# Permutations and Combinations.
# The Pigeonhole Principle.

# Get it together, team

In how many ways a chain of command (a linear hierarchical order) can be established for all 7 crew members?

Permutations

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# Get it together, team



In how many ways a chain of command (a linear hierarchical order) can be established for all 7 crew members?

$$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040.$$

# More Problems


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(a) In how many ways a chain of command (a linear hierarchical order) can be established for all 7 crew members?

$$(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040)$$

(b) If you are already the captain and the #1 by default?

(c) In how many ways *7 individual tasks* can be distributed among *7 crew members*?

# More Problems



(d) What if there are *100 tasks* and each person gets exactly one?

(e) Still *100 tasks*, but all must be assigned to at least someone?

(f) What if there are only *2 tasks*?

(g) Two tasks, but both should have *at least one person* assigned doing it?

# Factorial function

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$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3628800$$
$$= 10! \quad \text{in a convenient factorial notation.}$$

This function is called *factorial* and denoted by $n!$:

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$
$$\ldots$$
$$3! = 1 \cdot 2 \cdot 3$$
$$2! = 1 \cdot 2$$
$$1! = 1$$

and by convention,

$$0! = 1$$

# Permutations

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**Def.** A *permutation* of a set of distinct objects is an ordered arrangement of these objects.

The number of permutations of $n$ objects is

$$P(n) = n!$$

So, for example, given 6 pictures of cats, there are $6! = 720$ ways to arrange them in a row.

# Permutations



There are 15 different magazines on a table.


*You read three of them.* In how many ways it was possible to do?

# Permutations



There are 15 different magazines on a table.


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$$15 \cdot 14 \cdot 13 = 2730.$$

# Permutations




An arrangement of 3 objects out of $n$ is called a *3-permutation*.

**Def.** An ordered arrangement of $r$ elements from a set of $n$ is called an *r-permutation*.

$$P(n,r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

# Permutations


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**Def.** An ordered arrangement of $r$ elements from a set of $n$ is called an *$r$-permutation*.

$$P(n,r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}} = \frac{n!}{(n-r)!}$$

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# Permutations

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Let's prove the last formula.

The number of $r$-permutations of the set of $n$ elements:

$$P(n,r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

Multiply and divide by $(n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1$,

$$P(n,r) = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1} = \frac{n!}{(n-r)!}$$

# Permutations

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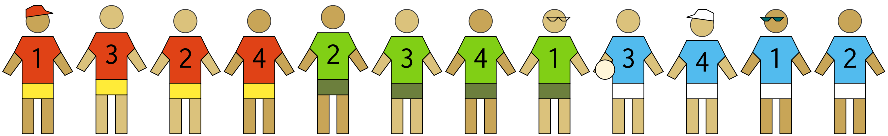
Three sport teams of four want to take a group photo.




In how may ways can they stand in a row so that all members of the same team are standing together?

# Have to open the door?

Permutations

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We must find the correct sequence to open the door:

**Up to 10 buttons** must be pressed in the right order.

But **each button, at most once.**

How many lock combinations do we have to try?

# Have to open the door?




$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 +$
$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 +$
$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 +$
$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 +$
$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 +$
$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 +$
$10 \cdot 9 \cdot 8 \cdot 7 +$
$10 \cdot 9 \cdot 8 +$
$10 \cdot 9 +$
$10 +$
$1$

Which is

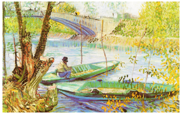
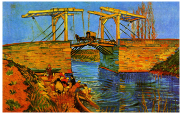
$$P(10,10) + P(10,9) + P(10,8) + P(10,7) + P(10,6) + P(10,5) + P(10,4) + P(10,3) + P(10,2) + P(10,1) + P(10,0) = 9\,864\,101.$$

If one attempt takes 1 second, it is more than 114 days of trying.

# Another problem



There are 4 paintings in a collection. In how many ways can we hang 3 of them on a wall?

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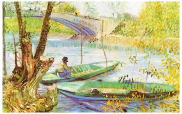
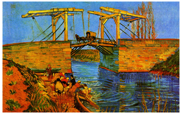



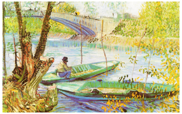
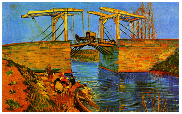
We can compute the number of 3-permutations of 4 objects, which is $P(4,3) = 4 \cdot 3 \cdot 2 = 24$.

$$
\begin{array}{cccccccc}
abc & acb & bac & bca & cab & cba & & \{a,\ b,\ c\} \\
abd & adb & bad & bda & dab & dba & & \{a,\ b,\ d\} \\
acd & acb & cad & cda & dac & dca & & \{a,\ c,\ d\} \\
bcd & bcb & cbd & cdb & dbc & dcb & & \{b,\ c,\ d\}
\end{array}
$$

Observe that there are 4 ways to pick a set of 3 paintings, and each of them can be arranged in $3! = 6$ many ways, and $4 \cdot 3! = 24$ too.

# Another problem



$$P(4,3) = 24 = 4 \cdot 3!$$

When we arrange $r$ objects out of $n$:

$$P(n,r) = \frac{n!}{(n-r)!}$$

But also,

$$P(n,r) = X \cdot r!$$

Where $X$ is the number of ways to *choose $r$ objects out of $n$ without assigning any order* to them. This is exactly what we did when we were selecting 3 paintings.

# Combinations

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$P(n,r) = X \cdot r!$$

$X$ is the number of ways to choose $r$ objects out of $n$ without assigning any order to them.

Knowing this $X$ can be useful. Can we find a formula for it?

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$$X = \frac{n!}{(n-r)!\, r!}$$

It is the number of so-called *$r$-combinations*.

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# Combinations

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$$X = \frac{n!}{(n-r)!\, r!}$$

It is the number of so-called *$r$-combinations*.

**Def.** An *$r$-combinations* is an unordered selection of $r$ objects from a set of $n$ objects.

We write $\qquad C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\, r!}$

Permutations
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# Combinations



The number of $r$-combinations:

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

(Note that $\binom{n}{r}$ reads as "$n$ choose $r$").

An $r$-combination is a selection of $r$ objects without specifying the order in which the objects are chosen.

**Exercise:**
There are three paintings, you'd like to buy some of them.

In how many ways one painting can be chosen? Two? Three?

# Combinations



In how many ways a team of 6 can be selected from a group of 12 students?

# Combinations

Permutations

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In how many ways a team of 6 can be selected from a group of 12 students?

$$\binom{12}{6} =$$

# Combinations

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In how many ways a team of 6 can be selected from a group of 12 students?

$$\binom{12}{6} = \frac{12!}{(12-6)!\,6!} = \frac{12!}{6!\cdot 6!} = \frac{7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12}{6!}$$

$$= \frac{7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} = \frac{7\cdot 11\cdot 12}{1} = 924.$$

# Example with cards



Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

# Example with cards



Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

# Example with cards



Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

More examples:

$$\binom{52}{2} = \frac{51 \cdot 52}{2} = 1326. \qquad \binom{52}{1} = 52.$$

# Combinations

Let's try to prove that for $1 \leq k \leq n$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Permutations

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# Summary



Given a set with $n$ elements.

The number of *permutations* of the elements the set:

$$P(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$

The number of *$r$-permutations* of the set:

$$P(n,r) = \frac{n!}{(n-r)!} = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

The number of unordered *$r$-combinations* ("$n$ choose $r$"):

$$\binom{n}{r} = \frac{P(n,r)}{P(r)} = \frac{n!}{(n-r)!\, r!}$$

# Solve



*Problem 1.*

How many permutations of the letters ABCDEFG contain

(a) the string BCD?

(b) the strings GCD and AB?

(c) the strings BAC and CED?

# Solve



*Problem 2.*

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

*Problem 3.*

How many bit strings contain exactly 8 zeros and 10 ones?

*Problem 4.*

How many bit strings contain exactly 8 zeros and 10 ones if every zero must be immediately followed by a one?

# Solve: “Knights of the Round Table”



**Def.** A *circular permutation* of $n$ people is their seating around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table.

**Def.** Similarly, a *circular $r$-permutation* of $n$ people is a seating of $r$ of these $n$ people around a circular table, where, again, the seatings that can be obtained by rotation are considered to be the same.

*Problem 5.*

In how many ways can King Arthur seat $n$ different knights at his round table?

*Problem 6.*

Count the number of circular $r$-permutations of $n$ people.

# “Knights of the Round Table”



*Problem 5.* In how many ways can King Arthur seat $n$ different knights at his round table?

**Answer:** Because $n$ normal permutations result in a single circular permutation, $\frac{n!}{n} = (n-1)!$

*Problem 6.*

Count the number of circular $r$-permutations of $n$ people.

**Answer:** $r$ normal $r$-permutations result in one circular $r$-permutation, so we get $\frac{n!}{(n-r)!\,r}$. Transform this formula:

$$\frac{n!}{(n-r)!\,r} = \frac{n!}{(n-r)!\,r!} \cdot \frac{r!}{r} = \binom{n}{r} \cdot \frac{r!}{r}$$

So, equivalently, we first choose $r$ knights out of $n$, and then count their circular permutations.

# A typical situation



A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

# A typical situation



A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

Four is enough.

# The Pigeonhole Principle



If you have 10 pigeons in 9 boxes, at least one box contains two birds.

*The pigeonhole principle.* If $k$ is a positive integer and $k + 1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more of the objects.

# The Pigeonhole Principle



There are 7 colleges in a city.

What is the minimum number of students to be invited for a survey to make sure that at least two are from the same college?

# The Pigeonhole Principle

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The Pigeonhole Principle

There are 7 colleges in a city.

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$$7 + 1 = 8 \text{ students.}$$

# Generalized Pigeonhole Principle



The ceiling function:

$$\lceil x \rceil = \text{the smallest integer not less that that } x$$

So, for example,

$$\lceil 2.0 \rceil = 2$$

$$\lceil 0.5 \rceil = 1$$

$$\lceil -3.5 \rceil = -3$$

*The Generalized Pigeonhole Principle.*

If $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

# Generalized Pigeonhole Principle



*The Generalized Pigeonhole Principle.*

If $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

**Example:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

# Generalized Pigeonhole Principle

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*The Generalized Pigeonhole Principle.*

If $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

**Example:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

"Birds" are cards. "Boxes" are suits, $k = 4$.
How many cards, $N$, should we take to guarantee that at least three of them fall in the same "box" (suit):

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{N}{4} \right\rceil \geq 3 > 2.$$

The smallest possible $N = 2 \cdot 4 + 1 = 9$. $\clubsuit\clubsuit\diamondsuit\diamondsuit\heartsuit\heartsuit\spadesuit\spadesuit$

# Solve



Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.