PENSION MATHEMATICS
with Numerical Illustrations

Second Edition

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Chapter 3

Basic Actuarial Functions

The purpose of this chapter is to introduce several actuarial functions used in the development of pension mathematics throughout the remainder of the book. The discussion begins with the composite survival function and interest function, perhaps the two most basic concepts in pension mathematics. Pension plan benefit functions are then presented, followed by a discussions of annuities, the latter representing a combination of interest and survival functions.

COMPOSITE SURVIVAL FUNCTION

The composite survival function represents the probability that an active plan participant survives in service for a given period, based on all of the decrement rates to which the employee is exposed. Whereas the probability of surviving one year in a single-decrement environment is equal to the complement of the rate of decrement, the probability of surviving one year in a multiple-decrement environment is equal to the product of such complements for each applicable rate of decrement. The probability of an active participant aged x surviving one year is

$$p_x^{(T)} = (1 - q_x^{\prime(m)})\,(1 - q_x^{\prime(t)})\,(1 - q_x^{\prime(d)})\,(1 - q_x^{\prime(r)})\,, \quad (3.1a)$$

or equivalently,

$$p_x^{(T)} = p_x^{\prime(m)}\, p_x^{\prime(t)}\, p_x^{\prime(d)}\, p_x^{\prime(r)}\,. \quad (3.1b)$$

This same probability can be expressed in terms of multiple-decrement probabilities:

$$p_x^{(T)} = 1 - (q_x^{(m)} + q_x^{(t)} + q_x^{(d)} + q_x^{(r)}). \quad (3.2a)$$

A common approximation for multiple-decrement probabilities is illustrated for the mortality probability as follows:

$$q_x^{(m)} \approx q_x^{\prime(m)} \left(1 - \tfrac{1}{2} q_x^{\prime(t)}\right) \left(1 - \tfrac{1}{2} q_x^{\prime(d)}\right) \left(1 - \tfrac{1}{2} q_x^{\prime(r)}\right). \tag{3.2b}$$

The probability of surviving in active service for n years is equal to the product of successive one-year composite survival probabilities:

$$_n p_x^{(T)} = \prod_{t=0}^{n-1} p_{x+t}^{(T)}. \tag{3.2c}$$

Table 3–1 shows the probability of surviving to age 65 from each attained age, based on nine different entry ages, under the various rates of decrement set out in Chapter 2. Since the termination decrement includes a 5-year select period, probabilities are given for each of the first five years subsequent to the nine entry ages. The chance of reaching retirement, at least under the illustrative assumptions, is quite low even up through attained age 50, where the probability is 0.62. Since retirement-related pension cost estimates are in direct proportion to the survival probability (i.e., the lower the probability the lower are such estimates), it is clear that discounting for the various decrements reduces pension costs significantly. Obviously, the cost reduction is mitigated to the extent that a benefit is provided with each type of decrement (e.g., a vested termination benefit, a death benefit, or a disability benefit).

The calculation of survival probabilities in accordance with the above definitions can be quite arduous. The probabilities, however, are readily obtainable from a *service table*. This table shows the hypothetical number of employees, from an arbitrary initial number, who survive to each future age. The initial number, or radix, is generally taken to be some large value, such as 100,000 or 1,000,000, and $l_x^{(T)}$ is the notation for the survivors at age x.[1] The total number of employees leaving active service during the year is denoted by $d_x^{(T)}$ and defined as

$$d_x^{(T)} = l_x^{(T)} q_x^{(T)}. \tag{3.3a}$$

[1] Although not indicated by the symbol $l_x^{(T)}$, it is understood to represent the number of survivors at age x who entered the plan at age y. At a later point the more general symbol $l_{x,y}^{(T)}$ will be used, which makes the entry age variable explicit.

TABLE 3–1
Probability of Surviving in Service to Age 65, $_{65-x}p_x^{(T)}$

Entry Age	Select Ages y	$y+1$	$y+2$	$y+3$	$y+4$	x	Attained Age
20	0.02	0.03	0.04	0.05	0.07	0.08	25
						0.10	26
						0.11	27
						0.13	28
						0.15	29
25	0.07	0.09	0.11	0.13	0.15	0.17	30
						0.19	31
						0.21	32
						0.23	33
						0.25	34
30	0.14	0.17	0.20	0.23	0.25	0.28	35
						0.30	36
						0.32	37
						0.34	38
						0.36	39
35	0.24	0.28	0.31	0.33	0.36	0.38	40
						0.40	41
						0.43	42
						0.45	43
						0.47	44
40	0.35	0.38	0.41	0.44	0.47	0.49	45
						0.52	46
						0.54	47
						0.57	48
						0.60	49
45	0.47	0.51	0.54	0.57	0.60	0.62	50
						0.65	51
						0.69	52
						0.72	53
						0.75	54
50	0.51	0.55	0.58	0.61	0.64	0.79	55
						0.80	56
						0.81	57
						0.82	58
						0.84	59
55	0.59	0.63	0.67	0.70	0.74	0.85	60
						0.87	61
						0.89	62
						0.92	63
						0.96	64
60	0.74	0.79	0.84	0.89	0.94	1.00	65

Total decrements from the active population equal the sum of each separate decrement:

$$d_x^{(T)} = d_x^{(m)} + d_x^{(t)} + d_x^{(d)} + d_x^{(r)} \tag{3.3b}$$

$$= l_x^{(T)} \left(q_x^{(m)} + q_x^{(t)} + q_x^{(d)} + q_x^{(r)}\right). \tag{3.3c}$$

Table 3–2 illustrates the concept of a service table for 1,000,000 entrants at age 20, based on the decrement assumptions specified in Chapter 2.[2] The probability of an age-20 entrant surviving in active service to age 65 is easily found from this table:

$$_{65-20}p_{20}^{(T)} = \frac{l_{65}^{(T)}}{l_{20}^{(T)}} = \frac{24{,}448}{1{,}000{,}000} = 0.0244. \tag{3.4a}$$

Similarly, the probability of an employee age 40 surviving to age 65 would be

$$_{25}p_{40}^{(T)} = \frac{l_{65}^{(T)}}{l_{40}^{(T)}} = \frac{24{,}448}{65{,}276} = 0.3745. \tag{3.4b}$$

Although service tables are an important source of computational efficiency, even when working with high speed computers, they are used only occasionally in presenting the theory of pension mathematics in this book.

INTEREST FUNCTION

The interest function is used to discount a future payment to the present time. It plays a crucial role in determining pension costs and, like the survival function of the previous section, it reduces such values. If i_t is the interest rate assumed for the tth year, the present value of one dollar due in n years is given by

$$\frac{1}{(1+i_1)(1+i_2)\cdots(1+i_n)}, \tag{3.5a}$$

and, if $i_1 = i_2 = \cdots = i_n$, we have

$$\frac{1}{(1+i)^n}. \tag{3.5b}$$

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[2] To the extent that any rates are entry-age dependent, such as termination rates, a separate service table is applicable to each entry age.

TABLE 3–2

Service Table

x	$l_x^{(T)}$	$d_x^{(m)}$	$d_x^{(t)}$	$d_x^{(d)}$	$d_x^{(r)}$	$d_x^{(T)}$
20	1,000,000	442	243,002	263	0	243,708
21	756,292	350	169,718	201	0	170,270
22	586,023	286	121,314	158	0	121,757
23	464,265	238	88,543	126	0	88,907
24	375,358	202	65,921	103	0	66,226
25	309,132	176	49,933	85	0	50,194
26	258,938	156	38,460	72	0	38,688
27	220,251	140	30,049	62	0	30,251
28	189,999	129	23,814	53	0	23,996
29	166,004	119	19,113	47	0	19,280
30	146,724	112	15,529	56	0	15,697
31	131,027	107	12,754	50	0	12,911
32	118,116	103	10,576	45	0	10,725
33	107,392	101	8,875	41	0	9,017
34	98,375	99	7,510	38	0	7,647
35	90,727	98	6,419	35	0	6,552
36	84,176	98	5,534	41	0	5,673
37	78,503	99	4,816	46	0	4,960
38	73,543	100	4,224	50	0	4,374
39	69,169	102	3,738	54	0	3,893
40	65,276	104	3,338	57	0	3,499
41	61,777	108	3,004	60	0	3,172
42	58,605	114	2,727	69	0	2,910
43	55,695	123	2,491	76	0	2,690
44	53,006	133	2,290	83	0	2,506
45	50,499	144	2,121	89	0	2,354
46	48,145	156	1,969	94	0	2,219
47	45,926	169	1,841	99	0	2,108
48	43,818	181	1,721	107	0	2,009
49	41,808	194	1,616	115	0	1,925
50	39,884	206	1,517	121	0	1,845
51	38,039	219	1,424	127	0	1,769
52	36,270	230	1,335	135	0	1,700
53	34,570	241	1,244	142	0	1,628
54	32,942	252	1,159	148	0	1,559
55	31,383	267	0	156	0	423
56	30,960	286	0	166	0	452
57	30,508	305	0	182	0	487
58	30,020	326	0	203	0	529
59	29,491	350	0	235	0	585
60	28,907	377	0	281	0	659
61	28,248	405	0	348	0	753
62	27,495	433	0	436	0	869
63	26,626	459	0	549	0	1,008
64	25,618	485	0	685	0	1,170
65	24,448	0	0	0	24,448	24,448

The following simplifying definition is used in connection with the present value function:

$$v = \frac{1}{(1+i)}. \tag{3.6}$$

Thus, v^n represents the present value of one dollar due in n years at an annual compound rate of interest equal to i.

The interest function, v^t, begins at a value of unity for $t = 0$ and approaches zero as t approaches infinity, provided $i > 0$. Also, v^t is inversely related to i, taking on the value of unity for $i = 0$ and approaching zero as i approaches infinity. Table 3–3 illustrates the significance of the interest factor. In addition to showing v^t for the illustrative interest assumption of 8 percent, the function is also evaluated for interest rates of 6 and 10 percent. Since the interest factor is associated with each entrant's potential future age, a period of about 70 years for entrants age 30 or younger, the interest discount function is tabulated at 5-year intervals for 70 years.

The significance of the interest rate function is readily apparent. For an age-30 entrant, v^{65-30} is 0.07 at an 8 percent rate of interest, 0.13 at 6 percent, and 0.04 at 10 percent. Retirement-related cost estimates are directly related to this factor and, since the interest discount extends beyond retirement, the total effect of the interest assumption is even greater than the effect of v^{r-y} alone.

Figure 3–1 shows the survival function from age x to age 65 and the interest function over this same age interval for an age-30 entrant under the model assumptions. This graph indicates that the interest function, based on an 8 percent rate, is smaller than the survival function at all ages, although this relationship may not always hold. The product of the interest and survival functions is frequently encountered in pension mathematics. Since both functions are less than unity, their product is quite small over most attained ages.

SALARY FUNCTION

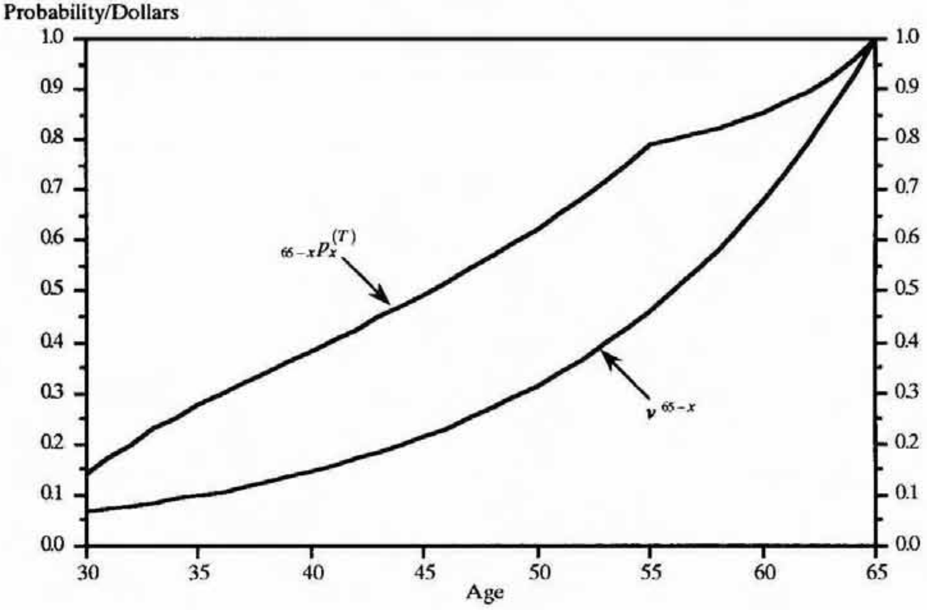
If a pension plan has benefits expressed in terms of salary, it is necessary to develop salary-related notation and procedures for estimating future salary. The current dollar salary for a par-

TABLE 3–3

Compound Interest Function, v^t

	Interest Rate		
t	6%	8%	10%
0	1.0000	1.0000	1.0000
5	0.7473	0.6806	0.6209
10	0.5584	0.4632	0.3855
15	0.4173	0.3152	0.2394
20	0.3118	0.2145	0.1486
25	0.2330	0.1460	0.0923
30	0.1741	0.0994	0.0573
35	0.1301	0.0676	0.0356
40	0.0972	0.0460	0.0221
45	0.0727	0.0313	0.0137
50	0.0543	0.0213	0.0085
55	0.0406	0.0145	0.0053
60	0.0303	0.0099	0.0033
65	0.0227	0.0067	0.0020
70	0.0169	0.0046	0.0013

FIGURE 3–1
Survival and Interest Functions from Age x to Age 65



ticipant age x is denoted by s_x, and S_x represents the *cumulative* salary from entry age y up to, but not including, age x.[3] Thus, for $x > y$ we have

$$S_x = \sum_{t=y}^{x-1} s_t. \tag{3.7}$$

In order to estimate the dollar salary at age x, based on the employee's age-y salary, the following formula is used:

$$s_x = s_y \frac{(SS)_x}{(SS)_y}[(1+I)(1+P)]^{(x-y)}, \tag{3.8a}$$

where

s_y = entry-age dollar salary

$(SS)_x$ = merit salary scale at age x

I = rate of inflation

P = rate of productivity reflected in the salary increases.

An age-y entrant's salary at age x can also be defined in terms of the age-z salary ($y < z < x$):

$$s_x = s_z \frac{(SS)_x}{(SS)_z}[(1+I)(1+P)]^{(x-z)}. \tag{3.8b}$$

If all of the salary increase assumptions were met from age y to age z, the employee's salary at age z would be equal to

$$s_z = s_y \frac{(SS)_z}{(SS)_y}[(1+I)(1+P)]^{(z-y)}. \tag{3.8c}$$

Substituting (3.8c) for s_z in (3.8b) reduces the latter to (3.8a), showing that s_x is identical, under the salary increase assumptions, whether derived from the entry age salary or the attained age salary.

Table 3–4 shows the salary function per dollar of entry-age salary based on the previously discussed merit scale, 1 percent

[3]It is to be understood throughout the remainder of this book that s_x and S_x are dependent on each participant's entry age y, despite the fact that y does not appear in the symbol. The symbols $s_{x,y}$ and $S_{x,y}$ could be used to denote these two functions, which have the virtue of making the entry age explicit. These symbols are used to make the entry age variable explicit at a later point where the equations require a summation over all combinations of x and y.

TABLE 3–4

Salary Function per Dollar of Entry Age Salary, $s_x \div s_y$

	Entry Age, y				
x	20	30	40	50	60
20	1 000				
21	1.097				
22	1.203				
23	1.317				
24	1.442				
25	1.575				
26	1.721				
27	1.877				
28	2.045				
29	2.228				
30	2.422	1 000			
31	2.632	1.087			
32	2.859	1.180			
33	3.100	1.280			
34	3.360	1.387			
35	3.636	1.501			
36	3.934	1.624			
37	4.249	1.754			
38	4.587	1.894			
39	4.948	2.043			
40	5.328	2.200	1 000		
41	5.736	2.368	1.077		
42	6.166	2.546	1.157		
43	6.625	2.735	1.244		
44	7.108	2.935	1.334		
45	7.619	3.146	1.430		
46	8.160	3.369	1.532		
47	8.733	3.605	1.639		
48	9.334	3.854	1.752		
49	9.969	4.116	1.871		
50	10.632	4.389	1.996	1 000	
51	11.331	4.678	2.127	1.066	
52	12.065	4.981	2.264	1.135	
53	12.833	5.298	2.409	1.207	
54	13.638	5.630	2.560	1.283	
55	14.474	5.976	2.717	1.361	
56	15.354	6.339	2.882	1.444	
57	16.262	6.714	3.052	1.530	
58	17.215	7.107	3.231	1.619	
59	18.203	7.515	3.417	1.712	
60	19.226	7.938	3.609	1.808	1 000
61	20.291	8.377	3.808	1.908	1.055
62	21.391	8.831	4.015	2.012	1.113
63	22.526	9.300	4.228	2.119	1.172
64	23.695	9.782	4.447	2.229	1.232

productivity factor, and 4 percent inflation rate, illustrated for decennial entry ages from 20 through 60. The age-64 salary is 24 times greater than the age-20 starting salary. At the other extreme, an age-60 entrant's initial salary is expected to increase by 23 percent by age 64. Retirement-related cost estimates are directly proportional to an employee's final 5-year average salary under the model benefit formula used to illustrate pension costs in this book; consequently, it is clear from Table 3–4 that the growth in a participant's future salary can increase pension cost estimates substantially. This is in contrast to the interest rate and decrement probabilities, both of which have a decreasing effect on pension cost estimates.

BENEFIT FUNCTION

The benefit function is used to determine the amount of benefits paid at retirement, vested termination, disablement, and death. This function, the interest function, and the survival function provide the basic components required to formulate pension costs, as shown in subsequent chapters. In this section consideration is given to the three most common types of benefit formulas used with defined benefit pension plans.

The symbol b_x denotes the annual benefit accrual during age x to age $x + 1$ for an age-y entrant, and is referred to as the *benefit accrual function*. The benefit accrual function can equal the formula accruals or, as discussed in this section, some other definition of accruals, such as a portion of the participant's projected retirement-age benefit. The accrued benefit, denoted by B_x, is equal to the sum of each attained age accrual up to, but not including, age x. This function is called the *accrued benefit function* and is defined for $x > y$ by

$$B_x = \sum_{t=y}^{x-1} b_t. \tag{3.9}$$

The convention of using lower and upper case letters, with the upper case denoting a summation of lower case functional values, was also used for the salary scale.[4]

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[4]The benefit functions, b_x and B_x, are dependent on the employee's entry age y, suggesting the notation $b_{x,y}$ and $B_{x,y}$, which is not used here for simplicity.

Flat Dollar Unit Benefit

Under a flat dollar unit benefit formula, b_x is equal to the annual benefit payable per year of service. These values are found to range from about $150 to $300 per year of service. Although the attained age subscript is shown here, it is unnecessary since the accrual is independent of age. The accrued benefit is a years-of-service multiple of the benefit accrual, that is,

$$B_x = (x - y)\, b_x . \tag{3.10}$$

Career Average

The career average benefit formula has the following definitions for the benefit accrual and the accrued benefit functions at age x:

$$b_x = k\, s_x , \tag{3.11a}$$

$$B_x = k\, S_x , \tag{3.11b}$$

where k denotes the proportion of attained age salary provided as an annual benefit accrual. The benefit functions under the career average formula follow precisely the pattern of the attained age salary and the cumulative salary discussed and illustrated previously.

Final Average

The final average benefit formula is somewhat more complicated. Let n denote the number of years over which the participant's salary prior to retirement is to be averaged, and let k equal the proportion of the average salary provided per year of service. The projected retirement benefit, assuming retirement occurs at the beginning of age r, is defined as

$$B_r = k\,(r - y)\,\tfrac{1}{n} \sum_{t=r-n}^{r-1} s_t , \tag{3.12a}$$

or more simply

$$B_r = k\,(r - y)\,\tfrac{1}{n}\,(S_r - S_{r-n}) . \tag{3.12b}$$

The attained age benefit accrual and accrued benefit functions can be defined in several ways under this benefit formula.

One approach is to defined B_x according to the benefit formula based on the participant's current salary average:

$$B_x = k\,(x-y)\tfrac{1}{n}\,(S_x - S_{x-n})\,, \tag{3.13}$$

where n is the smaller of the years specified in the benefit formula or $x-y$. The corresponding benefit accrual at age x can be determined by the following basic relationship:

$$b_x = B_{x+1} - B_x. \tag{3.14a}$$

Substituting the definition of B_x given in (3.13) and simplifying, we have:

$$b_x = k\,\tfrac{1}{n}\,(S_{x+1} - S_{x+1-n}) + k\,\tfrac{1}{n}\,(x-y)\big[(S_{x+1} - S_{x+1-n}) - (S_x - S_{x-n})\big] \tag{3.14b}$$

$$= k\,\tfrac{1}{n}\,(S_{x+1} - S_{x+1-n}) + k\,\tfrac{1}{n}\,(x-y)\big[s_x - s_{x-n}\big]. \tag{3.14c}$$

The first term on the right side of (3.14c) is the portion of the benefit accrual earned in the current year based on the participant's current n-year salary average, while the second term represents the portion earned as a result of the increase in the n-year salary average base. The increase in the salary base, which is represented by $\frac{1}{n}\,(s_x - s_{x-n})$ in (3.14c), is multiplied by years-of-service to date. Therefore, the benefit accrual during age x includes an implicit updating of the previous accruals which can cause b_x to be a steeply increasing function of x.

As will be shown later, steeply increasing benefit accrual and accrued benefit functions produce even steeper pension cost functions under some methods of determining pension costs and liabilities, characteristics that may be undesirable. Consequently, two modifications to benefit functions that are based on the plan's formula have been developed for funding purposes in order to mitigate this effect.

The *constant dollar* (CD) modification defines the benefit accrual function as a pro rata share of the participant's retirement-age projected benefit:

$${}^{CD}b_x = \frac{B_r}{(r-y)}\,, \qquad (y \le x < r) \tag{3.15a}$$

a constant for all x. The accrued benefit function is a years-of-service multiple of this constant:

$$^{CD}B_x = \frac{B_r}{(r-y)}(x - y). \qquad (y \le x \le r) \qquad (3.15b)$$

While the constant dollar modification is applicable to a career average benefit formula, it has no effect on a flat dollar unit benefit formula.

The *constant percent* (*CP*) modification defines b_x as a constant percent of salary. The appropriate percentage of attained age salary is found by dividing the projected benefit by the cumulative projected salary of the age-y entrant:

$$^{CP}b_x = \frac{B_r}{S_r}s_x, \qquad (y \le x < r) \qquad (3.16a)$$

$$^{CP}B_x = \frac{B_r}{S_r}S_x. \qquad (y \le x \le r) \qquad (3.16b)$$

The constant percent modification has no effect on a career average benefit formula, just as the constant dollar version has no effect on a flat dollar benefit formula.

Table 3–5 shows the benefit accrual and accrued benefit functions under the final average benefit formula used in this book (i.e., 1.5 percent of final 5-year average salary per year of service) for an age-30 entrant. The constant dollar and constant percent versions are illustrated along with the unmodified version of this benefit formula. In each case, the benefit functions are expressed as a percentage of the employee's projected retirement benefit, with retirement assumed to occur at the beginning of age 65. The constant percent version has only a minor effect as compared to the unmodified benefit functions; however, the constant dollar version produces significant differences. This modification allocates a constant 2.86 percent, or 1/35, of the projected benefit per year of service. The constant dollar version develops an accrued benefit equal to 50 percent of the projected benefit when one-half of the expected career has been completed, while the allocation under the constant percent is only 25 percent at this point, and the unmodified version is 18 percent. Thus, even though all of the accrued benefit functions begin at zero and attain B_r by age r, the modifications, and particularly the constant dollar version, cause the intermediate values to be different. Figures 3–2 and 3–3 show these relationships graphically.

TABLE 3–5

Benefit Accrual and Accrued Benefit Functions Expressed as a Percent of the Projected Benefit

	Unmodified		*Constant Percent*		*Constant Dollar*	
Age	b_x	B_x	${}^{CD}b_x$	${}^{CD}B_x$	${}^{CP}b_x$	${}^{CP}B_x$
30	0.32	0.00	0.67	0.00	2.86	0.00
31	0.35	0.32	0.73	0.67	2.86	2.86
32	0.38	0.67	0.79	1.41	2.86	5.71
33	0.41	1.06	0.86	2.20	2.86	8.57
34	0.45	1.47	0.93	3.06	2.86	11.43
35	0.58	1.92	1.01	4.00	2.86	14.29
36	0.66	2.49	1.09	5.01	2.86	17.14
37	0.75	3.15	1.18	6.10	2.86	20.00
38	0.84	3.90	1.28	7.28	2.86	22.86
39	0.95	4.74	1.38	8.56	2.86	25.71
40	1.07	5.70	1.48	9.93	2.86	28.57
41	1.19	6.76	1.59	11.42	2.86	31.43
42	1.33	7.95	1.71	13.01	2.86	34.29
43	1.47	9.28	1.84	14.73	2.86	37.14
44	1.63	10.75	1.98	16.57	2.86	40.00
45	1.80	12.39	2.12	18.54	2.86	42.86
46	1.99	14.19	2.27	20.66	2.86	45.71
47	2.18	16.18	2.43	22.93	2.86	48.57
48	2.39	18.36	2.60	25.36	2.86	51.43
49	2.62	20.75	2.77	27.95	2.86	54.29
50	2.86	23.37	2.96	30.73	2.86	57.14
51	3.11	26.23	3.15	33.68	2.86	60.00
52	3.38	29.34	3.35	36.83	2.86	62.86
53	3.66	32.71	3.57	40.19	2.86	65.71
54	3.96	36.38	3.79	43.76	2.86	68.57
55	4.28	40.34	4.02	47.55	2.86	71.43
56	4.61	44.62	4.27	51.57	2.86	74.29
57	4.96	49.23	4.52	55.84	2.86	77.14
58	5.32	54.19	4.79	60.36	2.86	80.00
59	5.71	59.51	5.06	65.15	2.86	82.86
60	6.10	65.22	5.35	70.21	2.86	85.71
61	6.51	71.32	5.64	75.56	2.86	88.57
62	6.95	77.83	5.95	81.20	2.86	91.43
63	7.38	84.78	6.26	87.15	2.86	94.29
64	7.84	92.16	6.59	93.41	2.86	97.14
65		100.00		100.00		100.00

FIGURE 3–2
Attained Age Benefit Accrual Functions as a Percent of the Projected Retirement Benefit

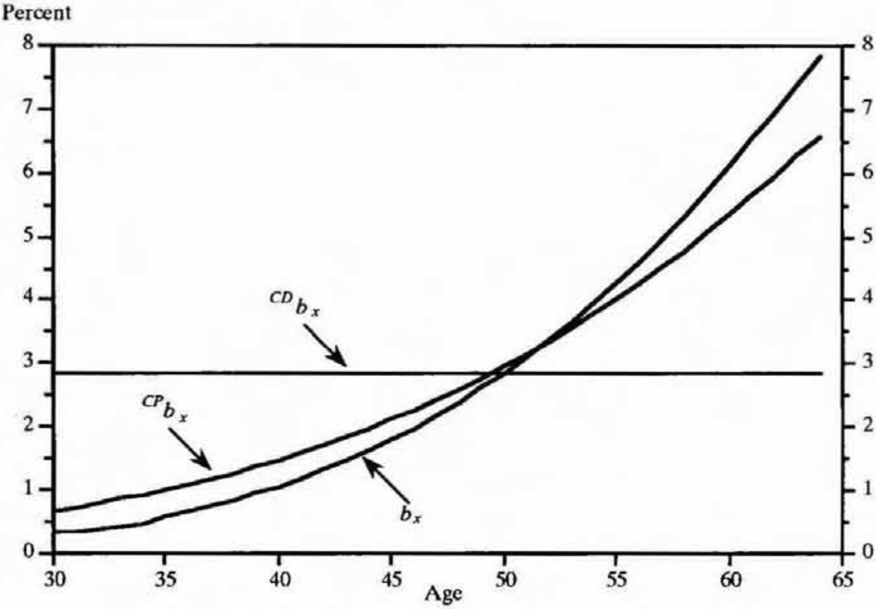
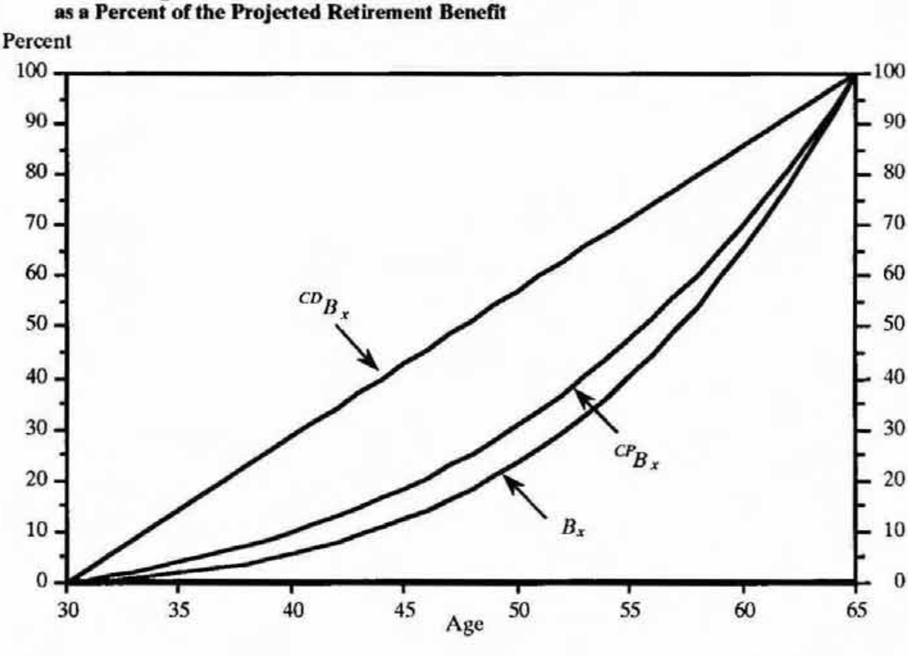


FIGURE 3–3
Attained Age Accrued Benefit Functions as a Percent of the Projected Retirement Benefit



ANNUITY FUNCTIONS

Annuities represent a combination of the survival and interest functions. Most annuities are based on the mortality-only survival function, and the material below reflects this emphasis. However, this section also defines a temporary employment-based annuity which uses the composite survival function, since this annuity is required for some funding methods.

Straight Life Annuity

If retirement benefits cease upon death, the annuity is called a *straight life annuity* and its present value, assuming an annual benefit of one dollar payable at the beginning of age x, is given by

$$\ddot{a}_x = \sum_{t=0}^{\infty} {}_tp_x^{(m)} v^t . \qquad (3.17a)$$

The infinity sign is used as the upper limit of the summation for simplicity, since ${}_tp_x^{(m)}$ becomes zero beyond some advanced age.[5]

A special case of the life annuity is when the interest rate is zero, in which case we have, simply, one plus the curtate life expectancy (i.e., based on whole years only) at age x. This is expressed notationally as

$$e_x = \left[\sum_{t=0}^{\infty} {}_tp_x^{(m)} \right] - 1 . \qquad (31.7b)$$

Period Certain Life Annuity

It is not uncommon to find an n-year *period certain life annuity* used as the basis for distributing pension benefits. During the certain period, benefits are payable whether or not the annuitant is alive. This type of annuity is a combination of an n-year *period certain annuity* plus an n-year *deferred life annuity*:

$$\ddot{a}_{\overline{x:\overline{n|}}} = \left[\sum_{t=0}^{n-1} v^t \right] + {}_np_x^{(m)} v^n \ddot{a}_{x+n}$$

[5] An approximation for the present value of an annuity payable m times a year, with payments at the beginning of each period, is found by subtracting $(m - 1)/2m$ from the annuity payable annually. Since retirement benefits are paid monthly, 11/24 should be subtracted from the above annuity to approximate a monthly payment of 1/12 of a dollar. The annual-pay annuity is assumed throughout this book for simplicity.

$$= \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x . \qquad (3.18)$$

These equations introduce notations deserving comment. The bar over the subscript $\overline{x{:}\overline{n}|}$ signifies that the annuity is paid until the last status fails, where the two statuses represent (1) the life of the plan member and (2) the n-year period. Thus, the period certain life annuity is technically a *last survivor annuity*. An annuity discussed subsequently is $\ddot{a}_{x:\overline{n}|}$ without the bar. This annuity is known as an n-year *temporary life annuity* which pays until the *first* of the two statuses fails and is technically a *joint annuity*. Finally, (3.18) uses two other symbols: $\ddot{a}_{\overline{n}|}$ indicating an n-year *period certain annuity* and ${}_{n|}\ddot{a}_x$ denoting the present value of an n-year *deferred life annuity*.

Joint and Survivor Annuity

Another type of annuity is known as the *joint and survivor annuity*. The term "joint" suggests that the payment amount is based on more than one status, and the term "survivor" suggests that it pays as least some amount until the last status fails. For example, a 50 percent joint and survivor annuity pays one dollar annually while both statuses are alive (usually husband and wife, but not necessarily restricted to couples), and reduces to 50 cents after the first death. Let x denote the age of the plan member, z the joint annuitant's age, and k the portion of the annual benefit paid to the survivor after the first death, regardless of who dies first. The $100k$ percent joint and survivor annuity may be represented as

$$\begin{aligned} {}^{k}\ddot{a}_{xz} = \sum_{t=0}^{\infty} v^{t}\big[{}_{t}p_{x}^{(m)}\,{}_{t}p_{z}^{(m)} + k\,{}_{t}p_{x}^{(m)}\,(1 - {}_{t}p_{z}^{(m)}) \\ + k\,{}_{t}p_{z}^{(m)}\,(1 - {}_{t}p_{x}^{(m)})\big]. \end{aligned} \qquad (3.19a)$$

The first term inside the brackets represents a payment of \$1 if both x and z are alive at time t, the second term represents a payment of \$$k$ if only x is alive, while the third term represents a payment of \$$k$ if only z is alive.

A widely used variation of this annuity is known as a *contingent* joint and survivor annuity. Under this form, the annuity benefit is reduced only if the plan member is the first to die. The survivor's benefit might be any portion, with one-half and two-thirds representing choices usually available. A $100k$ percent contingent joint and survivor annuity may be expressed as

$$ {}^{k}\ddot{a}_{\underset{1}{x}z} = \sum_{t=0}^{\infty} v^{t}\left[{}_{t}p_{x}^{(m)}\,{}_{t}p_{z}^{(m)} + {}_{t}p_{x}^{(m)}\left(1 - {}_{t}p_{z}^{(m)}\right) + k\,{}_{t}p_{z}^{(m)}\left(1 - {}_{t}p_{x}^{(m)}\right)\right], \tag{3.19b} $$

where the 1 under the x subscript stipulates that, if x is the first to die, only k dollars are continued to z. The bracketed expression represents a payment of \$1 if both x and z are alive at time t, a payment of \$1 if x is alive and z is not alive, and a payment of $\$k$ if z is alive and x is not alive. This expression reduces to

$$ {}^{k}\ddot{a}_{\underset{1}{x}z} = \sum_{t=0}^{\infty} v^{t}\left[{}_{t}p_{x}^{(m)} + k\,{}_{t}p_{z}^{(m)} - k\,{}_{t}p_{x}^{(m)}\,{}_{t}p_{z}^{(m)}\right]. \tag{3.19c} $$

In this form, the bracketed term represents a payment of \$1 to x regardless of whether or not z is alive, a payment of $\$k$ to z regardless of whether or not x is alive, and since this would result in a total payment of $\$(1 + k)$ in the event both are alive in year t, $\$k$ is subtracted if both are alive at time t.

Refund Annuities

Pension plans that require employee contributions frequently provide death benefits in retirement equal to the excess, if any, of the employee's accumulated contributions at retirement date over the cumulative benefits received up to the time of death. If the difference is paid in a lump sum, the annuity is termed a *modified cash refund annuity*, whereas if the difference is paid by continuing the benefit payments to a named beneficiary, the annuity is termed a *modified installment refund annuity*.[6]

Let C_r denote the employee's accumulated contributions at retirement and n' denote such contributions per dollar of retirement benefit (i.e., $n' = C_r \div B_r$). The present value at age r of the modified cash refund annuity may be written as

[6]Cash refund and installment annuities guarantee a return of the annuity's purchase cost. The term *modified* is used in connection with pension plans to indicate that the guarantee involves a return of accumulated employee contributions.

$$^{MCR}\ddot{a}_r = \sum_{t=0}^{\infty} v^t \, {}_tp_r^{(m)} \left[1 + v \, q_{r+t}^{(m)} \max\{(n' - t - 1), 0\} \right]. \qquad (3.20)$$

This formulation assumes that the lump sum payment, if any, will be made at the end of the year of death.

The modified installment refund annuity is equal to the sum of an n'-year annuity certain plus an n'-year deferred life annuity (i.e., a period certain life annuity, where the certain period is determined by the ratio of employee contributions at retirement to annual benefit payments):

$$^{MIR}\ddot{a}_r = \ddot{a}_{\overline{n'}|} + {}_{n'|}\ddot{a}_x . \qquad (3.21)$$

Table 3–6 shows numerical values for the annuities discussed up to this point under alternative interest rate, mortality rate, and attained-age assumptions. Life expectancies at age 55, 65, and 70 are shown in Section I. A 25 percent change in the underlying mortality rate is seen to affect this statistic by less than 25 percent. Section II of Table 3–6 shows annuity certain values, with the last row displaying an annuity certain for a period equal to an age-65 individual's life expectancy. Note that this value is greater than an age-65 life annuity shown in Section V. Figure 3–4 is useful in reasoning through this relationship. The area for both annuity representations is, of course, equal. Since their respective annuity values involve the product of each payment (or expected payment in the case of the life annuity) times v^t, for t ranging from zero to 45 (110 – 65), the life annuity payment stream involves much smaller values of v^t beyond age 65 + e_{65} than the values of v^t used with the annuity certain.

Section III of Table 3–6 shows the very substantial difference between a last survivor annuity and a temporary life annuity, both involving 10-year periods. Section IV illustrates the relatively small impact of providing a 100 percent contingent benefit versus a 50 or 75 percent contingency. Likewise, a 10-year difference in the age of the contingent annuitant has little impact on the annuity value.

Section V shows life annuity values for ages 55, 65, and 70. A 25 percent change in the underlying mortality rate has only slightly less impact than a 2 percentage point change in the interest rate. Finally, the modified cash refund and modified installment refund annuities are shown in Section VI of the table. Since employee contributions at retirement are assumed to be equal to 5 times annual benefits, the installment refund annuity is equal to a 5-year and age-65 last survivor annuity.

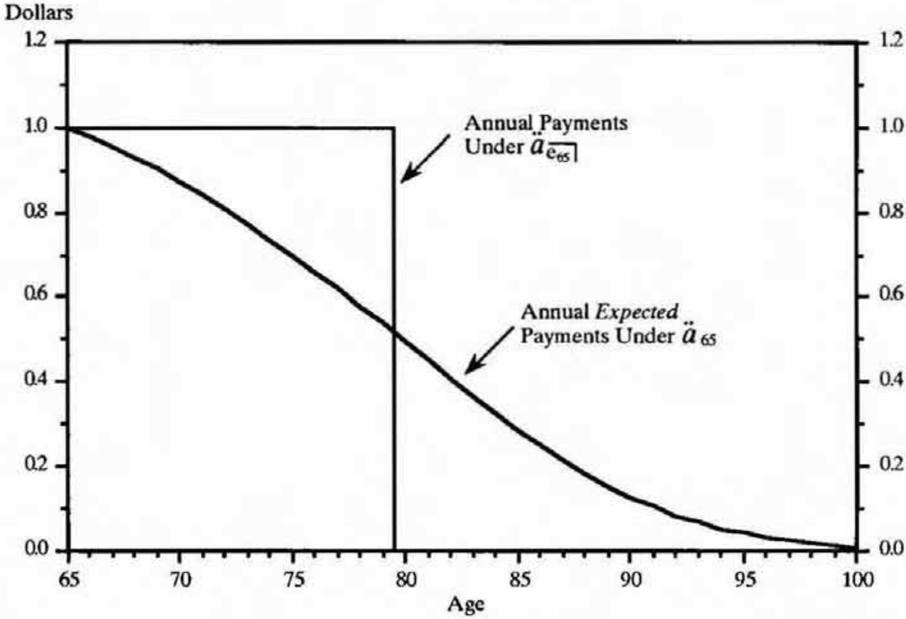
TABLE 3–6

Annuity Values Under Alternative Interest and Mortality Assumptions

Type of Annuity	Interest Rates: / Mortality as % of GAM-1971:	6% 100%	8% 75%	8% 100%	8% 125%	10% 100%
I. Life Expectancy	e_{55}		24.95	22.21	20.21	
	e_{65}		17.00	14.61	12.91	
	e_{70}		13.57	11.41	9.89	
II. Annuity Certain	$\ddot{a}_{\overline{5}\rceil}$	4.47		4.31		4.17
	$\ddot{a}_{\overline{10}\rceil}$	7.80		7.25		6.76
	$\ddot{a}_{\overline{15}\rceil}$	10.29		9.24		8.37
	$\ddot{a}_{\overline{e_{65}}\rceil}$	10.13	9.85	9.12	8.50	8.27
III. Last Survivor and Temporary Life Annuities	$\ddot{a}_{\overline{65:\overline{10}\rceil}}$	10.55	9.80	9.34	8.98	8.36
	$\ddot{a}_{65:\overline{10}\rceil}$	6.98	6.69	6.51	6.35	6.10
IV. Contingent Joint and Survivor Annuity	${}^{50}\ddot{a}_{65:60}$	11.09	10.22	9.65	9.17	8.53
	${}^{75}\ddot{a}_{65:60}$	11.78	10.71	10.18	9.72	8.94
	${}^{100}\ddot{a}_{65:60}$	12.46	11.21	10.70	10.26	9.35
	${}^{50}\ddot{a}_{65:55}$	11.46	10.43	9.89	9.44	8.70
	${}^{50}\ddot{a}_{65:65}$	10.75	10.02	9.41	8.90	8.36
V. Life Annuity	$\ddot{a}_{55}$	12.24	10.90	10.45	10.06	9.10
	$\ddot{a}_{65}$	9.73	9.24	8.60	8.08	7.71
	$\ddot{a}_{70}$	8.35	8.23	7.52	6.95	6.84
VI. Modified Cash and Installment Refund Annuities	${}^{MCR}\ddot{a}_{65}$	9.93	9.39	8.80	8.32	7.90
	${}^{MIR}\ddot{a}_{65}$	9.92	9.38	8.78	8.31	7.88

$n' = C_r \div B_r = 5$ for modified cash and installment refund annuities.

FIGURE 3–4
Life Annuity vs. Annuity Certain for Life Expectancy



Temporary Annuities

Temporary annuities, as will be shown in later chapters, are required for some actuarial cost methods. In this case, consideration is given to an employment-based annuity, subject to multiple decrements, rather than the typical retirement-related annuity involving only mortality. As noted earlier, the temporary annuity has $x{:}\overline{n}|$ as its subscript, indicating that the payments cease at the end of n years or, if sooner, at the time the status x fails. In addition, a superscript T is added to the annuity symbol to signify a multiple-decrement environment. Equation (3.22) sets forth the basic definition of this annuity:

$$\ddot{a}^{T}_{x:\overline{n}|} = \sum_{t=0}^{n-1} {}_{t}p_x^{(T)} v^{t}. \tag{3.22}$$

The value of n is often set at $r - x$ in the context of pension mathematics, resulting in an annuity running from the participant's attained age x up to, but not including, retirement age.

The withdrawal decrement may have a unique effect on this annuity, as shown in Table 3–7. Generally one would anticipate the present value of an annuity running to age 65 to become smaller at older ages. However, because of the withdrawal decrement, the value of the annuity for most entry ages reaches a maximum at some age in between the employee's entry age and retirement age. This maximum for an age-20 entrant is 8.18 at ages 41 and 42, with younger and older ages having smaller values. For example, at age 20, the 45-year annuity takes on a value of only 4.00.

An important variation of $\ddot{a}^T_{x:\overline{r-x}|}$ is represented by ${}^s\ddot{a}^T_{x:\overline{r-x}|}$, the superscript s denoting that the annuity is salary-based, as defined by

$$ {}^s\ddot{a}^T_{x:\overline{r-x}|} = \sum_{t=x}^{r-1} \frac{s_t}{s_x} \, {}_{t-x}p_x^{(T)} \, v^{\,t-x}. \qquad (3.23)$$

This formula represents the present value of an employee's future salary from age x to age r, per unit of salary at age x.

Table 3–8 replicates Table 3–7 based on the annuity defined in (3.23). This annuity, as one would expect, develops larger attained age values than the unit-based annuity given in Table 3–7. Whereas the unit-based annuity for an age-20 entrant reaches a maximum value of 8.18 at age 41, the salary-based annuity reaches a maximum of 14.35 at age 36. By age 64, however, both annuity values are equal to unity. As was the case for the annuity values shown in Table 3–7, the values given in Table 3–8 for the same attained age, but for different entry ages, are affected to some extent by the select period of the termination assumption and the elimination of termination rates after the first early retirement qualification age.

Figure 3–5 provides a graphical comparison of the annuities specified by equations (3.22) and (3.23) for the age-30 entrant. This graph shows that these annuities can take on unique shapes during an employee's career.

TABLE 3–7

Present Value of a Temporary Employment-Based Life Annuity from Age x to Age 65, $\ddot{a}^{T}_{x:\overline{65-x}|}$

	Entry Age, y				
x	20	30	40	50	60
20	4.00				
21	4.29				
22	4.58				
23	4.88				
24	5.19				
25	5.49				
26	5.79				
27	6.08				
28	6.36				
29	6.63				
30	6.88	6.07			
31	7.11	6.59			
32	7.32	7.03			
33	7.51	7.38			
34	7.67	7.63			
35	7.81	7.81			
36	7.93	7.93			
37	8.02	8.02			
38	8.09	8.09			
39	8.14	8.14			
40	8.17	8.17	7.55		
41	8.18	8.18	7.83		
42	8.18	8.18	8.00		
43	8.16	8.16	8.08		
44	8.12	8.12	8.09		
45	8.07	8.07	8.07		
46	8.00	8.00	8.00		
47	7.93	7.93	7.93		
48	7.84	7.84	7.84		
49	7.73	7.73	7.73		
50	7.62	7.62	7.62	7.09	
51	7.49	7.49	7.49	7.01	
52	7.35	7.35	7.35	6.86	
53	7.19	7.19	7.19	6.67	
54	7.01	7.01	7.01	6.44	
55	6.80	6.80	6.80	6.17	
56	6.34	6.34	6.34	5.86	
57	5.85	5.85	5.85	5.50	
58	5.32	5.32	5.32	5.09	
59	4.74	4.74	4.74	4.64	
60	4.12	4.12	4.12	4.12	3.84
61	3.44	3.44	3.44	3.44	3.30
62	2.70	2.70	2.70	2.70	2.64
63	1.89	1.89	1.89	1.89	1.87
64	1.00	1.00	1.00	1.00	1.00

TABLE 3–8

Present Value of a Temporary Employment-Based and Salary-Based Life Annuity from Age x to Age 65, ${}^{s}\ddot{a}^{T}_{x:\overline{65-x}|}$

	Entry Age, y				
x	20	30	40	50	60
20	6.68				
21	7.39				
22	8.13				
23	8.87				
24	9.61				
25	10.33				
26	11.01				
27	11.65				
28	12.24				
29	12.75				
30	13.20	11.42			
31	13.58	12.47			
32	13.88	13.27			
33	14.11	13.84			
34	14.25	14.18			
35	14.34	14.34			
36	14.35	14.35			
37	14.31	14.31			
38	14.22	14.22			
39	14.07	14.07			
40	13.89	13.89	12.72		
41	13.66	13.66	13.01		
42	13.40	13.40	13.08		
43	13.12	13.12	12.98		
44	12.81	12.81	12.77		
45	12.49	12.49	12.49		
46	12.15	12.15	12.15		
47	11.79	11.79	11.79		
48	11.42	11.42	11.42		
49	11.04	11.04	11.04		
50	10.65	10.65	10.65	9.75	
51	10.25	10.25	10.25	9.44	
52	9.83	9.83	9.83	9.06	
53	9.40	9.40	9.40	8.62	
54	8.95	8.95	8.95	8.14	
55	8.49	8.49	8.49	7.63	
56	7.72	7.72	7.72	7.08	
57	6.95	6.95	6.95	6.50	
58	6.16	6.16	6.16	5.89	
59	5.35	5.35	5.35	5.23	
60	4.54	4.54	4.54	4.54	4.22
61	3.70	3.70	3.70	3.70	3.55
62	2.83	2.83	2.83	2.83	2.77
63	1.94	1.94	1.94	1.94	1.92
64	1.00	1.00	1.00	1.00	1.00

FIGURE 3–5

Unit-Based and Salary-Based Temporary Annuity Values from Age *x* to Age 65