Permutation and Combinations

Read: 4.4

Next Class: 4.6

Motivations

- Different counting principles (multiplication and addition) provide a basis to calculate the number of possible outcomes or equivalently the size of a set constructed using basic set operations.
- Many counting problems have a common structure and can be cast as different variations of the problem of selecting *r* items from a set of *n* items.
- For these problems a set of closed-form solutions can be derived which simplify the calculation of the number of possibilities.

Permutations

- An ordered arrangement of objects is called a permutation.
  - Hence, a permutation of n distinct elements is an ordering of these n elements.
- It is denoted by $P(n,r)$ or ${}^{n}P_{r}$.
- Permutation problems are of the form where r distinct elements are drawn sequentially from a set of n objects. This implies that the order in which the different elements are drawn is important.
- This problem consists of a sequence of events and a solution thus involves the multiplication principle. In particular there are n possibilities for the first event, $n - 1$ possibilities for the second, and $n - r + 1$ for the last event.

Permutations (cont'd)

- Ordering of last four digits of a telephone number if digits are allowed to repeat
  - 10.10.10.10 = 10000
  - Ordering of four digits if repetition is not allowed = 10.9.8.7 = 5040 = 10!/6!

    where $n! = n*(n-1)*(n-2)*\ldots.*3*2*1$ and by definition $0! = 1$
- Hence, mathematically, for $r \le n$, an r-permutation from n objects is defined by

$$P(n,r) = n*(n-1)*(n-2)*\ldots..*(n-r+1) =$$

$$\Rightarrow \quad P(n,r) = \frac{n*(n-1)*(n-2)*\ldots..*(n-r+1)*(n-r)!}{(n-r)!}$$

$$\Rightarrow \quad = \frac{n!}{(n-r)!} \qquad \text{for } 0 \le r \le n$$

- Hence, $P(10,4) = 10! / (10-4)! = 10!/6! = 5040$

4

Permutations: Some special cases

- $P(n,0) = n! / n! = 1$
- This means that there is only one ordered arrangement of 0 objects, called the empty set.
- $P(n,1) = n!/(n-1)! = n$
- There are n ordered arrangements of one object (i.e. n ways of selecting one object from n objects).
- $P(n,n) = n!/(n-n)! = n!/0! = n!$
- This means that one can arrange n distinct objects in $n!$ ways, that is nothing but the multiplication principle.

5

Permutation Examples

1. Ten athletes compete in an Olympic event. Gold, silver and bronze medals are awarded to the first three in the event, respectively. How many ways can the awards be presented?

2. How many ways can six people be seated on six chairs?

3. How many permutations of the letters ABCDEF contain the letters DEF together in any order?

Permutation Examples

4. The professor's dilemma: how to arrange four books on OS, seven on programming, and three on data structures on a shelf such that books on the same subject must be together?

5. In how many ways can you park 4 cars in 7 parking spots?

6. How many 3 letter words can be formed from the letters in the word "english"?

Combinations

- Combination problems address the similar problem where r distinct elements are drawn simultaneously from a set of n objects, implying that the order in which they are drawn is irrelevant.
- Since there is no defined sequence of events, the multiplication principle can not directly be used. Furthermore the addition principle can not be applied because this problem does also not involve events of the type "A or B".

8

Combinations (cont'd)

- To solve the combination problem we can address it in the following way. First assume that the order actually matters. This can be solved using $P(n, r)$ but introduces many duplicates, i.e. sequences containing the same elements in a different order. To compensate for this it has then to be determined how many duplicates there are.
  - Given an unordered set of r distinct objects, it has to be determined how many different sequences of r elements can be formed. This is again a permutation problem and its solution is $P(r, r) = r!$
  - Using this, the number of possible permutations can be determined form the number of combinations $C(n, r)$ as $P(n, r) = C(n, r) * P(r, r) = C(n, r) * r!$.

9

Combinations (cont'd)

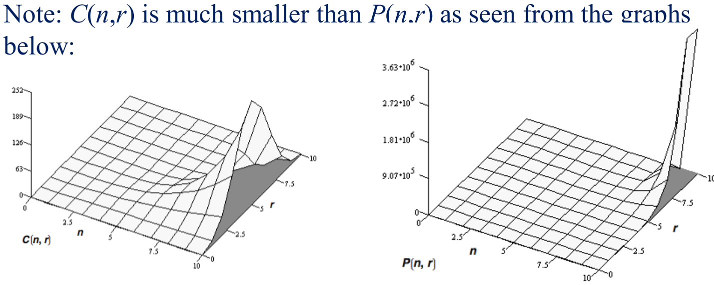
- The solution to the combination problem is then

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!} \text{ for } 0 \le r \le n$$

- When order in permutations becomes immaterial, i.e. we are just interested in selecting r objects from n distinct objects, we talk of combinations denoted by

$$C(n,r) \text{ or } {}^{n}C_{r}$$

- Note: $C(n,r)$ is much smaller than $P(n,r)$ as seen from the graphs below:



(a) (b)

Combinations: Special Cases

- $C(n,0) = 1$
- Only one way to choose 0 objects from *n* objects--chose the empty set

- $C(n,1) = n$
- Obvious, since *n* ways to choose one object from *n* objects

- $C(n,n) = 1$
- Only one way to choose *n* objects from *n* objects

Problem Solving Techniques

- Hint: Whenever a counting problem is presented, try to decide first if order is important, making it a permutation problem, or if it does not matter, making it a combination problem.

- To solve more complex problems, the permutation and combination techniques can be combined with each other or with the other counting principles.

- When reformulating and calculating the solution for such problems care has to be taken not to omit elements or to count them twice.

12

Combinations: Examples

- How many ways can we select a committee of three from 10?
- How many ways can a committee of two women and three men be selected from a group of five different women and six different men?
- How many five-card poker hands can be dealt from a standard 52-card deck?
- How many poker hands contain cards all of the same suit?

13

Combinations: Examples

- How many poker hands contain three cards of one denomination and two cards of another denomination?
- In how many ways can three athletes be declared winners from a group of 10 athletes who compete in an Olympic event?

14

Combinations: Examples

- 3 vehicles are selected from a parking lot of 12 cars and 9 trucks.
  - How many possibilities are there ?
  - How many selections contain 2 trucks ?

15

Combinations: Examples

- 3 vehicles are selected from a parking lot of 12 cars and 9 trucks. How many selections contain at least one truck ?
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Eliminating Duplicates

- Duplicates can also arise if the set from which elements are drawn contains not only distinct objects but also some identical ones.
- How many ways are there to form distinct rearrangements of n objects if each of the k kinds of objects o_i occurs n_i times ?
  - Given a specific sequence, identical sequences can be derived by interchanging identical objects. For each object "type" o_i, there are $P(n_i, n_i) = n_i!$ ways to do this. Therefore there every distinct sequence occurs $n_1! \; ::: \; n_k!$ times in the initial permutation.
  - The number of distinct rearrangements is thus

$$\frac{P(n,n)}{n_1!*n_2!*..n_k!} = \frac{n!}{n_1!*n_2!*..n_k!}$$

Eliminating Duplicates (cont'd)

- How many ways can a committee of two be chosen from four men and three women and it must include at least one man.
- How many distinct permutations can be made from the characters in the word FLORIDA?
- How many distinct permutations can be made from the characters in the word MISSISSIPPI?

18

Counting with Repetitions

- In all previous counting problems each chosen element was removed from the initial set as soon as it was chosen. A second set of problems arises if selected elements are not removed but choices stay the same throughout.
- Permutations with repetitions are problems where r elements are drawn sequentially from a set of n elements. Each element is immediately returned before the next object is selected.

  – This problem consists of a sequence of events, each of which has n possible outcomes. Its solution thus involves the multiplication principle.

  – The total number of possibilities for the permutation problem with repetitions is $n \times n \times \ldots \times n = n^r$

  – Examples:

  How many possible outcomes are there for a sequence of 3 rolls of a dice: $6^3 = 216$

19

Counting with Repetitions (cont'd)

- Combinations with repetitions correspond to the problem where a set of r elements is selected each of which can have one of n different, predefined types. As in all combination problems, the order in which the elements are selected is not important.
- To solve this problem it can be mapped into a different permutation problem.
  - Since all orders are identical in this problem, a specific order can be assumed for each possibility. In particular the sets can be ordered by the object type and thus have the form $\{o_i, o_j, \ldots o_k\}$ where $i \leq j \leq \ldots \leq k$.

Counting with Repetitions (cont'd)

- The most important aspect of the selected sets is how often each of the individual object types occurs. Counting this in a unary system the sets can be remapped, representing every object by a 0 and separating the different objects by 1s. To be able to uniquely determine each object and object type *r* 0s and *n-1* 1s are thus required (one to separate every two object types).

- For example if there are 3 possible object types to select from and 4 objects are drawn, sets would be remapped in the following way:

  $\{o_1, o_1, o_2, o_2\}$ ➔ 001001, $\{o_2, o_3, o_3, o_3\}$ ➔ 101000

Counting with Repetitions (cont'd)

- This remapping reduces the problem to "how many possibilities are there to rearrange a sequence containing *r* 0s and *n* -11s". This is a permutation problem with duplicates but without repetitions.
- The solution to the combination problem with repetitions is:

$$\frac{(r+(n-1))!}{r!(n-1)!} = C(r+n-1, r)$$

- Example:

  How many possible sets of 12 glass pearls can be formed if there are 5 possible colors ?

  C(12 + 4; 12)

Summary of Counting Techniques

Small problems with outcomes given by specific choices at each step	Draw a decision tree and count the leaves		
Outcomes of disjoint events	Use the addition principle		
Outcomes of overlapping events	Use the principle of inclusion and exclusion		
Outcomes of a sequence of events	Use the multiplication principle		
Ways to take r elements out of n distinct objects		Without repetitions	With repetitions
	Order matters	Permutation $P(n,r) = \frac{n!}{(n-r)!}$	Permutation with repetitions n^r
	Order does not matter	Combination $C(n,r) = \frac{n!}{(n-r)!r!}$	Combination with repetitions $C(r+n-1,r) = \frac{(r+n-1)!}{r!(n-1)!}$
Number of distinct rearrangements of n objects with duplicates	Use the formula $\frac{n!}{n_1!n_2!\ldots n_k!}$		

Class Exercises

- 1. How many permutations of the characters in the word COMPUTER are there? How many of these end in a vowel?

  8!

  3x7!
- 2. How many distinct permutations of the characters in ERROR are there?

  5!/3!
- 3. In how many ways can you seat 11 men and eight women in a row if no two women are to sit together?

  11!*C(12,8)*8!
- 4. A set of four coins is selected from a box containing five dimes and seven quarters.

  C(12,4) = 495
- 5. Find the number of sets which has two dimes and two quarters.

  C(5,2)*C(7,2) = 10*21 = 210
- 6. Find the number of sets composed of all dimes or all quarters.

  C(5,4) + C(7,4) = 5 + 35 = 40