Physical Interpretation of the Cole-Cole Model in Viscoelasticity

Wubing Deng and Igor B. Morozov

University of Saskatchewan, wubing.deng@usask.ca, igor.morozov@usask.ca

Summary

The basis of the popular Cole-Cole rheological model in viscoelasticity is investigated by using the first-principle physics. The Cole-Cole model is usually, viewed a convenient way to fit the observed frequency-dependent attenuation and velocity-dispersion spectra, but its time-domain and numerical formulations are excessively complex and physically inconsistent (for example, requiring special mathematical “operators” such as fractional derivatives and memory variables). However, we show that the Cole-Cole spectra can be naturally explained by standard Lagrangian mechanics with nonlinear energy dissipation, and therefore no such mathematical extensions are required. For linear dissipation, the Lagrangian model extends to many other rheologies, such as all viscoelastic linear solids and Biot’s and double-porosity poroelasticity. The new model also provides straightforward ways for numerical modeling of waves and any other deformations of media with Cole-Cole attenuation spectra.

Introduction

Seismic-wave attenuation spectra often exhibit broad peaks with frequency, which are caused by various elastic or inelastic mechanisms due to the complexity of the Earth. The mechanisms of internal friction include grain sliding, poroelasticity and multiple-porosity effects, solid viscosity, a broad group of wave-induced fluid-flow processes in mesoscopic heterogeneities, and numerous scattering effects such as produced by finely-layered elastic structures. These attenuation mechanisms are often phenomenologically described by linear viscoelastic (VE) models, such as the standard linear solid (SLS) for attenuation within a narrow frequency range and the generalized SLS (GSLS) or Cole-Cole models describing broader frequency bands of seismic attenuation. These VE models are broadly used for interpreting experimental data and for numerical forward modeling of seismic waveforms.

While comparing several types of mathematical VE models, several authors (Chapter 3 in Wang, 2008; Picotti and Carcione, 2017) argue that the Cole-Cole model gives a better description of physics of the attenuation and dispersion. However, the physical reasons for preferring the Cole-Cole models have still not been clearly stated, and this preference is usually based on empirical observations, such as variable and often broader attenuation ($Q^{-1}(f)$) peaks predicted by the Cole-Cole model. Usually, a larger number of Maxwell’s bodies is required in order to fit an observed spectrum by a GSLS than by a Cole-Cole model. Thus, the Cole-Cole model seems to better represent an individual observed peak in $Q^{-1}(f)$, which is often viewed as a “relaxation mechanism”. The Cole-Cole model can also be fit to $Q^{-1}(f)$ spectra varying with frequency steeper than those for an SLS. Nevertheless, these are still not “physical” reasons but preferences for the shapes of the observed spectra.

Thus, it should be useful to identify the physical principles of the GSLS and Cole-Cole models and get a clearer understanding about what kinds of the “relaxation mechanisms” may be identified in these models. The procedure for constructing physical theories is well known in theoretical physics (e.g., Landau and Lifshitz, 1986), and in this paper, we employ this procedure in order to suggest the physical origins of the GSLS and Cole-Cole models used in seismology and materials science.

In the following sections, we compare two definitions of the Cole-Cole model. In the first approach, the Cole-Cole model of order η (which is equivalent to an SLS for $\eta = 1$) is interpreted as a mathematical relation between the time- or frequency-dependent stress and strain measured in some experiments. This mathematical relation can be described empirically and without any knowledge of the physical interactions involved. By contrast, the second approach focuses on finding the physical laws for a medium that would lead to the Cole-Cole (or SLS) relations between the strain and stress. This approach reveals that the case $\eta \neq 1$ requires a nonlinearity of internal friction, such as nonlinear viscosity.

Cole-Cole Model

The general time-retarded stress-strain relation can be presented by using partial derivatives in time as (Tschoegl, 1989)

$$\sum_{i=0}^{h} \tau_\sigma^i \frac{\partial^i \sigma}{\partial t^i} = M_0 \sum_{j=0}^{k} \tau_\varepsilon^j \frac{\partial^j \varepsilon}{\partial t^j}, \tag{1}$$

where M_0 is the static (relaxed) modulus. τ_σ and τ_ε are the stress -relaxation and strain-retardation times, respectively. For linear VE models such as the SLS and GSLS, $h = k = 1$ are taken in eq. (1). For the Cole-Cole model, the corresponding differential stress-strain relation is (Tschoegl, 1989)

$$\sigma + \tau_\sigma^\eta \frac{\partial^\eta \sigma}{\partial t^\eta} = M_0 \left(\varepsilon + \tau_\varepsilon^\eta \frac{\partial^\eta \varepsilon}{\partial t^\eta} \right), \tag{2a}$$

where η is a parameter ranging from 0 to 2, operator $\partial^\eta f \equiv \mathrm{F}^{-1}\left[\left(-i\omega \right)^\eta \mathrm{F}\left[f \right] \right]$ denotes the fractional derivative of order η, and F and F^{-1} are the forward and inverse Fourier transforms, respectively. The complex-valued modulus $M(\omega)$ is obtained by transforming eq. (2a) into the frequency domain (Jones, 1986):

$$\frac{\sigma(\omega)}{\varepsilon(\omega)} \equiv M(\omega) = M_0 \frac{1 + \left(-i\omega\tau_\varepsilon \right)^\eta}{1 + \left(-i\omega\tau_\sigma \right)^\eta}. \tag{2b}$$

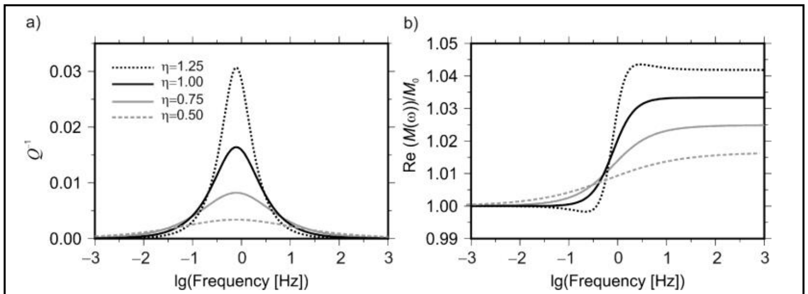

Figure 1. Complex moduli for a Cole-Cole body with different values of order η. a) Inverse Q, b) modulus dispersion.

In eq. (2b), we use a negative sign in $-i\omega\tau$ corresponding to the selection of the form of $\exp\left(-i\omega t \right)$ for harmonic oscillations, where t is the time. When $\eta = 1$, Eq. (2b) describe the standard linear solid (SLS).

The dependences of the modulus M and attenuation Q^{-1} on frequency for the Cole-Cole model with several values of η are shown in Figure 1. For fixed τ_σ, τ_ε, and consequently the frequency of the attenuation peak in the Cole-Cole model, the magnitude of modulus dispersion and attenuation increases with increasing η. For $\eta > 1$, the interval of strong attenuation and positive dispersion is narrow (about one octave in frequency) and is flanked by intervals of weak negative dispersion (black dotted lines in Figure 1).

Elastic Medium with Linear Viscous Friction (General Linear Solid)

In eqs. (2), the strain $\varepsilon(t)$ and stress $\sigma(t)$ may have various meanings and can even be taken at different points within the body. However, in order to understand this ratio as a “rheological law” for the material, we need to find the actual “causal” (physical) relations between $\varepsilon(t)$ and $\sigma(t)$. More generally and precisely, we need to find the mechanical equations of motion governing the displacement, $\mathbf{u}(\mathbf{x},t)$ of every point in the medium. To perform this task, we need to specify the procedure of measurement of $\varepsilon(t)$ and

$\sigma(t)$ more completely. Let us assume that $\varepsilon(t)$ and $\sigma(t)$ represent measurements conducted at the same point in a uniform and isotropic medium, and try determining the mechanical properties of this medium.

In classical mechanics, the dynamics of the medium can be described by giving its Lagrangian function L (kinetic and elastic energy densities) plus the dissipation pseudo-potential D if the medium is lossy (Landau and Lifshitz, 1986). In a model called the General Linear Solid (GLS), Morozov and Deng (2016) proposed the following forms of these functions:

$$\begin{cases} L = \int_V \left[\frac{1}{2} \dot{\mathbf{u}}_i^T \boldsymbol{\rho} \dot{\mathbf{u}}_i - \left(\frac{1}{2} \boldsymbol{\Delta}^T \mathbf{K} \boldsymbol{\Delta} - \tilde{\boldsymbol{\varepsilon}}_{ij}^T \boldsymbol{\mu} \tilde{\boldsymbol{\varepsilon}}_{ij} \right) \right] dV, \\ D = \int_V \left(\frac{1}{2} \dot{\mathbf{u}}_i^T \mathbf{d} \dot{\mathbf{u}}_i + \frac{1}{2} \dot{\boldsymbol{\Delta}}^T \boldsymbol{\eta}_K \dot{\boldsymbol{\Delta}} + \dot{\tilde{\boldsymbol{\varepsilon}}}_{ij}^T \boldsymbol{\eta}_\mu \dot{\tilde{\boldsymbol{\varepsilon}}}_{ij} \right) dV. \end{cases} \quad (3)$$

Here, V is the volume of the body, the overdots denote the time derivatives, the spatial coordinates are indicated by subscripts i and ^{T} denotes the matrix transpose. Model vector $\mathbf{u}$ contains $N \geq 1$ elements, of which the first element is the observable displacement of the rock and the rest correspond to internal degrees of freedom, such as filtration fluid flows or displacements of grain assemblages. By using standard relations from the theory of elasticity, the bulk strain $\boldsymbol{\Delta}$ and deviatoric strain $\tilde{\boldsymbol{\varepsilon}}_{ij}$ are derived from vector $\mathbf{u}$ (Morozov and Deng, 2016). The mechanical properties of the material are described by the constitutive parameter matrices in eqs. (3): $\mathbf{K}$ and $\boldsymbol{\mu}$ are the bulk and shear moduli, $\boldsymbol{\eta}_k$ and $\boldsymbol{\eta}_\mu$ are the corresponding bulk and shear viscosities, and matrix $\mathbf{d}$ describes Darcy friction of the pore fluid. In this paper, we do not consider pore fluids and therefore set $\mathbf{d} = \mathbf{0}$.

By using the Euler-Lagrange equations, all equations of motion are obtained from eqs. (3):

$$\begin{cases} \boldsymbol{\rho} \ddot{\mathbf{u}}_i = -\mathbf{d} \dot{\mathbf{u}}_i + \partial_j \boldsymbol{\sigma}_{ij} \\ \boldsymbol{\sigma}_{ij} = \mathbf{K} \boldsymbol{\Delta} \delta_{ij} + 2 \boldsymbol{\mu} \tilde{\boldsymbol{\varepsilon}}_{ij} + \boldsymbol{\eta}_K \dot{\boldsymbol{\Delta}} \delta_{ij} + 2 \boldsymbol{\eta}_\mu \dot{\tilde{\boldsymbol{\varepsilon}}}_{ij} \end{cases}. \quad (4)$$

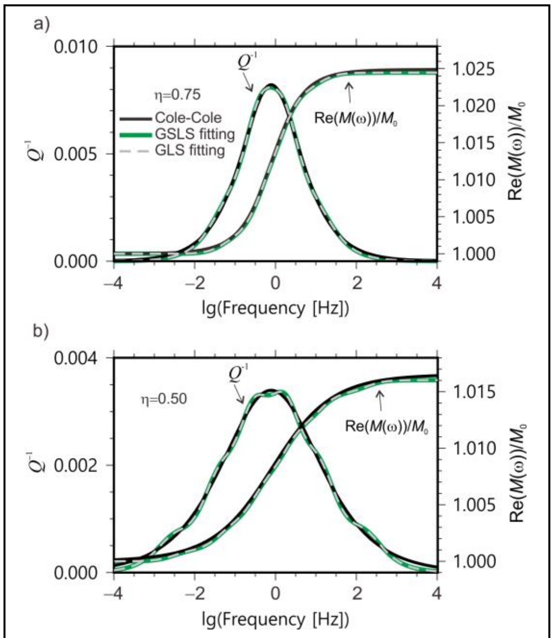

Figure 2. GLS and GSLS approximation of Cole-Cole model with a) $\eta = 0.75$ and b) $\eta = 0.50$.

Eqs. (4) can be used to describe all types of media while using strictly macroscopic variables. For example, Biot's poroelasticity is obtained by taking $N = 2$ and setting $\boldsymbol{\eta}_K \equiv \mathbf{0}$, $\boldsymbol{\eta}_\mu \equiv \mathbf{0}$ and $\mathbf{d} \neq \mathbf{0}$. All types of VE models (such as SLS or GSLS) are obtained by setting $\boldsymbol{\eta}_K \neq \mathbf{0}$, $\boldsymbol{\eta}_\mu \neq \mathbf{0}$ and $\mathbf{d} \equiv \mathbf{0}$; and finally, an elastic medium in any of these cases is obtained by taking $\boldsymbol{\eta}_K \equiv \mathbf{0}$, $\boldsymbol{\eta}_\mu \equiv \mathbf{0}$ and $\mathbf{d} \equiv \mathbf{0}$ (Morozov and Deng, 2016). Eqs. (3)-(4) can be used to model propagation of any waves or the behavior of a rock sample in any experiment (Morozov and Deng, 2016a).

For broad attenuation peaks ($\eta \leq 1$), the Cole-Cole spectra can be predicted with good accuracy by GLS models (eqs. (3)-(4)) with multiple internal variables. For example, Figure 2 shows such an approximation for $\eta = 0.5$ and 0.75 by GLS (linear-viscosity) media with $N = 6$ and 5, respectively. The required dimension N increases with decreasing Cole-Cole model parameter η. Such prediction of the Cole-Cole spectra by the GLS rheology (eq. (3)) is equivalent to formally approximating these spectra by those of a GSLS. However, the important difference of the present approach is in providing a rigorous physical meaning for all variables and complete differential equations of motion (4) without hypothetical "material memory" and frequency-dependent material properties.

Nonlinear Viscous Friction (Cole-Cole)

It is often considered physically beneficial (Picotti and Carcione, 2017; Szewczyk et al., 2017) to approximate the observed $\sigma(\omega)/\varepsilon(\omega)$ ratios by a single Cole-Cole spectrum rather than by a GSLS with multiple elements. However, this reduction of the number of variables is achieved by using multiple orders of derivatives in the Hooke's law (eq. (1)) or fractional derivatives (eq. (2a)), which in fact represent integral, "non-instantaneous" operators. Such operators are highly undesirable in the mechanical framework (eqs. (3)), in which the functions L and D would then become dependent on multiple orders of derivatives or on time integrals (Landau and Lifshitz, 1986). The key mechanical concepts such as the elastic energy and energy dissipation rate would become double integrals in time that would be difficult to measure and interpret. Nevertheless, a simple a natural alternative to such complicated mechanics exists in the form of the model (3) with a nonlinear dissipation function D (Coulman et al., 2013):

$$D=\int_V \frac{1}{\tau_r^2}\left[\frac{1}{2}\left(\tau_r \breve{\boldsymbol{\Delta}}^{1-\boldsymbol{\nu}_\Delta,\boldsymbol{\nu}_\Delta}\right)^T \boldsymbol{\eta}_K \breve{\boldsymbol{\Delta}}^{1-\boldsymbol{\nu}_\Delta,\boldsymbol{\nu}_\Delta}\tau_r + \left(\tau_r \breve{\boldsymbol{\varepsilon}}^{1-\boldsymbol{\nu}_\varepsilon,\boldsymbol{\nu}_\varepsilon}\right)^T \boldsymbol{\eta}_\mu \breve{\boldsymbol{\varepsilon}}^{1-\boldsymbol{\nu}_\varepsilon,\boldsymbol{\nu}_\varepsilon}\tau_r\right] dV \ . \tag{5}$$

The construction of this dissipation function is based on three simple principles: A) isotropy and dependence on the invariants of the strain and strain-rate tensors (Landau and Lifshitz, 1986); B) power-law dependence on strains and strain rates, so that with $\nu_\Delta = \nu_\varepsilon = 1$, the quadratic function Δ (second eq. (3)) is obtained; and C) this function D is linear with respect to the total strain amplitude. Parameter τ_r is the reference time scale that maintains the correct dimensionality of D, and it can be set equal 1 s by selecting the appropriate units for viscosity (Coulman et al., 2013). Parameters ν_Δ and ν_ε are the bulk and shear exponents for non-Newtonian viscosity, respectively, and notation $\breve{\boldsymbol{\Delta}}^{1-\boldsymbol{\nu}_\Delta,\boldsymbol{\nu}_\Delta}$ means a vector whose l-th element equals $\breve{\Delta}_l^{a,b} \equiv \Delta_l^a \dot{\Delta}_l^b$, and similarly for $\breve{\boldsymbol{\varepsilon}}^{a,b}$.

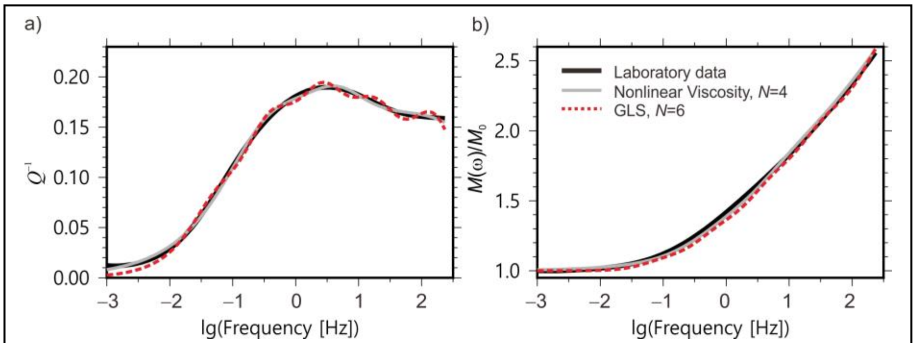

Figure 3. Approximation of a) bitumen sand's P-wave attenuation and b) modulus dispersion by using different model.

Clearly, the non-quadratic function D in eq. (5) is only one possible example of non-Newtonian viscosity. Other forms of D can be considered yielding similar results. In particular, we might relax the requirement C) above by selecting more general power-law dependencies $\boldsymbol{\Delta}^a$ and $\dot{\boldsymbol{\Delta}}^b$ in eq. (5). With any of such choices, the important common observation is that once function D has a power-law dependence on $\dot{\boldsymbol{\Delta}}$ and/or $\dot{\boldsymbol{\varepsilon}}$, the resulting equations of motion lead to Cole-Cole spectra for frequency-dependent stress-strain ratios. When $\nu = 1$, the energy dissipation rate is independent of the strain but is proportional to the square of the strain rate. Such dependence is typical for linear viscosity (Landau and Lifshitz, 1986). For $\nu < 1$, the dissipation increases slower with strain rate, but it also increases with strain. As noted above, this dependence on strain is only inspired by the requirement C) and is not significant for the model. The non-quadratic dependence of D strain rate (which leads to viscosity dependent on strain rate; Coulman et al., 2013) is characteristic for non-Newtonian viscosity.

By using the nonlinear D in eq. (5), the equations of motion (4) become

$$\begin{cases} \boldsymbol{\rho}\ddot{\mathbf{u}}_i = \partial_j \boldsymbol{\sigma}_{ij}, \\ \boldsymbol{\sigma}_{ij} = \mathbf{K}\boldsymbol{\Delta}\delta_{ij} + 2\boldsymbol{\mu}\tilde{\boldsymbol{\varepsilon}}_{ij} + \nu\left(\boldsymbol{\eta}_K \breve{\boldsymbol{\Delta}}^{2-2\nu,2\nu-1}\delta_{ij} + 2\boldsymbol{\eta}_\mu \breve{\boldsymbol{\varepsilon}}_{ij}^{2-2\nu,2\nu-1}\right), \end{cases} . \tag{6}$$

If we select $2\nu - 1 = \eta$, then by solving eqs. (6) for a harmonic P-wave, the exact Cole-Cole spectra shown in Figure 2 are obtained with $N=2$. Interestingly, Coulman et al. (2013) estimated the rheologic exponent ν ranging from 0.56 to 0.79 in Earth materials, which leads to the corresponding Cole-Cole parameters $\eta = 0.12$ to 0.58.

The attenuation and dispersion of bitumen sands are approximated by solving eqs. (6) and by using GLS model respectively (Figure 3). Compare to the fitting by GLS with N=6, nonlinear viscosity model with $N = 4$ can provide a more accurate approximation. The reduction of N indicates that nonlinear viscosity is a more physical meaningful interpretation for the attenuation and dispersion of the bitumen sands.

Conclusions

Lagrangian mechanics with nonlinear energy dissipation helps explaining the popular Cole-Cole model in a rigorous and purely mechanical manner. In contrast to the conventional Cole-Cole model, the Lagrangian model directly relates the experimental data to physical properties, such as elastic moduli and viscosity, without the use of fractional derivatives or "memory variables". This model also leads to generalizations of the Cole-Cole mode to more complex systems and to new algorithms for numerical modeling of seismic wavefields.

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