Permutation Groups and the Solution of German Enigma Cipher

Jirí Tuma

Charles University, Prague, Czech Republic

How do cipher systems work?

The sender starts with a **plain text**. Then with the help of a **key** he produces a **cipher text**. The cipher text is created from the plain text and the key by an algorithm. This process is called **encryption**.

The authorized addressee knows the encryption algorithm and the key and deciphers the plain text from the cipher text and the key. The reverse process is called **decryption**.

An adversary may intercept the cipher text and attempt to recover the original plain text from it. This is called **solving the cipher**, informally **codebreaking**.

Maxims of cryptology

Never use the same key to encrypt two different messages;

Never encrypt the same message twice with two different keys;

Always assume that the adversary knows the encryption algorithm;

Never underrate the adversary.

Indicators

Security of a cipher system depends mainly on **key-exchange**, the way in which the sender and the authorized addressee exchange the message key.

They agree in advance on a position in the cipher text that will contain information from which the message key can be recovered. This information is called an **indicator**. The indicator is most often placed at the beginning of the ciphertext. The indicator is usually not given in plain text but it is also enciphered in some way.

Thus a cipher text usually has the form

`indicator messagetext`

Enigma, first contacts

From 1928 the German army *Wehrmacht* started to test a new cipher system. This caused panic in the ranks of the Polish Secret Service. After the end of the First World War France and Great Britain believed that Germany was no longer a threat to their security. However, Poland never shared this opinion.

Attempts to solve the new cipher had been completely unsuccessful for several years. Even parapsychology was involved but in vain. Finally, someone in the Polish Secret Service got the idea that mathematicians could be useful.
A course in cryptanalysis was organized for students of mathematics at the University of Poznan.

Young Polish mathematicians

Three of the best graduates of the course,

Marian Rejewski (1905-1980),
Henryk Zygalski (1906-1978) and
Jerzy Rózycki (1907-1942)

then accepted the offer to work on cryptoanalysis of the new cipher.

First results

Various statistical tests were applied to the intercepted cipher texts. As a result it became clear that the **first six letters** of each cipher text **formed the indicator**.

The statistical tests also suggested that the cipher is very probably a **polyalphabetic cipher**. This means that every letter of the plain text is replaced by a single corresponding letter in the cipher text. The cipher letter depends on the position of the plain letter in the plain text.

First results

A similar cipher was produced by a commercial machine **Enigma** that had been on sale since 1926. So the Polish Intelligence Service hypothesized that the new cipher was produced by a military version of the **Enigma** machine.

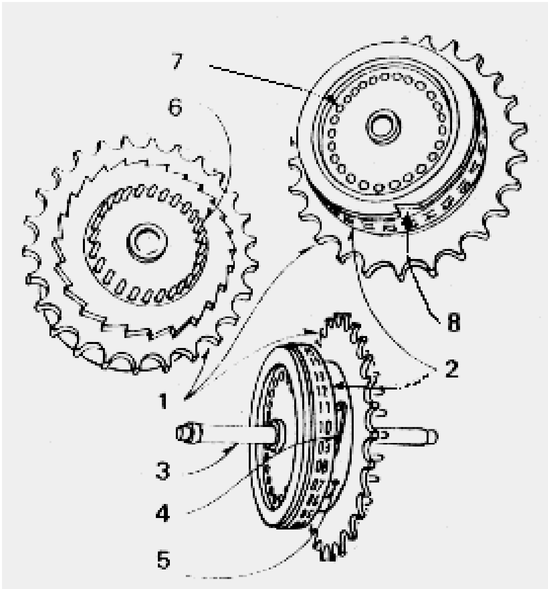
Military Enigma

Important parts of the machine are

- keyboard,
- bulbs,
- plugboard (stecker board),
- scrambler,
- entry wheel,
- rotor,
- reflector (Umkehrwalze).

Rotors

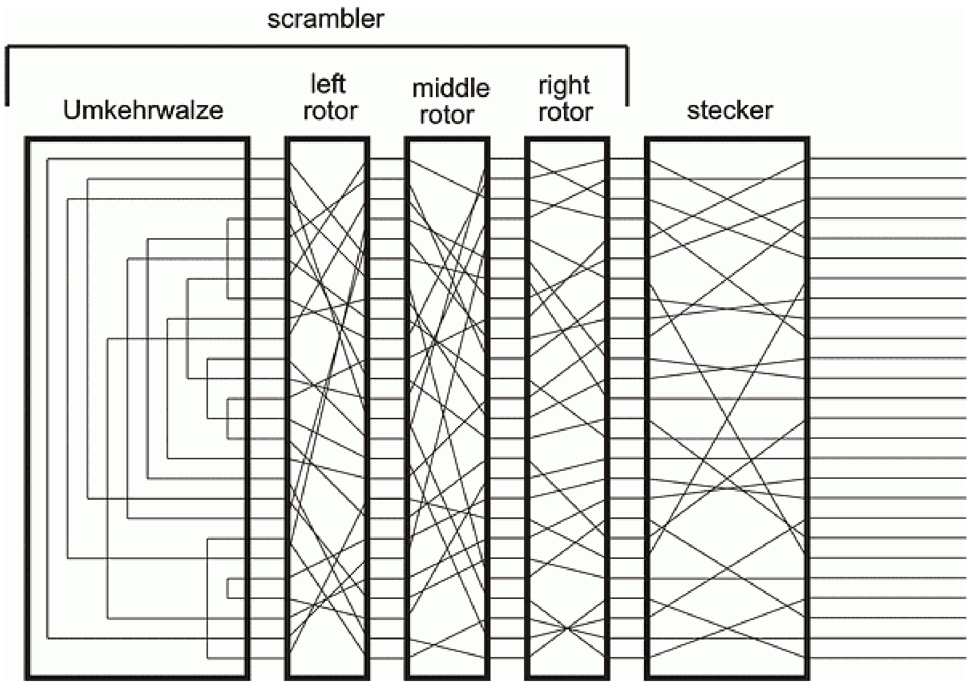
- Here is the structure of the rotors
- finger notches,
- alphabet ring,
- shaft,
- catch,
- core containing
- cross-wirings,
- spring loaded contacts,
- discs,
- carry notch.



(figure from: http://www.codesandciphers.org.uk/bletchleypark/)

Wiring of Enigma

The wiring of the military version of the Enigma machine can be schematized in the following way.

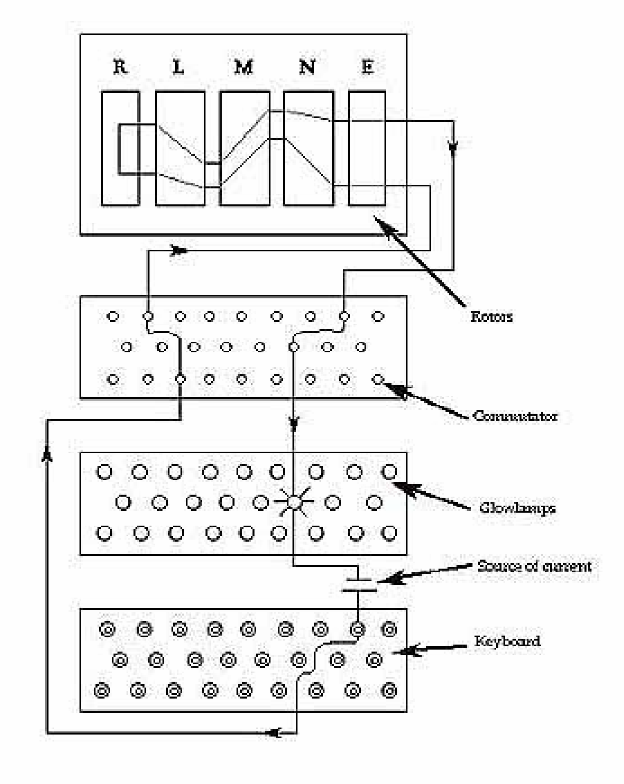


(picture from http://www.ellsbury.com/)

© Jirí Tuma, tuma@karlin.mff.cuni.cz, 2003

Flow of current

This is how the current flows through the Enigma machine after pressing a key.



Operation manual

The operation manual was obtained through French espionage and passed to the Polish Intelligence Service in December 1932.
It contained the information needed to set up the machine using the **daily key**.

The daily key consisted of

- the order of the rotors: e.g. *II,III,I*;
- the position of the rings on the rotors: e.g. *KUB*;
- the plug board connections: e.g. *AU,CR,DK,JZ,LN,PS*;
- the basic setting, letters visible in the little windows: e.g. *UFW*.

© Jirí Tuma, tuma@karlin.mff.cuni.cz, 2003

The message key

After setting up the machine with the use of the daily key, the operator was supposed to choose randomly three letters called the **message key**, e.g. *HTS*.

Then he wrote the message key twice, i.e. *HTS HTS*.

Then he encrypted these six letters by pressing the corresponding six keys on the keyboard and wrote down the cipher text. He obtained the cipher version of the message key, the indicator, e.g. the message key *HTS HTS* was encrypted as *NEV GWY*.

Then he turned the rotors to see the message key letters in the small windows, and continued with encrypting the message text from the position given by the message key.

Thus the message *HELLO* was encrypted as *BPTQS*.

Violation of cryptological maxims

Finally, he passed the indicator and the cipher text to the radio operator. The whole encrypted message HELLO was thus transmitted as

NEV GWY BPTQS

The rules violated two of the cryptological maxims.
All message keys during the same day were encrypted with the **same daily key**.

Each particular message key was encrypted **twice with different keys**.
The violation of two cryptological maxims was the starting point of a mathematical analysis of the Enigma cipher.

Could this open the door to solve the cipher?

The situation in December 1932

Let us summarize the information available to Marian Rejewski at that time:

- a commercial version of the Enigma machine (without the plugboard and certainly with different wirings inside the rotors and the reflector),
- the operation manual,
- daily keys for September and October 1932.

Important:

the daily keys were from two different quarters of the year, and many intercepted encrypted messages not only from these two months but from many other months.

Permutations

The (unknown) wiring inside a rotor (or the reflector) can be described by the mathematical concept of a **permutation**.

Definition. A one-to-one mapping on a set X is called a **permutation** on the set X.

The value of a permutation P at a point x will be written as xP. Every permutation P on a set X uniquely defines the **inverse permutation** P^{-1}. This is determined by the property

$$(xP)P^{-1} = x$$

for every $x \in X$.

Product of permutations

Any two permutations P,Q on the same set X can be composed (as mappings) to get the **composition** or **product** PQ of the two permutations. Its value at a given element $x \in X$ is

$$x(PQ) = (xP)Q.$$

The **identity permutation** I on X is defined by

$$xI = x$$

for every $x \in X$.

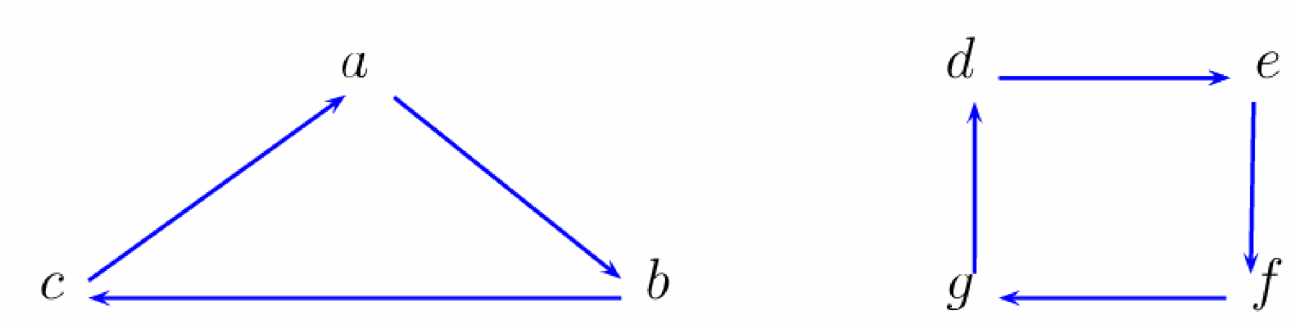
Thus $PP^{-1} = I = P^{-1}P$ for every permutation P on X.

Graph of a permutation

We can visualize permutations by drawing their **graphs**.
For example, the permutation P on the set $\{a, b, c, d, e, f, g\}$ defined by

$$aP = b,\ bP = c,\ cP = a,\ dP = e,\ eP = f,\ fP = g,\ gP = d,$$

can be visualized as



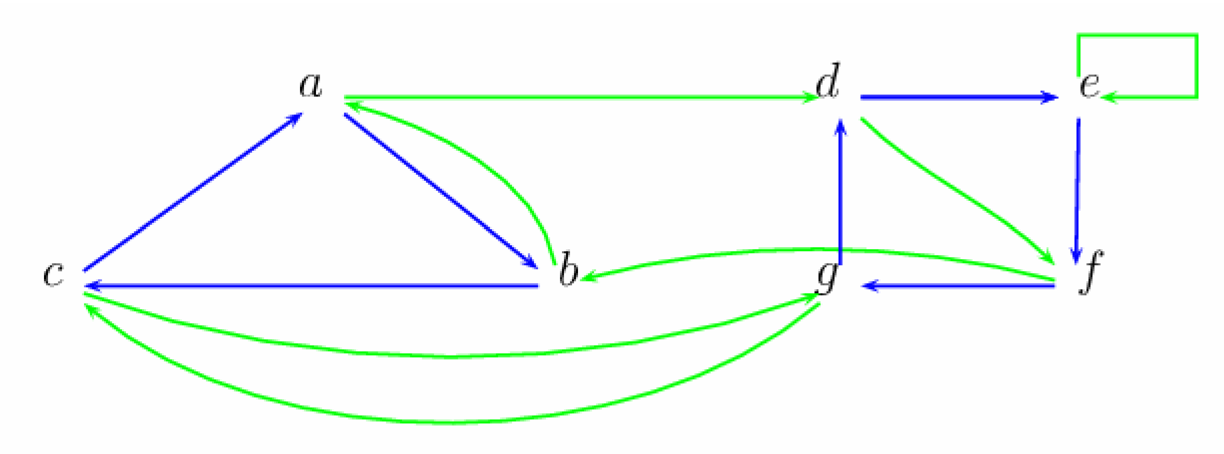
Graph of the composition (I)

If we want to find the graph of the product PQ of the permutation p with another permutation Q defined by

$$aQ = d,\ bQ = a,\ cQ = g,\ dQ = f,\ eQ = e,\ fQ = b,\ gQ = c,$$

we first draw the graph of P (blue arrows).

Then we draw the graph of Q to it (green arrows).

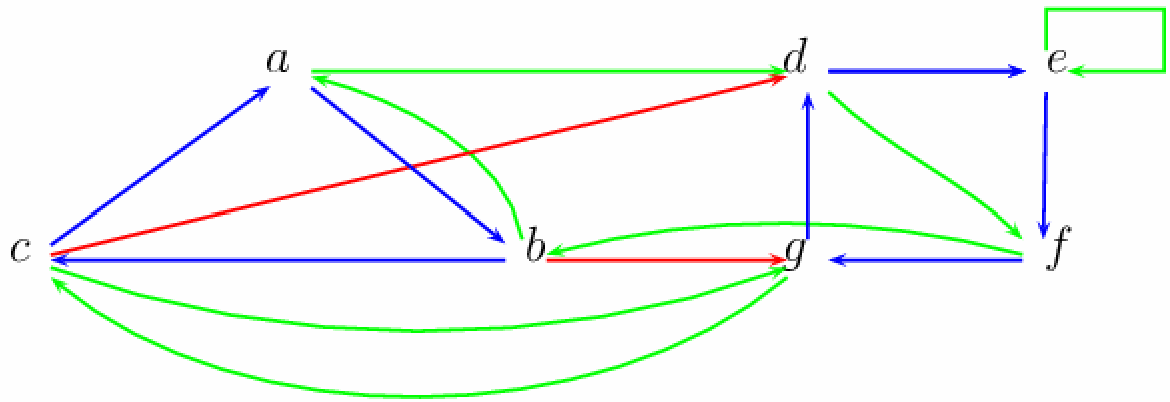


Graph of the composition (II)

To get an arrow in the graph of the product PQ from a given element, we first follow the arrow of P and then continue with the arrow of Q.

Here is how it works for the element c (redarrow $c \to d$).

And for the element b (red line $b \to g$).



Math. model of rotors and reflector

If we take a rotor, we may denote the spring contacts on one side of the rotor by letters of the alphabet $a, b, c, \ldots, x, y, z$. Opposite to each spring contact there is a disc, through which the current flows out of the rotor. We denote it by the same letter as the opposite spring contact.

Thus the wires inside the rotor define a one-to-one mapping, or a permutation, on the alphabet $\{a, b, c, \ldots, x, y, z\}$.

Let us denote the permutations that describe the wirings of the left, middle and right rotors by L, M and N.

There are also thirteen wires inside the reflector, each connecting two springs. Thus the wiring inside the reflector can be described by another permutation R on the alphabet $\{a, b, c, \ldots, x, y, z\}$. This time the permutation Q is not arbitrary but has to have 13 cycles of length 2.

Static model of the scrambler

The passage of the electrical current through the scrambler now can be described as the composition of permutations

$$NMLRL^{-1}M^{-1}N^{-1}.$$

It should be emphasized that none of the four permutations involved was known to Rejewski.

This model does not take into account that the right rotor moves first when we press a key and only then the current flows through the closed circuit.

Dynamic model of the scrambler

If the right rotor moves first, then the current from the disc a of the entry wheel does not flow to the spring a of the right rotor, but to the spring b of the right rotor. Similarly, from the disc b of the entry wheel it flows to the spring c of the right rotor, etc.
We denote by P the cyclic permutation

$$\begin{tikzcd} a \arrow[r] & b \arrow[r] & c \arrow[r] & d & \cdots & x \arrow[r] & y \arrow[r] & z \arrow[lllllll, bend left] \end{tikzcd}$$

It maps every letter of the alphabet $\{a, b, c, \ldots, x, y, z\}$ to the subsequent one and the last letter z to the first letter a.

Dynamic model of Enigma (I)

The passage of the current through the scrambler can now be described by the permutation

$$PNP^{-1}MLRL^{-1}M^{-1}PN^{-1}P^{-1}.$$

Only the cyclic permutation P is known.

There might have been also a cross-wiring between the plug board and the entry wheel. That there, in fact, was no cross-wiring there was unknown at the end of 1932, so another unknown permutation H must be added to the dynamic model.

Dynamic model (II)

Thus the wiring in the Enigma machine together with the movement of the right rotor can be described by a single permutation

$$HPNP^{-1}MLRL^{-1}M^{-1}PN^{-1}P^{-1}H^{-1}.$$

And cables in the plug board determine another permutation S of the alphabet $\{a, b, c, \ldots, x, y, z\}$. The permutation S has some cycles of length 2 and some of length 1 depending on the number of cables used to make the connections in the plug board. It was determined by the plugboard connections in the daily key used to encrypt all message keys during a day.

Complete model

Thus the whole dynamic model of the operation of the Enigma machine can be described by the permutation

$$SHPNP^{-1}MLRL^{-1}M^{-1}PN^{-1}P^{-1}H^{-1}S^{-1}.$$

The cyclic permutation P is known, the unknown permutations L,M,N and H describe the unknown internal structure of the Enigma machine. The permutation S changes day by day and is given by the corresponding daily key.

Permutations of the day

The first six letters of each message transmitted during the same day were encrypted by the same key given by the setup of the machine for that day. So we can denote by A the permutation of the first letters of messages transmitted that day.

Similarly, we denote by B the permutation of the second letters of messages transmitted the same day, by C the permutation of the third letters, by D, E and F the permutations of the fourth, fifth and sixth letters.

All the six permutations A, B, C, D, E, F were also unknown.

We may call them the **permutations of the day**.

Connection to the dynamic model (I)

We have already found another description of the permutation determined by the setup of the machine for the day. It was

$$SHPNP^{-1}MLRL^{-1}M^{-1}PN^{-1}P^{-1}H^{-1}S^{-1}.$$

So we get the equation

$$A = SHPNP^{-1}MLRL^{-1}M^{-1}PN^{-1}P^{-1}H^{-1}S^{-1}.$$

Connection (II)

We can find a similar expression for the second permutation of the day B. We only have to take into account that after pressing the second key the right rotor has already turned twice. So the equation is

$$B = SHP^{2}NP^{-2}MLRL^{-1}M^{-1}P^{2}N^{-1}P^{-2}H^{-1}S^{-1}.$$

The remaining four permutations of the day can be expressed as

$$C = SHP^{3}NP^{-3}MLRL^{-1}M^{-1}P^{3}N^{-1}P^{-3}H^{-1}S^{-1},$$
$$D = SHP^{4}NP^{-4}MLRL^{-1}M^{-}1P^{4}N^{-1}P^{-4}H^{-1}S^{-1},$$
$$E = SHP^{5}NP^{-5}MLRL^{-1}M^{-}1P^{5}N^{-1}P^{-5}H^{-1}S^{-1},$$
$$F = SHP^{6}NP^{-6}MLRL^{-1}M^{-}1P^{6}N^{-1}P^{-6}H^{-1}S^{-1}.$$

Conjugated permutations

It should be emphasized that these equations are valid only under the assumption that the only rotor that moved during the encryption of the six letters of the message keys during the given day was the right one. But this happened on average in 20 out of 26 days. Quite often.

All the permutations in the previous expressions of the permutations of the day were unknown except P. But something important was known!

Before we proceed let us state an important definition.
Definition. Two permutations K,L on the same set X are called **conjugated** if there exists another permutation P on the set X such that

$$K = PLP^{-1}.$$

The theorem that won WWII

And one more definition.

Definition. The list of lengths of all cycles in a permutation K is called the **cyclic structure** of the permutation K.

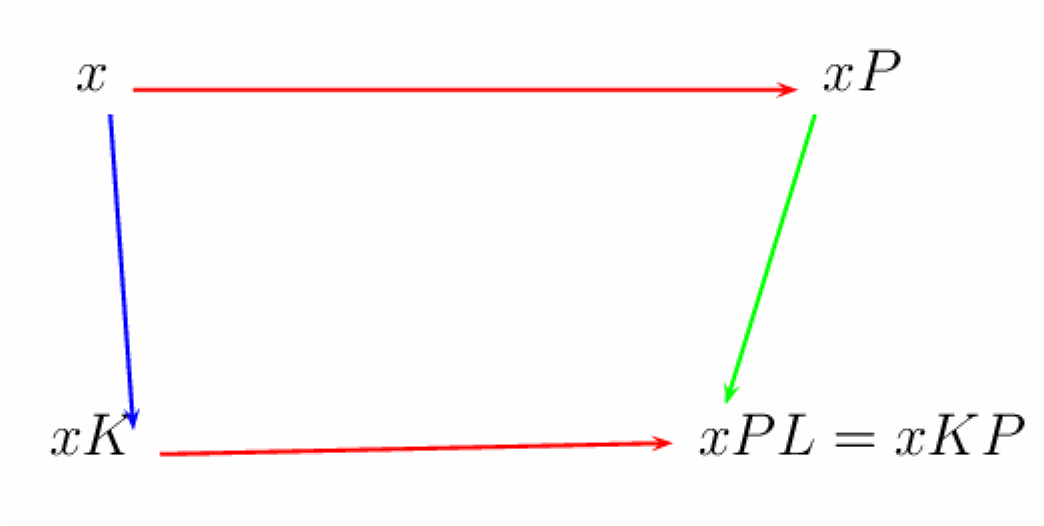
Theorem. Two permutations K,L on the same set X are conjugated if and only if they have the same cyclic structure.

We can get an idea why the theorem is true by drawing the graphs of permutations. So assume that permutations K,L are conjugated and let P be a permutation such that

$$K = PLP^{-1}.$$

Proof (I)

Now choose an arbitrary element $x \in X$ and look at the following part of the graphs of the three permutations. The arrows of the permutation K are blue, the arrows of L are green, and the arrows of P are red.



© Jirí Tuma, tuma@karlin.mff.cuni.cz, 2003

Proof (II)

Thus the permutation P maps each cycle of the permutation K to a cycle of the permutation L with the same length. Conjugated permutations must have the same cyclic structures.

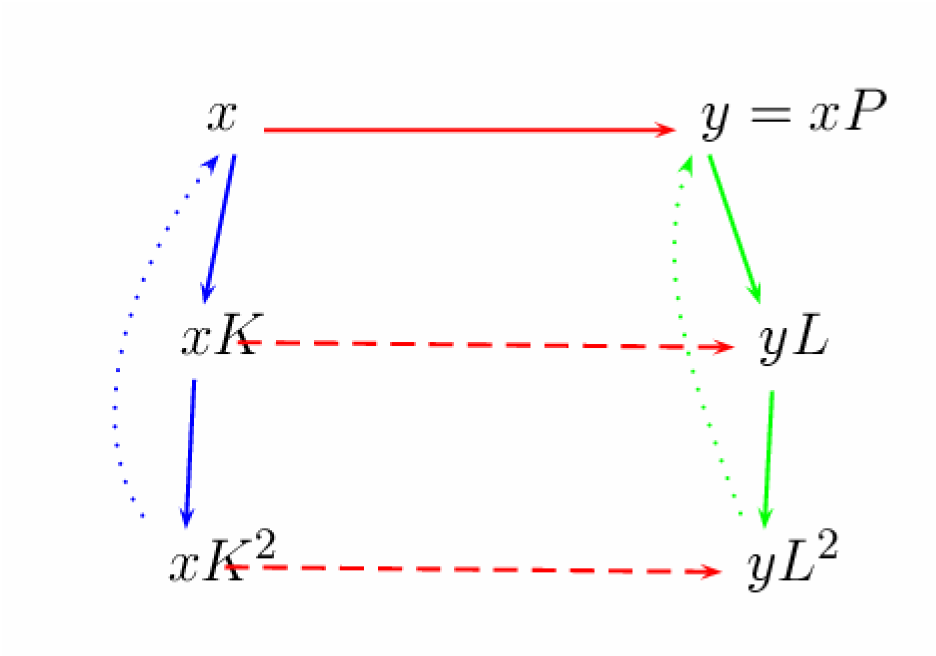
Now assume conversely that two permutations K,L have the same cyclic structure. Choose a cycle in the permutation K and a cycle in the permutation L of the same length. Further, choose an element x in the chosen cycle of K and an element y in the chosen cycle of the permutation L. Try to set $xP = y$.

We search for a permutation P satisfying the equation

$$K = PLP^{-1}.$$

Proof (III)

Look at the following part of the graphs of all three permutations.



End of the proof

Thus the choice of x and y uniquely determines the values of the permutation P on the chosen cycle of the permutation K.
Observe also that this method enables us to find all possible permutations P that satisfy the equation

$$K = PLP^{-1}$$

if the given permutations K,L have the same cyclic structure.

Characteristics of the day

The permutation R describing the wiring of the reflector has all cycles of length 2. This is the reason why $R^2 = I$, or

$$R^{-1} = R.$$

All permutations A,B,C,D,E,F of the day are conjugated to R. So they all have only cycles of length 2. Thus we get

$$A^2 = B^2 = C^2 = D^2 = E^2 = F^2 = I,$$

or stated otherwise, each of the six permutations is equal to its inverse.

We do not know the permutations A,B,C,D,E,F yet. But we do know the products AD, BE, CF if there are enough intercepted messages for the day. Rejewski called them the **characteristics of the day**.

Finding characteristics

You will recall that the intercepted indicators were obtained by enciphering message keys twice. Stated otherwise, by enciphering messages of the form

$$xyzxyz,$$

where x, y, z can be arbitrary letters.

If $xA = u$ and $xD = v$, then we get

$$uAD = v,$$

since $A^{-1} = A$.

But various pairs of the letters $u, v = uAD$ are known! They are the first and fourth letters of the intercepted messages.

A busy manoeuver day (I)

So if there are enough intercepted messages from a given day, we do know the characteristics *AD*, *BE* and *CF* of the day.

As an example, we find the characteristics of a busy manoeuver day, when the following indicators were intercepted.

The following table shows 64 intercepted characteristics from this manoeuver day. There are enough of them to find all three permutations *AD*, *BE* and *CF*.

A busy manoeuver day (II)

1. AUQ AMN	17. KHB XJV	33. RJL WPX	49. VII PZK
2. BNH CHL	18. KHB XJV	34. RFC WQQ	50. VII PZK
3. BCT CGJ	19. LDR HDE	35. SYX SCW	51. VQZ PVR
4. CIK BZT	20. LDR HDE	36. SYX SCW	52. VQZ PVR
5. DDB VDV	21. MAW UXP	37. SYX SCW	53. WTM RAO
6. EJP IPS	22. MAW UXP	38. SYX SCW	54. WTM RAO
7. GPB ZSV	23. NXD QTU	39. SYX SCW	55. WTM RAO
8. GPB ZSV	24. NXD QTU	40. SJM SPO	56. WKI RKK
9. HNO THD	25. NLU QFZ	41. SJM SPO	57. XRS GNM
10. HNO THD	26. OBU DLZ	42. SJM SPO	58. XRS GNM
11. HXV TTI	27. PVJ FEG	43. SUG SMF	59. XOI GUK
12. IKG JKF	28. QGA LYB	44. SUG SMF	60. XYW GCP
13. IKG JKF	29. QGA LYB	45. TMN EBY	61. YPC OSQ
14. IND JHU	30. RJL WPX	46. TMN EBY	62. ZZY YRA
15. JWF MIC	31. RJL WPX	47. TAA EXB	63. ZEF YOC
16. JWF MIC	32. RJL WPX	48. USE NWH	64. ZSJ YWG

Characteristics of the manoeuver day

From the first indicator we find for example that

$$aAD = a,\ uBE = m,\ qCF = n.$$

Similarly, the second indicator gives

$$bAD = c,\ nBE = h,\ hCF = l.$$

The following table lists the cycles of all three characteristics of the manoeuver day.

$$AD = (a), (s), (bc), (rw), (dvpfkxgzyo), (eijmunqlht),$$
$$BE = (axt), (blfqveoum), (cgy), (d), (hjpswizrn), (k),$$
$$CF = (abviktjgfcqny), (duzrehlxwpsmo).$$

⑦ Jirí Tuma, tuma@karlin.mff.cuni.cz, 2003

More equations

Putting together the earlier found equations for the permutations $A,B,C,D,E,$ F of the day we get the following
system of equations:

$$AD = SHPNP^{-1}MLRL^{-1}M^{-1}PN^{-1}P^{3}NP^{-4}MLRL^{-1}M^{-1}P^{4}N^{-1}P^{-4}H^{-1}S^{-1},$$

$$BE = SHP^{2}NP^{-2}MLRL^{-1}M^{-1}P^{2}N^{-1}P^{3}NP^{-5}MLRL^{-1}M^{-1}P^{5}N^{-1}P^{-5}H^{-1}S^{-1},$$

$$CF = SHP^{3}NP^{-3}MLRL^{-1}M^{-1}P^{3}N^{-1}P^{3}NP^{-6}MLRL^{-1}M^{-1}P^{6}N^{-1}P^{-6}H^{-1}S^{-1}.$$

How to simplify the system?

Now, not only the cyclic permutation was known, but also the three characteristics of the day on the left hand sides of the equations. But the system is certainly unsolvable for the unknown wirings of the three rotors and the reflector.

The system can be formally simplified by substituting $Q = MLRL^{-1}M^{-1}$ into the three equations. The permutation Q is the wiring of a virtual reflector consisting of the reflector and the left and middle rotors that were fixed during the day. The substitution only leads to a slightly simplified system of equations on the next slide.

A slightly simplified system

$$AD = SHPNP^{-1}QPN^{-1}P^{3}NP^{-4}QP^{4}N^{-1}P^{-4}H^{-1}S^{-1},$$
$$BE = SHP^{2}NP^{-2}QP^{2}N^{-1}P^{3}NP^{-5}QP^{5}N^{-1}P^{-5}H^{-1}S^{-1},$$
$$CF = SHP^{3}NP^{-3}QP^{3}N^{-1}P^{3}NP^{-6}QP^{6}N^{-1}P^{-6}H^{-1}S^{-1}.$$

This system is still certainly unsolvable for the unknown wirings N,Q even if all the other permutations are known. It has to be further simplified.
Now comes the first of Marian Rejewski's ingenious ideas.

Blunders of German operators

When studying the table of intercepted messages he observed that the message keys were certainly not chosen randomly as the manual stated.

But if the operators did not choose the message keys randomly, what were the probable non random choices?

Marian Rejewski first proved the following simple theorem that helped him to understand the relationship between the two permutations A,D of a day and their composition, the characteristic AD of the same day. You will remember that each of the two permutations A and D contained only cycles of length 2.

One more theorem on permutations

Theorem. A permutation K on a set Z can be expressed as the composition of two permutations X, Y with cycles of length two if and only if it contains an even number of cycles of each length.

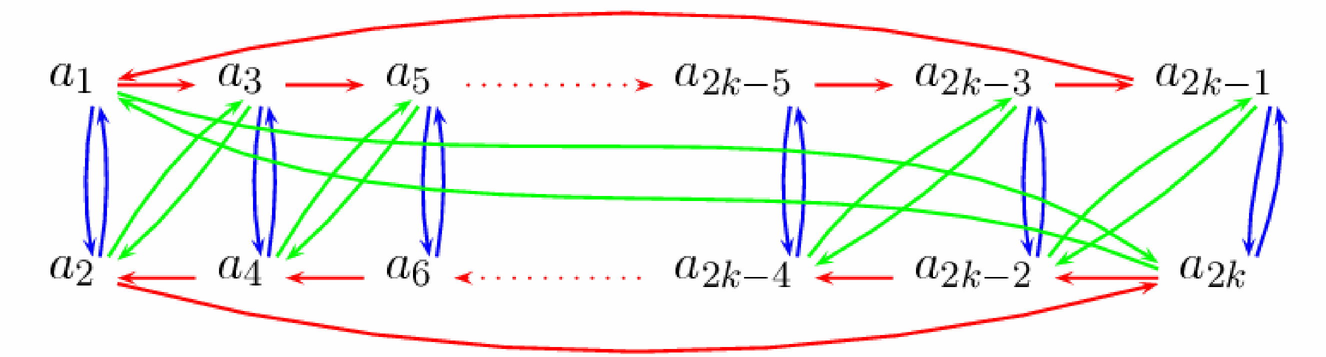
In order to understand why the theorem holds we first assume that $K = XY$ where both permutations X and Y have all cycles of length two.

Proof (I)

We take a cycle (a_1, a_2) of the permutation X and investigate what are the lengths of the cycles of the product XY containing the elements a_1 and a_2. A picture will help.

We first draw the graph of X and then the graph of Y.

Finally, we draw the graph of XY.



Proof (II)

We see that the elements a_1 and a_2 belong to two different cycles of the same length. They are $(a_1 a_3 \cdots a_{2k-3} a_{2k-1})$ and $(a_2 a_{2k} a_{2k-2} \cdots a_4)$.

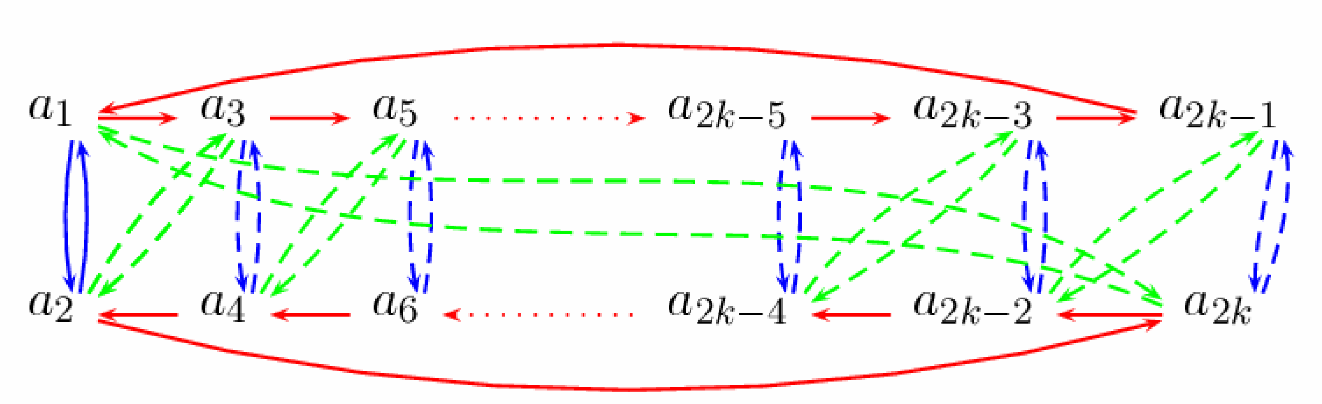
Thus the composition XY has always even number of cycles of any given length.

To prove the converse we assume that a permutation K has even number of cycles of any given length. We will show how to find all possible expressions of K as a composition XY of two permutations which have all cycles of length 2.

We choose two cycles of the same length, say k, and an arbitrary element a_1 in one of the cycles and an element a_2 in the other cycle. We may assume that (a_1, a_2) is one of the cycles in X.

Proof (III)

We denote the two cycles $(a_1 a_3 \cdots a_{2k-3} a_{2k-1})$ and $(a_2 a_{2k} a_{2k-2} \cdots a_4)$, and draw them one under another and in the opposite direction. Then we add the chosen cycle (a_1, a_2) of permutation X. Since we want to have $K = XY$, the permutation Y must map the element a_2 to a_3. From the same reason, a_3 must be mapped to a_4 in X. And so on.



End of the proof

This procedure can be used for any pair of cycles of the same length. And since the permutation K has an even number of cycles of any given length, we may define in this way the permutations X and Y on all elements of the set Z.

Note also that this procedure gives us a method how to find all decompositions of K as the product $K = XY$ of permutations with all cycles of length 2.

The number of possibilities

For example, the three characteristics of the busy manoeuvre day

AD = (a), (s), (bc), (rw), (dvpfkxgzyo), (eijmunqlht),
BE = (axt), (blfqveoum), (cgy), (d), (hjpswizrn), (k),
CF = (abviktjgfcqny), (duzrehlxwpsmo)

give 13 possibilities for the permutations C and F, 3 × 9 possibilities for the permutations B and E and 2 × 10 possibilities for the permutations A and D. All together 20 × 27 × 13 = 7020 possibilities for the permutations of the day.

Reducing the number of possibilities

To reduce the number of possibilities Marian Rejewski guessed that the operators had probably chosen message keys consisting of the same three letters or perhaps three neighbouring letters on the keyboard.

He tried various guesses of this sort for the message key that came encrypted as *SYX SCW*. This indicator was most suspicious since it appeared five times during the day.

When he tried the possibility that the suspicious pattern *SYX SCW* was in fact the encryption of the message key *AAA AAA* all of a sudden he was able to reconstruct the permutations *A,B,C,D,E,F* that gave the plain indicators of the right form *xyz xyz*.

A miracle

This assumption means that $aA = s$, $aB = y$ and $aC = x$.
Since the characteristic CF has only two cycles of length 13, the assumption determines uniquely the permutations C and F.

Since the elements a and y belong to different cycles of the same length 3 in the characteristic BE, this guess also determines the values of B and E on the six elements of the two cycles. The letters a and s form cycles of length 1 in AD, thus the cycle (as) belongs to both A and D.

With another two guesses of this form Marian Rejewski was eventually able to reconstruct all message keys used during this day. They are listed in the following table.

The message keys

AUQ AMN:	sss	IKG JKF:	ddd	QGA LYB:	xxx	VQZ PVR:	ert
BNH CHL:	rfv	IND JHU:	dfg	RJL WPX:	bbb	WTM RAO:	ccc
BCT CGJ:	rtz	JWF MIC:	ooo	RFC WQQ:	bnm	WKI RKK:	cde
CIK BZT:	wer	KHB XJV:	lll	SYX SCW:	aaa	XRS GNM:	qqq
DDB VDV:	ikl	LDR HDE:	kkk	SJM SPO:	abc	XOI GUK:	qwe
EJP IPS:	vbn	MAW UXP:	yyy	SUG SMF:	asd	XYW GCP:	qay
FBR KLE:	hjk	NXD QTU:	ggg	TMN EBY:	ppp	YPC OSQ:	mmm
GPB ZSV:	nml	NLU QFZ:	ghj	TAA EXB:	pyx	ZZY YRA:	uvw
HNO THD:	fff	OBU DLZ:	jjj	USE NWH:	zui	ZEF YOC:	uio
HXV TTI:	fgh	PVJ FEG:	tzu	VII PZK:	eee	ZSJ YWG:	uuu

With the exception of two message keys *abc* and *uvw* all the remaining ones are either triples of the same letters or triples of letters on neighbouring keys on the Enigma keyboard. And these two message keys are also far from being random.

A simplified system of equations

The deep psychological insight into the habits of German operators enabled him to simplify the original system of equations to the following one. Now the permutations A,B,C,D,E,F were known.

$$A = SHP^{1}NP^{-1}QP^{1}N^{-1}P^{-1}H^{-1}S^{-1}$$
$$B = SHP^{2}NP^{-2}QP^{2}N^{-1}P^{-2}H^{-1}S^{-1}$$
$$C = SHP^{3}NP^{-3}QP^{3}N^{-1}P^{-3}H^{-1}S^{-1}$$
$$D = SHP^{4}NP^{-4}QP^{4}N^{-1}P^{-4}H^{-1}S^{-1}$$
$$E = SHP^{5}NP^{-5}QP^{5}N^{-1}P^{-5}H^{-1}S^{-1}$$
$$F = SHP^{6}NP^{-6}QP^{6}N^{-1}P^{-6}H^{-1}S^{-1}.$$

The daily keys were known!

But the busy manoeuvre day was in September 1932 and Marian Rejewski had the daily keys from this month. So he in fact knew also the permutation S. He could move it to the right hand sides of the equations among the already known permutations. It gave him the following system.

$$
\begin{aligned}
S^{-1}AS &= HP^{1}NP^{-1}QP^{1}N^{-1}P^{-1}H^{-1}\\
S^{-1}BS &= HP^{2}NP^{-2}QP^{2}N^{-1}P^{-2}H^{-1}\\
S^{-1}CS &= HP^{3}NP^{-3}QP^{3}N^{-1}P^{-3}H^{-1}\\
S^{-1}DS &= HP^{4}NP^{-4}QP^{4}N^{-1}P^{-4}H^{-1}\\
S^{-1}ES &= HP^{5}NP^{-5}QP^{5}N^{-1}P^{-5}H^{-1}\\
S^{-1}FS &= HP^{6}NP^{-6}QP^{6}N^{-1}P^{-6}H^{-1}.
\end{aligned}
$$

Germans like order

Now only the permutations H,N and Q were unknown. Rejewski first tried the same wiring between the plug board and the entry wheel that was used in the commercial Enigma. But it went nowhere.

This unsuccessful attempt led him to the second ingenious insight into the psychology, this time of the constructors of the Enigma machine. Rejewski observed that the wiring between the plug board and the entry wheel in the commercial Enigma machine was very regular. The plugs were connected to the entry wheel in the order of the keys on the keyboard. So he tried another regular wiring, this time in the order of the alphabet. This means that $H = I$, the identity permutation. So the permutation H could be eliminated from the equations.

Number of unknowns is getting smaller

Now only two permutations N and Q remained unknown. Rejewski rewrote the system of six equations in the following form.

$$
\begin{aligned}
T &= P^{-1}S^{-1}ASP^{1} = NP^{-1}QP^{1}N^{-1},\\
U &= P^{-2}S^{-1}BSP^{2} = NP^{-2}QP^{2}N^{-1},\\
W &= P^{-3}S^{-1}CSP^{3} = NP^{-3}QP^{3}N^{-1},\\
X &= P^{-4}S^{-1}DSP^{4} = NP^{-4}QP^{4}N^{-1},\\
Y &= P^{-5}S^{-1}ESP^{5} = NP^{-5}QP^{5}N^{-1},\\
Z &= P^{-6}S^{-1}DSP^{6} = NP^{-6}QP^{6}N^{-1}.
\end{aligned}
$$

Further calculations

By multiplying the pairs of subsequent equations he obtained the following system of five equations in the two unknowns N and Q.

$$
\begin{aligned}
TU &= NP^{-1}(QP^{-1}QP)PN^{-1},\\
UW &= NP^{-2}(QP^{-1}QP)P^{2}N^{-1},\\
WX &= NP^{-3}(QP^{-1}QP)P^{3}N^{-1},\\
XY &= NP^{-4}(QP^{-1}QP)P^{4}N^{-1},\\
YZ &= NP^{-5}(QP^{-1}QP)P^{5}N^{-1}.
\end{aligned}
$$

Only one unknown remained

From this system he eliminated the common expression $QP^{-1}QP$ and obtained the following system of four equations in one unknown N, or still better, in the unknown $V = NP^{-1}N^{-1}$.

$$
\begin{aligned}
UW &= NP^{-1}N^{-1}(TU)NPN^{-1} = V\,(TU)V^{-1},\\
WX &= NP^{-1}N^{-1}(UW)NPN^{-1} = V\,(UW)V^{-1},\\
XY &= NP^{-1}N^{-1}(WX)NPN^{-1} = V\,(WX)V^{-1},\\
Y\,Z &= NP^{-1}N^{-1}(XY\,)NPN^{-1} = V\,(XY\,)V^{-1}.
\end{aligned}
$$

Moreover, all four equations have the familiar form

$$J = V\,KV^{-1}.$$

The solution

From the proof of the theorem that won the WWII we already know how to find all solutions V of each of the four equations. There must be a common solution V of the four equations that is a cyclic permutation, since it is conjugated to the cyclic permutation P^{-1}. Hence he could also find the permutation N describing the wiring of the right rotor.

The wiring of one of the rotors thus became known. In those times the German army changed the order of rotors in daily keys every quarter of the year. Since September and October are in different quarters of the year, the same method led to the discovery of the wiring of another rotor.
This was the rotor used as the right (fast) rotor in the other quarter of the year 1932 from which the daily keys were known.

A replica of Enigma was built

Then it was easy to calculate the wiring of the remaining rotor and also of the reflector. At the end of January 1933 a replica of the military Enigma machine was built.

Hitler came to power in Germany in January 1933. From almost the same time the Polish Intelligence Service was able to read most of the top secret German army communications.

In July 1939, when it became clear that a new war in Europe was inevitable, the Polish Intelligence Service organized a meeting near Warsaw. There it passed the replicas and all other information to its French and British counterparts. This is how the first replica of the military Enigma machine got to Bletchley Park, the centre of the British cryptanalysis of that time.

References

❋Rejewski, Marian, *An application of the theory of permutations in breaking the Enigma cipher*, Applicationes Mathematicae **16**, No. 4, Warsaw 1980, in http://mad.home.cern.ch/frode/crypto/rew80.pdf

❋ Rejewski, Marian, *How Polish mathematicians broke the Enigma cipher*, IEEE Annals of the history of computing, July 1981, 213-234

❋ Kozaczuk, Wladyslaw, *Enigma*, London: Arms and Armour, 1984,

❋ Bauer, Friedrich L., *Decrypted Secrets, Methods and Maxims of Cryptology*, 2nd ed. Springer-Verlag, Berlin Heidelberg 2000.

Enigma simulators

Enigma simulators can be found e.g. at

- http://www.xat.nl/enigma/
- http://www.ugrad.cs.jhu.edu/ russell/classes/enigma/
- http://frode.home.cern.ch/frode/crypto/simula/m3/

Printable version by Enrico Grigolon (November 2003).
Final editing by Frode Weierud (November 2003).