# Permutations, Combinations and the Binomial Theorem

October 27, 2011

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## Answer

*This task can be performed in* $40 \cdot 39 \cdot 38$ *different ways.*

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### Remark

*If we exchange $a_i$ and $a_{i+1}$ we remove 1 inversion.*

*So on the average, we'll have to perform $\frac{n(n-1)}{4}$ such exchanges.*

*Better sorting programs compare records that are far apart thus capable of removing more inversions in one exchange.*

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*Can we design a sorting algorithm that will sort any given* 5 *objects in no more than* 7 *comparisons?*

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*For a fixed integer $n$ what is the smallest number of comparisons a sorting algorithm needs to execute to sort any input list of $n$ objects?*

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We do have sorting algorithms that execute about $c \cdot n \log n$ comparisons.

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## Theorem

*Every integer m has a unique representation:*
$m = \sum_{k=0}^{s} d_k \cdot k! \quad 0 \leq d_k \leq k.$

Proof.

First recall that $\sum_{k=1}^{s} k \cdot k! = (s+1)! - 1$ so by the previous remark the representation is unique.

We now proceed by induction to prove that every integer has a Cantor Digits representation.

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First recall that $\sum_{k=1}^{s} k \cdot k! = (s+1)! - 1$ so by the previous remark the representation is unique.

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6. $m + 1 = (d_k + 1) \cdot k! + \ldots$.

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# Enumerating Permutations

Given an n- permutation $\pi = a_1 a_2 \ldots a_n$ we asociate with it the integer $f(\pi) = \sum_{k=1}^{n-1} d_k \cdot k!$.

The coefficients $d_k$ are calculated as follows:

Let $a_j = k + 1$. Then $d_k = |\{a_{i_m} | i_m > j \text{ and } (k+1) = a_j > a_{i_m}\}|$

In words: $d_k$ is the number of entries in the permutation $\pi$ that are to the right of $k+1$ and are smaller than $k+1$.

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**Example**

*Let* $\pi = 7\ 5\ 4\ 6\ 1\ 3\ 2\ 8.$

$d_1 = 0,\ d_2 = 1,\ d_3 = 3,\ d_4 = 4, d_5 = 3, d_6 = 6.$
*So* $\quad f(\pi) = 6 \cdot 6! + 3 \cdot 5! + 4 \cdot 4! + 3 \cdot 3! + 2!$

## Example

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6. *In our example:* $f^{-1}(20000) = 7\ 1\ 6\ 5\ 8\ 3\ 4\ 2$.

# Efficient Generation of Permutations and Combinations

Permutations can be generated either by the lexicographic order or by the Cantor-Digits enumeration.
There is another method called *The Arrow* algorithm.

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3. Stop when no arrow above an entry points to a smaller entry.

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## Remark (Generating Combinations)

*We wish to generate all r-combinations of an n-set* $\{a_1, a_2, \ldots, a_n\}$. *We shall proceed lexicographically:* $\{a_1, a_2, \ldots a_r\}$ *will be the first ("smallest") and* $\{a_{n-r+1}, \ldots, a_n\}$ *be the last ("largest").*

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*What is the 4-subset of $\{1, 2, \ldots, 8\}$ following $\{3, 5, 7, 8\}$?*

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Ans: $\{3, 6, 7, 8\}$.

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To simplify the notation, we shall assume that our universal set is $\{1, 2, \ldots, n\}$ and the numbers in the $r$ subsets are sorted.

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**Example**

*The 4-combination following the combination* $\{3, 5, 7, 10\}$ *in* $\binom{\{1,2,\ldots,10\}}{4}$ *is:* $\{3, 5, 8, 9\}$.

# The Binomial theorem

You probably know a few proofs of the classical binoial theorem:

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There are many interesting relations among the binomial coefficieints. We shall briefly explore them and also see the technique of *double counting* used to prove many combinatorial identities. We start with **Pascal's idenitity:**

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof.

Here is a simple combinatorial (double counting) proof:

□

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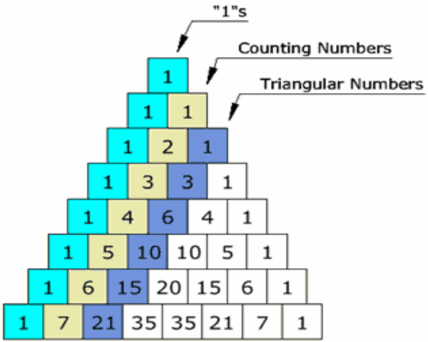
This relation among the binomial coefficient is traditionally encapsulated in the famous Pascal's triangle.

# Pascal's Triangle

Pascal's Triangle contains many patterns and relations.



# A Sample of Combinatorial Identies

There are literally thousands of combinatorial identities based on the binomial coefficients. We shall look at a small sample.

1. $$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

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2. $$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i-1}$$

   (or the number of ditinct subsets of even order is equal to the number of subset of odd order). Proof: $(1-1)^n = 0$.

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Both sides count the number of ways to select a team of $n$ students from a class with $n$ male students and $n$ females.

2. Vandermonde's Identity:

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# A tribute to Gauss

**Question**

*An urn contains* $100$ *balls numbered* $1, 2, \ldots, 100$*.* $100$ *persons draw a ball, note the number on it and return it to the urn. What is the probability that no two persons draw the same ball?*

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## Answer

*There are* $100^{100}$ *different ways to draw* $100$ *balls. There are only* $100!$ *ways to draw different balls. So the probability that no two persons will draw the same ball is* $\frac{100!}{100^{100}}$. *So we need to estimate this number.*

# Estimates

1. Simplest estimates:

$$n! = \prod_{i=1}^{n} i \leq \prod_{i=1}^{n} n = n^n \qquad n! = \prod_{i=1}^{n} i \geq \prod_{i=1}^{n} 2 = 2^n$$

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2. Slightly better estimates:

$$n! \geq \prod_{i=n/2}^{n} i \geq \prod_{i=n/2}^{n} n/2 = \left(\frac{n}{2}\right)^{\frac{n}{2}} \quad n! \leq \left(\prod_{i=1}^{n/2} \frac{n}{2}\right)\left(\prod_{i=n/2}^{n} n\right) = \frac{n^n}{2^{\frac{n}{2}}}$$

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## Remark

*So the probability that each person will see a different number is* $< 2^{-50}$ *or just about no chance!*

*Even though it looks as if the estimates assume that n is even, it is not difficult to show that they hold for odd n.*

# Gauss' nice estimates

Theorem (Gauss)

$$n^{\frac{n}{2}} \leq n! \leq \left(\frac{n+1}{2}\right)^n$$

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$i(n+1-i) \geq n \Rightarrow n! \geq \sqrt{n^n}$

We conclude by mentioning a very famous and beautiful approximation: Stirling's Formula.
It uses two of the most famous constants in mathematics: $\pi$ and $e$ in one expression involving an approximation of the integer valued function $n!$.

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

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$\binom{2n}{n} \sim \frac{4^n}{\sqrt{2\pi}}$ Is another useful approximation.