PEARSON EDEXCEL INTERNATIONAL A LEVEL

PURE MATHEMATICS 3

Student Book

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ABOUT THIS BOOK

The following three themes have been fully integrated throughout the Pearson Edexcel International Advanced Level in Mathematics series, so they can be applied alongside your learning.

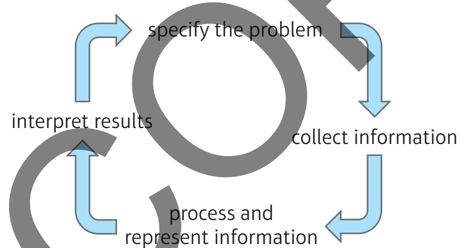
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols

2. Mathematical problem-solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Challenge questions provide extra stretch

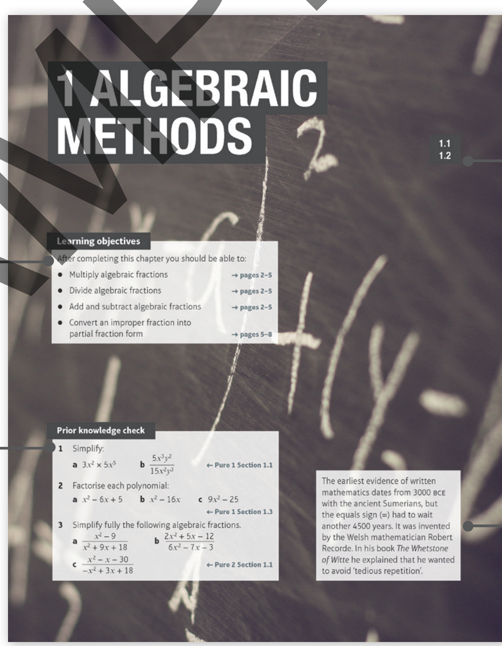
The Mathematical Problem-Solving Cycle



3. Transferable skills

- Transferable skills are embedded throughout this book, in the exercises and in some examples
- These skills are signposted to show students which skills they are using and developing

Finding your way around the book



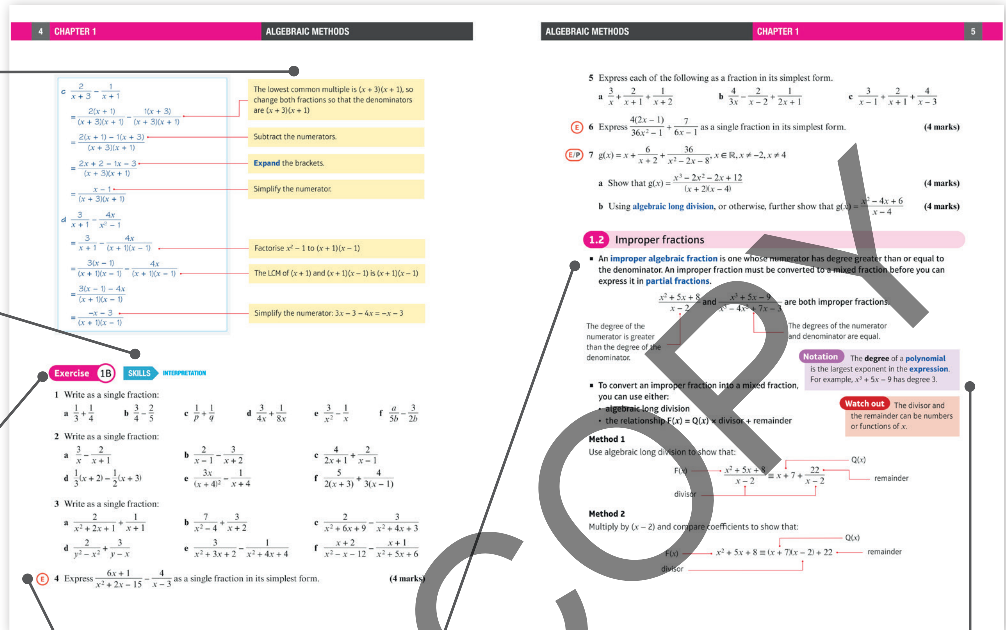
Each chapter is mapped to the specification content for easy reference

Each chapter starts with a list of *Learning objectives*

The *Prior knowledge check* helps make sure you are ready to start the chapter

The real world applications of the maths you are about to learn are highlighted at the start of the chapter.

Glossary terms will be identified by bold blue text on their first appearance.



Step-by-step worked examples focus on the key types of questions you'll need to tackle

Transferable skills are signposted where they naturally occur in the exercises and examples

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard

Exercises are packed with exam-style questions to ensure you are ready for the exams

Each section begins with explanation and key learning points

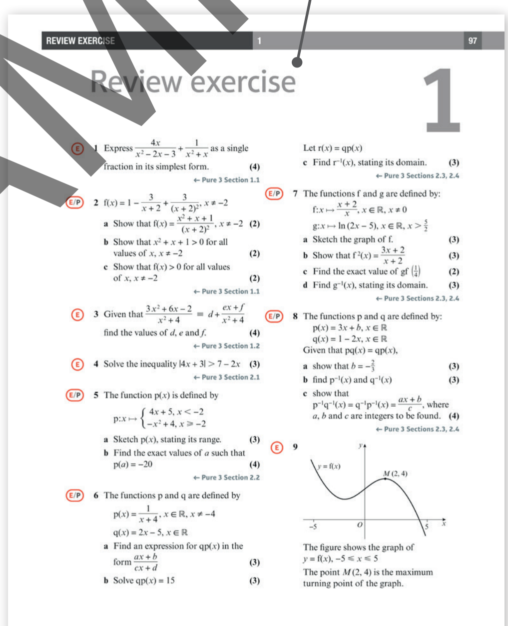
Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

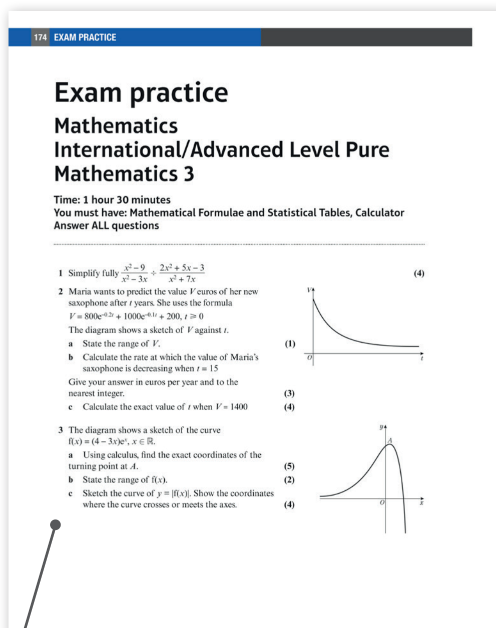
Exam-style questions are flagged with Ⓔ

Problem-solving questions are flagged with Ⓟ

Each chapter ends with a *Chapter review* and a *Summary of key points*

After every few chapters, a *Review exercise* helps you consolidate your learning with lots of exam-style questions





A full practice paper at the back of the book helps you prepare for the real thing

QUALIFICATION AND ASSESSMENT OVERVIEW

Qualification and content overview

Pure Mathematics 3 (P3) is a **compulsory** unit in the following qualifications:

International Advanced Level in Mathematics

International Advanced Level in Pure Mathematics

Assessment overview

The following table gives an overview of the assessment for this unit.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

Unit	Percentage	Mark	Time	Availability
P3: Pure Mathematics 3 Paper code WMA13/01	$16\frac{2}{3}$% of IAL	75	1 hour 30 min	January, June and October First assessment June 2020

IAL: International Advanced A Level.

Assessment objectives and weightings

		Minimum weighting in IAS and IAL
AO1	Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of contexts.	30%
AO2	Construct rigorous mathematical arguments and proofs through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions, including the construction of extended arguments for handling substantial problems presented in unstructured form.	30%
AO3	Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinement of such models.	10%
AO4	Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.	5%
AO5	Use contemporary calculator technology and other permitted resources (such as formulae booklets or statistical tables) accurately and efficiently; understand when not to use such technology, and its limitations. Give answers to appropriate accuracy.	5%

Relationship of assessment objectives to units

P3	Assessment objective				
	AO1	AO2	AO3	AO4	AO5
Marks out of 75	25–30	25–30	5–10	5–10	5–10
%	$33\frac{1}{3}$–40	$33\frac{1}{3}$–40	$6\frac{2}{3}$–$13\frac{1}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$

Calculators

Students may use a calculator in assessments for these qualifications. Centres are responsible for making sure that calculators used by their students meet the requirements given in the table below.

Students are expected to have available a calculator with at least the following keys: +, –, ×, ÷, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, ln x, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory.

Prohibitions

Calculators with any of the following facilities are prohibited in all examinations:

- databanks
- retrieval of text or formulae
- built-in symbolic algebra manipulations
- symbolic differentiation and/or integration
- language translators
- communication with other machines or the internet

Extra online content

Whenever you see an *Online* box, it means that there is extra online content available to support you.

SolutionBank

SolutionBank provides a full worked solution for questions in the book. Download all the solutions as a PDF or quickly find the solution you need online.

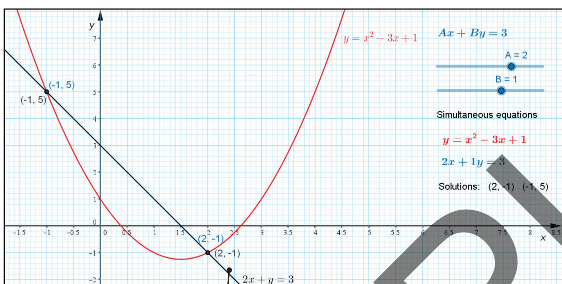
Use of technology

Explore topics in more detail, visualise problems and consolidate your understanding. Use pre-made GeoGebra activities or Casio resources for a graphic calculator.

Online Find the point of intersection graphically using technology.

GeoGebra

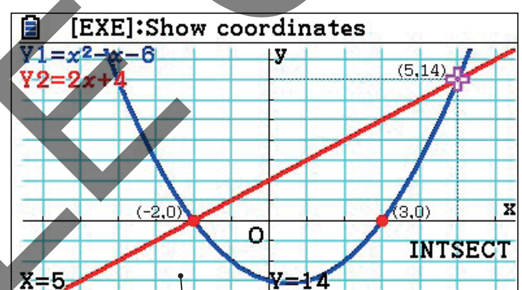
GeoGebra-powered interactives



Interact with the mathematics you are learning using GeoGebra's easy-to-use tools

CASIO

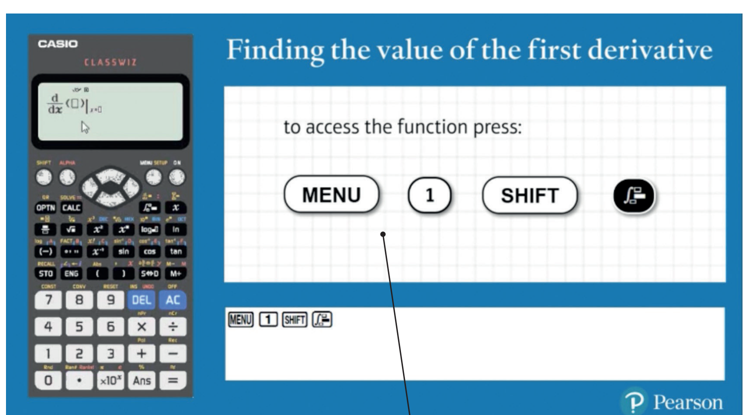
Graphic calculator interactives



Explore the mathematics you are learning and gain confidence in using a graphic calculator

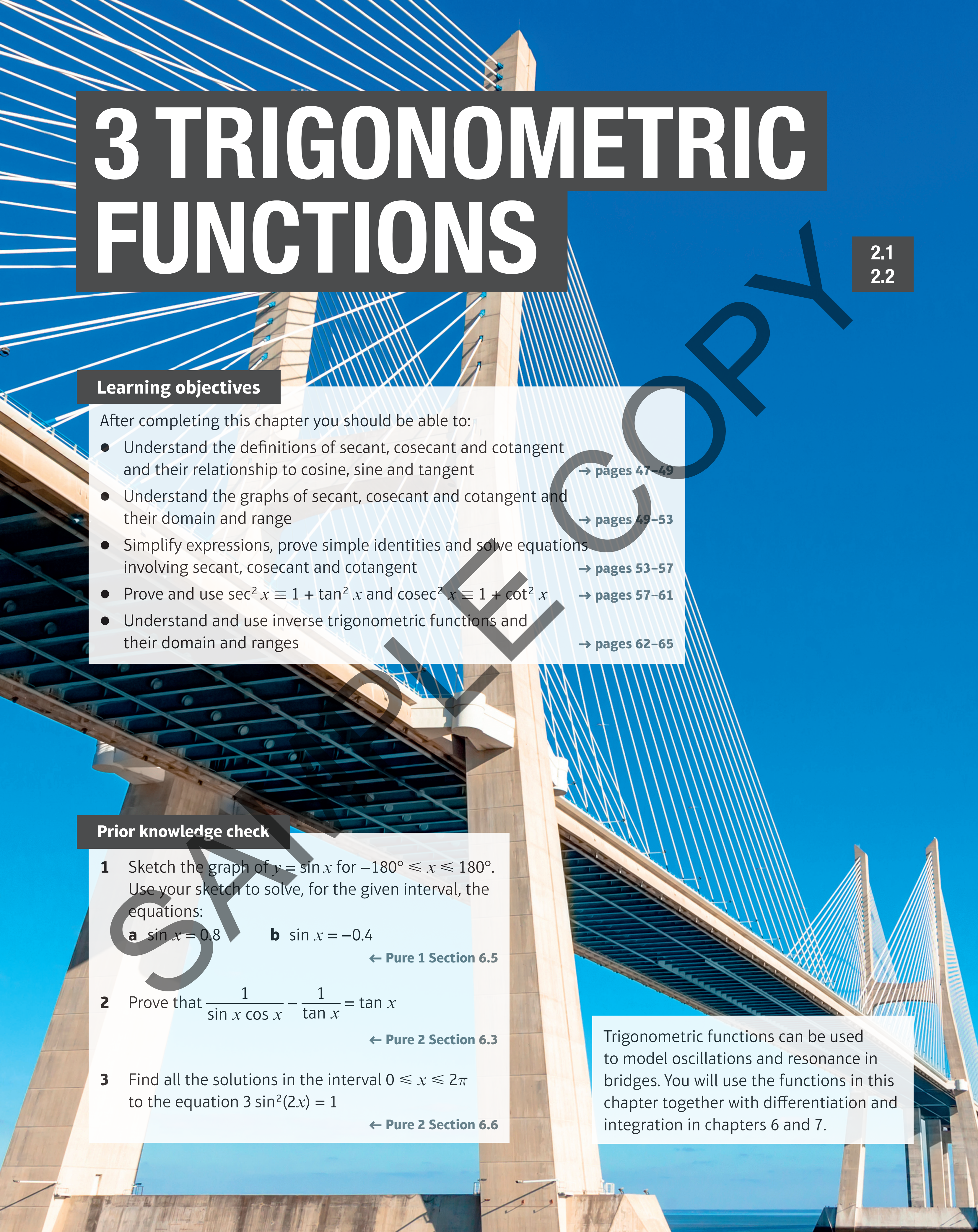
Calculator tutorials

Our helpful video tutorials will guide you through how to use your calculator in the exams. They cover both Casio's scientific and colour graphic calculators.



Online Work out each coefficient quickly using the ${}^nC_r$ and power functions on your calculator.

Step-by-step guide with audio instructions on exactly which buttons to press and what should appear on your calculator's screen



3 TRIGONOMETRIC FUNCTIONS

2.1
2.2

Learning objectives

After completing this chapter you should be able to:

- Understand the definitions of secant, cosecant and cotangent and their relationship to cosine, sine and tangent → pages 47–49
- Understand the graphs of secant, cosecant and cotangent and their domain and range → pages 49–53
- Simplify expressions, prove simple identities and solve equations involving secant, cosecant and cotangent → pages 53–57
- Prove and use $\sec^2 x \equiv 1 + \tan^2 x$ and $\text{cosec}^2 x \equiv 1 + \cot^2 x$ → pages 57–61
- Understand and use inverse trigonometric functions and their domain and ranges → pages 62–65

Prior knowledge check

1 Sketch the graph of $y = \sin x$ for $-180° \leqslant x \leqslant 180°$. Use your sketch to solve, for the given interval, the equations:

a $\sin x = 0.8$ **b** $\sin x = -0.4$

← Pure 1 Section 6.5

2 Prove that $\dfrac{1}{\sin x \cos x} - \dfrac{1}{\tan x} = \tan x$

← Pure 2 Section 6.3

3 Find all the solutions in the interval $0 \leqslant x \leqslant 2\pi$ to the equation $3\sin^2(2x) = 1$

← Pure 2 Section 6.6

Trigonometric functions can be used to model oscillations and resonance in bridges. You will use the functions in this chapter together with differentiation and integration in chapters 6 and 7.

3.1 Secant, cosecant and cotangent

Secant (sec), cosecant (cosec) and cotangent (cot) are known as the **reciprocal** trigonometric functions.

- $\sec x = \dfrac{1}{\cos x}$ (**undefined** for values of x for which $\cos x = 0$)
- $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (undefined for values of x for which $\sin x = 0$)
- $\cot x = \dfrac{1}{\tan x}$ (undefined for values of x for which $\tan x = 0$)

You can also write $\cot x$ in terms of $\sin x$ and $\cos x$.

- $\cot x = \dfrac{\cos x}{\sin x}$

Example 1

Use your calculator to write down the values of:

a sec 280° **b** cot 115°

a $\sec 280° = \dfrac{1}{\cos 280°} = 5.76$ (3 s.f.)

b $\cot 115° = \dfrac{1}{\tan 115°} = -0.466$ (3 s.f.)

Make sure your calculator is in degrees mode.

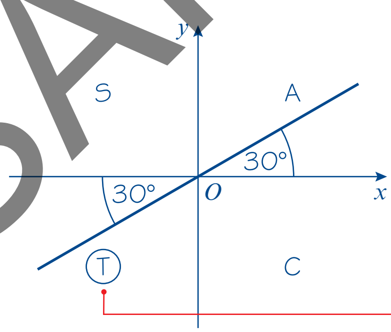
Example 2

Work out the exact values of:

a sec 210° **b** $\operatorname{cosec} \dfrac{3\pi}{4}$

Exact here means give in surd form.

a $\sec 210° = \dfrac{1}{\cos 210°}$



210° is in the 3rd quadrant, so cos 210° = –cos 30°

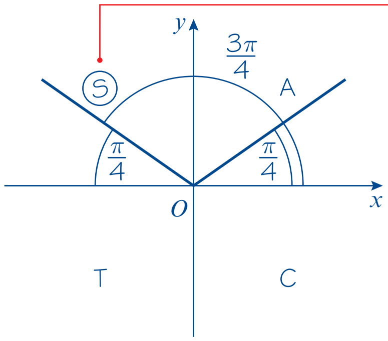
$\cos 30° = \dfrac{\sqrt{3}}{2}$ so $-\cos 30° = -\dfrac{\sqrt{3}}{2}$

So $\sec 210° = -\dfrac{2}{\sqrt{3}}$

Or $\sec 210° = -\dfrac{2\sqrt{3}}{3}$ if you rationalise the denominator.

b $\text{cosec}\,\frac{3\pi}{4} = \frac{1}{\sin\left(\frac{3\pi}{4}\right)}$

$\frac{3\pi}{4}$ is in the 2nd quadrant, so $\sin\frac{3\pi}{4} = +\sin\frac{\pi}{4}$



So $\text{cosec}\,\frac{3\pi}{4} = \frac{1}{\sin\left(\frac{\pi}{4}\right)}$

$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

So $\text{cosec}\left(\frac{3\pi}{4}\right) = \sqrt{2}$

Exercise 3A

SKILLS ANALYSIS

1 Without using your calculator, write down the sign of:

a sec 300° b cosec 190° c cot 110°
d cot 200° e sec 95°

2 Use your calculator to find, to 3 significant figures, the values of:

a sec 100° b cosec 260° c cosec 280°
d cot 550° e $\cot\frac{4\pi}{3}$ f sec 2.4 rad
g $\text{cosec}\,\frac{11\pi}{10}$ h sec 6 rad

3 Find the exact value (as an integer, fraction or surd) of each of the following:

a cosec 90° b cot 135° c sec 180°
d sec 240° e cosec 300° f cot (–45°)
g sec 60° h cosec (–210°) i sec 225°
j $\cot\frac{4\pi}{3}$ k $\sec\frac{11\pi}{6}$ l $\text{cosec}\left(-\frac{3\pi}{4}\right)$

(P) 4 Prove that $\text{cosec}(\pi - x) \equiv \text{cosec}\,x$

(P) 5 Show that cot 30° sec 30° = 2

(P) 6 Show that $\text{cosec}\,\frac{2\pi}{3} + \sec\frac{2\pi}{3} = a + b\sqrt{3}$, where a and b are real numbers to be found.

Challenge

SKILLS CREATIVITY

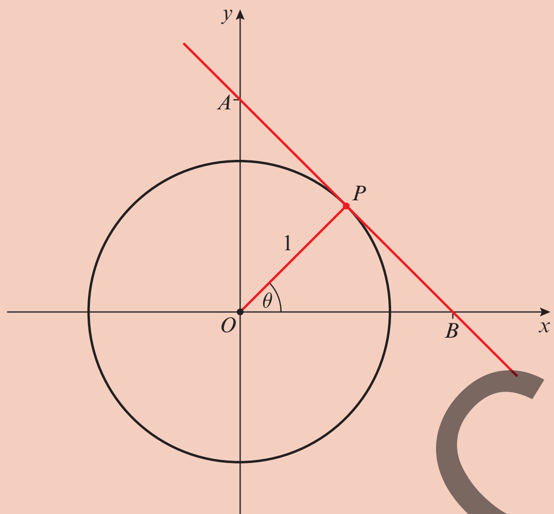
The point P lies on the unit circle, centre O. The radius OP makes an **acute angle** of θ with the positive x-axis. The tangent to the circle at P intersects the coordinate axes at points A and B.

Prove that:

a $OB = \sec\theta$

b $OA = \operatorname{cosec}\theta$

c $AP = \cot\theta$

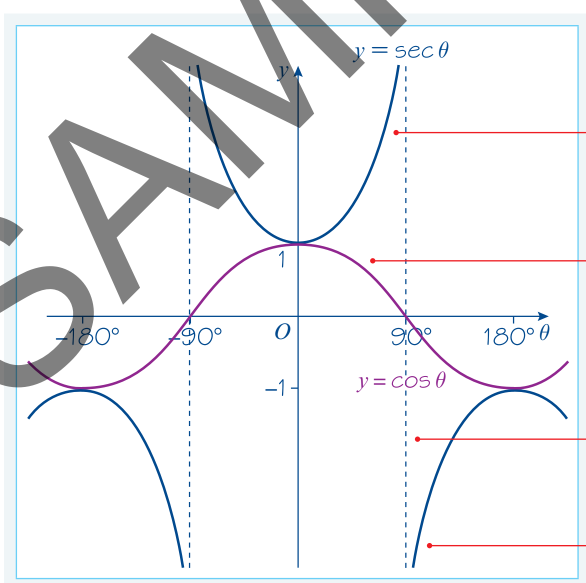


3.2 Graphs of sec x, cosec x and cot x

You can use the graphs of $y = \cos x$, $y = \sin x$ and $y = \tan x$ to sketch the graphs of their reciprocal functions.

Example 3 SKILLS INTERPRETATION

Sketch, in the **interval** $-180° \leqslant \theta \leqslant 180°$, the graph of $y = \sec\theta$



First draw the graph $y = \cos\theta$

For each value of θ, the value of $\sec\theta$ is the reciprocal of the corresponding value of $\cos\theta$.

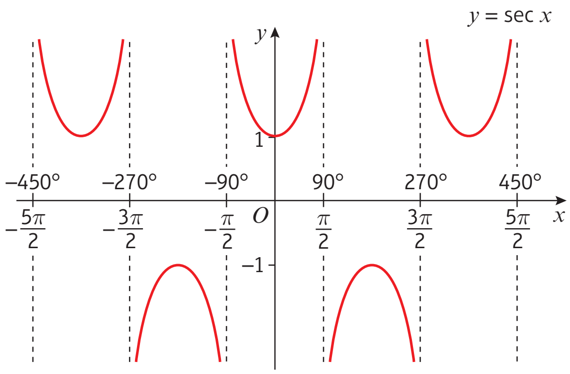
In particular: $\cos 0° = 1$, so $\sec 0° = 1$;
and $\cos 180° = -1$, so $\sec 180° = -1$

As θ approaches 90° from the left, $\cos\theta$ is +ve but approaches zero, and so $\sec\theta$ is +ve but becomes increasingly large.

At $\theta = 90°$, $\sec\theta$ is undefined and there is a vertical asymptote. This is also true for $\theta = -90°$

As θ approaches 90° from the right, $\cos\theta$ is −ve but approaches zero, and so $\sec\theta$ is −ve but becomes increasingly large negative.

- The graph of $y = \sec x$, $x \in \mathbb{R}$, has **symmetry** in the y-axis and has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$

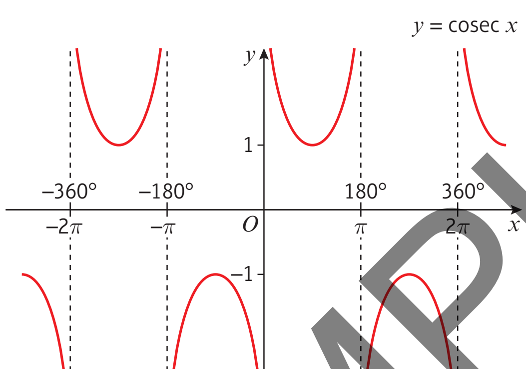


Notation The domain can also be given as $x \in \mathbb{R}, x \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$

$\mathbb{Z}$ is the symbol used for **integers**, which are the positive and negative whole numbers including 0.

- The domain of $y = \sec x$ is $x \in \mathbb{R}$, $x \neq 90°, 270°, 450°, \dots$ or any odd multiple of 90°
- In radians the domain is $x \in \mathbb{R}$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$ or any odd multiple of $\frac{\pi}{2}$
- The range of $y = \sec x$ is $y \leqslant -1$ or $y \geqslant 1$

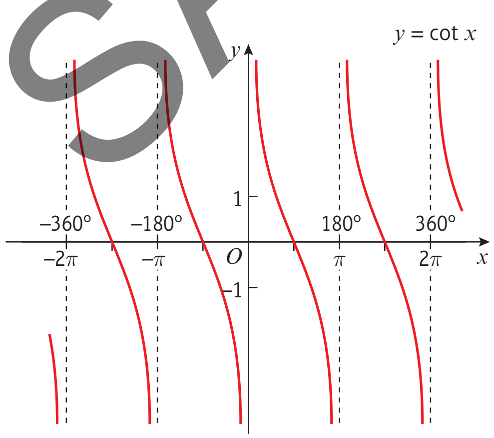
- The graph of $y = \operatorname{cosec} x$, $x \in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\sin x = 0$



Notation The domain can also be given as $x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$

- The domain of $y = \operatorname{cosec} x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \dots$ or any multiple of 180°
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \dots$ or any multiple of π
- The range of $y = \operatorname{cosec} x$ is $y \leqslant -1$ or $y \geqslant 1$

- The graph of $y = \cot x$, $x \in \mathbb{R}$, has period 180° or π radians. It has vertical asymptotes at all the values of x for which $\tan x = 0$



- The domain of $y = \cot x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of π
- The range of $y = \cot x$ is $y \in \mathbb{R}$

Notation The domain can also be given as $x \in \mathbb{R}, x \neq n\pi, n \in \mathbb{Z}$

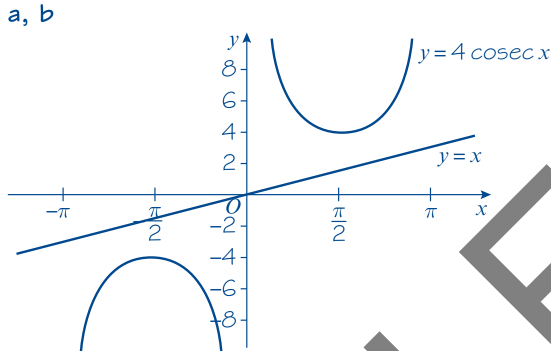
Example 4

a Sketch the graph of $y = 4\operatorname{cosec} x$, $-\pi \leqslant x \leqslant \pi$

b On the same axes, sketch the line $y = x$

c State the number of solutions to the equation $4\operatorname{cosec} x - x = 0$, $-\pi \leqslant x \leqslant \pi$

a, b



$y = 4\operatorname{cosec} x$ is a stretch of the graph of $y = \operatorname{cosec} x$, scale factor 4 in the y-direction. You only need to draw the graph for $-\pi \leqslant x \leqslant \pi$

c $4\operatorname{cosec} x - x = 0$

$4\operatorname{cosec} x = x$

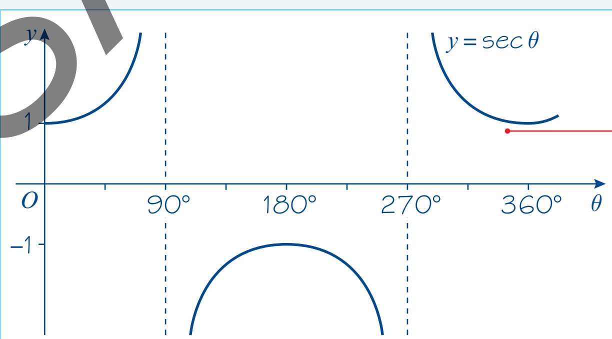
$y = 4\operatorname{cosec} x$ and $y = x$ do not intersect for $-\pi \leqslant x \leqslant \pi$ so the equation has no solutions in the given range.

Problem-solving

The solutions to the equation $f(x) = g(x)$ correspond to the points of intersection of the graphs of $y = f(x)$ and $y = g(x)$

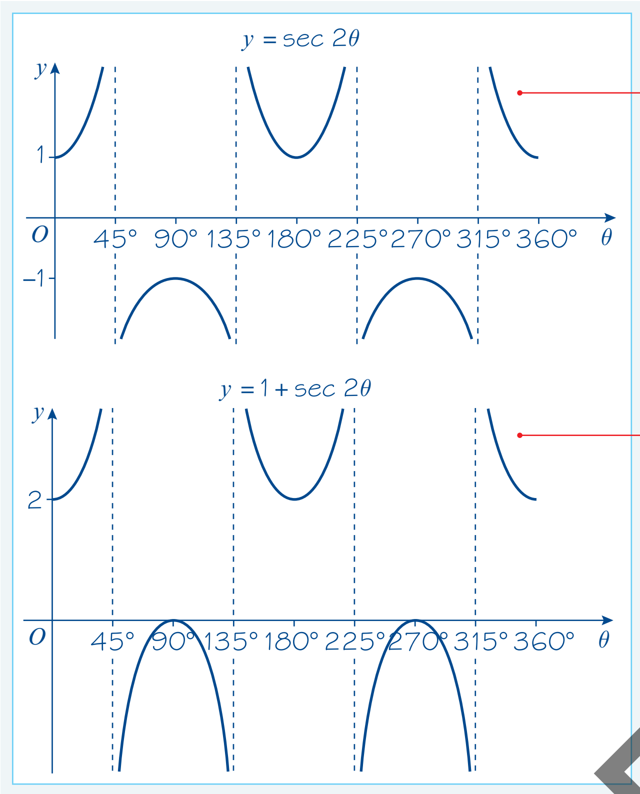
Example 5

Sketch, in the interval $0° \leqslant \theta \leqslant 360°$, the graph of $y = 1 + \sec 2\theta$



Online Explore transformations of the graphs of reciprocal trigonometric functions using technology.

Step 1
Draw the graph of $y = \sec\theta$



Step 2
Stretch in the θ-direction with scale factor $\frac{1}{2}$

Step 3
Translate by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Exercise 3B

SKILLS INTERPRETATION

1 Sketch, in the interval $-540° \leqslant \theta \leqslant 540°$, the graphs of:

a $y = \sec\theta$ **b** $y = \operatorname{cosec}\theta$ **c** $y = \cot\theta$

2 **a** Sketch, on the same set of axes, in the interval $-\pi \leqslant x \leqslant \pi$, the graphs of $y = \cot x$ and $y = -x$

b **Deduce** the number of solutions of the equation $\cot x + x = 0$ in the interval $-\pi \leqslant x \leqslant \pi$

3 **a** Sketch, on the same set of axes, in the interval $0° \leqslant \theta \leqslant 360°$, the graphs of $y = \sec\theta$ and $y = -\cos\theta$

b Explain how your graphs show that $\sec\theta = -\cos\theta$ has no solutions.

4 **a** Sketch, on the same set of axes, in the interval $0° \leqslant \theta \leqslant 360°$, the graphs of $y = \cot\theta$ and $y = \sin 2\theta$

b Deduce the number of solutions of the equation $\cot\theta = \sin 2\theta$ in the interval $0° \leqslant \theta \leqslant 360°$

5 **a** Sketch on separate axes, in the interval $0° \leqslant \theta \leqslant 360°$, the graphs of $y = \tan\theta$ and $y = \cot(\theta + 90°)$

b Hence, state a relationship between $\tan\theta$ and $\cot(\theta + 90°)$

(P) **6 a** Describe the relationships between the graphs of:

i $y = \tan\left(\theta + \frac{\pi}{2}\right)$ and $y = \tan\theta$ **ii** $y = \cot(-\theta)$ and $y = \cot\theta$

iii $y = \text{cosec}\left(\theta + \frac{\pi}{4}\right)$ and $y = \text{cosec}\,\theta$ **iv** $y = \sec\left(\theta - \frac{\pi}{4}\right)$ and $y = \sec\theta$

b By considering the graphs of $y = \tan\left(\theta + \frac{\pi}{2}\right)$, $y = \cot(-\theta)$, $y = \text{cosec}\left(\theta + \frac{\pi}{4}\right)$ and $y = \sec\left(\theta - \frac{\pi}{4}\right)$, state which pairs of functions are equal.

(P) **7** Sketch on separate axes, in the interval $0° \leqslant \theta \leqslant 360°$, the graphs of:

a $y = \sec 2\theta$ **b** $y = -\text{cosec}\,\theta$ **c** $y = 1 + \sec\theta$

d $y = \text{cosec}(\theta - 30°)$ **e** $y = 2\sec(\theta - 60°)$ **f** $y = \text{cosec}(2\theta + 60°)$

g $y = -\cot(2\theta)$ **h** $y = 1 - 2\sec\theta$

In each case, show the coordinates of any maximum and minimum points, and of any points at which the curve meets the axes.

8 Write down the periods of the following functions. Give your answers in terms of π.

a $\sec 3\theta$ **b** $\text{cosec}\,\frac{1}{2}\theta$ **c** $2\cot\theta$ **d** $\sec(-\theta)$

(E/P) **9 a** Sketch, in the interval $-2\pi \leqslant x \leqslant 2\pi$, the graph of $y = 3 + 5\,\text{cosec}\,x$ **(3 marks)**

b Hence deduce the range of values of k for which the equation $3 + 5\,\text{cosec}\,x = k$ has no solutions. **(2 marks)**

(E/P) **10 a** Sketch the graph of $y = 1 + 2\sec\theta$ in the interval $-\pi \leqslant \theta \leqslant 2\pi$ **(3 marks)**

b Write down the θ-coordinates of points at which the **gradient** is zero. **(2 marks)**

c Deduce the maximum and minimum values of $\frac{1}{1 + 2\sec\theta}$ and give the smallest positive values of θ at which they occur. **(4 marks)**

3.3 Using sec x, cosec x and cot x

You need to be able to simplify expressions, prove identities and solve equations involving sec x, cosec x and cot x.

■ **sec $x = k$ and cosec $x = k$ have no solutions for $-1 < k < 1$**

Example 6

Simplify:

a $\sin\theta\cot\theta\sec\theta$

b $\sin\theta\cos\theta(\sec\theta + \text{cosec}\,\theta)$

a $\sin\theta\cot\theta\sec\theta$

$\equiv \sin\theta \times \frac{\cos\theta}{\sin\theta} \times \frac{1}{\cos\theta}$

$\equiv 1$

b $\sec\theta + \csc\theta \equiv \frac{1}{\cos\theta} + \frac{1}{\sin\theta}$

$\equiv \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta}$

So $\sin\theta\cos\theta(\sec\theta + \csc\theta) = \sin\theta + \cos\theta$

Write the expression in terms of sin and cos, using $\cot\theta \equiv \frac{\cos\theta}{\sin\theta}$ and $\sec\theta \equiv \frac{1}{\cos\theta}$

Write the expression in terms of sin and cos, using $\sec\theta \equiv \frac{1}{\cos\theta}$ and $\csc\theta \equiv \frac{1}{\sin\theta}$

Put over a common denominator.

Multiply both sides by $\sin\theta\cos\theta$.

Example 7

a Prove that $\frac{\cot\theta\csc\theta}{\sec^2\theta + \csc^2\theta} \equiv \cos^3\theta$

b Hence explain why the equation $\frac{\cot\theta\csc\theta}{\sec^2\theta + \csc^2\theta} = 8$ has no solutions.

a Consider the LHS:

The numerator $\cot\theta\csc\theta$

$\equiv \frac{\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta} \equiv \frac{\cos\theta}{\sin^2\theta}$

The denominator $\sec^2\theta + \csc^2\theta$

$\equiv \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$

$\equiv \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$

$\equiv \frac{1}{\cos^2\theta\sin^2\theta}$

So $\frac{\cot\theta\csc\theta}{\sec^2\theta + \csc^2\theta}$

$\equiv \left(\frac{\cos\theta}{\sin^2\theta}\right) \div \left(\frac{1}{\cos^2\theta\sin^2\theta}\right)$

$\equiv \frac{\cos\theta}{\sin^2\theta} \times \frac{\cos^2\theta\sin^2\theta}{1}$

$\equiv \cos^3\theta$

b Since $\frac{\cot\theta\csc\theta}{\sec^2\theta + \csc^2\theta} \equiv \cos^3\theta$ we are required to solve the equation $\cos^3\theta = 8$

$\cos^3\theta = 8 \Rightarrow \cos\theta = 2$ which has no solutions since $-1 \leqslant \cos\theta \leqslant 1$

Write the expression in terms of sin and cos, using $\cot\theta \equiv \frac{\cos\theta}{\sin\theta}$ and $\csc\theta = \frac{1}{\sin\theta}$

Write the expression in terms of sin and cos, using $\sec^2\theta \equiv \left(\frac{1}{\cos\theta}\right)^2 \equiv \frac{1}{\cos^2\theta}$ and $\csc^2\theta \equiv \frac{1}{\sin^2\theta}$

Remember that $\sin^2\theta + \cos^2\theta \equiv 1$

Remember to invert the fraction when changing from ÷ sign to ×.

Problem-solving

Write down the equivalent equation, and state the range of possible values for $\cos\theta$.

Example 8

Solve the equations:

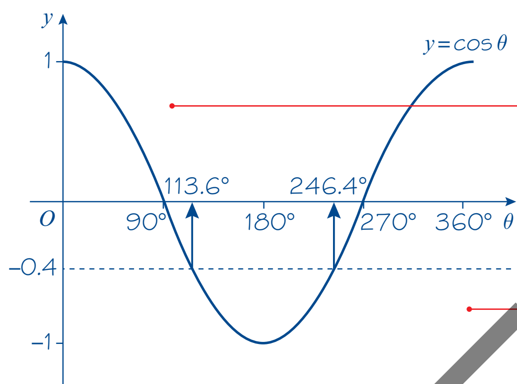
a $\sec\theta = -2.5$ **b** $\cot 2\theta = 0.6$

in the interval $0° \leqslant \theta \leqslant 360°$

a $\frac{1}{\cos\theta} = -2.5$

Substitute $\frac{1}{\cos\theta}$ for $\sec\theta$ and then simplify to get an equation in the form $\cos\theta = k$

$\cos\theta = \frac{1}{-2.5} = -0.4$



Sketch the graph of $y = \cos\theta$ for the given interval. The graph is **symmetrical** about $\theta = 180°$
Find the principal value using your calculator then subtract this from 360° to find the second solution.

You could also find all the solutions using a CAST diagram. This method is shown for part **b** below.

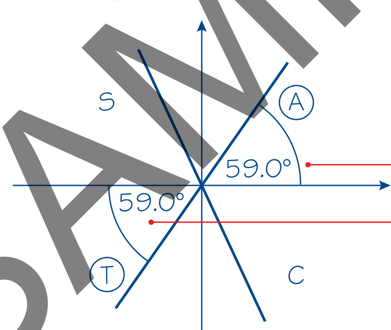
$\theta = 113.6°, 246.4° = 114°, 246°$ (3 s.f.)

b $\frac{1}{\tan 2\theta} = 0.6$

Substitute $\frac{1}{\tan 2\theta}$ for $\cot 2\theta$ and then simplify to get an equation in the form $\tan 2\theta = k$

$\tan 2\theta = \frac{1}{0.6} = \frac{5}{3}$

Let $X = 2\theta$, so that you are solving $\tan X = \frac{5}{3}$, in the interval $0° \leqslant X \leqslant 720°$



Draw the CAST diagram, with the acute angle $X = \tan^{-1}\left(\frac{5}{3}\right)$ drawn to the horizontal in the 1st and 3rd quadrants.

$X = 59.0°, 239.0°, 419.0°, 599.0°$

Remember that $X = 2\theta$

so $\theta = 29.5°, 120°, 210°, 300°$ (3 s.f.)

Exercise 3C

SKILLS ANALYSIS

1 Rewrite the following as powers of $\sec\theta$, $\operatorname{cosec}\theta$ or $\cot\theta$.

a $\frac{1}{\sin^3\theta}$ **b** $\frac{4}{\tan^6\theta}$ **c** $\frac{1}{2\cos^2\theta}$ **d** $\frac{1-\sin^2\theta}{\sin^2\theta}$

e $\frac{\sec\theta}{\cos^4\theta}$ **f** $\sqrt{\operatorname{cosec}^3\theta\cot\theta\sec\theta}$ **g** $\frac{2}{\sqrt{\tan\theta}}$ **h** $\frac{\operatorname{cosec}^2\theta\tan^2\theta}{\cos\theta}$

2 Write down the value(s) of $\cot x$ in each of the following equations:

a $5\sin x = 4\cos x$ **b** $\tan x = -2$ **c** $\frac{3\sin x}{\cos x} = \frac{\cos x}{\sin x}$

3 Using the definitions of sec, cosec, cot and tan, simplify the following expressions.

a $\sin\theta\cot\theta$ **b** $\tan\theta\cot\theta$

c $\tan 2\theta\operatorname{cosec} 2\theta$ **d** $\cos\theta\sin\theta(\cot\theta+\tan\theta)$

e $\sin^3 x\operatorname{cosec} x + \cos^3 x\sec x$ **f** $\sec A - \sec A\sin^2 A$

g $\sec^2 x\cos^5 x + \cot x\operatorname{cosec} x\sin^4 x$

4 Prove that:

a $\cos\theta + \sin\theta\tan\theta \equiv \sec\theta$ **b** $\cot\theta + \tan\theta \equiv \operatorname{cosec}\theta\sec\theta$

c $\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta$ **d** $(1-\cos x)(1+\sec x) \equiv \sin x\tan x$

e $\frac{\cos x}{1-\sin x} + \frac{1-\sin x}{\cos x} \equiv 2\sec x$ **f** $\frac{\cos\theta}{1+\cot\theta} \equiv \frac{\sin\theta}{1+\tan\theta}$

5 Solve the following equations for values of θ in the interval $0° \leqslant \theta \leqslant 360°$
Give your answers to 3 significant figures where necessary.

a $\sec\theta = \sqrt{2}$ **b** $\operatorname{cosec}\theta = -3$ **c** $5\cot\theta = -2$ **d** $\operatorname{cosec}\theta = 2$

e $3\sec^2\theta - 4 = 0$ **f** $5\cos\theta = 3\cot\theta$ **g** $\cot^2\theta - 8\tan\theta = 0$ **h** $2\sin\theta = \operatorname{cosec}\theta$

6 Solve the following equations for values of θ in the interval $-180° \leqslant \theta \leqslant 180°$

a $\operatorname{cosec}\theta = 1$ **b** $\sec\theta = -3$ **c** $\cot\theta = 3.45$

d $2\operatorname{cosec}^2\theta - 3\operatorname{cosec}\theta = 0$ **e** $\sec\theta = 2\cos\theta$ **f** $3\cot\theta = 2\sin\theta$

g $\operatorname{cosec} 2\theta = 4$ **h** $2\cot^2\theta - \cot\theta - 5 = 0$

7 Solve the following equations for values of θ in the interval $0 \leqslant \theta \leqslant 2\pi$
Give your answers in terms of π.

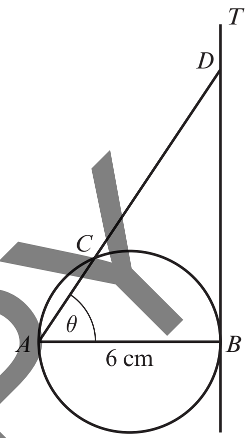
a $\sec\theta = -1$ **b** $\cot\theta = -\sqrt{3}$

c $\operatorname{cosec}\frac{\theta}{2} = \frac{2\sqrt{3}}{3}$ **d** $\sec\theta = \sqrt{2}\tan\theta,\ \theta \neq \frac{\pi}{2},\ \theta \neq \frac{3\pi}{2}$

E/P **8** In the diagram, $AB = 6$ cm is the diameter of the circle and BT is the tangent to the circle at B. The **chord** AC is extended to meet this tangent at D and $\angle DAB = \theta$

a Show that $CD = 6(\sec\theta - \cos\theta)$ cm. **(4 marks)**

b Given that $CD = 16$ cm, calculate the length of the chord AC. **(3 marks)**



Problem-solving

AB is the diameter of the circle, so $\angle ACB = 90°$

E/P **9** **a** Prove that $\frac{\operatorname{cosec} x - \cot x}{1 - \cos x} \equiv \operatorname{cosec} x$ **(4 marks)**

b Hence solve, in the interval $-\pi \leqslant x \leqslant \pi$, the equation $\frac{\operatorname{cosec} x - \cot x}{1 - \cos x} = 2$ **(3 marks)**

E/P **10** **a** Prove that $\frac{\sin x \tan x}{1 - \cos x} - 1 \equiv \sec x$ **(4 marks)**

b Hence explain why the equation $\frac{\sin x \tan x}{1 - \cos x} - 1 = -\frac{1}{2}$ has no solutions. **(1 mark)**

E/P **11** Solve, in the interval $0° \leqslant x \leqslant 360°$, the equation $\frac{1 + \cot x}{1 + \tan x} = 5$ **(8 marks)**

Problem-solving

Use the relationship $\cot x = \frac{1}{\tan x}$ to form a quadratic equation in $\tan x$.

← Pure 1 Section 2.1

3.4 Trigonometric identities

You can use the identity $\sin^2 x + \cos^2 x \equiv 1$ to prove the following identities.

- **$1 + \tan^2 x \equiv \sec^2 x$**
- **$1 + \cot^2 x \equiv \operatorname{cosec}^2 x$**

Example 9 SKILLS ANALYSIS

a Prove that $1 + \tan^2 x \equiv \sec^2 x$

b Prove that $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

a $\sin^2 x + \cos^2 x \equiv 1$

Unless otherwise stated, you can assume the identity $\sin^2 x + \cos^2 x \equiv 1$ in proofs involving cosec, sec and cot in your exam.

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} \equiv \frac{1}{\cos^2 x}$$

Divide both sides of the identity by $\cos^2 x$.

$$\left(\frac{\sin x}{\cos x}\right)^2 + 1 \equiv \left(\frac{1}{\cos x}\right)^2$$

Use $\tan x \equiv \frac{\sin x}{\cos x}$ and $\sec x \equiv \frac{1}{\cos x}$

so $1 + \tan^2 x \equiv \sec^2 x$

b $\sin^2 x + \cos^2 x \equiv 1$

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} \equiv \frac{1}{\sin^2 x}$$

Divide both sides of the identity by $\sin^2 x$.

$$1 + \left(\frac{\cos x}{\sin x}\right)^2 \equiv \left(\frac{1}{\sin x}\right)^2$$

Use $\cot x \equiv \frac{\cos x}{\sin x}$ and $\operatorname{cosec} x \equiv \frac{1}{\sin x}$

so $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

Example 10

Given that $\tan A = -\frac{5}{12}$, and that angle A is **obtuse**, find the exact values of:

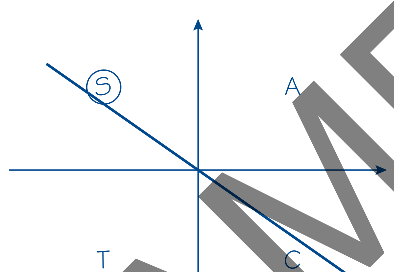
a $\sec A$ **b** $\sin A$

a Using $1 + \tan^2 A \equiv \sec^2 A$

$\tan^2 A = \frac{25}{144}$

$$\sec^2 A = 1 + \frac{25}{144} = \frac{169}{144}$$

$$\sec A = \pm\frac{13}{12}$$



Problem-solving

You are told that A is obtuse. This means it lies in the second quadrant, so $\cos A$ is negative, and $\sec A$ is also negative.

$$\sec A = -\frac{13}{12}$$

b Using $\tan A \equiv \frac{\sin A}{\cos A}$

$$\sin A \equiv \tan A \cos A$$

So $\sin A = \left(-\frac{5}{12}\right) \times \left(-\frac{12}{13}\right)$

$\cos A = -\frac{12}{13}$, since $\cos A = \frac{1}{\sec A}$

$$= \frac{5}{13}$$

Example 11

Prove the identities:

a $\operatorname{cosec}^4\theta - \cot^4\theta \equiv \frac{1+\cos^2\theta}{1-\cos^2\theta}$

b $\sec^2\theta - \cos^2\theta \equiv \sin^2\theta(1+\sec^2\theta)$

a LHS $= \operatorname{cosec}^4\theta - \cot^4\theta$

$\equiv (\operatorname{cosec}^2\theta + \cot^2\theta)(\operatorname{cosec}^2\theta - \cot^2\theta)$

$\equiv \operatorname{cosec}^2\theta + \cot^2\theta$

$\equiv \frac{1}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}$

$\equiv \frac{1+\cos^2\theta}{\sin^2\theta}$

$\equiv \frac{1+\cos^2\theta}{1-\cos^2\theta} =$ RHS

b RHS $= \sin^2\theta + \sin^2\theta\sec^2\theta$

$\equiv \sin^2\theta + \frac{\sin^2\theta}{\cos^2\theta}$

$\equiv \sin^2\theta + \tan^2\theta$

$\equiv (1-\cos^2\theta) + (\sec^2\theta - 1)$

$\equiv \sec^2\theta - \cos^2\theta$

$\equiv$ LHS

This is the difference of two squares, so factorise.

As $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$, so $\operatorname{cosec}^2\theta - \cot^2\theta \equiv 1$

Using $\operatorname{cosec}\theta \equiv \frac{1}{\sin\theta}$, $\cot\theta \equiv \frac{\cos\theta}{\sin\theta}$

Using $\sin^2\theta + \cos^2\theta \equiv 1$

Write in terms of $\sin\theta$ and $\cos\theta$.

Use $\sec\theta \equiv \frac{1}{\cos\theta}$

$\frac{\sin^2\theta}{\cos^2\theta} \equiv \left(\frac{\sin\theta}{\cos\theta}\right)^2 \equiv \tan^2\theta$

Look at LHS. It is in terms of $\cos^2\theta$ and $\sec^2\theta$, so use $\sin^2\theta + \cos^2\theta \equiv 1$ and $1 + \tan^2\theta \equiv \sec^2\theta$

Problem-solving

You can start from either the LHS or the RHS when proving an identity. Try starting with the LHS using $\cos^2\theta \equiv 1 - \sin^2\theta$ and $\sec^2\theta \equiv 1 + \tan^2\theta$

Example 12

Solve the equation $4\operatorname{cosec}^2\theta - 9 = \cot\theta$ in the interval $0° \leqslant \theta \leqslant 360°$

The equation can be rewritten as

$4(1 + \cot^2\theta) - 9 = \cot\theta$

So $4\cot^2\theta - \cot\theta - 5 = 0$

$(4\cot\theta - 5)(\cot\theta + 1) = 0$

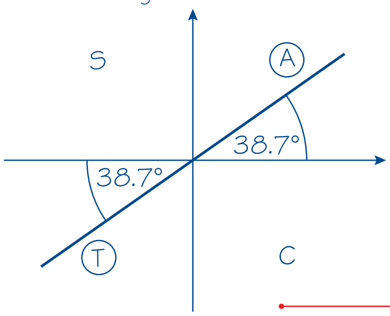
So $\cot\theta = \frac{5}{4}$ or $\cot\theta = -1$

$\therefore \tan\theta = \frac{4}{5}$ or $\tan\theta = -1$

This is a quadratic equation. You need to write it in terms of one trigonometric function only, so use $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Factorise, or solve using the quadratic formula.

For $\tan\theta = \frac{4}{5}$

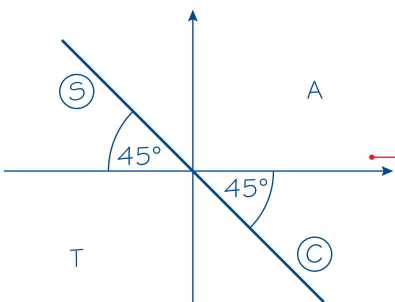


As $\tan\theta$ is +ve, θ is in the 1st and 3rd quadrants. The acute angle to the horizontal is $\tan^{-1}\left(\frac{4}{5}\right) = 38.7°$

If α is the value the calculator gives for $\tan^{-1}\left(\frac{4}{5}\right)$, then the solutions are α and $(180° + \alpha)$

$\theta = 38.7°, 219°$ (3 s.f.)

For $\tan\theta = -1$



As $\tan\theta$ is –ve, θ is in the 2nd and 4th quadrants. The acute angle to the horizontal is $\tan^{-1} 1 = 45°$

If α is the value the calculator gives for $\tan^{-1}(-1)$, then the solutions are $(180° + \alpha)$ and $(360° + \alpha)$, as α is not in the given interval.

$\theta = 135°, 315°$

Online Solve this equation numerically using your calculator.

Exercise 3D

SKILLS ANALYSIS

Give answers to 3 significant figures where necessary.

1 Simplify each of the following expressions.

a $1 + \tan^2\left(\frac{\theta}{2}\right)$ **b** $(\sec\theta - 1)(\sec\theta + 1)$ **c** $\tan^2\theta(\text{cosec}^2\theta - 1)$

d $(\sec^2\theta - 1)\cot\theta$ **e** $(\text{cosec}^2\theta - \cot^2\theta)^2$ **f** $2 - \tan^2\theta + \sec^2\theta$

g $\dfrac{\tan\theta\sec\theta}{1 + \tan^2\theta}$ **h** $(1 - \sin^2\theta)(1 + \tan^2\theta)$ **i** $\dfrac{\text{cosec}\,\theta\cot\theta}{1 + \cot^2\theta}$

j $(\sec^4\theta - 2\sec^2\theta\tan^2\theta + \tan^4\theta)$ **k** $4\,\text{cosec}^2 2\theta + 4\,\text{cosec}^2 2\theta\cot^2 2\theta$

(P) **2** Given that $\text{cosec}\, x = \dfrac{k}{\text{cosec}\, x}$, where $k > 1$, find, in terms of k, possible values of $\cot x$.

3 Given that $\cot\theta = -\sqrt{3}$, and that $90° < \theta < 180°$, find the exact values of:

a $\sin\theta$ **b** $\cos\theta$

4 Given that $\tan\theta = \frac{3}{4}$, and that $180° < \theta < 270°$, find the exact values of:

a $\sec\theta$ **b** $\cos\theta$ **c** $\sin\theta$

5 Given that $\cos\theta = \frac{24}{25}$, and that θ is a reflex angle, find the exact values of:

a $\tan\theta$ **b** $\text{cosec}\,\theta$

P **6** Prove the following identities:

a $\sec^4\theta - \tan^4\theta \equiv \sec^2\theta + \tan^2\theta$

b $\text{cosec}^2 x - \sin^2 x \equiv \cot^2 x + \cos^2 x$

c $\sec^2 A(\cot^2 A - \cos^2 A) \equiv \cot^2 A$

d $1 - \cos^2\theta \equiv (\sec^2\theta - 1)(1 - \sin^2\theta)$

e $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} \equiv 1 - 2\sin^2 A$

f $\sec^2\theta + \text{cosec}^2\theta \equiv \sec^2\theta\,\text{cosec}^2\theta$

g $\text{cosec}\,A\sec^2 A \equiv \text{cosec}\,A + \tan A\sec A$

h $(\sec\theta - \sin\theta)(\sec\theta + \sin\theta) \equiv \tan^2\theta + \cos^2\theta$

P **7** Given that $3\tan^2\theta + 4\sec^2\theta = 5$, and that θ is obtuse, find the exact value of $\sin\theta$.

P **8** Solve the following equations in the given intervals:

a $\sec^2\theta = 3\tan\theta$, $0° \leqslant \theta \leqslant 360°$

b $\tan^2\theta - 2\sec\theta + 1 = 0$, $-\pi \leqslant \theta \leqslant \pi$

c $\text{cosec}^2\theta + 1 = 3\cot\theta$, $-180° \leqslant \theta \leqslant 180°$

d $\cot\theta = 1 - \text{cosec}^2\theta$, $0 \leqslant \theta \leqslant 2\pi$

e $3\sec\frac{1}{2}\theta = 2\tan^2\frac{1}{2}\theta$, $0° \leqslant \theta \leqslant 360°$

f $(\sec\theta - \cos\theta)^2 = \tan\theta - \sin^2\theta$, $0 \leqslant \theta \leqslant \pi$

g $\tan^2 2\theta = \sec 2\theta - 1$, $0° \leqslant \theta \leqslant 180°$

h $\sec^2\theta - (1 + \sqrt{3})\tan\theta + \sqrt{3} = 1$, $0 \leqslant \theta \leqslant 2\pi$

E/P **9** Given that $\tan^2 k = 2\sec k$,

a find the value of $\sec k$ **(4 marks)**

b deduce that $\cos k = \sqrt{2} - 1$. **(2 marks)**

c Hence solve, in the interval $0° \leqslant k \leqslant 360°$, $\tan^2 k = 2\sec k$, giving your answers to 1 decimal place. **(3 marks)**

E/P **10** Given that $a = 4\sec x$, $b = \cos x$ and $c = \cot x$,

a express b in terms of a **(2 marks)**

b show that $c^2 = \dfrac{16}{a^2 - 16}$ **(3 marks)**

E/P **11** Given that $x = \sec\theta + \tan\theta$,

a show that $\dfrac{1}{x} = \sec\theta - \tan\theta$ **(3 marks)**

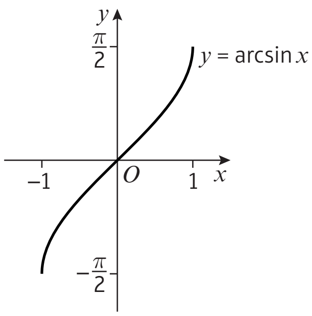
b Hence express $x^2 + \dfrac{1}{x^2} + 2$ in terms of θ, in its simplest form. **(5 marks)**

E/P **12** Given that $2\sec^2\theta - \tan^2\theta = p$, show that $\text{cosec}^2\theta = \dfrac{p-1}{p-2}$, $p \neq 2$ **(5 marks)**

3.5 Inverse trigonometric functions

You need to understand and use the inverse trigonometric functions arcsin x, arccos x and arctan x and their graphs.

■ **The inverse function of sin x is called arcsin x.**



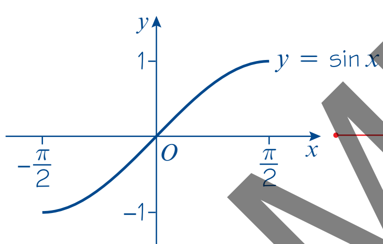
Hint The $\sin^{-1}$ function on your calculator will give principal values in the same range as arcsin.

- **The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$**
- **The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arcsin x \leqslant 90°$**

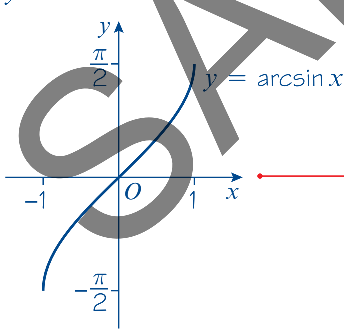
Example 13

Sketch the graph of $y = \arcsin x$

$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$



$y = \arcsin x$



Step 1
Draw the graph of $y = \sin x$, with the restricted domain of $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Restricting the domain ensures that the inverse function exists since $y = \sin x$ is a **one-to-one** function for the restricted domain. Only one-to-one functions have inverses. ← Pure 1 Section 2.3

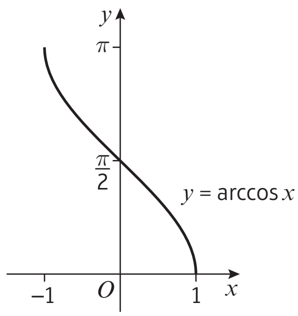
Step 2
Reflect in the line $y = x$

The domain of arcsin x is $-1 \leqslant x \leqslant 1$; the range is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$

Remember that the x and y coordinates of points interchange (swap) when reflecting in $y = x$
For example:

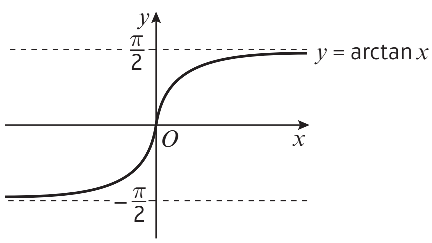
$$\left(\frac{\pi}{2}, 1\right) \rightarrow \left(1, \frac{\pi}{2}\right)$$

■ The inverse function of $\cos x$ is called $\arccos x$.



- The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arccos x$ is $0 \leqslant \arccos x \leqslant \pi$ or $0° \leqslant \arccos x \leqslant 180°$

■ The inverse function of $\tan x$ is called $\arctan x$.



Watch out Unlike $\arcsin x$ and $\arccos x$, the function $\arctan x$ is defined for all real values of x.

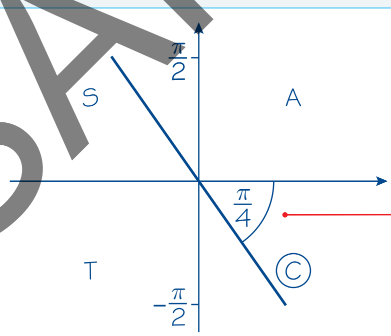
- The domain of $y = \arctan x$ is $x \in \mathbb{R}$
- The range of $y = \arctan x$ is $-\frac{\pi}{2} \leqslant \arctan x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arctan x \leqslant 90°$

Example 14

Work out, in radians, the values of:

a $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ **b** $\arccos(-1)$ **c** $\arctan(\sqrt{3})$

a



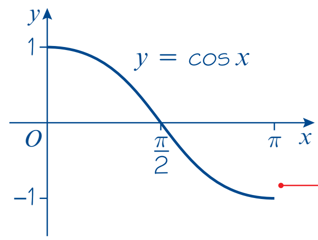
$\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

You need to solve, in the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, the equation $\sin x = -\frac{\sqrt{2}}{2}$

The angle to the horizontal is $\frac{\pi}{4}$ and, as sin is –ve, it is in the 4th quadrant.

Online Use your calculator to evaluate inverse trigonometric functions in radians.

b

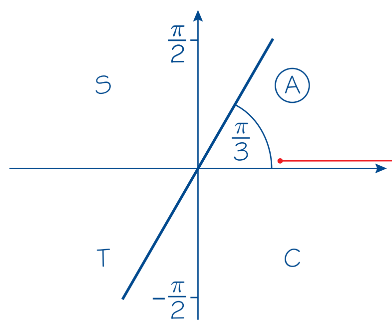


$\arccos(-1) = \pi$

You need to solve, in the interval $0 \leqslant x \leqslant \pi$, the equation $\cos x = -1$

Draw the graph of $y = \cos x$

c



$\arctan(\sqrt{3}) = \frac{\pi}{3}$

You need to solve, in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the equation $\tan x = \sqrt{3}$

The angle to the horizontal is $\frac{\pi}{3}$ and, as tan is +ve, it is in the 1st quadrant.

You can verify these results using the $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ functions on your calculator.

Exercise 3E

SKILLS INTERPRETATION

In this exercise, all angles are given in radians.

1 Without using a calculator, work out, giving your answer in terms of π:

a $\arccos(0)$ **b** $\arcsin(1)$ **c** $\arctan(-1)$ **d** $\arcsin\left(-\frac{1}{2}\right)$

e $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ **f** $\arctan-\frac{1}{\sqrt{3}}$ **g** $\arcsin\left(\sin\frac{\pi}{3}\right)$ **h** $\arcsin\left(\sin\frac{2\pi}{3}\right)$

2 Find:

a $\arcsin\left(\frac{1}{2}\right) + \arcsin\left(-\frac{1}{2}\right)$ **b** $\arccos\left(\frac{1}{2}\right) - \arccos\left(-\frac{1}{2}\right)$ **c** $\arctan(1) - \arctan(-1)$

P **3** Without using a calculator, work out the values of:

a $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$ **b** $\sin\left(\arcsin\left(-\frac{1}{2}\right)\right)$

c $\tan(\arctan(-1))$ **d** $\cos(\arccos 0)$

P **4** Without using a calculator, work out the exact values of:

a $\sin\left(\arccos\left(\frac{1}{2}\right)\right)$ **b** $\cos\left(\arcsin\left(-\frac{1}{2}\right)\right)$ **c** $\tan\left(\arccos\left(-\frac{\sqrt{2}}{2}\right)\right)$

d $\sec(\arctan(\sqrt{3}))$ **e** $\text{cosec}(\arcsin(-1))$ **f** $\sin\left(2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right)$

P **5** Given that $\arcsin k = \alpha$, where $0 < k < 1$, write down the first two positive values of x satisfying the equation $\sin x = k$

E/P **6** Given that x satisfies $\arcsin x = k$, where $0 < k < \frac{\pi}{2}$,

a state the range of possible values of x **(1 mark)**

b express, in terms of x,

i $\cos k$ **ii** $\tan k$ **(4 marks)**

Given, instead, that $-\frac{\pi}{2} < k < 0$,

c how, if at all, are your answers to part **b** affected? **(2 marks)**

P **7** Sketch the graphs of:

a $y = \frac{\pi}{2} + 2\arcsin x$ **b** $y = \pi - \arctan x$

c $y = \arccos(2x + 1)$ **d** $y = -2\arcsin(-x)$

E/P **8** The function f is defined as $\mathrm{f}: x \mapsto \arcsin x, -1 \leqslant x \leqslant 1$, and the function g is such that $\mathrm{g}(x) = \mathrm{f}(2x)$

a Sketch the graph of $y = \mathrm{f}(x)$ and state the range of f. **(3 marks)**

b Sketch the graph of $y = \mathrm{g}(x)$ **(2 marks)**

c Define g in the form $\mathrm{g}: x \mapsto \ldots$ and give the domain of g. **(3 marks)**

d Define g^{-1} in the form $\mathrm{g}^{-1}: x \mapsto \ldots$ **(2 marks)**

E/P **9** **a** Prove that for $0 \leqslant x \leqslant 1$, $\arccos x = \arcsin\sqrt{1 - x^2}$ **(4 marks)**

b Give a reason why this result is not true for $-1 \leqslant x \leqslant 0$ **(2 marks)**

Challenge

SKILLS INTERPRETATION

a Sketch the graph of $y = \sec x$, with the restricted domain $0 \leqslant x \leqslant \pi$, $x \neq \frac{\pi}{2}$

b Given that arcsec x is the inverse function of $\sec x$, $0 \leqslant x \leqslant \pi$, $x \neq \frac{\pi}{2}$, sketch the graph of $y = \text{arcsec}\, x$ and state the range of arcsec x.

Chapter review 3

Give any non-exact answers to equations to 1 decimal place.

E/P **1** Solve $\tan x = 2\cot x$, in the interval $-180° \leqslant x \leqslant 90°$ **(4 marks)**

E/P **2** Given that $p = 2\sec\theta$ and $q = 4\cos\theta$, express p in terms of q. **(4 marks)**

E/P **3** Given that $p = \sin\theta$ and $q = 4\cot\theta$, show that $p^2q^2 = 16(1 - p^2)$ **(4 marks)**

P **4** **a** Solve, in the interval $0° < \theta < 180°$,

i $\operatorname{cosec}\theta = 2\cot\theta$ **ii** $2\cot^2\theta = 7\operatorname{cosec}\theta - 8$

b Solve, in the interval $0° \leqslant \theta \leqslant 360°$,

i $\sec(2\theta - 15°) = \operatorname{cosec}135°$ **ii** $\sec^2\theta + \tan\theta = 3$

c Solve, in the interval $0 \leqslant x \leqslant 2\pi$,

i $\operatorname{cosec}\left(x + \frac{\pi}{15}\right) = -\sqrt{2}$ **ii** $\sec^2 x = \frac{4}{3}$

E/P **5** Given that $5\sin x\cos y + 4\cos x\sin y = 0$, and that $\cot x = 2$, find the value of $\cot y$. **(5 marks)**

P **6** Prove that:

a $(\tan\theta + \cot\theta)(\sin\theta + \cos\theta) \equiv \sec\theta + \operatorname{cosec}\theta$

b $\dfrac{\operatorname{cosec} x}{\operatorname{cosec} x - \sin x} \equiv \sec^2 x$

c $(1 - \sin x)(1 + \operatorname{cosec} x) \equiv \cos x\cot x$

d $\dfrac{\cot x}{\operatorname{cosec} x - 1} - \dfrac{\cos x}{1 + \sin x} \equiv 2\tan x$

e $\dfrac{1}{\operatorname{cosec}\theta - 1} + \dfrac{1}{\operatorname{cosec}\theta + 1} \equiv 2\sec\theta\tan\theta$

f $\dfrac{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}{1 + \tan^2\theta} \equiv \cos^2\theta$

E/P **7** **a** Prove that $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} \equiv 2\operatorname{cosec} x$ **(4 marks)**

b Hence solve, in the interval $-2\pi \leqslant x \leqslant 2\pi$, $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = -\dfrac{4}{\sqrt{3}}$ **(4 marks)**

E/P **8** Prove that $\dfrac{1 + \cos\theta}{1 - \cos\theta} \equiv (\operatorname{cosec}\theta + \cot\theta)^2$ **(4 marks)**

E **9** Given that $\sec A = -3$, where $\frac{\pi}{2} < A < \pi$,

a calculate the exact value of $\tan A$ **(3 marks)**

b show that $\operatorname{cosec} A = \dfrac{3\sqrt{2}}{4}$ **(3 marks)**

10 Given that $\sec\theta = k$, $|k| \geqslant 1$, and that θ is obtuse, express in terms of k:

a $\cos\theta$ **b** $\tan^2\theta$ **c** $\cot\theta$ **d** $\operatorname{cosec}\theta$

E **11** Solve, in the interval $0 \leqslant x \leqslant 2\pi$, the equation $\sec\left(x + \frac{\pi}{4}\right) = 2$, giving your answers in terms of π. **(5 marks)**

E/P **12** Find, in terms of π, the value of $\arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right)$ **(4 marks)**

E/P **13** Solve, in the interval $0 \leqslant x \leqslant 2\pi$, the equation $\sec^2 x - \frac{2\sqrt{3}}{3}\tan x - 2 = 0$, giving your answers in terms of π. **(5 marks)**

E/P **14** **a** Factorise $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2$ **(2 marks)**

b Hence solve $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2 = 0$ in the interval $0° \leqslant x \leqslant 360°$ **(4 marks)**

E/P **15** Given that $\arctan(x - 2) = -\frac{\pi}{3}$, find the value of x. **(3 marks)**

E **16** On the same set of axes, sketch the graphs of $y = \cos x$, $0 \leqslant x \leqslant \pi$, and $y = \arccos x$, $-1 \leqslant x \leqslant 1$, showing the coordinates of points at which the curves meet the axes. **(4 marks)**

E/P **17** **a** Given that $\sec x + \tan x = -3$, use the identity $1 + \tan^2 x \equiv \sec^2 x$ to find the value of $\sec x - \tan x$ **(3 marks)**

b Deduce the values of:

i $\sec x$ **ii** $\tan x$ **(3 marks)**

c Hence solve, in the interval $-180° \leqslant x \leqslant 180°$, $\sec x + \tan x = -3$ **(3 marks)**

E/P **18** Given that $p = \sec\theta - \tan\theta$ and $q = \sec\theta + \tan\theta$, show that $p = \frac{1}{q}$ **(4 marks)**

E/P **19** **a** Prove that $\sec^4\theta - \tan^4\theta = \sec^2\theta + \tan^2\theta$ **(3 marks)**

b Hence solve, in the interval $-180° \leqslant \theta \leqslant 180°$, $\sec^4\theta = \tan^4\theta + 3\tan\theta$ **(4 marks)**

P **20** **a** Sketch the graph of $y = \sin x$ and shade in the area representing $\int_0^{\frac{\pi}{2}} \sin x \, dx$.

b Sketch the graph of $y = \arcsin x$ and shade in the area representing $\int_0^1 \arcsin x \, dx$.

c By considering the shaded areas, explain why $\int_0^{\frac{\pi}{2}} \sin x \, dx + \int_0^1 \arcsin x \, dx = \frac{\pi}{2}$

P **21** Show that $\cot 60° \sec 60° = \frac{2\sqrt{3}}{3}$

E/P **22** **a** Sketch, in the interval $-2\pi \leqslant x \leqslant 2\pi$, the graph of $y = 2 - 3\sec x$ **(3 marks)**

b Hence deduce the range of values of k for which the equation $2 - 3\sec x = k$ has no solutions. **(2 marks)**

P **23** **a** Sketch the graph of $y = 3\arcsin x - \frac{\pi}{2}$, showing clearly the exact coordinates of the end-points of the curve. **(4 marks)**

b Find the exact coordinates of the point where the curve crosses the x-axis. **(3 marks)**

24 **a** Prove that for $0 < x \leqslant 1$, $\arccos x = \arctan \dfrac{\sqrt{1-x^2}}{x}$

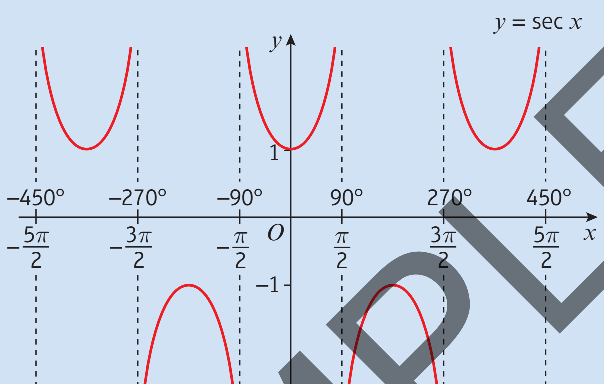
b Prove that for $-1 \leqslant x < 0$, $\arccos x = k + \arctan \dfrac{\sqrt{1-x^2}}{x}$, where k is a constant to be found.

Summary of key points

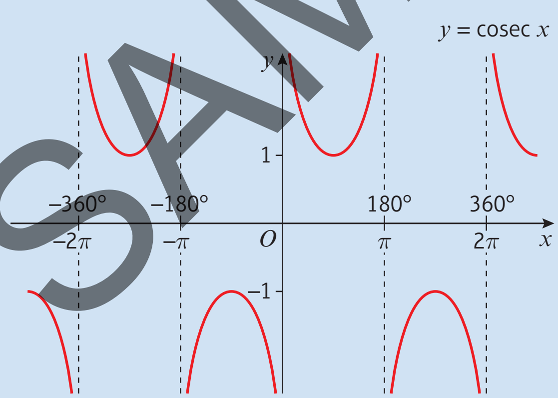
1

- $\sec x = \dfrac{1}{\cos x}$ (undefined for values of x for which $\cos x = 0$)
- $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (undefined for values of x for which $\sin x = 0$)
- $\cot x = \dfrac{1}{\tan x}$ (undefined for values of x for which $\tan x = 0$)
- $\cot x = \dfrac{\cos x}{\sin x}$

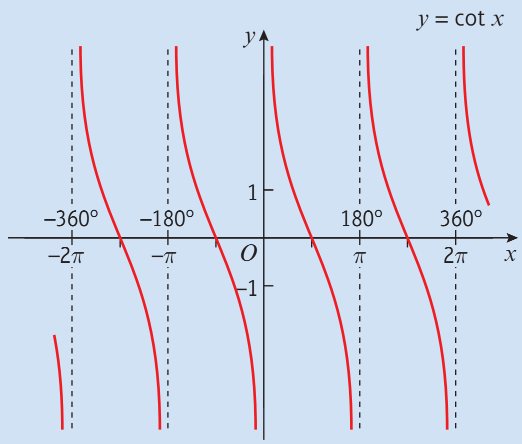
2 The graph of $\boldsymbol{y = \sec x}$, $x \in \mathbb{R}$, has symmetry in the y-axis and has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$



3 The graph of $\boldsymbol{y = \operatorname{cosec} x}$, $x \in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\sin x = 0$



4 The graph of $\boldsymbol{y = \cot x}$, $x \in \mathbb{R}$, has period 180° or π radians. It has vertical asymptotes at all the values of x for which $\tan x = 0$

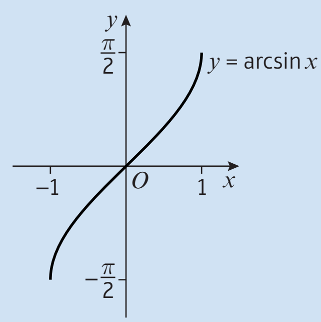


5 You can use the identity $\sin^2 x + \cos^2 x \equiv 1$ to prove the following identities:

- $1 + \tan^2 x \equiv \sec^2 x$
- $1 + \cot^2 x \equiv \text{cosec}^2 x$

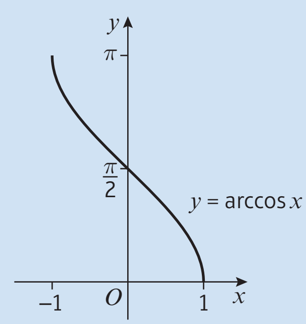
6 The **inverse function** of $\sin x$ is called **arcsin** $\boldsymbol{x}$.

- The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arcsin x \leqslant 90°$



7 The inverse function of $\cos x$ is called **arccos** $\boldsymbol{x}$.

- The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arccos x$ is $0 \leqslant \arccos x \leqslant \pi$ or $0° \leqslant \arccos x \leqslant 180°$



8 The inverse function of $\tan x$ is called **arctan** $\boldsymbol{x}$.

- The domain of $y = \arctan x$ is $x \in \mathbb{R}$
- The range of $y = \arctan x$ is $-\frac{\pi}{2} \leqslant \arctan x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arctan x \leqslant 90°$