Phaseless Inverse Scattering Problems and Global Convergence

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This talk reflects my research activity in 2015-2016

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PHASELESS INVERSE SCATTERING PROBLEM:

Let $u(x,k)$ be the complex valued wave field, where k is the wave number, $x \in \mathbb{R}^3$.
Determine the scatterer, given $|u(x,k)|$ outside of this scatterer.

APPLICATIONS:

Imaging of nanostructures and biological cells
Sizes: 0.1 micron range
The wavelength $\lambda \leq 1$ micron

OUR FOCUS:

Reconstruction of coefficients of Schrödinger and generalized Helmholtz equations from phaseless data

In parallel R.G. Novikov has developed methods for phase reconstruction, including uniqueness theorems. His statements of problems are different from ours

- The phaseless inverse scattering problem for the Schrödinger equation was posed in the book of K. Chadan and P.C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, Springer-Verlag, New York, 1977
- It was also implicitly posed in the book of R.G. Newton, *Inverse Schrödinger Scattering in Three Dimensions*, Springer, New York, 1989
- Works of M.V. Klibanov, V.G. Romanov and R.G. Novikov (2014-2016) provided the first full solution of this problem

QUESTIONS TO ADDRESS

1. Uniqueness
2. Reconstruction procedure
3. Numerical procedure

UNIQUENESS FOR THE CASE OF THE SCHRODINGER EQUATION

$$\Delta_x u + k^2 u - q(x)\, u = -\delta(x - x_0),\, x \in \mathbb{R}^3, \tag{1}$$

$$\partial_r u(x, x_0, k) - iku(x, x_0, k) = o\left(\frac{1}{r}\right), r = |x - x_0| \to \infty. \tag{2}$$

Let $\Omega, G \subset \mathbb{R}^3$ be two bounded domains, $\Omega \subset G$,

$$S = \partial G, dist(S, \partial\Omega) \geq 2\varepsilon = const. > 0.$$

For an arbitrary point $y \in \mathbb{R}^3$ and for an arbitrary number $\rho > 0$ denote $B_\rho(y) = \{x : |x - y| < \rho\}$.

The potential $q(x)$ is a real valued function satisfying the following conditions

$$q(x) \in C^2(\mathbb{R}^3), q(x) = 0 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega, \tag{3}$$

$$q(x) \geq 0. \tag{4}$$

Phaseless Inverse Scattering Problem 1 (PISP1).
Suppose that the function $q(x)$ is unknown. Also, assume that the following function $f_1(x, x_0, k)$ is known

$$f_1(x, x_0, k) = |u(x, x_0, k)|, \forall x_0 \in S, \forall x \in B_\varepsilon(x_0), x \neq x_0, \forall k \in (a, b),$$

where $(a, b) \subset \mathbb{R}$ is an arbitrary interval. Determine the function $q(x)$ for $x \in \Omega$.

Theorem 1. *Let $u_1(x, x_0, k)$ and $u_2(x, x_0, k)$ be solutions of the problem (1), (2) with corresponding potentials $q_1(x)$* and *$q_2(x)$ satisfying conditions (3), (4). Assume that*

$$\begin{aligned} |u_1(x, x_0, k)| = |u_2(x, x_0, k)| = f_1(x, x_0, k), \forall x_0 \in S, \\ \forall x \in B_\varepsilon(x_0), x \neq x_0, \forall k \in (a, b). \end{aligned} \tag{5}$$

Then $q_1(x) \equiv q_2(x)$.

$$u(x,x_0,k) = u_0(x,x_0,k) + u_{sc}(x,x_0,k),$$

$$u_0 = \frac{\exp(ik|x-x_0|)}{4\pi|x-x_0|}.$$

$u_0(x,x_0,k)$ is the incident spherical wave and $u_{sc}(x,x_0,k)$ is the scattered wave.

Phaseless Inverse Scattering Problem 2 (PISP2)
Suppose that the function $q(x)$ is unknown. Also, assume that the following function $f_2(x,x_0,k)$ is known

$$f_2(x,x_0,k) = |u_{sc}(x,x_0,k)|, \forall x_0 \in S, \forall x \in B_\varepsilon(x_0), x \neq x_0, \forall k \in (a,b)$$

Determine the function $q(x)$ for $x \in \Omega$.

Let $G_1 \subset \mathbb{R}^3$ be another bounded domain,
$G \subset G_1, S \cap \partial G_1 = \varnothing$.

Theorem 2. *Assume that all conditions of Theorem 1 hold, except that (5) is replaced with*

$$\left|u_{sc,1}(x,x_0,k)\right| = \left|u_{sc,2}(x,x_0,k)\right| = f_2(x,x_0,k), \forall x_0 \in S,$$
$$\forall x \in B_{\varepsilon}(x_0), x \neq x_0, \forall k \in (a,b),$$

where $u_{sc,j} = u_j - u_0, j = 1, 2$. *In addition, assume that* $q(x) \neq 0$, $\forall x \in S$ and $q(x) = 0$ for $x \in \mathbb{R}^3 \diagdown G_1$. *Then* $q_1(x) \equiv q_2(x)$.

UNIQUENESS FOR THE CASE OF THE GENERALIZED HELMHOLTZ EQUATION

$$c \in C^{15}(\mathbb{R}^3), \quad c(x) \geq c_0 = const. > 0, \tag{6}$$

$$c(x) = 1 \quad \text{for} \;\; x \in \mathbb{R}^3 \setminus \Omega. \tag{7}$$

The conformal Riemannian metric generated by the function $c(x)$ is

$$d\tau = \sqrt{c\,(x)}\,|dx|\,, |dx| = \sqrt{(dx_1)^2 + (dx_2)^2 + (dx_3)^2}. \tag{8}$$

Let $\Gamma\,(x, y)$ be the geodesic line connecting points x and y.

Assumption 1. *We assume that geodesic lines of the metric (8) satisfy the regularity condition, i.e. for each two points* $x, y \in \mathbb{R}^3$ *there exists a single geodesic line* $\Gamma\,(x, y)$ *connecting these points.*

A sufficient condition for the validity of Assumption is (V.G. Romanov, 2014):

$$\sum_{i,j=1}^{3} \frac{\partial^2 \ln c(x)}{\partial x_i \partial x_j} \xi_i \xi_j \geq 0, \ \forall \xi \in \mathbb{R}^3, \forall x \in \overline{\Omega}.$$

The function $\tau(x, y)$ is the travel time from y to x and is the solution of the eikonal equation,

$$|\nabla_x \tau(x, y)|^2 = c(x),$$

$$\tau(x, y) = O\left(|x - y|\right) \ \text{ as } \ x \to y,$$

$$\tau\left(x, y\right) = \int\limits_{\Gamma(x,y)} \sqrt{c\left(\xi\right)} d\sigma.$$

GENERALIZED HELMHOLTZ EQUATION:

$$\Delta u + k^2 c(x) u = -\delta(x-y), \quad x \in \mathbb{R}^3, \tag{9}$$

$$\frac{\partial u}{\partial r} - iku = o(r^{-1}) \text{ as } r = |x-y| \to \infty. \tag{10}$$

Phaseless Inverse Scattering Problem 3 (PISP3). *Let $u(x, y, k)$ be the solution of the problem (9), (10). Assume that the following function $f_3(x, y, k)$ is known*

$$f_3(x, y, k) = |u(x, y, k)|, \forall y \in S, \forall x \in B_\varepsilon(y), x \neq y, \forall k \in (a, b),$$

where $(a, b) \subset \{z > 0\}$ is a certain interval. Determine the function $c(x)$.

Theorem 3. *Consider an arbitrary pair of points $y \in S, x \in B_\varepsilon(y), x \neq y$. And consider the function $g_{x,y}(k) = f_3(x, y, k)$ as the function of the variable k. Then the function $\varphi_{x,y}(k) = u(x, y, k)$ of the variable k is reconstructed uniquely, as soon as the function $g_{x,y}(k)$ is given for all $k \in (a, b)$. The PISP3 has at most one solution.*

RECONSTRUCTION PROCEDURE FOR PISP4

M.V. Klibanov and V.G. Romanov (2016)

Consider the case when the modulus of the scattered wave is measured.
Incident spherical wave $u_0(x, y, k)$,

$$u_0(x, y, k) = \frac{\exp(ik|x-y|)}{4\pi|x-y|}.$$

Scattered wave $u_{sc}(x, y, k)$,

$$u_{sc}(x, y, k) = u(x, y, k) - u_0(x, y, k) = u(x, y, k) - \frac{\exp(ik|x-y|)}{4\pi|x-y|}.$$

Phaseless Inverse Scattering Problem 4 (PISP4).
Suppose that the following function $f_4(x, y, k)$ *is known*

$$f_4(x, k, y) = |u_{sc}(x, y, k)|^2, \ \forall (x, y) \in S \times S, \ \forall k \geq k_0,$$

where $k_0 = const. > 0$. *Determine the function* $c(x)$.

Associated Cauchy problem

$$c(x)v_{tt} = \Delta v + \delta(x-y,t), \quad (x,t) \in \mathbb{R}^4,$$

$$v|_{t<0} \equiv 0.$$

Fourier transform (results of B.R. Vainberg: 1980 and earlier):

$$u(x,y,k) = \int_0^\infty v(x,y,t)\, e^{ikt} dt.$$

The function v can be represented as

$$v(x,y,t) = A(x,y)\delta(t-\tau(x,y)) + \hat{v}(x,y,t)H(t-\tau(x,y)),$$

$$H(t) = \begin{cases} 1, t>0, \\ 0, t<0, \end{cases}$$

$$A(x,y) > 0,$$

$\hat{v}(x,y,t)$ is sufficiently smooth.

Hence, the following asymptotic behavior takes place in any bounded domain of $\mathbb{R}^3$

$$u(x,y,k) = A(x,y)\, e^{ik\tau(x,y)} + O(1/k), k \to \infty.$$

Hence,

$$f_4(x,y,k) = A^2(x,y) + \frac{1}{16\pi^2 |x-y|^2} - \\ - \frac{A(x,y)}{2\pi|x-y|} \cos\left[k\left(\tau(x,y) - |x-y|\right)\right] + O\left(\frac{1}{k}\right), k \to \infty,$$

Ignore $O(1/k)$,

$$f_4(x,y,k) = A^2(x,y) + \frac{1}{16\pi^2 |x-y|^2} - \\ - \frac{A(x,y)}{2\pi|x-y|} \cos\left[k\left(\tau(x,y) - |x-y|\right)\right], k \to \infty.$$

Consider $k \geq k_1$

$$f_4^*(x,y) = f_4(x,y,k_2) = \max_{k \geq k_1} f_4(x,k,y) = \left(A(x,y) + \frac{1}{4\pi|x-y|}\right)^2$$

Hence, we find the number $A(x,y)$ as

$$A(x,y)=\sqrt{f_4^*(x,y)}-\frac{1}{4\pi|x-y|}.$$

Assume that $\tau(x,y)\neq|x-y|$. Hence, since $\beta\left(x\right)\geq0$, then $\tau(x,y)>\left|x-y\right|$.
There exists the number $k_3>k_2$ such that

$$k_3=\min\left\{k:k>k_2,f_4\left(x,y,k\right)=f_4^*(x,y)\right\}.$$

Hence,

$$k_3\left(\tau(x,y)-|x-y|\right)=k_2\left(\tau(x,y)-|x-y|\right)+2\pi.$$

Thus,

$$\tau(x,y)=|x-y|+\frac{2\pi}{k_3-k_2}.$$

Inverse Kinematic Problem (IKP, 1960-ies-1980ies: V.G. Romanov, R. Mukhometov and then many others)=**Travel Time Tomography Problem**. *Given the function* $\tau(x,y), \forall x,y \in S$, *find the function* $c(x)$.
Uniqueness of IKP is well known: Romanov, Mukhometov.
Numerical method is still unclear.
The most recent numerical result: U. Schröder and T. Schuster, *Inverse Problems*, 32, 085009, 2016.

LINEARIZATION

$$\begin{aligned} c(x) &= \beta(x) + 1, \\ \beta(x) &\geq 0, \beta(x) = 0 \text{ for } x \notin \overline{\Omega}. \end{aligned}$$

Assume that

$$||\beta||_{C^1(\overline{\Omega})} << 1.$$

Then the linearization of the function $\tau(x, y)$ with respect to the function β leads to

$$\tau(x, y) = |x - y| + \int\limits_{L(x,y)} \beta(\xi)\, d\sigma,$$

$L(x, y)$ is the straight line connecting points x and y.
We got the problem of the inversion of the 2-D Radon transform.

NUMERICAL STUDY

M.V. Klibanov, L.H. Nguyen, K. Pan, 2015.

Above formulae work only for a sufficiently large $k-$interval
Also, they work only for sufficiently large values of k.
QUESTION: Does this method work for realistic values of k?

ANSWER: Yes. Imaging of nanostructures. The range of our wavelengths is $\lambda \in [0.078, 0.126]\, \mu m$. Dimensionless $k = 2\pi/\lambda$:

$$k \in [50, 80] = [k_1, k_2]\,.$$

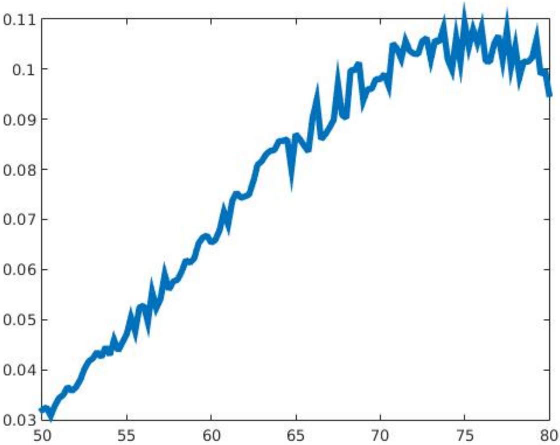


Figure 1: *Noisy data* $f_4(x, y, k), k \in [50, 80]$. *The* $k-$*interval is too short to apply the above procedure.*

- The above procedure was essentially modified to work with smaller $k-$intervals.

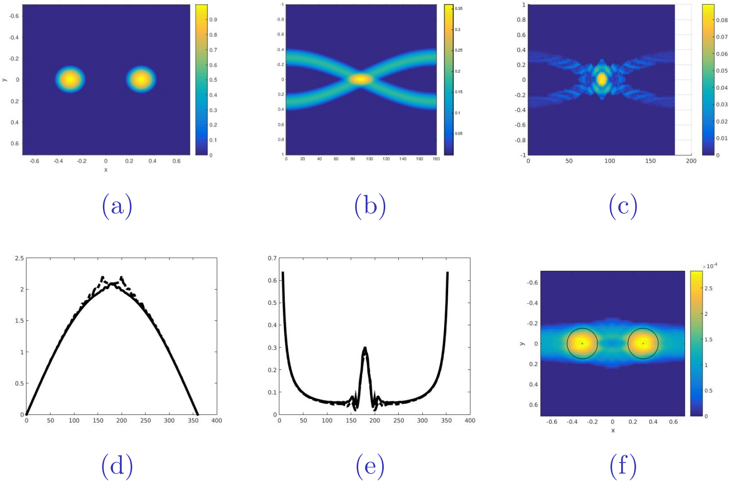


Figure 2: *A sample of the image obtained by our modified reconstruction procedure. a) True image. b) The 2-D Radon transform of the central* $z-$*cross-section. c) The reconstructed function* $\tau(x,y) - |x-y|$ *in the central* $z-$*cross-section. d) The reconstructed (solid) and true functions* $\tau(x,y)$ *for a fixed source position* y *when* x *runs over the opposite side of the square. e) The same for* $A(x,y)$. *f) The reconstructed image.*

CONCLUSIONS

1. Shapes and locations of targets are reconstructed well.
2. However, values of the unknown coefficient $c(x)$ inside the targets are reconstructed poorly.
3. Therefore, the two-stage imaging procedure should take place.
4. Stage 1: The same as above.
5. Stage 2: A globally convergent numerical method for a Coefficient Inverse Problem should be applied. The result of stage 1 should be taken as the first guess for the so-called "tail function".
6. By our experience, that globally convergent method should provide accurate values of $c(x)$ inside the targets.

A GLOBALLY CONVERGENT NUMERICAL METHOD FOR A COEFFICIENT INVERSE PROBLEM

DEFINITION 1. We call a numerical method for a CIP *globally convergent* if a theorem is proved, which claims that this method delivers at least one point in a sufficiently small neighborhood of the exact solution without any advanced knowledge of this neighborhood. In addition, this theorem must be confirmed computationally.

DEFINITION 2. We call a numerical method for a CIP *locally convergent* if its convergence to the exact solution cannot be guaranteed unless its starting point is located in a sufficiently small neighborhood of this solution.

A. CIPs with single measurement data.

- Two types of globally convergent numerical methods: Klibanov and his group, 1997-2016.
- We focus on one of them, since it is verified on experimental data.

B. CIPs with many measurements: either many sources or many directions of the incident plane wave.

- Methods of M.I. Belishev and S.I. Kabanikhin.

ALTERNATIVES TO GLOBALLY CONVERGENT METHODS:

- Various versions of the least squares minimization method. Locally convergent. No interest: local minima and ravines.
- Small perturbation methods, e.g. Born-like series. Local convergence.

THE METHOD

M.V. Klibanov, L.H. Nguyen, H. Liu

$$\Delta u + k^2 c(x) u = 0, \quad x \in \mathbb{R}^3.$$

The incident plane wave:

$$u_0\left(x, k\right) = \exp\left(ikx_3\right).$$

The total wave field:

$$u\left(x, k\right) = u_0\left(x, k\right) + u_{\mathrm{sc}}\left(x, k\right),$$

$$\frac{\partial u_{\mathrm{sc}}}{\partial r} - iku_{\mathrm{sc}} = o(r^{-1}), \;\; r = |x| \to \infty.$$

$$\Omega = \{|x| < R\} \subset \mathbb{R}^3.$$

Geodesic Lines:

$$\begin{cases} (\nabla\tau(x))^2 = c(x), \\ \tau(x) = x_3 \text{ for } x_3 \leq -R. \end{cases}$$

$$\tau(x) = \int\limits_{\Gamma(x)} \sqrt{c\,(\xi)}d\sigma.$$

The line $\Gamma\,(x)$ intersects the plane $P = \{x_3 = -R\}$ orthogonally.

Assumption 2. *We assume that geodesic lines satisfy the regularity condition in* $\mathbb{R}^3$. *In other words, for each point* $x \in \mathbb{R}^3$ *there exists a single geodesic line* $\Gamma(x)$ *connecting* x *with the plane* P *such that* $\Gamma(x)$ *intersects* $P = \{x_3 = -R\}$ *orthogonally.*

Coefficient Inverse Problem (CIP). *Let* $\underline{k} > 0$ *and* $\overline{k} > 0$ *be two sufficiently large numbers and* $0 < \underline{k} < \overline{k}$. *Assume that the function* $g(x,k)$ *is known, where*

$$g(x,k) = u(x,k), \quad x \in \partial\Omega, k \in \left[\underline{k}, \overline{k}\right].$$

Determine the function $c(x)$ *for* $x \in \Omega$.

ASYMPTOTIC BEHAVIOR

$$u\left(x,k\right)=A(x)e^{-ik\tau(x)}\left(1+O\left(1/k\right)\right), k\rightarrow\infty, x\in\Omega, \tag{11}$$

$$\left|O\left(1/k\right)\right|\leq B/k, \forall x\in\overline{\Omega},$$

$$A(x)>0.$$

Using the asymptotic behavior (11), one can prove that there exists unique function $v\left(x,k\right), k\geq\underline{k}$ such that

$$u\left(x,k\right)=e^{v(x,k)}, x\in\Omega, k\geq\underline{k}.$$

Hence,

$$\nabla v=\frac{\nabla u}{u}, \quad \partial_k v=\frac{\partial_k u}{u}.$$

$$\Delta v(x,k)+(\nabla v(x,k))^2=-k^2c(x). \tag{12}$$

Let

$$q(x,k) = \partial_k v(x,k) = \frac{\partial_k u(x,k)}{u(x,k)}, \quad x \in \Omega, k \in \left(\underline{k}, \overline{k}\right).$$

$$v(x,k) = -\int_k^{\overline{k}} q(x,\kappa) d\kappa + V(x), \quad x \in \Omega, k \in (\underline{k}, \overline{k}). \tag{13}$$

$$V\left(x\right) = v\left(x, \overline{k}\right) = \log u\left(x, \overline{k}\right). \tag{14}$$

We call $V\left(x\right)$ the "tail function".

The differentiation of (12) with respect to k leads to

$$\Delta q(x,k) + 2\nabla q(x,k)\nabla v(x,k) = -2kc(x) = 2(\Delta v + (\nabla v)^2)/k.$$

This and (13) imply that for all $k \in \left[\underline{k}, \overline{k}\right]$

$$\begin{aligned} k\Delta q(x,k) + 2k\nabla q(x,k)\nabla \left(-\int_k^{\overline{k}} q(x,\kappa)d\kappa + V(x)\right) \\ = 2\Delta\left(-\int_k^{\overline{k}} q(x,\kappa)d\kappa + V(x)\right) + \\ +2\left(\nabla\left(-\int_k^{\overline{k}} q(x,\kappa)d\kappa + V(x)\right)\right)^2. \end{aligned}$$

Boundary condition:

$$q(x,k) = \frac{\partial_k g(x,k)}{g(x,k)} =: \psi(x,k) \quad \text{on } \partial\Omega, \forall k \in \left[\underline{k}, \overline{k}\right].$$

Let $h > 0$ be the partition step size of a uniform partition of the frequency interval $\left[\underline{k}, \overline{k}\right]$,

$$\underline{k} = k_N < k_{N-1} < ... < k_1 < k_0 = \overline{k}, k_{j-1} - k_j = h.$$

Approximate the function $q(x,k)$ as a piecewise constant function with respect to $k \in \left[\underline{k}, \overline{k}\right]$.
Let

$$q(x,k) = q_n(x), \psi(x,k) = \psi_n(x), k \in [k_n, k_{n-1}), n = 1, ..., N.$$

We set $q_0(x) \equiv 0$. Denote

$$\overline{q_{n-1}} = \sum_{j=0}^{n-1} q_j(x).$$

We obtain

$$\begin{gathered}\Delta q_n - A_n h \nabla \overline{q_{n-1}} \nabla q_n = \\ -A_n \nabla q_{n-1} \nabla V_{n-1} + 2\left(\Delta V_{n-1} + \left(\nabla V_{n-1}\right)^2\right)/k_{n-1} \\ -4\nabla V_{n-1} h \nabla \overline{q_{n-1}}/k_{n-1} - 2h\Delta \overline{q_{n-1}}/k_{n-1}, \ x \in \Omega, \\ q_n \mid_{\partial\Omega} = \psi_n(x),\end{gathered} \tag{15}$$

- Solve Dirichlet boundary value problems (15) sequentially.
- Similar to the Predictor-Corrector scheme: V_{n-1} is Predictor and q_n is Corrector.
- For each n we also update the tail function V_n.
- The discrete analog of the integral (13) is:

$$\nabla v_n(x) = -\left(h\nabla q_n(x) + h\nabla \overline{q_{n-1}}(x)\right) + \nabla V_n(x), \ x \in \Omega.$$

Using $\Delta v_n = \operatorname{div}(\nabla v_n)$, calculate the approximation $c_n(x) \in C^{\alpha}\left(\overline{\Omega}\right)$ for the target coefficient $c(x)$ as

$$c_n(x) = \max\left(1, -\frac{1}{k_n^2} Re(\Delta v_n(x) + (\nabla v_n(x))^2)\right), x \in \Omega.$$

- Solve the forward problem with $c(x) := c_n(x)$.
- Update the gradient of the tail function as

$$\nabla V_n(x) = \frac{\nabla u_n(x, \overline{k})}{u_n(x, \overline{k})}.$$

The stopping criterion is developed computationally.

IMPORTANT QUESTION: HOW TO OBTAIN THE FIRST APPROXIMATION $V_0(x)$ FOR THE TAIL FUNCTION?

Let $c^*(x)$ be the EXACT solution of CIP with idealized noiseless data.
$\overline{k}$ is sufficiently large.
For all $k \geq \overline{k}$ drop the term $O(1/k)$ in the asymptotics expansion (11).
Hence,

$$u^*(x,k) \approx A^*(x)\, e^{-ik\tau^*(x)}, k \geq \overline{k}.$$

Set

$$\log u^*(x,k) = \ln A^*(x) - ik\tau^*(x) \text{ for } k \geq \overline{k}.$$

Hence,

$$\log u^*(x,k) = -ik\tau^*(x)\left(1 + O\left(\frac{1}{k}\right)\right), k \to \infty. \tag{16}$$

Drop again the term $O\left(1/k\right)$ in (16). Next, set $k=\overline{k}$.
Exact tail function

$$V^{*}\left(x\right)=\log u^{*}\left(x,\overline{k}\right)$$

Hence, we approximate the exact tail function $V^{*}\left(x\right)$ for $k=\overline{k}$ as

$$V^{*}\left(x\right)=-i\overline{k}\tau^{*}\left(x\right). \tag{17}$$

Since $q\left(x,k\right)=\partial_{k}V\left(x,k\right)$, then

$$q^{*}\left(x,\overline{k}\right)=-i\tau^{*}\left(x\right). \tag{18}$$

Set in equation (15) $k:=\overline{k},q\left(x,\overline{k}\right):=q^{*}\left(x,\overline{k}\right),V(x):=V^{*}(x)$.
Next, substitute in the resulting equation formulae (17) and (18),

$$\begin{aligned}\Delta\tau^{*}&=0\text{ in }\Omega,\\ \tau^{*}\mid_{\partial\Omega}&=i\psi^{*}\left(x,\overline{k}\right).\end{aligned} \tag{19}$$

THE FIRST APPROXIMATION FOR THE TAIL FUNCTION

$V_0(x)$:

$$V_0(x) = -i\overline{k}\tau(x), \tag{20}$$

$\tau(x)$ is the $C^{2+\alpha}\left(\overline{\Omega}\right)$ –solution of the following analog of:

$$\begin{array}{c} \Delta\tau = 0 \text{ in } \Omega, \\ \tau\mid_{\partial\Omega} = i\psi\left(x, \overline{k}\right). \end{array} \tag{21}$$

$$c_0(x) = -\frac{1}{\overline{k}^2} Re\left(\Delta V_0 + \overline{k}^2 \left(\nabla V_0\right)^2\right) \tag{22}$$

By (19), (21) and the Schauder theorem

$$\|V_0 - V^*\|_{C^{2+\alpha}\left(\overline{\Omega}\right)} \leq C\left\|\psi\left(x, \overline{k}\right) - \psi^*\left(x, \overline{k}\right)\right\|_{C^{2+\alpha}(\partial\Omega)}. \tag{23}$$

By (22) and (23)

$$\|c_0 - c^*\|_{C^{\alpha}\left(\overline{\Omega}\right)} \leq C\left\|\psi\left(x, \overline{k}\right) - \psi^*\left(x, \overline{k}\right)\right\|_{C^{2+\alpha}(\partial\Omega)}. \tag{24}$$

CONCLUSIONS:

- By (24) we obtain a good approximation for the target coefficient c^* already on the zero iteration of our method.
- This is the **global convergence**.
- Still, numerical experience tells us that we need to do more iterations.
- The stopping criterion is selected computationally.

GLOBAL CONVERGENCE THEOREM

σ is the level of the error in the boundary data,

$$\|\psi_n - \psi^*\|_{C^{2+\alpha}(\partial\Omega)} \leq \sigma,$$

h is the step size in k, $h = k_{n-1} - k_n$
The error parameter:

$$\eta = h + \sigma.$$

Theorem 4 (global convergence). *Assume that the first approximation $V_0(x)$ for the tail function is constructed as above. subsection 5.3. Let numbers $\overline{k} > \underline{k} > 1$. Then there exist sufficiently small numbers*

$$a = \overline{k} - \underline{k}, \theta > 0$$

and a sufficiently large number M,

$$M > 1,$$

all numbers depend only on some parameters of the problem, such that if the error parameter $\eta = h + \sigma$ is so small that

$$\eta \in (0, \eta_0), \ \eta_0 = \frac{\theta}{M^{20N-12}}, \tag{25}$$

then for $n \in [1, N]$

$$\|c_n - c^*\|_{C^\alpha(\overline{\Omega})} \leq M^{10n-6}\eta, \tag{26}$$

or, by (25)

$$\|c_n - c^*\|_{C^\alpha(\overline{\Omega})} \leq \sqrt{\eta}. \tag{27}$$

- Thus, the optimal number of iterations is $N = 1$. However, our numerical experience tells us that we need $N = 7 - 8$.

- This is the **GLOBAL CONVERGENCE**, since (25)-(27) guarantee that for $n \in [1, N]$ all functions c_n are located in a sufficiently small neighborhood of the exact solution c^*.

PERFORMANCE OF THE GLOBALLY CONVERGENT NUMERICAL METHOD ON EXPERIMENTAL DATA

D.-L. Nguyen, M.V. Klibanov, A.E. Kolesov, M.A. Fiddy and H. Liu

- The data were collected by Professor M.A. Fiddy (Optoelectronic Center of our university)
- Device: Virtual Network Analyzer.
- Frequency range: 1GHz-10GHz.
- Target application: Imaging of spatially distributed dielectric constants of explosives, including improvised explosive devices and antipersonnel land mines
- Because of this application, only the **BACKSCATTERING** data were collected using the **SINGLE** location of the source.

COMPETING EXPERIMENTAL DATA OF FRESNEL INSTITUTE (FRANCE)

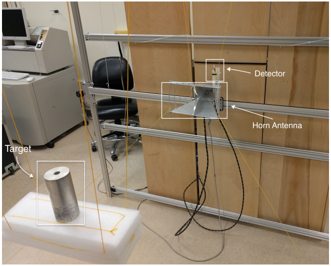
- Anechoic chamber was used
- Their data are not corrupted by unwanted reflections from objects in the room
- Thus, their data match simulated data very well.
- Inclusions/background contrasts do not exceed 2.3

- Over-determined data: many sources

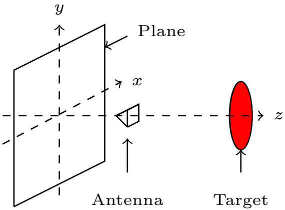
OUR DATA FIT THE DESIRED APPLICATION

- No anechoic chamber
- Regular room: unwanted reflections are in place
- Our data have a large discrepancy with simulated data
- Pre-processing of the data is necessary

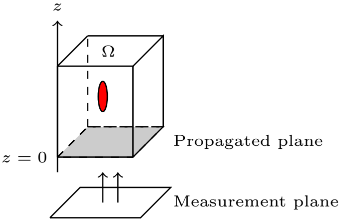
- Non-overdetermined



(a)



(b)



(c)

Figure 3: *a) A photograph of the experimental arrangement. b) Schematic diagram of measurements. c) Schematic diagram of data propagation*

DATA PROPAGATION

- This is a very important step of the pre-processing procedure
- The measurement plane is too far from targets to be imaged: ≈1 meter
- We need to move the data closer to the targets
- Angular spectrum representation method
- $g(\mathbf{x}, k)$ is the experimental data on the measurement plane

$$P_m = \{\mathbf{x} : -5 < x < 5, -5 < y < 5, z = b\}.$$

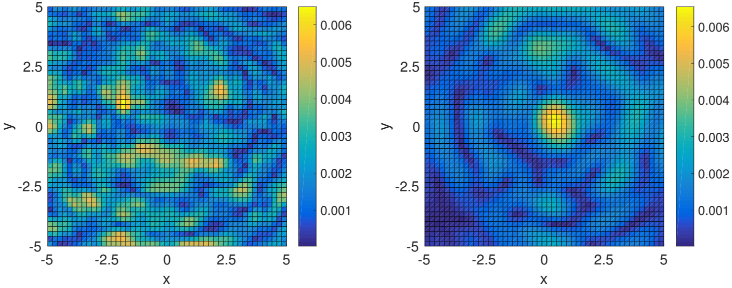
- $f(\mathbf{x}, k)$ is the propagated data on the propagated plane

$$P_p = \{\mathbf{x} : -5 < x < 5, -5 < y < 5, z = a\}.$$

- Then

$$\hat{g}(k_x, k_y, k) = \iint_{\mathbb{R}^2} g(x, y, b, k) e^{-i(k_x x + k_y y)} dx dy,$$

$$f(x, y, a, k) = \iint_{k_x^2 + k_y^2 < k^2} \hat{g}(k_x, k_y, k) e^{i[k_x x + k_y y - k_z(a-b)]} dk_x dk_y.$$



(a) (b)

Figure 4: *a) Modulus of measured data at a certain frequency k. b) Modulus of propagated data*

CENTRAL FREQUENCY: 2.6 GHz

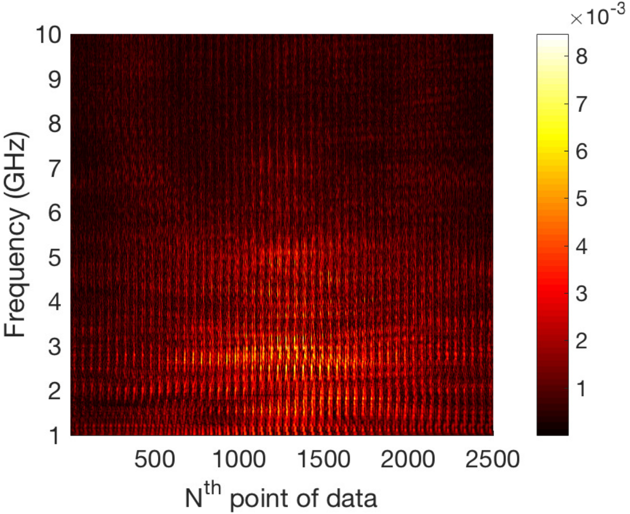


Figure 5: *Central frequency of the signal*

TABLES OF MEASURED AND COMPUTED DIELECTRIC CONSTANTS

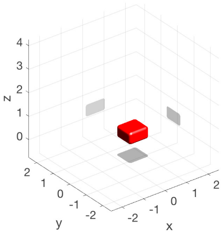
Table 1: Targets

Object ID	Name of the target
1	A piece of yellow pine
2	A piece of wet wood
3	A geode
4	A tennis ball
5	A baseball

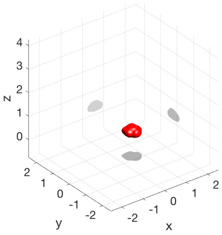
Table 2: Measured and computed dielectric constant of the targets

Obj. ID	Measured ε_r (error)	Computed ε_r	Relative error
1	5.30 (1.6%)	5.44	2.6%
2	8.48 (4.9%)	7.60	10.3%
3	5.44 (1.1%)	5.55	2.0%
4	3.80 (13%)	4.00	5.2%
5	not available	4.76	n/a

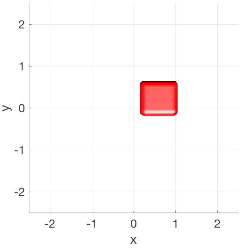
IMAGES



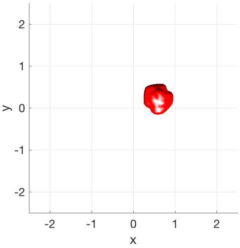
(a) Target 1



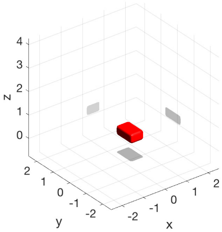
(b) Reconstructed target 1



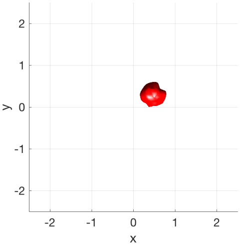
(c) Top view of (a)



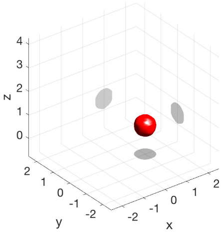
(d) Top view of (b)



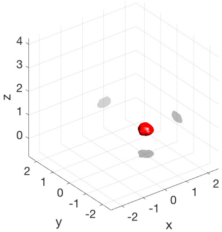
(a) Target 2



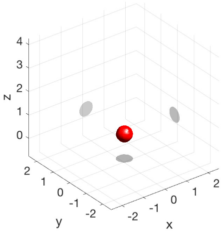
(b) Reconstructed target 2



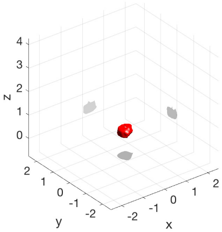
(c) Target 3



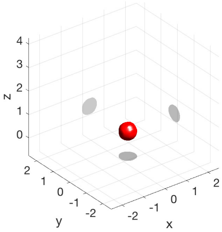
(d) Reconstructed target 3



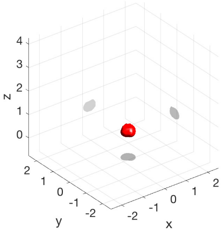
(a) Target 4



(b) Reconstructed target 4



(c) Target 5



(d) Reconstructed target 5