PHY2053 Lecture 17

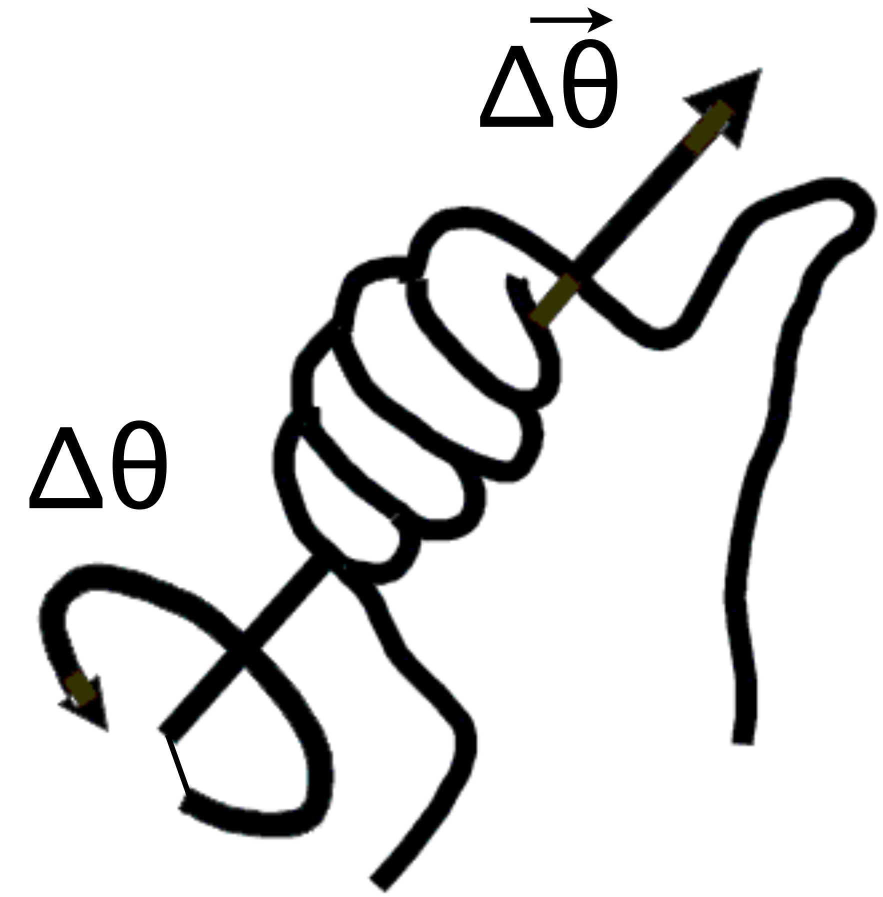
Ch. 8.4, 8.6 (skip 8.5) Rotational Equilibrium,
Rotational Formulation of Newton's Laws

Linear vs Rotational Variables Transformation Table

Linear Variable	→ Rotational Variable
distance	angle
velocity	angular velocity
acceleration	angular acceleration
mass [inertia]	moment of inertia
momentum $p = mv$	angular momentum, $L = I\omega$
force, $F = ma$	torque, $\tau = I\alpha$
$K_{trans} = \frac{1}{2}mv^2$	$K_{rotation} = \frac{1}{2}I\omega^2$

"Right Hand Rule" for rotational systems

- change of angle ($\Delta\theta$) is a vector along the axis of rotation, and its direction is determined by the right hand rule:
- orient your right hand so that the fingers point in the direction in which the angle is changing; your thumb points in the direction of the rotational vector



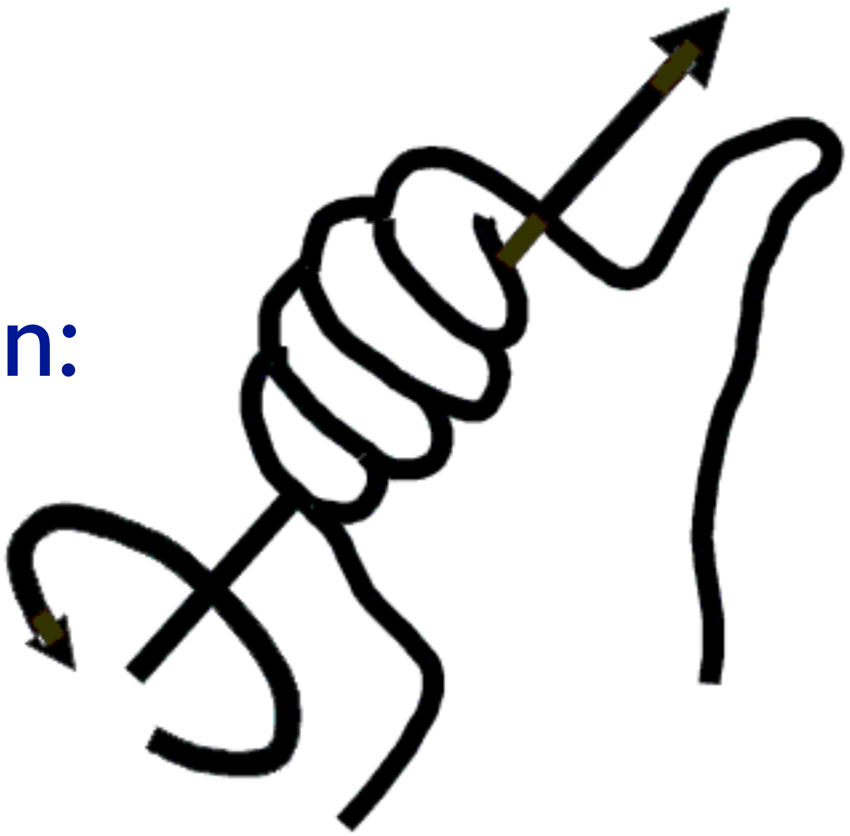
Reminder: from angle to angular acceleration

- instantaneous angular velocity:

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\theta}}{\Delta t}$$

- instantaneous angular acceleration:

$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\omega}}{\Delta t}$$



Torque

- The equivalent of force, for rotational systems (somewhat hand-waving derivation of equivalence)

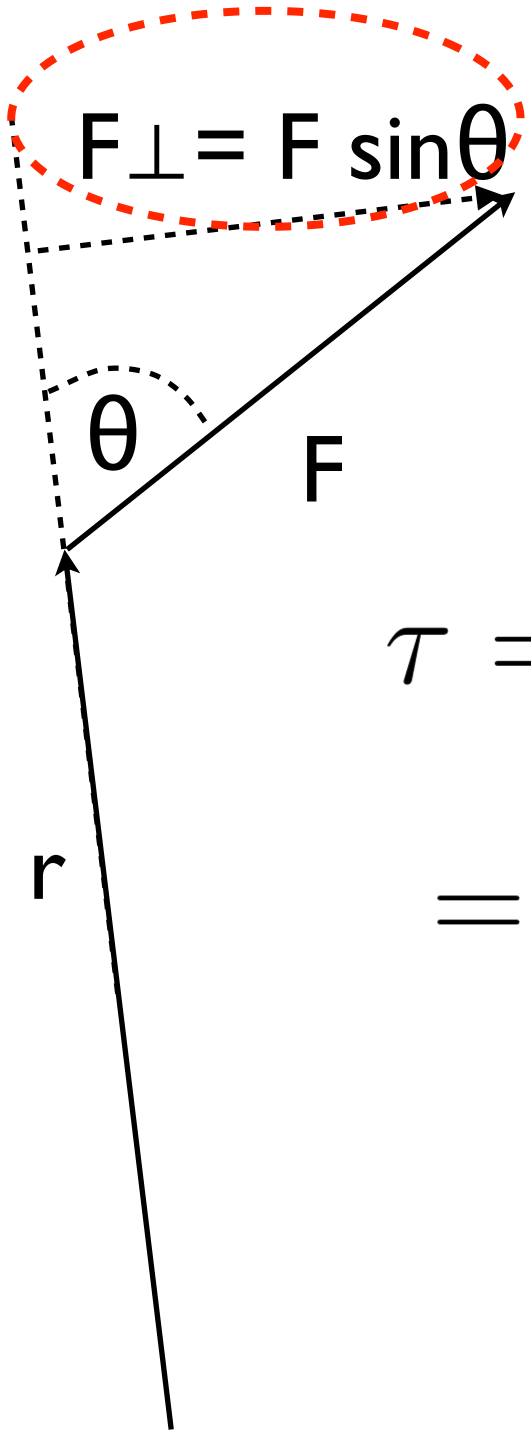
$$F_i = m_i a_i \quad | r_i, \sum_i \quad \rightarrow \quad \sum_i r_i F_i = \sum_i m_i r_i a_i$$

$$[a_i = \alpha r_i] \quad \sum_i r_i F_i = \sum_i m_i r_i^2 \alpha$$

torque = momentum of inertia × angular acceleration

- not quite so trivial, need to account for the orientation of force wrt "lever arm" of object i, "useful force"

$$\tau_i = r_i F_i \sin(\alpha_i)$$

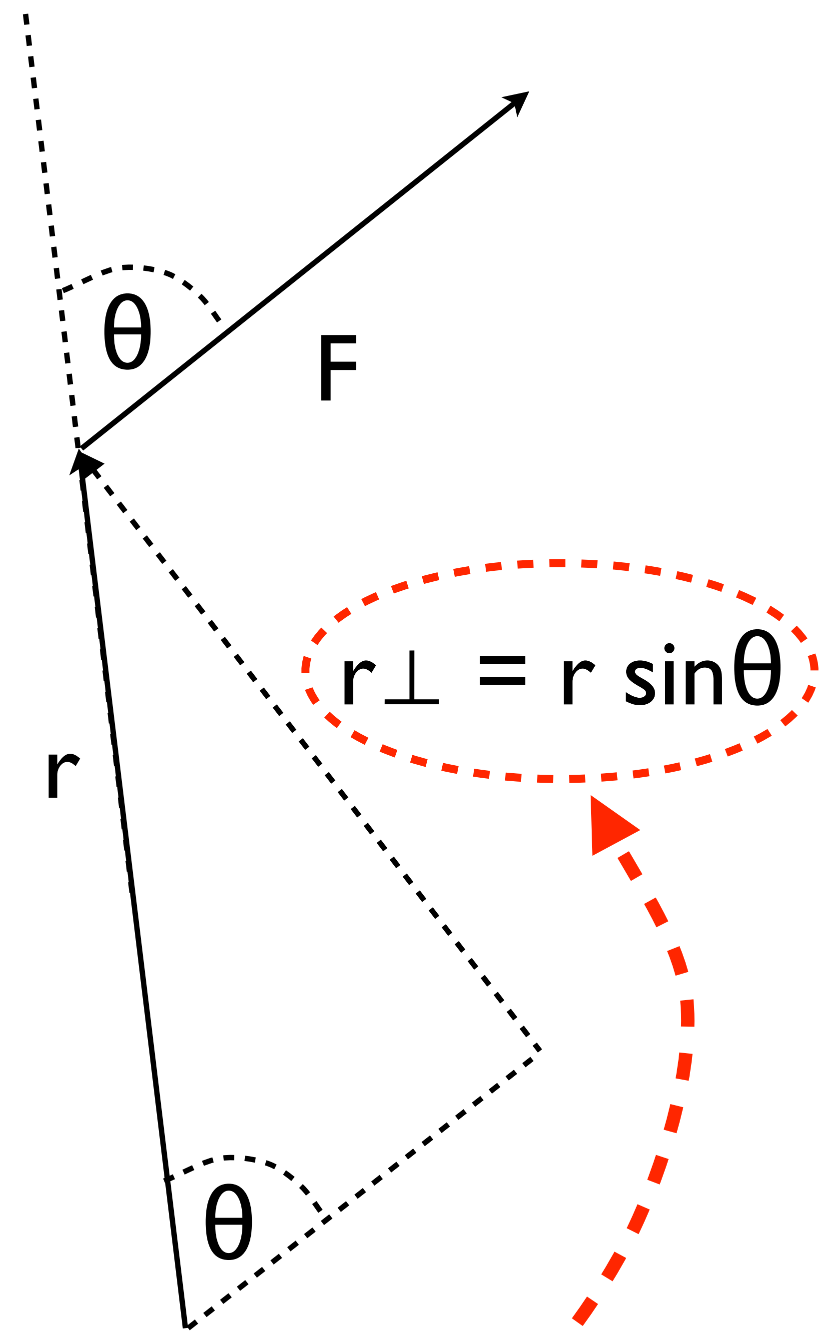


"Perpendicular force"

$$\tau = (F \sin \theta) r = F_{\perp} r$$

$$= F (r \sin \theta) = F r_{\perp}$$

"lever arm"



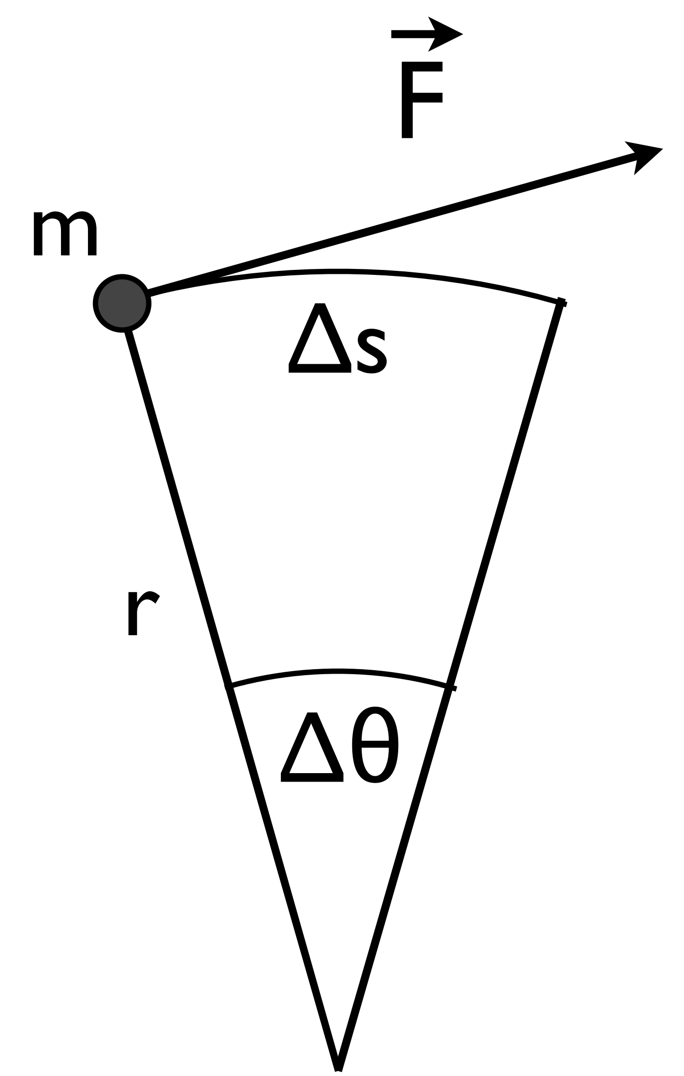
- for the same force, the torque is directly proportional to the lever arm (projection of r perpendicular to F)

Work done by torque

- Start from definition of work, for rotational systems:

$$W = F\Delta s; \quad [\Delta s = r\Delta\theta]$$

$$W = Fr\Delta\theta = \tau\Delta\theta$$

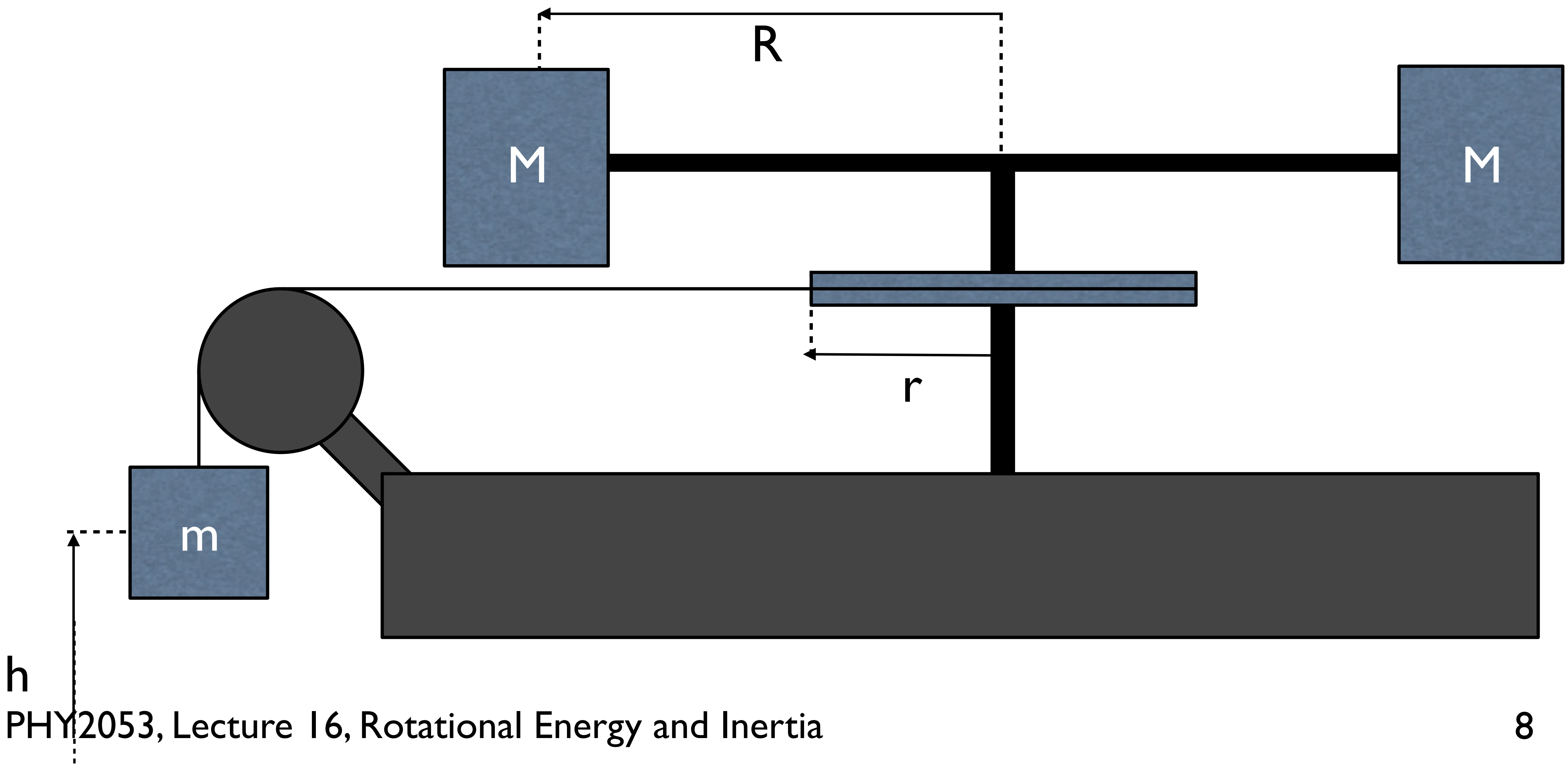


- Hooke's law, for rotational systems (torsion spring):

$$\tau = -\kappa\Delta\theta$$

Unusual Atwood's Machine

In the machine below, the small weight (mass m) is connected via an ideal pulley to the massless horizontal wheel of radius r. Ignore the moments of inertia of the connecting rods; the two masses M are both R away from the axis of rotation. What is the velocity of mass m as it falls h from its initial position, where it was at rest?



Discussion: Unusual Atwood's Machine, via energy conservation

Utilize $E_i = E_f$ $\qquad E_i = K_i + U_i \rightarrow U_i = mgh$

$\hookrightarrow$ from rest, $K_i = 0$

$U_f = 0$

$$E_f = U_f + K_f = 0 + \frac{mv^2}{2} + \frac{I\omega^2}{2}$$

, however $v = \omega r$ ← radius of pulley

$$\omega = \frac{v}{r}$$

$$E_f = \frac{mv^2}{2} + \frac{2MR^2}{2} \cdot \left(\frac{v}{r}\right)^2$$

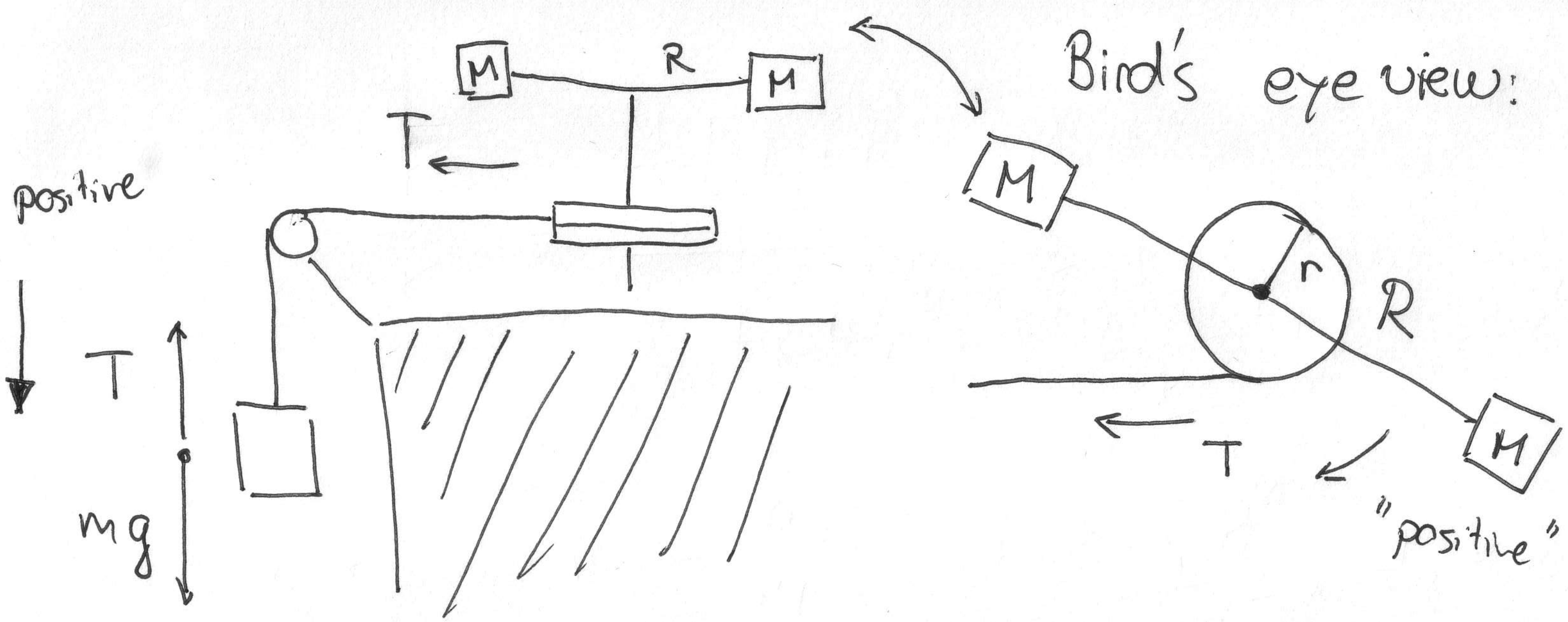
$$= \frac{mv^2}{2} + MR^2 \cdot \frac{v^2}{r^2} = \frac{mv^2}{2} + M\frac{R^2}{r^2}v^2 = v^2\left[\frac{m}{2} + M\frac{R^2}{r^2}\right]$$

$$E_i = E_f \rightarrow U_i = K_f \rightarrow mgh = v^2\left[\frac{m}{2} + M\frac{R^2}{r^2}\right]$$

$$mgh = mv^2\left[\frac{1}{2} + \frac{MR^2}{mr^2}\right]$$

$$v^2 = \frac{gh}{\frac{1}{2} + \frac{MR^2}{mr^2}} \rightarrow v = \sqrt{\frac{gh}{\frac{1}{2} + \frac{MR^2}{mr^2}}}$$

Discussion: Unusual Atwood's Machine, via force, tension and torque



Newton II for the weight:

$$mg - T = ma \quad [\Sigma F = ma]$$

Rotational Newton II for spindle: $[\Sigma \tau = I\alpha]$

$$rT = 2MR^2 \cdot \alpha \, ; \quad \alpha = \frac{a}{r}$$

$$rT = 2MR^2 \cdot \frac{a}{r} \quad \Big| \cdot \frac{1}{r}$$

$$T = \frac{2MR^2}{r^2} \cdot a$$

$$mg - \frac{2MR^2}{r^2} a = ma$$

$$mg = a\left[\frac{2MR^2}{r^2} + m\right]$$

$$a = \frac{mg}{m + \frac{2MR^2}{r^2}} = \frac{g}{1 + \frac{2MR^2}{mr^2}}$$

Discussion: Unusual Atwood's Machine, via force, tension and torque 2

we know that for constant acceleration,

$$v^2 = 2a\Delta x \rightarrow v_f^2 = 2ah$$

$$v_f^2 = 2h \cdot \frac{g}{1 + \frac{2MR^2}{mr^2}} = \frac{2gh}{1 + \frac{2MR^2}{mr^2}} \cdot \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{gh}{\frac{1}{2} + \frac{MR^2}{mr^2}}$$

$$v_f = \sqrt{\frac{gh}{\frac{1}{2} + \frac{MR^2}{mr^2}}}$$

Same answer as when using energy conservation.

Statics, Rotational Equilibrium

- Condition 1: Newton's 1st law: a stationary object that has no net force acting on it remains stationary

$$\sum_i \vec{F}_i = 0$$

- Condition 2: For the object not to start rotating about an axis, the net torque on the object must be zero

$$\sum_i \vec{\tau}_i = 0$$

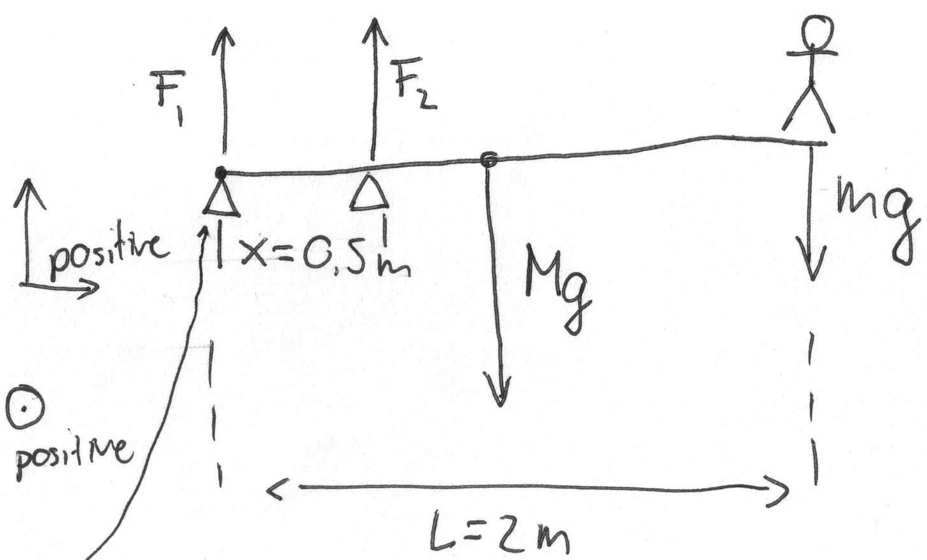
- Important: Conditions of static equilibrium, once established for one axis, apply to any arbitrary axis

Example: Springboard

- A diver weighing 80 kg is standing on the end of a 2 m long springboard which weighs 240 kg. The springboard is supported by two springs on the other end. One spring is at the very end of the board, and the other is 0.5 m toward the center of the board. Compute the forces generated by the springs if the system is in static equilibrium. Are the springs compressed or stretched?

Discussion: Springboard

First, we need a schematic with all the forces. I will make a guess about their directions, we will find out if it was correct [positive values mean correct guess]



Choice of reference point for torque is arbitrary; I pick the end far from the diver

$\sum \vec{F} = 0$:

$$F_1 + F_2 - Mg - mg = 0 \quad (*)$$

$\sum \vec{\tau} = 0$:

$$F_1 \cdot 0 + F_2 x - Mg\frac{L}{2} - mg\,L = 0$$

$$F_2 x = \left(\frac{M}{2} + m\right) g L$$

$$F_2 = \left(\frac{M}{2} + m\right) g \frac{L}{x} \qquad \frac{L}{x} = \frac{2m}{0{,}5m} = 4$$

Discussion: Springboard 2

$$F_2 = \Big(\underbrace{\frac{240\,\text{kg}}{2}}_{120} + 80\,\text{kg}\Big)\cdot g \cdot 4 = 4\cdot 200\overset{\text{kg}}{g} = 800\overset{\text{kg}}{g} \simeq \underline{8\,\text{kN}}\,//$$

F_2 is positive, I guessed the right direction. Spring 2 is therefore compressed.

From (*): $F_1 = -F_2 + Mg + mg = (-800g + 240g + 80g)\,\text{kg}$

$$= [-480\,\text{kg}]\cdot g \simeq \underline{-4.8\,\text{kN}}\,//$$

F_1 is negative, I guessed the direction wrong: F_1 is a downward force, and spring 1 is therefore stretched.

Angular Momentum

- Momentum-impulse theorem:

$$F_i \Delta t = m_i \Delta v_i \quad |r_i, \sum_i \quad \rightarrow \quad \sum_i r_i F_i \Delta t = \sum_i m_i r_i \Delta v_i; \quad [v_i = \omega r_i]$$

$$\sum_i \tau_i \Delta t = \sum_i r_i F_i \Delta t = \sum_i m_i r_i^2 \Delta\omega = I \Delta\omega$$

- by analogy with p = mv, define angular momentum:

$$\vec{L} = I\vec{\omega}$$

- angular momentum-impulse theorem:

$$\vec{\tau}\Delta t = \Delta\vec{L} = I\Delta\vec{\omega}$$

$$\vec{\tau} = 0 \quad \rightarrow \quad \vec{L} = \vec{const}$$

Example: "Angular Collision"

- A disk of mass m_1 and radius R is spinning with an angular frequency ω_i . A second disk, of mass $3m_1$ is dropped onto disk 1 so that they share the same axis of rotation. After a brief period of friction, the two disks are rotating with the same angular frequency ω_f.
- What is the ratio ω_f/ω_i ?
- What fraction of the initial kinetic energy was wasted through friction?

Discussion: Angular Collision

The angular momentum of the system is unchanged:

$$L_i = L_f \qquad L_i = I_1 \omega_i \qquad L_f = [I_1 + I_2]\omega_f$$

$$[I_1 + I_2]\omega_f = I_1 \omega_i \rightarrow \omega_f = \omega_i \frac{I_1}{I_1 + I_2}$$

$$I_1 = m_1 R^2 \cdot \frac{1}{2} \ ; \quad I_2 = m_2 R^2 \cdot \frac{1}{2} \ ; \quad m_2 = 3 m_1$$

$$\omega_f = \omega_i \cdot \frac{m_1 \cdot \frac{R^2}{2}}{m_1 \frac{R^2}{2} + m_2 \frac{R^2}{2}} = \omega_i \cdot \frac{m_1}{m_1 + m_2} = \omega_i \cdot \frac{m_i}{4 m_i} = \frac{\omega_i}{4}$$

The initial and final kinetic energies are:

$$K_i = \frac{I_1}{2} \omega_i^2 \qquad K_f = (I_1 + I_2) \cdot \frac{1}{2} \cdot \omega_f^2$$

Discussion: Angular Collision 2

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}[I_1+I_2]\omega_f^2}{\frac{1}{2} I_1 \omega_i^2} = \left(\frac{I_1+I_2}{I_1}\right)\frac{\omega_f^2}{\omega_i^2}$$

we know $\omega_f = \frac{\omega_i}{4}$, $I_1 = m_1\frac{R^2}{2}$, $I_2 = 3m_1\frac{R^2}{2}$

$$\frac{K_f}{K_i} = \left[\frac{I_1}{I_1} + \frac{I_2}{I_1}\right]\cdot\left(\frac{\omega_f}{\omega_i}\right)^2 = \left[1+\frac{3m_1\frac{R^2}{2}}{m_1\frac{R^2}{2}}\right]\cdot\left(\frac{1}{4}\right)^2 =$$

$$= [1+3]\cdot\frac{1}{16} = \frac{4}{16} = \frac{1}{4} = 25\%$$

75% of the initial kinetic energy went into friction as the second disk is dropped onto the first.

The vector nature of torque and angular momentum

Next Lecture