulations
- symbolic differentiation and/or integration
- language translators
- communication with other machines or the internet

Extra online content

Whenever you see an *Online* box, it means that there is extra online content available to support you.

SolutionBank

SolutionBank provides worked solutions for questions in the book. Download all the solutions as a PDF or quickly find the solution you need online.

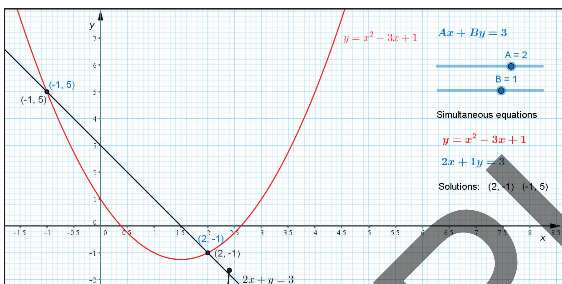
Use of technology

Explore topics in more detail, visualise problems and consolidate your understanding. Use pre-made GeoGebra activities or Casio resources for a graphic calculator.

Online Find the point of intersection graphically using technology.

GeoGebra

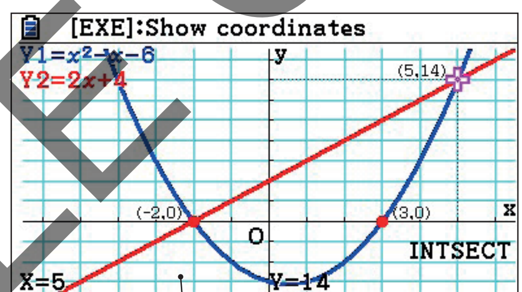
GeoGebra-powered interactives



Interact with the mathematics you are learning using GeoGebra's easy-to-use tools

CASIO

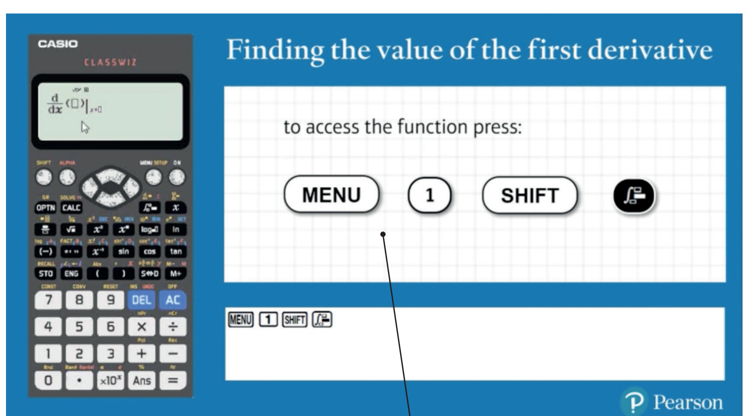
Graphic calculator interactives



Explore the mathematics you are learning and gain confidence in using a graphic calculator

Calculator tutorials

Our helpful video tutorials will guide you through how to use your calculator in the exams. They cover both Casio's scientific and colour graphic calculators.



Step-by-step guide with audio instructions on exactly which buttons to press and what should appear on your calculator's screen

Online Work out each coefficient quickly using the ${}^nC_r$ and power functions on your calculator.

1 PROJECTILES

1.1
1.2

Learning objectives

After completing this chapter you should be able to:

- Use the constant acceleration formulae to solve problems involving vertical motion under gravity → pages 2–5
- Model motion under gravity for an object projected horizontally → pages 2–5
- Resolve velocity into components → pages 5–7
- Solve problems involving particles projected at an angle → pages 8–14
- Derive the formulae for time of flight, range and greatest height, and the equation of the path of a projectile → pages 14–19

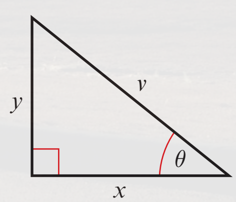
Prior knowledge check

1 A small ball is projected vertically upwards from a point P with speed 15 m s^{-1}. The ball is modelled as a particle moving freely under gravity. Find:

a the maximum height of the ball

b the time taken for the ball to return to P. ← Mechanics 1 Section 2.6

2 Use the diagram to write expressions for x and y in terms of v and θ. ← International GCSE Mathematics



3 **a** Given $\sin\theta = \frac{5}{13}$, find:

i $\cos\theta$ **ii** $\tan\theta$

b Given $\tan\theta = \frac{8}{15}$, find:

i $\sin\theta$ **ii** $\cos\theta$ ← Pure 2 Section 6.2

A particle moving in a vertical plane under the action of gravity is sometimes called a **projectile**. You can use projectile motion to model the flight of a basketball.

1.1 Horizontal projection

You can **model** the **motion** of a **projectile** as a **particle** being acted on by a single **force**, **gravity**. In this model you ignore the effects of **air resistance** and any rotational movement (i.e. spinning) on the particle.

You can analyse the motion of a projectile by considering its **horizontal** motion and its **vertical** motion separately. Because gravity acts vertically downwards, there is **no force** acting on the particle in the horizontal direction.

- **The horizontal motion of a projectile is modelled as having constant velocity ($a = 0$). You can use the formula $s = vt$.**

  The force due to gravity is modelled as being constant, so the vertical acceleration is constant.

- **The vertical motion of a projectile is modelled as having constant acceleration due to gravity ($a = g$).**

  Use $g = 9.8\ \text{m s}^{-2}$ unless the question specifies a different value.

Links You can use the **constant acceleration** formulae for the vertical motion of a projectile:

$v = u + at$

$s = \left(\frac{u + v}{2}\right)t$ $s = ut + \frac{1}{2}at^2$

$v^2 = u^2 + 2as$ $s = vt - \frac{1}{2}at^2$

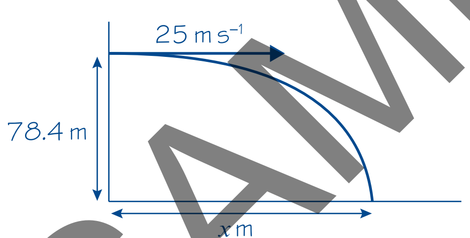
← Mechanics 1 Section 2.5

Example 1 SKILLS PROBLEM-SOLVING

A particle is **projected** horizontally at $25\ \text{m s}^{-1}$ from a point 78.4 metres above a horizontal surface. Find:

a the time taken by the particle to reach the surface

b the horizontal distance travelled in that time.



First draw a diagram showing all the information given in the question.

Projected horizontally, R(→), $u_x = 25$

Notation u_x is the initial horizontal velocity. u_y is the initial vertical velocity.

Taking the downward direction as positive, R(↓), $u_y = 0$

The particle is projected horizontally so $u_y = 0$.

a R(↓), $u = 0$, $s = 78.4$, $a = 9.8$, $t = ?$

$s = ut + \frac{1}{2}at^2$

$78.4 = 0 + \frac{1}{2} \times 9.8 \times t^2$

$78.4 = 4.9t^2$

$\frac{78.4}{4.9} = t^2$

$t^2 = 16$ so $t = 4\text{ s}$

b R(→), $u = 25$, $s = x$, $t = 4$

$s = vt$

$x = 25 \times 4$ so $x = 100\text{ m}$

Watch out The sign of g (**positive** or **negative**) depends on which direction is chosen as positive.
Positive direction downwards: $g = 9.8\text{ m s}^{-2}$
Positive direction upwards: $g = -9.8\text{ m s}^{-2}$

The time taken must be positive so choose the positive square root.

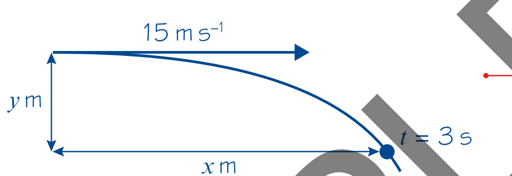
Your answer to part **a** tells you the time taken for the particle to hit the surface. The horizontal motion has constant velocity so you can use: distance = **speed** × time.

Example 2

A particle is projected horizontally with a velocity of 15 m s^{-1}. Find:

a the horizontal and vertical **components** of the **displacement** of the particle from the point of projection after 3 seconds

b the distance of the particle from the point of projection after 3 seconds.



Draw a diagram based on the information in the question.

Projected horizontally, R(→), $u_x = 15$

Taking the downward direction as positive, R(↓), $u_y = 0$

a R(↓), $u = 0$, $s = y$, $a = 9.8$, $t = 3$

$s = ut + \frac{1}{2}at^2$

$y = 0 + \frac{1}{2} \times 9.8 \times 3^2$

$y = 44.1\text{ m}$

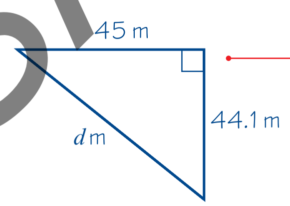
R(→), $u = 15$, $s = x$, $t = 3$

$s = vt$

$x = 15 \times 3 = 45\text{ m}$

Use $s = ut + \frac{1}{2}at^2$ to find the vertical distance. This is the same distance as the particle would travel in 3 seconds if it was dropped and fell under the action of gravity.

b



$d = \sqrt{45^2 + 44.1^2}$

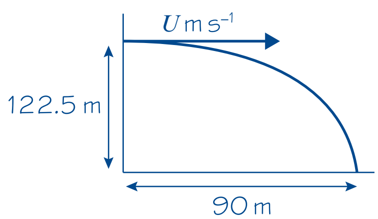
$= \sqrt{3969.81}$

$d = 63.0\text{ m}$ (3 s.f.)

The distance travelled is the **magnitude** of the displacement **vector**. Sketch a right-angled triangle showing the components and use Pythagoras' Theorem.

Example 3

A particle is projected horizontally with a speed of U m s^{-1} from a point 122.5 m above a horizontal **plane**. The particle hits the plane at a point which is at a horizontal distance of 90 m from the starting point. Find the initial speed of the particle.



Projected horizontally, R(→), $u_x = U$

Taking the downward direction as positive, R(↓), $u_y = 0$

R(↓), $u = 0$, $s = 122.5$, $a = 9.8$, $t = ?$

$s = ut + \frac{1}{2}at^2$

$122.5 = 0 + \frac{1}{2} \times 9.8 \times t^2$

$122.5 = 4.9t^2$

$t^2 = 25$ so $t = 5$ s

R(→), $v = U$, $s = 90$, $t = 5$

$s = vt$

$90 = U \times 5$

$U = 90 \div 5$ so $U = 18$ m s^{-1}

Substitute $t = 5$ into the equation for horizontal motion to find U.

Problem-solving

Many projectile problems can be solved by first using the **vertical motion** to find the total time taken.

Exercise 1A SKILLS PROBLEM-SOLVING

1 A particle is projected horizontally at 20 m s^{-1} from a point h metres above horizontal ground. It lands on the ground 5 seconds later. Find:

a the value of h

b the horizontal distance travelled between the time the particle is projected and the time it hits the ground.

2 A particle is projected horizontally with a velocity of 18 m s^{-1}. Find:

a the horizontal and vertical components of the displacement of the particle from the point of projection after 2 seconds

b the distance of the particle from the point of projection after 2 seconds.

3 A particle is projected horizontally with a speed of U m s^{-1} from a point 160 m above a horizontal plane. The particle hits the plane at a point which is at a horizontal distance of 95 m from the point of projection. Find the initial speed of the particle.

4 A particle is projected horizontally from a point A which is 16 m above horizontal ground. The particle strikes the ground at a point B which is at a horizontal distance of 140 m from A. Find the **speed of projection** of the particle.

(P) **5** A particle is projected horizontally with velocity $20\,\text{m}\,\text{s}^{-1}$ along a flat **smooth** table-top from a point 2 m from the table edge. The particle then leaves the table-top which is at a height of 1.2 m from the floor. Work out the total time taken for the particle to travel from the point of projection until it lands on the floor.

(E) **6** A darts player throws darts at a dartboard which hangs vertically. The motion of a dart is modelled as that of a particle moving freely under gravity. The darts move in a vertical plane which is **perpendicular** to the plane of the dartboard. A dart is thrown horizontally with an initial velocity of $14\,\text{m}\,\text{s}^{-1}$. It hits the dartboard at a point which is 9 cm below the level from which it was thrown.

Find the horizontal distance from the point where the dart was thrown to the dartboard. **(4 marks)**

(E/P) **7** A particle of **mass** 2.5 kg is projected along a horizontal **rough surface** with a velocity of $5\,\text{m}\,\text{s}^{-1}$. After travelling a distance of 2 m the ball leaves the rough surface as a projectile and lands on the ground which is 1.2 m vertically below. Given that the total time taken for the ball to travel from the initial point of projection to the point when it lands is 1.0 seconds, find:

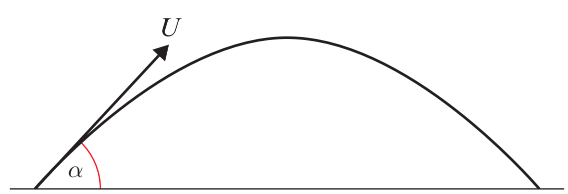
a the time for which the particle is in contact with the surface **(4 marks)**

b the **coefficient of friction** between the particle and the surface **(6 marks)**

c the horizontal distance travelled from the point of projection to the point where the particle hits the ground. **(3 marks)**

1.2 Horizontal and vertical components

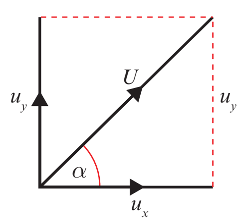
Suppose a particle is projected with initial velocity U, at an angle α above the horizontal. The angle α is called the **angle of projection**.



You can **resolve** the velocity into **components** that act horizontally and vertically:

$$\cos\alpha = \frac{u_x}{U} \quad \text{so} \quad u_x = U\cos\alpha$$

$$\sin\alpha = \frac{u_y}{U} \quad \text{so} \quad u_y = U\sin\alpha$$



Links This is the same technique as you use to resolve forces into components.
← Mechanics 1 Section 5.1

■ **When a particle is projected with initial velocity U, at an angle α above the horizontal:**

- **The horizontal component of the initial velocity is $U\cos\alpha$.**
- **The vertical component of the initial velocity is $U\sin\alpha$.**

Example 4

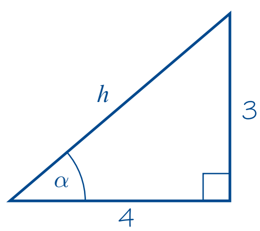
SKILLS PROBLEM-SOLVING

A particle is projected from a point on a horizontal plane with an initial velocity of $40\,\mathrm{m\,s^{-1}}$ at an angle α above the horizontal, where $\tan\alpha = \frac{3}{4}$.

a Find the horizontal and vertical components of the initial velocity.

Given that the vectors **i** and **j** are unit vectors acting in a plane, horizontally and vertically respectively,

b express the initial velocity as a vector in terms of **i** and **j**.



a $\tan\alpha = \frac{3}{4}$ so $h = \sqrt{3^2 + 4^2} = 5$

$\sin\alpha = \frac{3}{5}$ $\cos\alpha = \frac{4}{5}$

R(→), $u_x = u\cos\alpha = 40 \times \frac{4}{5} = 32\,\mathrm{m\,s^{-1}}$

R(↑), $u_y = u\sin\alpha = 40 \times \frac{3}{5} = 24\,\mathrm{m\,s^{-1}}$

b $U = (32\mathbf{i} + 24\mathbf{j})\,\mathrm{m\,s^{-1}}$

Problem-solving

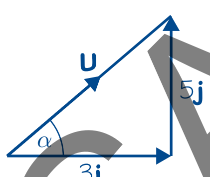
When you are given a value for $\tan\alpha$, you can find the values of $\cos\alpha$ and $\sin\alpha$ without working out the value of α. Here $\tan\alpha = \frac{3}{4} = \frac{\text{opp}}{\text{adj}}$, so sketch a right-angled triangle with **opposite** side = 3 and **adjacent** side = 4.

Online Find $\cos\alpha$ and $\sin\alpha$ using your calculator.

You can write velocity as a vector using **i**–**j** notation. Remember to include units.

Example 5

A particle is projected with velocity $\mathbf{U} = (3\mathbf{i} + 5\mathbf{j})\,\mathrm{m\,s^{-1}}$, where **i** and **j** are the unit vectors in the horizontal and vertical directions respectively. Find the initial speed of the particle and its angle of projection.



Speed, $|\mathbf{U}| = \sqrt{3^2 + 5^2} = \sqrt{34}\,\mathrm{m\,s^{-1}}$

$\tan\alpha = \frac{5}{3}$

$\alpha = 59.0°$ (3 s.f.)

Initial speed is $\sqrt{34}\,\mathrm{m\,s^{-1}}$ and the particle is projected at an angle of 59.0° above the horizontal.

Speed is the magnitude of the velocity vector. If the initial velocity is $(p\mathbf{i} + q\mathbf{j})\,\mathrm{m\,s^{-1}}$, the initial speed is $\sqrt{p^2 + q^2}$.

When an initial velocity is given in the form $(p\mathbf{i} + q\mathbf{j})\,\mathrm{m\,s^{-1}}$, the values of p and q are the horizontal and vertical components of the velocity respectively.

Exercise 1B

SKILLS PROBLEM-SOLVING

In this exercise, **i** and **j** are unit vectors acting in a vertical plane, horizontally and vertically respectively.

1 A particle is projected from a point on a horizontal plane with an initial velocity of $25\,\text{m}\,\text{s}^{-1}$ at an angle of 40° above the horizontal.

a Find the horizontal and vertical components of the initial velocity.

b Express the initial velocity as a vector in the form $(p\mathbf{i} + q\mathbf{j})\,\text{m}\,\text{s}^{-1}$.

2 A particle is projected from a cliff top with an initial velocity of $18\,\text{m}\,\text{s}^{-1}$ at an angle of 20° below the horizontal.

a Find the horizontal and vertical components of the initial velocity.

b Express the initial velocity as a vector in the form $(p\mathbf{i} + q\mathbf{j})\,\text{m}\,\text{s}^{-1}$.

3 A particle is projected from a point on level ground with an initial velocity of $35\,\text{m}\,\text{s}^{-1}$ at an angle α above the horizontal, where $\tan\alpha = \frac{5}{12}$.

a Find the horizontal and vertical components of the initial velocity.

b Express the initial velocity as a vector in terms of **i** and **j**.

4 A particle is projected from the top of a building with an initial velocity of $28\,\text{m}\,\text{s}^{-1}$ at an angle θ below the horizontal, where $\tan\theta = \frac{7}{24}$.

a Find the horizontal and vertical components of the initial velocity.

b Express the initial velocity as a vector in terms of **i** and **j**.

5 A particle is projected with initial velocity $\mathbf{U} = (6\mathbf{i} + 9\mathbf{j})\,\text{m}\,\text{s}^{-1}$.
Find the initial speed of the particle and its angle of projection.

6 A particle is projected with initial velocity $\mathbf{U} = (4\mathbf{i} - 5\mathbf{j})\,\text{m}\,\text{s}^{-1}$.
Find the initial speed of the particle and its angle of projection.

(P) **7** A particle is projected with initial velocity $\mathbf{U} = (3k\mathbf{i} + 2k\mathbf{j})\,\text{m}\,\text{s}^{-1}$.

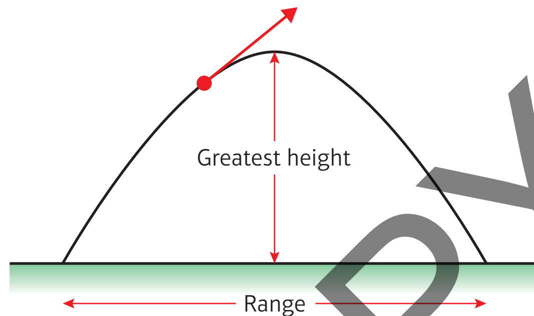
a Find the angle of projection.

Given the initial speed is $3\sqrt{13}\,\text{m}\,\text{s}^{-1}$,

b find the value of k.

1.3 Projection at any angle

You can solve problems involving particles projected at any angle by resolving the initial velocity into horizontal and vertical components.



The distance from the point where the particle was projected to the point where it strikes the horizontal plane is called the **range**.

The time the particle takes to move from its point of projection to the point where it strikes the horizontal plane is called the **time of flight** of the particle.

- **A projectile reaches its point of greatest height when the vertical component of its velocity is equal to 0.**

Example 6 SKILLS PROBLEM-SOLVING

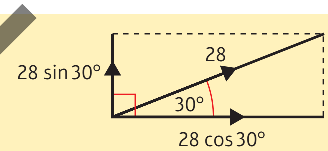
A particle P is projected from a point O on a horizontal plane with speed 28 m s^{-1} and with angle of **elevation** 30°. After projection, the particle moves freely under gravity until it strikes the plane at a point A. Find:

a the greatest height above the plane reached by P **b** the time of flight of P **c** the distance OA.

Resolving the velocity of projection horizontally and vertically:

R(→), $u_x = 28\cos 30° = 24.248\ldots$

R(↑), $u_y = 28\sin 30° = 14$



a Taking the upward direction as positive:

R(↑), $u = 14,\ v = 0,\ a = -9.8,\ s = ?$

At the highest point the vertical component of the velocity is zero.

$v^2 = u^2 + 2as$

$0^2 = 14^2 - 2 \times 9.8 \times s$

The vertical motion is motion with constant acceleration.

$s = \dfrac{14^2}{2 \times 9.8} = 10$

The greatest height above the plane reached by P is 10 m.

When the particle strikes the plane, it is at the same height (zero) as when it started.

b R(↑), $s = 0,\ u = 14,\ a = -9.8,\ t = ?$

$s = ut + \frac{1}{2}at^2$

$0 = 14t - 4.9t^2$

$= t(14 - 4.9t)$

$t = 0$ corresponds to the point where P was projected and can be ignored.

$t = 0$ or $t = \dfrac{14}{4.9} = 2.857\ldots$

The time of flight is 2.9 s (2 s.f.)

Watch out In this example the value for g was given to 2 significant figures so your answers should be given to 2 significant figures.

c R(→), distance = speed × time

There is no horizontal acceleration.

$= 28\cos 30° \times 2.857\ldots$

Use the unrounded value for the time of flight.

$= 69.282\ldots$

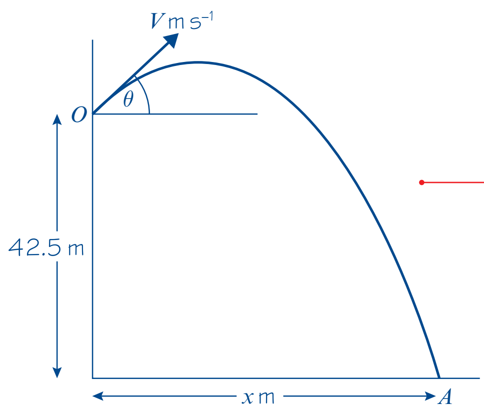
$OA = 69$ m (2 s.f.)

Example 7

A particle is projected from a point O with speed $V\,\text{m s}^{-1}$ and at an angle of elevation of θ, where $\tan\theta = \frac{4}{3}$. The point O is 42.5 m above a horizontal plane. The particle strikes the plane at a point A, 5 s after it is projected.

a Show that $V = 20$.

b Find the distance between O and A.



Start by drawing a diagram.

Resolving the velocity of projection horizontally and vertically:

R(→), $u_x = V\cos\theta = \frac{3}{5}V$

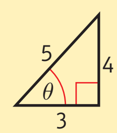
R(↑), $u_y = V\sin\theta = \frac{4}{5}V$

You will need $\sin\theta$ and $\cos\theta$ to resolve the initial velocity. When you know $\tan\theta$ you can draw a triangle to find $\cos\theta$ and $\sin\theta$.

$\tan\theta = \frac{4}{3}$

$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$



a Taking the upward direction as positive:

R(↑), $s = -42.5,\ u = \frac{4}{5}V,\ g = -9.8,\ t = 5$

$s = ut + \frac{1}{2}at^2$

$-42.5 = \frac{4}{5}V \times 5 - 4.9 \times 25$

$4V = 4.9 \times 25 - 42.5 = 80$

$V = \frac{80}{4} = 20$, as required.

Use the formula $s = ut + \frac{1}{2}at^2$ to obtain an equation in V.

b Let the horizontal distance moved be x m:

R(→), distance = speed × time

$x = \frac{3}{5}V \times 5 = 3V = 60$

Use the value of V found in part **a** to find the horizontal distance moved by the particle.

Using Pythagoras' Theorem:

$OA^2 = 42.5^2 + 60^2 = 5406.25$

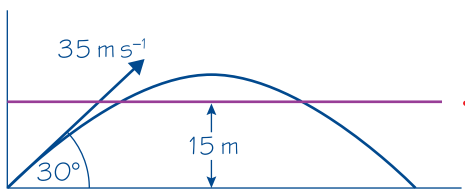
$OA = \sqrt{5406.25} = 73.527\ldots$

The distance between O and A is 74 m, to 2 significant figures.

Example 8

A particle is projected from a point O with speed $35\,\text{m s}^{-1}$ at an angle of elevation of 30°. The particle moves freely under gravity.

Find the length of time for which the particle is 15 m or more above O.



The particle is 15 m above O twice. First on the way up and then on the way down.

Resolving the initial velocity vertically:

R(↑), $u_y = 35 \sin 30° = 17.5$

$s = 15,\ u = 17.5,\ a = -9.8,\ t = ?$

$$s = ut + \tfrac{1}{2}at^2$$

$$15 = 17.5t - 4.9t^2$$

In this example the horizontal component of the initial velocity is not used.

$4.9t^2 - 17.5t + 15 = 0$

Form a quadratic equation in t to find the two times when the particle is 15 m above O. Between these two times, the particle will be more than 15 m above O.

Multiplying by 10:

$49t^2 - 175t + 150 = 0$

$(7t - 10)(7t - 15) = 0$

$$t = \frac{10}{7}, \frac{15}{7}$$

$$\frac{15}{7} - \frac{10}{7} = \frac{5}{7} = 0.71\text{ s (2 s.f.)}$$

Online Use your calculator to solve a quadratic equation.

The particle is 15 m or more above O for 0.71 s (2 s.f.).

You should give this answer as a decimal to 2 significant figures, 0.71 s, following previous use of $g = 9.8$ m s⁻².

Example 9

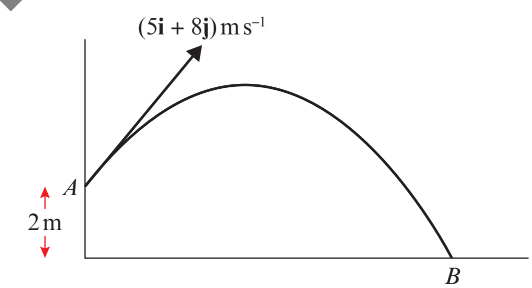
A ball is struck by a racket at a point A which is 2 m above horizontal ground. Immediately after being struck, the ball has velocity $(5\mathbf{i} + 8\mathbf{j})$ m s⁻¹, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors horizontally and vertically respectively. After being struck, the ball travels freely under gravity until it strikes the ground at the point B, as shown in the diagram.

Find:

a the greatest height above the ground reached by the ball

b the speed of the ball as it reaches B

c the angle the velocity of the ball makes with the ground as the ball reaches B.



a Taking the upward direction as positive:

R(↑), $u = 8,\ v = 0,\ a = -9.8,\ s = ?$

The velocity of projection has been given as a vector in terms of $\mathbf{i}$ and $\mathbf{j}$. The horizontal component is 5 and the vertical component is 8.

$v^2 = u^2 + 2as$

$0^2 = 8^2 - 2 \times 9.8 \times s$

$$s = \frac{64}{19.6} = 3.265\ldots$$

This is the greatest height above the point of projection. You need to add 2 m to find the height above the ground.

The greatest height above the ground reached by the ball is $2 + 3.265\ldots = 5.3$ m, to 2 significant figures.

b The horizontal component of the velocity of the ball at B is $5\,\text{m s}^{-1}$.

The vertical component of the velocity of the ball at B is given by:

$R(\uparrow)$, $s = -2$, $u = 8$, $a = -9.8$, $v = ?$

$v^2 = u^2 + 2as$

$\quad = 8^2 + 2 \times (-9.8) \times (-2) = 103.2$

The speed at B is given by:

$v^2 = 5^2 + 103.2 = 128.2$

$v = \sqrt{128.2}$

The speed of the ball as it reaches B is $11\,\text{m s}^{-1}$, to 2 significant figures.

c The angle is given by:

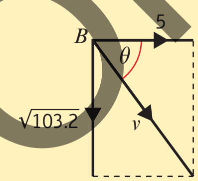
$\tan\theta = \dfrac{\sqrt{103.2}}{5} \Rightarrow \theta = 64°$ (2 s.f.)

The angle the velocity of the ball makes with the ground as the ball reaches B is 64°, to the nearest degree.

The horizontal motion is motion with constant speed, so the horizontal component of the velocity never changes.

There is no need to find the square root of 103.2 at this point, as you need v^2 in the next stage of the calculation.

As the ball reaches B, its velocity has two components as shown below.



The magnitude (speed) and direction of the velocity are found using trigonometry and Pythagoras' Theorem.

Exercise 1C

SKILLS PROBLEM-SOLVING

In this exercise, **i** and **j** are unit vectors acting in a vertical plane, horizontally and vertically respectively.

Whenever a **numerical value** of g is required, take $g = 9.8\,\text{m s}^{-2}$ unless otherwise stated.

1 A particle is projected with speed $35\,\text{m s}^{-1}$ at an angle of elevation of 60°. Find the time the particle takes to reach its greatest height.

2 A ball is projected from a point 5 m above horizontal ground with speed $18\,\text{m s}^{-1}$ at an angle of elevation of 40°. Find the height of the ball above the ground 2 s after projection.

3 A stone is projected from a point above horizontal ground with speed $32\,\text{m s}^{-1}$, at an angle of 10° below the horizontal. The stone takes 2.5 s to reach the ground. Find:

a the height of the point of projection above the ground

b the distance from the point on the ground vertically below the point of projection to the point where the stone reaches the ground.

4 A projectile is launched from a point on horizontal ground with speed $150\,\text{m s}^{-1}$ at an angle of 10° above the horizontal. Find:

a the time the projectile takes to reach its highest point above the ground

b the range of the projectile.

5 A particle is projected from a point O on a horizontal plane with speed 20 m s^{-1} at an angle of elevation of $45°$. The particle moves freely under gravity until it strikes the ground at a point X. Find:

a the greatest height above the plane reached by the particle

b the distance OX.

P **6** A ball is projected from a point A on level ground with speed 24 m s^{-1}. The ball is projected at an angle θ to the horizontal where $\sin\theta = \frac{4}{5}$. The ball moves freely under gravity until it strikes the ground at a point B. Find:

a the time of flight of the ball

b the distance from A to B.

P **7** A particle is projected with speed 21 m s^{-1} at an angle of elevation α. Given that the greatest height reached above the point of projection is 15 m, find the value of α, giving your answer to the nearest degree.

8 A particle P is projected from the origin with velocity $(12\mathbf{i} + 24\mathbf{j})\text{ m s}^{-1}$, where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical unit vectors respectively. The particle moves freely under gravity. Find:

a the position vector of P after 3 s

b the speed of P after 3 s.

P **9** A stone is thrown with speed 30 m s^{-1} from a window which is 20 m above horizontal ground. The stone hits the ground 3.5 s later. Find:

a the angle of projection of the stone

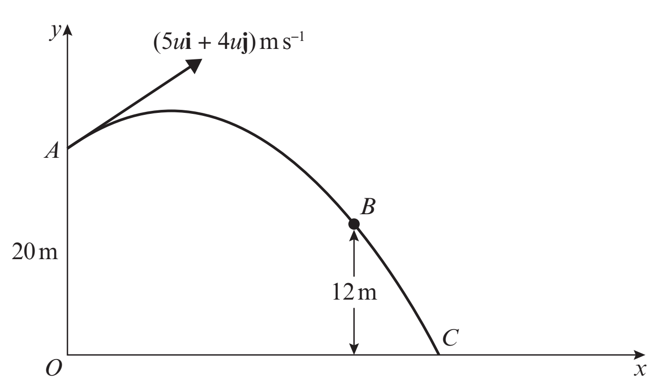
b the horizontal distance from the window to the point where the stone hits the ground.

E/P **10** A ball is thrown from a point O on horizontal ground with speed $U\text{ m s}^{-1}$ at an angle of elevation of θ, where $\tan\theta = \frac{3}{4}$. The ball strikes a vertical wall which is 20 m from O at a point which is 3 m above the ground. Find:

a the value of U **(6 marks)**

b the time from the instant the ball is thrown to the instant that it strikes the wall. **(2 marks)**

E/P **11** A particle P is projected from a point A with position vector $20\mathbf{j}$ m with respect to a fixed origin O. The velocity of projection is $(5u\mathbf{i} + 4u\mathbf{j})\text{ m s}^{-1}$. The particle moves freely under gravity, passing through a point B, which has position vector $(k\mathbf{i} + 12\mathbf{j})$ m, where k is a constant, before reaching the point C on the x-axis, as shown in the diagram.



The particle takes 4 s to move from A to B. Find:

a the value of u **(4 marks)**

b the value of k **(2 marks)**

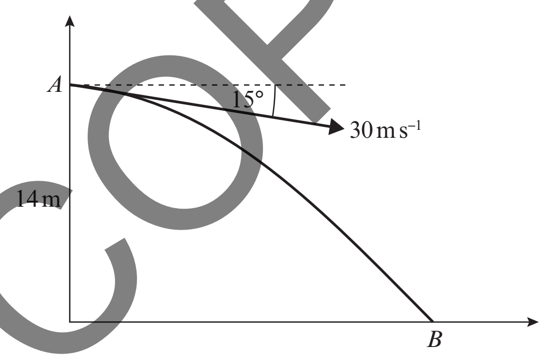
c the angle the velocity of P makes with the x-axis as it reaches C. **(6 marks)**

Watch out When finding a square root involving use of $g = 9.8\text{ m s}^{-2}$ to work out an answer, an exact surd (irrational number) answer is **not** acceptable.

E **12** A stone is thrown from a point A with speed 30 m s^{-1} at an angle of 15° below the horizontal. The point A is 14 m above horizontal ground. The stone strikes the ground at the point B, as shown in the diagram. Find:

a the time the stone takes to travel from A to B **(6 marks)**

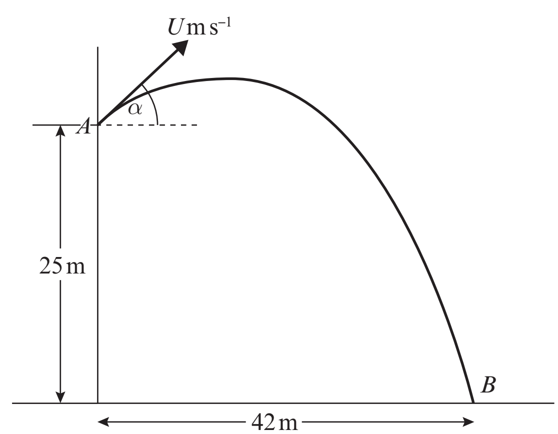
b the distance AB. **(2 marks)**



E/P **13** A particle is projected from a point on level ground with speed $U\text{ m s}^{-1}$ and angle of elevation α. The **maximum** height reached by the particle is 42 m above the ground and the particle hits the ground 196 m from its point of projection.

Find the value of α and the value of U. **(9 marks)**

E/P **14** In this question use $g = 10\text{ m s}^{-2}$.
An object is projected with speed $U\text{ m s}^{-1}$ from a point A at the top of a vertical building. The point A is 25 m above the ground. The object is projected at an angle α above the horizontal, where $\tan\alpha = \frac{5}{12}$. The object hits the ground at the point B, which is at a horizontal distance of 42 m from the foot of the building, as shown in the diagram. The object is modelled as a particle moving freely under gravity.



Find:

a the value of U **(6 marks)**

b the time taken by the object to travel from A to B **(2 marks)**

c the speed of the object when it is 12.4 m above the ground, giving your answer to 2 significant figures. **(5 marks)**

E/P **15** An object is projected from a fixed origin O with velocity $(4\mathbf{i} + 5\mathbf{j})$ m s^{-1}. The particle moves freely under gravity and passes through the point P with position vector $k(\mathbf{i} - \mathbf{j})$ m, where k is a positive constant.

a Find the value of k. **(6 marks)**

b Find:

i the speed of the object at the instant when it passes through P

ii the direction of motion of the object at the instant when it passes through P. **(7 marks)**

E/P **16** A basketball player is standing on the floor 10 m from the basket. The height of the basket is 3.05 m, and he shoots the ball from a height of 2 m, at an angle of 40° above the horizontal.

The basketball can be modelled as a particle moving in a vertical plane. Given that the ball passes through the basket,

a find the speed with which the basketball is thrown. **(6 marks)**

b State two factors that can be ignored by modelling the basketball as a particle. **(2 marks)**

Challenge

A vertical tower is 85 m high. A stone is projected at a speed of 20 m s^{-1} from the top of a tower at an angle of α below the horizontal. At the same time, a second stone is projected horizontally at a speed of 12 m s^{-1} from a window in the tower 45 m above the ground.

Given that the two stones move freely under gravity in the same vertical plane, and that they collide in mid-air, show that the time that elapses between the moment they are projected and the moment they collide is 2.5 s.

1.4 Projectile motion formulae

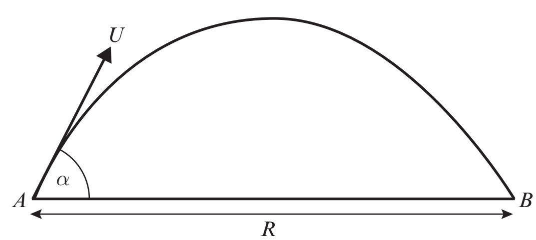
You need to be able to derive general formulae related to the motion of a particle which is projected from a point on a horizontal plane and moves freely under gravity.

Example 10 SKILLS REASONING/ARGUMENTATION

A particle is projected from a point on a horizontal plane with an initial velocity U, at an angle α above the horizontal, and moves freely under gravity until it hits the plane at point B. Given that the acceleration due to gravity is g, find expressions for:

a the time of flight, T

b the range, R, on the horizontal plane.



Taking the upward direction as positive and resolving the velocity of projection:

R(↑), $u_y = U\sin\alpha$

R(→), $u_x = U\cos\alpha$

a Considering vertical motion:

R(↑), $u = U\sin\alpha,\ s = 0,\ a = -g,\ t = T$

$s = ut + \frac{1}{2}at^2$

$0 = (U\sin\alpha)T - \frac{1}{2} \times g \times T^2$

$0 = T\left(U\sin\alpha - \frac{gT}{2}\right)$

either $T = 0$ (at A) or $U\sin\alpha - \frac{gT}{2} = 0$

so $T = \frac{2U\sin\alpha}{g}$

b Considering horizontal motion:

R(→), $v_x = U\cos\alpha,\ s = R,\ t = T$

$s = v_x t$

$R = U\cos\alpha \times T$ using $T = \frac{2U\sin\alpha}{g}$

$R = U\cos\alpha \times \frac{2U\sin\alpha}{g} = \frac{2U^2\sin\alpha\cos\alpha}{g}$

Using $2\sin\alpha\cos\alpha \equiv \sin 2\alpha$:

$R = \frac{U^2\sin 2\alpha}{g}$

Online Explore the parametric equations for the path of a particle and their Cartesian form, both algebraically and graphically using technology.

When the particle reaches the horizontal plane, the vertical displacement is 0.

Taking out the factor T, one solution is $T = 0$ which is at the start of the motion.

Problem-solving Follow the same steps as you would if you were given values of U and α and asked to find the time of flight and the range. The answer will be an algebraic expression in terms of U and α instead of a numerical value.

Substitute for T in the equation $R = U\cos\alpha \times T$

$U\cos\alpha \times \frac{2U\sin\alpha}{g} = \frac{U\cos\alpha}{1} \times \frac{2U\sin\alpha}{g}$

Use the double-angle formula for $\sin 2\alpha$.

Notation g is usually left as a letter in the formulae for projectile motion.

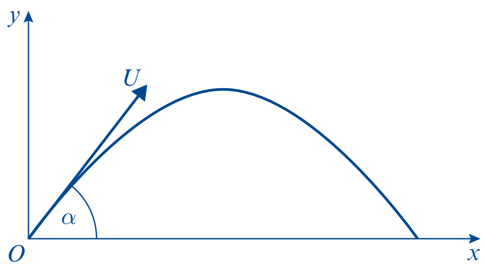
Example 11

A particle is projected from a point with speed U at an angle of elevation α and moves freely under gravity. When the particle has moved a horizontal distance x, its height above the point of projection is y.

a Show that $y = x\tan\alpha - \frac{gx^2}{2u^2}(1 + \tan^2\alpha)$.

A particle is projected from a point O on a horizontal plane, with speed $28\ \text{m s}^{-1}$ at an angle of elevation α. The particle passes through a point B, which is at a horizontal distance of 32 m from O and at a height of 8 m above the plane.

b Find the two possible values of α, giving your answers to the nearest degree.



a R(→), $u_x = U\cos\alpha$

R(↑), $u_y = U\sin\alpha$

For the horizontal motion:

R(→), $s = vt$

$x = u\cos\alpha \times t$ **(1)**

For the vertical motion, taking upward as positive:

R(↑), $s = ut + \frac{1}{2}at^2$

$y = U\sin\alpha \times t - \frac{1}{2}gt^2$ **(2)**

Rearranging **(1)** to make t the subject of the formula:

$t = \dfrac{x}{U\cos\alpha}$ **(3)**

Substituting **(3)** into **(2)**:

$$y = U\sin\alpha \times \frac{x}{U\cos\alpha} - \frac{1}{2}g\left(\frac{x}{U\cos\alpha}\right)^2$$

Using $\tan\alpha \equiv \dfrac{\sin\alpha}{\cos\alpha}$ and $\dfrac{1}{\cos\alpha} \equiv \sec\alpha$,

$$y = x\tan\alpha - \frac{gx^2}{2U^2}\sec^2\alpha$$

Using $\sec^2\alpha \equiv 1 + \tan^2\alpha$,

$$y = x\tan\alpha - \frac{gx^2}{2u^2}(1 + \tan^2\alpha)$$, as required.

b Using the result in **a** with $U = 28$, $x = 32$, $y = 8$ and $g = 9.8$

$8 = 32\tan\alpha - 6.4(1 + \tan^2\alpha)$

Rearranging as a quadratic in $\tan\alpha$:

$6.4\tan^2\alpha - 32\tan\alpha + 14.4 = 0$

$4\tan^2\alpha - 20\tan\alpha + 9 = 0$

$(2\tan\alpha - 1)(2\tan\alpha - 9) = 0$

$\tan\alpha = \frac{1}{2}, \frac{9}{2}$

$\alpha = 27°$ and $77°$, to the nearest degree

Resolve the velocity of projection horizontally and vertically.

You have obtained two equations, labelled **(1)** and **(2)**. Both equations contain t and the result you have been asked to show has no t in it. You must eliminate (i.e. remove) t using substitution.

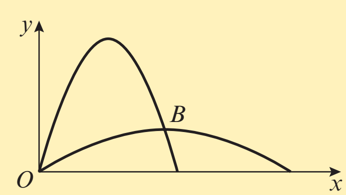
If the upward direction is taken as positive, the vertical acceleration is $-g$.

(1) and **(2)** are parametric equations describing the path of the particle. You can eliminate the parameter, t, to find the Cartesian form of the path.

To obtain a quadratic expression in $\tan\alpha$, you need to use the identity $\sec^2\alpha \equiv 1 + \tan^2\alpha$.

You should use your calculator to check the solutions to this equation.

There are two possible angles of elevation for which the particle will pass through B. This sketch illustrates the two paths.



- For a particle which is projected from a point on a horizontal plane with an initial velocity U at an angle α above the horizontal, and that moves freely under gravity:
  - Time of flight $= \frac{2U\sin\alpha}{g}$
  - Time to reach greatest height $= \frac{U\sin\alpha}{g}$
  - Range on horizontal plane $= \frac{U^2\sin 2\alpha}{g}$
  - Equation of trajectory: $y = x\tan\alpha - gx^2\frac{(1+\tan^2\alpha)}{2U^2}$

    where y is the vertical height of the particle, x is the horizontal distance from the point of projection, and g is the acceleration due to gravity.

Watch out You need to know how to derive the equations. But be careful of using them in projectile problems. They are hard to memorise, and it is usually safer to answer projectile problems using the techniques covered in Section 1.3.

Hint The equation for the trajectory of the particle is a **quadratic** equation for y in x. This proves that the path of a projectile moving freely under gravity is a quadratic curve, or **parabola**.

Exercise 1D

SKILLS REASONING/ARGUMENTATION

Whenever a numerical value of g is required, take $g = 9.8\text{ m s}^{-2}$ unless otherwise stated.

P **1** A particle is launched from a point on a horizontal plane with initial velocity $U\text{ m s}^{-1}$ at an angle of elevation α. The particle moves freely under gravity until it strikes the plane. The greatest height of the particle is h m.

Show that $h = \frac{U^2\sin^2\alpha}{2g}$.

P **2** A particle is projected from a point with speed 21 m s^{-1} at an angle of elevation α and moves freely under gravity. When the particle has moved a horizontal distance x m, its height above the point of projection is y m.

a Show that $y = x\tan\alpha - \frac{x^2}{90\cos^2\alpha}$.

b Given that $y = 8.1$ when $x = 36$, find the value of $\tan\alpha$.

P **3** A particle is launched from a point on a horizontal plane with initial speed $U\text{ m s}^{-1}$ at an angle of elevation α. The particle moves freely under gravity until it strikes the plane. The range of the particle is R m.

a Show that the time of flight of the particle is $\frac{2U\sin\alpha}{g}$ seconds.

b Show that $R = \frac{U^2\sin 2\alpha}{g}$.

c Deduce that, for a fixed u, the greatest possible range is when $\alpha = 45°$.

d Given that $R = \frac{2U^2}{5g}$, find the two possible values of the angle of elevation at which the particle could have been launched.

P **4** A firework is launched vertically with a speed of $v\,\text{m}\,\text{s}^{-1}$. When it reaches its maximum height, the firework explodes into two parts, which are projected horizontally in opposite directions, each with speed $2v\,\text{m}\,\text{s}^{-1}$.
Show that the two parts of the firework land a distance $\frac{4v^2}{g}$ m apart.

E/P **5** In this question use $g = 10\,\text{m}\,\text{s}^{-2}$.
A particle is projected from a point O with speed U at an angle of elevation α above the horizontal and moves freely under gravity. When the particle has moved a horizontal distance x, its height above O is y.

a Show that $y = x\tan\alpha - \dfrac{gx^2}{2U^2\cos^2\alpha}$. **(4 marks)**

A boy throws a stone from a point P at the end of a pier. The point P is 15 m above sea level. The stone is projected with a speed of $8\,\text{m}\,\text{s}^{-1}$ at an angle of elevation of 40°. By modelling the stone as a particle moving freely under gravity,

b find the horizontal distance of the stone from P when the stone is 2 m above sea level. **(5 marks)**

E/P **6** A particle is projected from a point with speed U at an angle of elevation α above the horizontal and moves freely under gravity. When it has moved a horizontal distance x, its height above the point of projection is y.

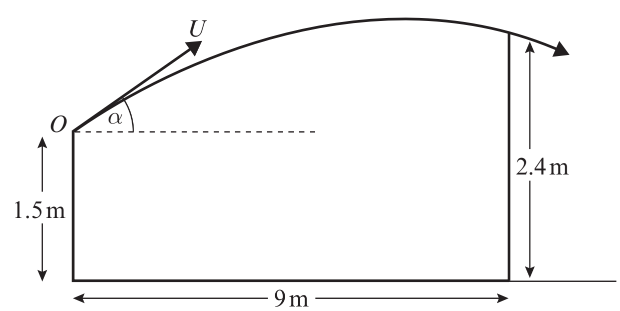
a Show that $y = x\tan\alpha - \dfrac{gx^2}{2U^2}(1 + \tan^2\alpha)$. **(5 marks)**

An athlete throws a javelin from a point P at a height of 2 m above horizontal ground. The javelin is projected at an angle of elevation of 45° with a speed of $30\,\text{m}\,\text{s}^{-1}$.
By modelling the javelin as a particle moving freely under gravity,

b find, to 3 significant figures, the horizontal distance of the javelin from P when it hits the ground **(5 marks)**

c find, to 2 significant figures, the time elapsed from the point the javelin is thrown to the point when it hits the ground. **(2 marks)**

E/P **7** A girl playing volleyball on horizontal ground hits the ball towards the net 9 m away from a point 1.5 m above the ground. The ball moves in a vertical plane which is perpendicular to the net. The ball just passes over the top of the net, which is 2.4 m above the ground, as shown in the diagram.
The ball is modelled as a particle projected with initial speed $U\,\text{m}\,\text{s}^{-1}$ from point O, 1.5 m above the ground at an angle α to the horizontal.



a By writing down expressions for the horizontal and vertical distances from O to the ball, t seconds after it was hit, show that when the ball passes over the net:

$$0.9 = 9\tan\alpha - \frac{81g}{2U^2\cos^2\alpha}$$ **(6 marks)**

Given that $\alpha = 30°$,

b find the speed of the ball as it passes over the net. **(6 marks)**

E/P **8** In this question, **i** and **j** are unit vectors in a horizontal and upward vertical direction respectively. An object is projected from a fixed point A on horizontal ground with velocity $(k\mathbf{i} + 2k\mathbf{j})\,\text{m s}^{-1}$, where k is a positive constant. The object moves freely under gravity until it strikes the ground at B, where it immediately comes to rest. **Relative to** O, the position vector of a point on the path of the object is $(x\mathbf{i} + y\mathbf{j})$ m.

a Show that $y = 2x - \dfrac{gx^2}{2k^2}$. **(5 marks)**

Given that $AB = R$ m and the maximum vertical height of the object above the ground is H m,

b using the result in part **a**, or otherwise, find, in terms of k and g,

i R **ii** H **(6 marks)**

Challenge

A stone is projected from a point on a straight sloping hill. Given that the hill slopes downwards at an angle of 45°, and that the stone is projected at an angle of 45° above the horizontal with speed $U\,\text{m s}^{-1}$, show that the stone lands a distance $\dfrac{2\sqrt{2}\,U^2}{g}$ m down the hill.

Chapter review 1

Whenever a numerical value of g is required, take $g = 9.8\,\text{m s}^{-2}$ unless otherwise stated.

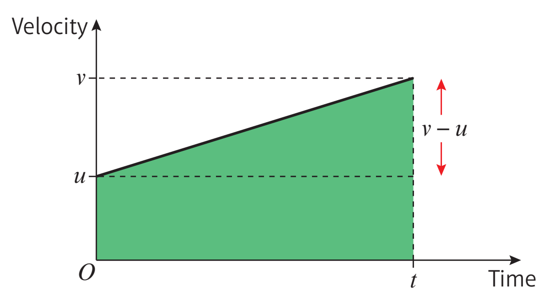
1 A ball is thrown vertically downwards from the top of a tower with speed $6\,\text{m s}^{-1}$. The ball strikes the ground with speed $25\,\text{m s}^{-1}$. Find the time the ball takes to move from the top of the tower to the ground.

2 A child drops a ball from a point at the top of a cliff which is 82 m above the sea. The ball is initially **at rest**. Find:

a the time taken for the ball to reach the sea **b** the speed with which the ball hits the sea.

c State one physical factor which has been ignored in making your calculation.

P **3** The velocity–time graph represents the motion of a particle moving in a straight line, accelerating from velocity u at time 0 to velocity v at time t.



a Use the graph to show that:

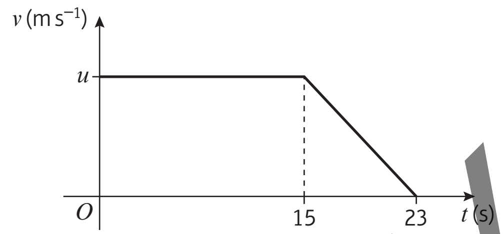
i $v = u + at$ **ii** $s = \left(\dfrac{u + v}{2}\right)t$

b Hence show that:

i $v^2 = u^2 + 2as$ **ii** $s = ut + \frac{1}{2}at^2$ **iii** $s = vt - \frac{1}{2}at^2$

E/P **4** A particle is projected vertically upwards with a speed of $30\,\text{m s}^{-1}$ from a point A. The point B is h metres above A. The particle moves freely under gravity and is above B for 2.4 s. Calculate the value of h. **(5 marks)**

P **5** The diagram is a velocity–time graph representing the motion of a cyclist along a straight road.
At time $t = 0$ s, the cyclist is moving with velocity u m s^{-1}. The velocity is maintained until time $t = 15$ s, when she slows down with constant **deceleration**, coming to rest when $t = 23$ s.
The total distance she travels in 23 s is 152 m.
Find the value of u.



6 A particle P is projected from a point O on a horizontal plane with speed 42 m s^{-1} and with angle of elevation 45°. After projection, the particle moves freely under gravity until it strikes the plane. Find:

a the greatest height above the plane reached by P

b the time of flight of P.

7 A stone is thrown horizontally with speed 21 m s^{-1} from a point P on the edge of a cliff h metres above sea level. The stone lands in the sea at a point Q, where the horizontal distance of Q from the cliff is 56 m.
Calculate the value of h.

E **8** A ball is thrown from a window above a horizontal lawn. The velocity of projection is 15 m s^{-1} and the angle of elevation is α, where $\tan\alpha = \frac{4}{3}$. The ball takes 4 s to reach the lawn. Find:

a the horizontal distance between the point of projection and the point where the ball hits the lawn **(3 marks)**

b the vertical height above the lawn from which the ball was thrown. **(3 marks)**

E **9** A projectile is fired with velocity 40 m s^{-1} at an angle of elevation of 30° from a point A on horizontal ground. The projectile moves freely under gravity until it reaches the ground at the point B. Find:

a the distance AB **(5 marks)**

b the speed of the projectile at the first instant when it is 15 m above the ground. **(5 marks)**

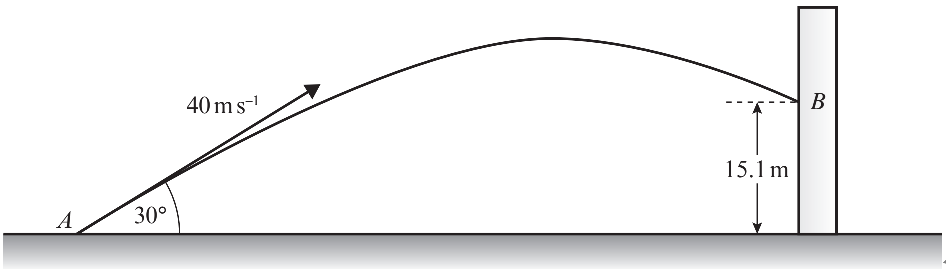
E/P **10** A projectile P is projected from a point on a horizontal plane with speed U at an angle of elevation θ.

a Show that the range of the projectile is $\frac{U^2 \sin 2\theta}{g}$. **(6 marks)**

b Hence find, as θ varies, the maximum range of the projectile. **(2 marks)**

c Given that the range of the projectile is $\frac{2U^2}{3g}$, find the two possible value of θ.
Give your answers to the nearest 0.1°. **(3 marks)**

(E) **11**



A golf ball is driven from a point A with a speed of $40\,\mathrm{m\,s^{-1}}$ at an angle of elevation of 30°. On its downward flight, the ball hits a tree at a height 15.1 m above the level of A, as shown in the diagram above. Find:

a the time taken by the ball to reach its greatest height above A **(3 marks)**

b the time taken by the ball to travel from A to B **(6 marks)**

c the speed with which the ball hits the tree. **(5 marks)**

(E/P) **12** A particle P is projected from a fixed origin O with velocity $(12\mathbf{i} + 5\mathbf{j})\,\mathrm{m\,s^{-1}}$. The particle moves freely under gravity and passes through the point A with position vector $\lambda(2\mathbf{i} - \mathbf{j})$ m, where λ is a positive constant.

a Find the value of λ. **(6 marks)**

b Find:

i the speed of P at the instant when it passes through A

ii the direction of motion of P at the instant when it passes through A. **(7 marks)**

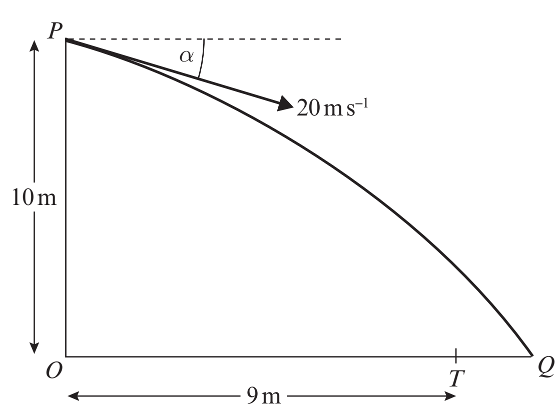
(E/P) **13** In this question use $g = 10\,\mathrm{m\,s^{-2}}$.
A boy plays a game at a fairground. He needs to throw a ball through a hole in a vertical target to win a prize. The motion of the ball is modelled as that of a particle moving freely under gravity. The ball moves in a vertical plane which is perpendicular to the plane of the target. The boy throws the ball horizontally at the same height as the hole with a speed of $10\,\mathrm{m\,s^{-1}}$. It hits the target at a point 20 cm below the hole.

a Find the horizontal distance from the point where the ball was thrown to the target. **(4 marks)**

The boy throws the ball again with the same speed and at the same distance from the target.

b Work out the possible angles above the horizontal the boy could throw the ball so that it passes through the hole. **(6 marks)**

(E/P) **14** In this question use $g = 10\,\mathrm{m\,s^{-2}}$.
A stone is thrown from a point P at a target, which is on horizontal ground. The point P is 10 m above the point O on the ground. The stone is thrown from P with speed $20\,\mathrm{m\,s^{-1}}$ at an angle of α below the horizontal, where $\tan\alpha = \frac{3}{4}$.



The stone is modelled as a particle and the target as a point T. The distance OT is 9 m. The stone misses the target and hits the ground at the point Q, where OTQ is a straight line, as shown in the diagram. Find:

a the time taken by the stone to travel from P to Q **(5 marks)**

b the distance TQ. **(4 marks)**

The point A is on the path of the stone vertically above T.

c Find the speed of the stone at A. **(5 marks)**

E/P **15** A vertical mast is 32 m high. Two balls P and Q are projected at the same time. Ball P is projected horizontally from the top of the mast with speed 18 m s^{-1}. Ball Q is projected from the bottom of the mast with speed 30 m s^{-1} at an angle α above the horizontal. The balls move freely under gravity in the same vertical plane and collide in mid-air. By considering the horizontal motion of each ball,

a prove that $\cos\alpha = \frac{3}{5}$. **(4 marks)**

b Find the time which elapses between the instant when the balls are projected and the instant when they collide. **(4 marks)**

Challenge

A cruise ship is 250 m long, and is accelerating forwards in a straight line at a constant rate of 1.5 m s^{-2}. A golfer stands at the stern (back) of the cruise ship and hits a golf ball towards the bow (front). Given that the golfer hits the golf ball at an angle of elevation of 60°, and that the ball lands directly on the bow of the cruise ship, find the speed, v, with which the golfer hits the ball.

Problem-solving

You need to calculate the initial speed of the ball relative to the golfer. This is the speed the ball would appear to be travelling at if you were standing on the ship.

Summary of key points

1 The force of **gravity** causes all objects to accelerate towards the Earth. If you ignore the effects of air resistance, this acceleration is constant. It does not depend on the mass of the object.

2 An object moving vertically freely under gravity can be modelled as a particle with a constant downward acceleration of $g = 9.8\text{ m s}^{-2}$.

3 The **horizontal** motion of a projectile is modelled as having **constant velocity** ($a = 0$). You can use the formula $s = vt$.

4 The **vertical** motion of a projectile is modelled as having **constant acceleration** due to gravity ($a = g$).

5 When a particle is projected with initial velocity U, at an angle α above the horizontal:
- The **horizontal component** of the initial velocity is $U\cos\alpha$.
- The **vertical component** of the initial velocity is $U\sin\alpha$.

6 A projectile reaches its point of greatest height when the vertical component of its velocity is equal to 0.

7 For a particle which is projected from a point on a horizontal plane with an initial velocity U at an angle α above the horizontal, and that moves freely under gravity:

- Time of flight = $\dfrac{2U\sin\alpha}{g}$
- Time to reach greatest height = $\dfrac{U\sin\alpha}{g}$
- Range on horizontal plane = $\dfrac{U^2\sin 2\alpha}{g}$
- Equation of trajectory: $y = x\tan\alpha - gx^2\dfrac{(1+\tan^2\alpha)}{2U^2}$

  where y is the vertical height of the particle, x is the horizontal distance from the point of projection, and g is the acceleration due to gravity.