5.1 Perpendiculars and Bisectors

What *you should learn*

GOAL 1 Use properties of perpendicular bisectors.

GOAL 2 Use properties of angle bisectors to identify equal distances, such as the lengths of beams in a roof truss in **Example 3**.

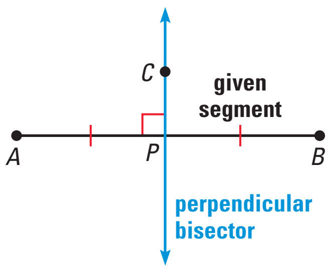
Why *you should learn it*

▼ To solve **real-life** problems, such as deciding where a hockey goalie should be positioned in **Exs. 33–35**.

REAL LIFE

GOAL 1 USING PROPERTIES OF PERPENDICULAR BISECTORS

In Lesson 1.5, you learned that a segment bisector intersects a segment at its midpoint. A segment, ray, line, or plane that is perpendicular to a segment at its midpoint is called a **perpendicular bisector**.



$\overleftrightarrow{CP}$ is a $\perp$ bisector of $\overline{AB}$.

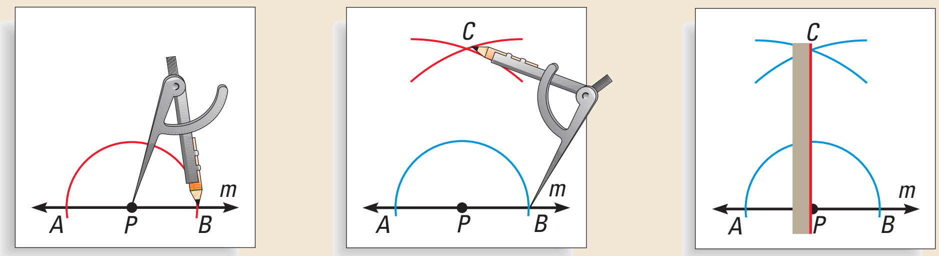
The construction below shows how to draw a line that is perpendicular to a given line or segment at a point P. You can use this method to construct a perpendicular bisector of a segment, as described below the activity.

ACTIVITY

Construction

Perpendicular Through a Point on a Line

Use these steps to construct a line that is perpendicular to a given line m and that passes through a given point P on m.



1. Place the compass point at P. Draw an arc that intersects line m twice. Label the intersections as A and B.
2. Use a compass setting greater than AP. Draw an arc from A. With the same setting, draw an arc from B. Label the intersection of the arcs as C.
3. Use a straightedge to draw $\overleftrightarrow{CP}$. This line is perpendicular to line m and passes through P.

You can measure $\angle CPA$ on your construction to verify that the constructed line is perpendicular to the given line m. In the construction, $\overleftrightarrow{CP} \perp \overline{AB}$ and $PA = PB$, so $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$.

A point is **equidistant from two points** if its distance from each point is the same. In the construction above, C is equidistant from A and B because C was drawn so that $CA = CB$.

STUDENT HELP

Look Back

For a construction of a perpendicular to a line through a point not on the given line, see p. 130.

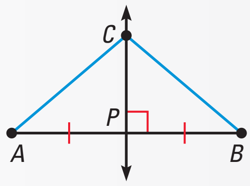
Theorem 5.1 below states that *any* point on the perpendicular bisector $\overleftrightarrow{CP}$ in the construction is equidistant from A and B, the endpoints of the segment. The converse helps you prove that a given point lies on a perpendicular bisector.

THEOREMS

THEOREM 5.1 ***Perpendicular Bisector Theorem***

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

If $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$, then $CA = CB$.

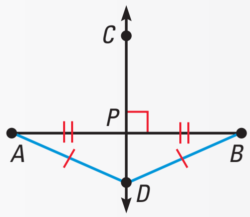


CA* = *CB

THEOREM 5.2 ***Converse of the Perpendicular Bisector Theorem***

If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.

If $DA = DB$, then D lies on the perpendicular bisector of $\overline{AB}$.



***D* is on $\overleftrightarrow{CP}$.**

Proof

Plan for Proof of Theorem 5.1 Refer to the diagram for Theorem 5.1 above. Suppose that you are given that $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$. Show that right triangles $\triangle APC$ and $\triangle BPC$ are congruent using the SAS Congruence Postulate. Then show that $\overline{CA} \cong \overline{CB}$.

Exercise 28 asks you to write a two-column proof of Theorem 5.1 using this plan for proof. Exercise 29 asks you to write a proof of Theorem 5.2.

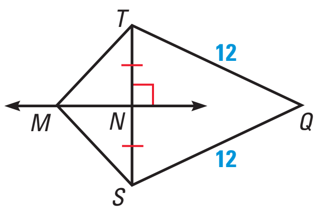
Logical Reasoning

EXAMPLE 1 *Using Perpendicular Bisectors*

In the diagram shown, $\overleftrightarrow{MN}$ is the perpendicular bisector of $\overline{ST}$.

a. What segment lengths in the diagram are equal?

b. Explain why Q is on $\overleftrightarrow{MN}$.



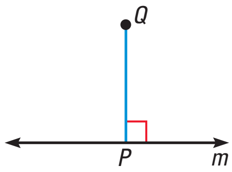
SOLUTION

a. $\overleftrightarrow{MN}$ bisects $\overline{ST}$, so $NS = NT$. Because M is on the perpendicular bisector of $\overline{ST}$, $MS = MT$ (by Theorem 5.1). The diagram shows that $QS = QT = 12$.

b. $QS = QT$, so Q is equidistant from S and T. By Theorem 5.2, Q is on the perpendicular bisector of $\overline{ST}$, which is $\overleftrightarrow{MN}$.

GOAL 2 USING PROPERTIES OF ANGLE BISECTORS

The **distance from a point to a line** is defined as the length of the perpendicular segment from the point to the line. For instance, in the diagram shown, the distance between the point Q and the line m is QP.



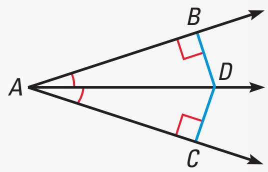
When a point is the same distance from one line as it is from another line, then the point is **equidistant from the two lines** (or rays or segments). The theorems below show that a point in the interior of an angle is equidistant from the sides of the angle if and only if the point is on the bisector of the angle.

THEOREMS

THEOREM 5.3 ***Angle Bisector Theorem***

If a point is on the bisector of an angle, then it is equidistant from the two sides of the angle.

If $m\angle BAD = m\angle CAD$, then $DB = DC$.

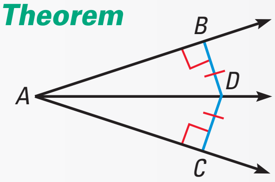


$DB = DC$

THEOREM 5.4 ***Converse of the Angle Bisector Theorem***

If a point is in the interior of an angle and is equidistant from the sides of the angle, then it lies on the bisector of the angle.

If $DB = DC$, then $m\angle BAD = m\angle CAD$.



$m\angle BAD = m\angle CAD$

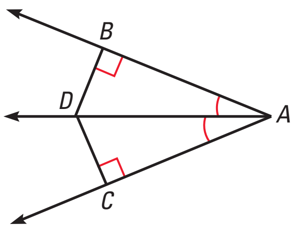
A paragraph proof of Theorem 5.3 is given in Example 2. Exercise 32 asks you to write a proof of Theorem 5.4.

EXAMPLE 2 *Proof of Theorem 5.3*

Proof

GIVEN ▶ D is on the bisector of $\angle BAC$.
$\overline{DB} \perp \overrightarrow{AB}$, $\overline{DC} \perp \overrightarrow{AC}$

PROVE ▶ $DB = DC$



Plan for Proof Prove that $\triangle ADB \cong \triangle ADC$. Then conclude that $\overline{DB} \cong \overline{DC}$, so $DB = DC$.

SOLUTION

Paragraph Proof By the definition of an angle bisector, $\angle BAD \cong \angle CAD$. Because $\angle ABD$ and $\angle ACD$ are right angles, $\angle ABD \cong \angle ACD$. By the Reflexive Property of Congruence, $\overline{AD} \cong \overline{AD}$. Then $\triangle ADB \cong \triangle ADC$ by the AAS Congruence Theorem. Because corresponding parts of congruent triangles are congruent, $\overline{DB} \cong \overline{DC}$. By the definition of congruent segments, $DB = DC$.

FOCUS ON CAREERS

ENGINEERING TECHNICIAN
In manufacturing, engineering technicians prepare specifications for products such as roof trusses, and devise and run tests for quality control.

CAREER LINK
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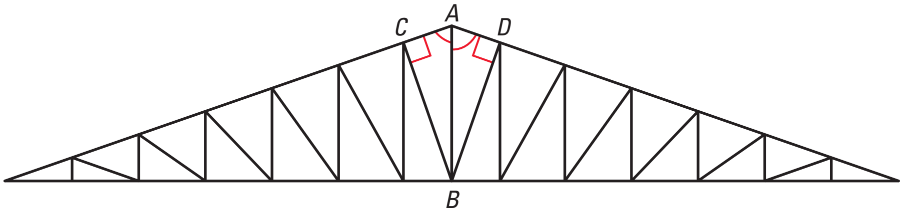
EXAMPLE 3 *Using Angle Bisectors*

ROOF TRUSSES Some roofs are built with wooden trusses that are assembled in a factory and shipped to the building site. In the diagram of the roof truss shown below, you are given that $\overrightarrow{AB}$ bisects $\angle CAD$ and that $\angle ACB$ and $\angle ADB$ are right angles. What can you say about $\overline{BC}$ and $\overline{BD}$?



SOLUTION

Because $\overline{BC}$ and $\overline{BD}$ meet $\overline{AC}$ and $\overline{AD}$ at right angles, they are perpendicular segments to the sides of $\angle CAD$. This implies that their lengths represent the distances from the point B to $\overrightarrow{AC}$ and $\overrightarrow{AD}$. Because point B is on the bisector of $\angle CAD$, it is equidistant from the sides of the angle.

▶ So, $BC = BD$, and you can conclude that $\overline{BC} \cong \overline{BD}$.

GUIDED PRACTICE

Vocabulary Check ✓

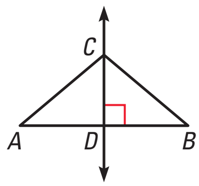
1. If D is on the __?__ of $\overline{AB}$, then D is *equidistant* from A and B.

Concept Check ✓

2. Point G is in the interior of $\angle HJK$ and is equidistant from the sides of the angle, $\overrightarrow{JH}$ and $\overrightarrow{JK}$. What can you conclude about G? Use a sketch to support your answer.

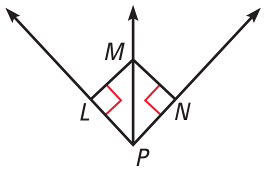
Skill Check ✓

In the diagram, $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$.



3. What is the relationship between $\overline{AD}$ and $\overline{BD}$?
4. What is the relationship between $\angle ADC$ and $\angle BDC$?
5. What is the relationship between $\overline{AC}$ and $\overline{BC}$? Explain your answer.

In the diagram, $\overrightarrow{PM}$ is the bisector of $\angle LPN$.



6. What is the relationship between $\angle LPM$ and $\angle NPM$?
7. How is the distance between point M and $\overrightarrow{PL}$ related to the distance between point M and $\overrightarrow{PN}$?

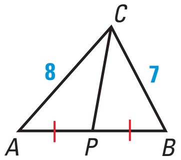
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 811.

LOGICAL REASONING **Tell whether the information in the diagram allows you to conclude that *C* is on the perpendicular bisector of $\overline{AB}$. Explain your reasoning.**

8.



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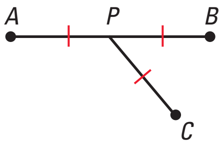

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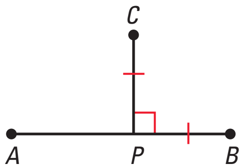
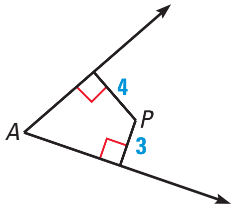
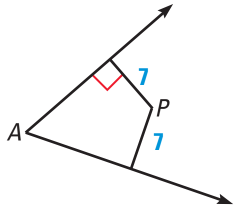


LOGICAL REASONING **In Exercises 11–13, tell whether the information in the diagram allows you to conclude that *P* is on the bisector of $\angle A$. Explain.**

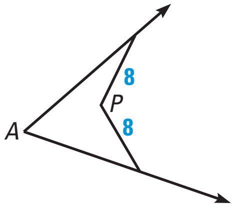
11.



12.



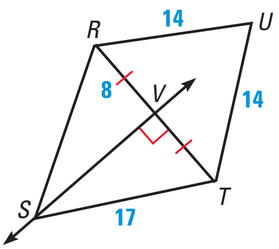
13.



14. **CONSTRUCTION** Draw $\overline{AB}$ with a length of 8 centimeters. Construct a perpendicular bisector and draw a point D on the bisector so that the distance between D and $\overline{AB}$ is 3 centimeters. Measure $\overline{AD}$ and $\overline{BD}$.

15. **CONSTRUCTION** Draw a large $\angle A$ with a measure of 60°. Construct the angle bisector and draw a point D on the bisector so that $AD = 3$ inches. Draw perpendicular segments from D to the sides of $\angle A$. Measure these segments to find the distance between D and the sides of $\angle A$.

USING PERPENDICULAR BISECTORS **Use the diagram shown.**



16. In the diagram, $\overleftrightarrow{SV} \perp \overline{RT}$ and $\overline{VR} \cong \overline{VT}$. Find VT.

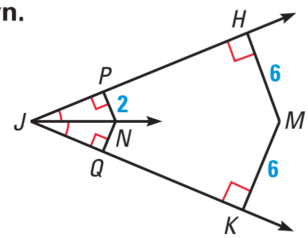
17. In the diagram, $\overleftrightarrow{SV} \perp \overline{RT}$ and $\overline{VR} \cong \overline{VT}$. Find SR.

18. In the diagram, $\overleftrightarrow{SV}$ is the perpendicular bisector of $\overline{RT}$. Because $UR = UT = 14$, what can you conclude about point U?

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 8–10, 14, 16–18, 21–26
Example 2: Exs. 11–13, 15, 19, 20, 21–26
Example 3: Exs. 31, 33–35

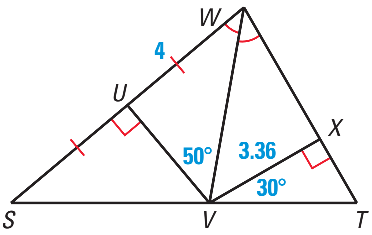
USING ANGLE BISECTORS **Use the diagram shown.**



19. In the diagram, $\overrightarrow{JN}$ bisects $\angle HJK$, $\overline{NP} \perp \overrightarrow{JP}$, $\overline{NQ} \perp \overrightarrow{JQ}$, and $NP = 2$. Find NQ.

20. In the diagram, $\overrightarrow{JN}$ bisects $\angle HJK$, $\overline{MH} \perp \overrightarrow{JH}$, $\overline{MK} \perp \overrightarrow{JK}$, and $MH = MK = 6$. What can you conclude about point M?

USING BISECTOR THEOREMS **In Exercises 21–26, match the angle measure or segment length described with its correct value.**



A. $60°$ **B.** 8

C. $40°$ **D.** 4

E. $50°$ **F.** 3.36

21. SW **22.** $m\angle XTV$

23. $m\angle VWX$ **24.** VU

25. WX **26.** $m\angle WVX$

27. **PROVING A CONSTRUCTION** Write a proof to verify that $\overleftrightarrow{CP} \perp \overline{AB}$ in the construction on page 264.

STUDENT HELP

Look Back
For help with proving that constructions are valid, see p. 231.

28. **PROVING THEOREM 5.1** Write a proof of Theorem 5.1, the Perpendicular Bisector Theorem. You may want to use the plan for proof given on page 265.

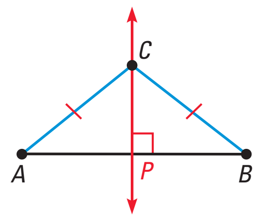
GIVEN ▶ $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$.

PROVE ▶ C is equidistant from A and B.

29. **PROVING THEOREM 5.2** Use the diagram shown to write a two-column proof of Theorem 5.2, the Converse of the Perpendicular Bisector Theorem.

GIVEN ▶ C is equidistant from A and B.

PROVE ▶ C is on the perpendicular bisector of $\overline{AB}$.

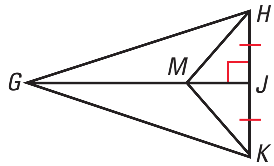


Plan for Proof Use the Perpendicular Postulate to draw $\overleftrightarrow{CP} \perp \overleftrightarrow{AB}$. Show that $\triangle APC \cong \triangle BPC$ by the HL Congruence Theorem. Then $\overline{AP} \cong \overline{BP}$, so $AP = BP$.

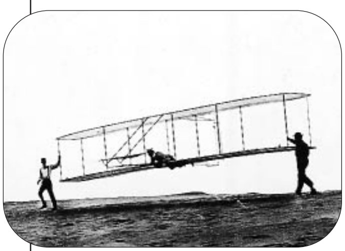
30. **PROOF** Use the diagram shown.

GIVEN ▶ $\overline{GJ}$ is the perpendicular bisector of $\overline{HK}$.

PROVE ▶ $\triangle GHM \cong \triangle GKM$

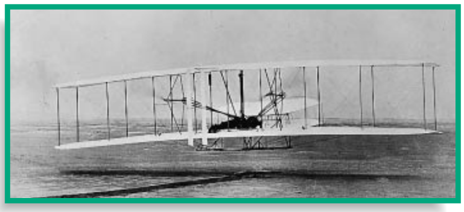


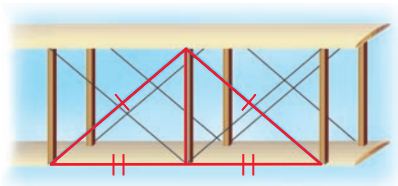
FOCUS ON PEOPLE



THE WRIGHT BROTHERS
In Kitty Hawk, North Carolina, on December 17, 1903, Orville and Wilbur Wright became the first people to successfully fly an engine-driven, heavier-than-air machine.

31. **EARLY AIRCRAFT** On many of the earliest airplanes, wires connected vertical posts to the edges of the wings, which were wooden frames covered with cloth. Suppose the lengths of the wires from the top of a post to the edges of the frame are the same and the distances from the bottom of the post to the ends of the two wires are the same. What does that tell you about the post and the section of frame between the ends of the wires?

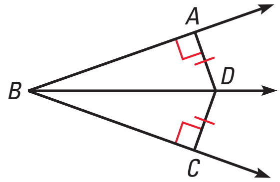




32. **DEVELOPING PROOF** Use the diagram to complete the proof of Theorem 5.4, the Converse of the Angle Bisector Theorem.

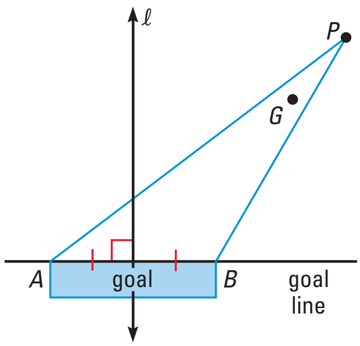
GIVEN ▶ D is in the interior of $\angle ABC$ and is equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

PROVE ▶ D lies on the angle bisector of $\angle ABC$.



Statements	Reasons
1. D is in the interior of $\angle ABC$.	**1.** ___?___
2. D is ___?___ from $\overrightarrow{BA}$ and $\overrightarrow{BC}$.	**2.** Given
3. ___?___ = ___?___	**3.** Definition of equidistant
4. $\overline{DA} \perp$ ___?___, ___?___ $\perp \overrightarrow{BC}$	**4.** Definition of distance from a point to a line
5. ___?___	**5.** If 2 lines are ⊥, then they form 4 rt. ∠s.
6. ___?___	**6.** Definition of right triangle
7. $\overline{BD} \cong \overline{BD}$	**7.** ___?___
8. ___?___	**8.** HL Congruence Thm.
9. $\angle ABD \cong \angle CBD$	**9.** ___?___
10. $\overrightarrow{BD}$ bisects $\angle ABC$ and point D is on the bisector of $\angle ABC$.	**10.** ___?___

ICE HOCKEY **In Exercises 33–35, use the following information.** In the diagram, the goalie is at point G and the puck is at point P. The goalie's job is to prevent the puck from entering the goal.

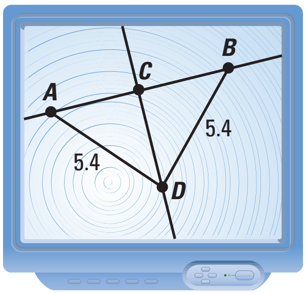


33. When the puck is at the other end of the rink, the goalie is likely to be standing on line ℓ. How is ℓ related to $\overline{AB}$?

34. As an opposing player with the puck skates toward the goal, the goalie is likely to move from line ℓ to other places on the ice. What should be the relationship between $\overrightarrow{PG}$ and $\angle APB$?

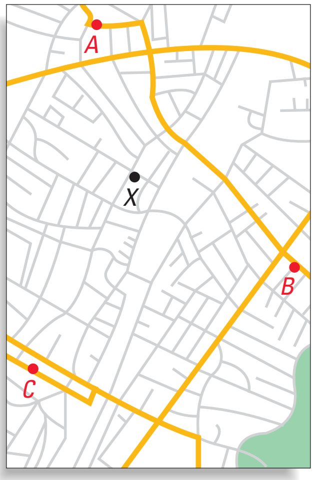
35. How does $m\angle APB$ change as the puck gets closer to the goal? Does this change make it easier or more difficult for the goalie to defend the goal? Explain.

36. **TECHNOLOGY** Use geometry software to construct $\overline{AB}$. Find the midpoint C. Draw the perpendicular bisector of $\overline{AB}$ through C. Construct a point D along the perpendicular bisector and measure $\overline{DA}$ and $\overline{DB}$. Move D along the perpendicular bisector. What theorem does this construction demonstrate?



Test Preparation

37. MULTI-STEP PROBLEM Use the map shown and the following information. A town planner is trying to decide whether a new household X should be covered by fire station A, B, or C.



a. Trace the map and draw the segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

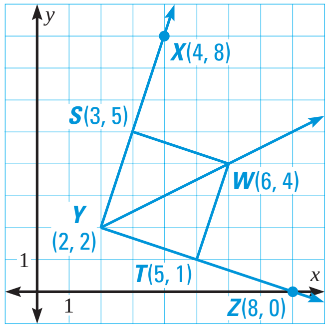
b. Construct the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. Do the perpendicular bisectors meet at a point?

c. The perpendicular bisectors divide the town into regions. Shade the region closest to fire station A red. Shade the region closest to fire station B blue. Shade the region closest to fire station C gray.

d. *Writing* In an emergency at household X, which fire station should respond? Explain your choice.

★ Challenge

USING ALGEBRA **Use the graph at the right.**



38. Use slopes to show that $\overline{WS} \perp \overrightarrow{YX}$ and that $\overline{WT} \perp \overrightarrow{YZ}$.

39. Find WS and WT.

40. Explain how you know that $\overrightarrow{YW}$ bisects $\angle XYZ$.

EXTRA CHALLENGE
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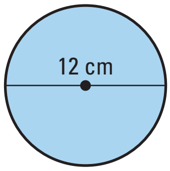
MIXED REVIEW

CIRCLES **Find the missing measurement for the circle shown. Use 3.14 as an approximation for π.** (Review 1.7 for 5.2)



41. radius **42.** circumference **43.** area

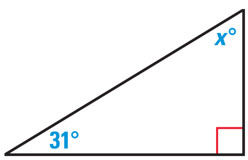
CALCULATING SLOPE **Find the slope of the line that passes through the given points.** (Review 3.6)

44. $A(-1, 5)$, $B(-2, 10)$ **45.** $C(4, -3)$, $D(-6, 5)$ **46.** $E(4, 5)$, $F(9, 5)$

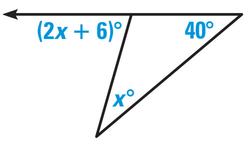
47. $G(0, 8)$, $H(-7, 0)$ **48.** $J(3, 11)$, $K(-10, 12)$ **49.** $L(-3, -8)$, $M(8, -8)$

USING ALGEBRA **Find the value of x.** (Review 4.1)

50.



51.



52.