Physics 103 - Discussion Notes #1

Approximations in Physics

- Almost all physically interesting problems can't be solved exactly - need to make approximations in order to solve them!
- In fact, a big part of doing physics is learning to use physical insights from a problem and turn them into mathematical approximations.
- As Keith says in the notes, there are two main ways to approximately solve problems in physics - perturbatively, or numerically.
- We'll begin with the perturbative approach, but first we'll briefly review the concept of Taylor series.

Taylor Series

- Taylor's Theorem states that *most* functions (and in this class we'll almost certainly never deal with functions that are exceptions) can be expressed as an infinite sum of polynomials

  $$f\left(x\right) = \sum_{n=0}^{\infty} c_n \left(x - a\right)^n$$

  where the c_n are given by

  $$c_n = \frac{f^{(n)}\left(a\right)}{n!}$$

  where $f^{(n)}$ denotes the nth derivative of the function f.
- I've written the formula like this to emphasize that this is a sum of *polynomials* - don't let the weird form of the coefficients distract you from this fact, the c_n are just numbers.

Quick Example - Taylor Series of e^x about $a = 0$

- You've probably all seen this example at some point, but just as a quick refresher let's calculate the Taylor series of e^x about $a = 0$. We need to calculate $c_n = \frac{f^{(n)}(a)}{n!}$. For e^x we have simply

$$\frac{d^n}{dx^n} e^x = e^x$$

so

$$c_n = \frac{e^0}{n!} = \frac{1}{n!}$$

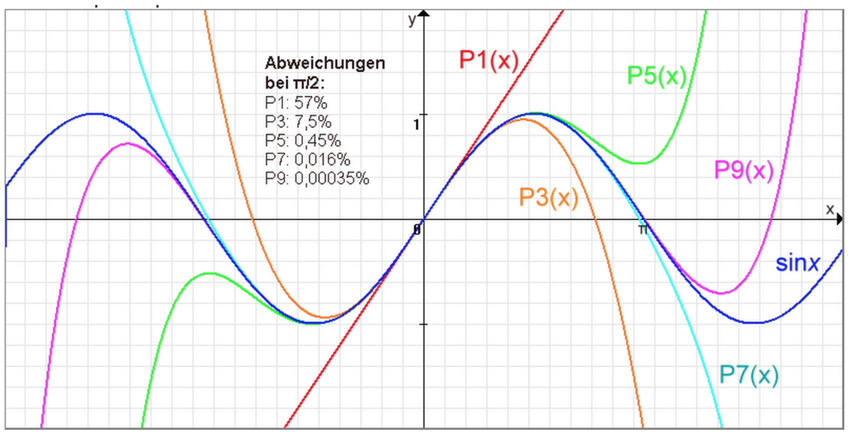
Thus

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots}$$

- In practice we rarely explicitly calculate Taylor series - most problems in physics involve functions whose Taylor series have been tabulated and can just be looked up. Example include e^x, $\sin x$, and $\ln\left(1+x\right)$, which actually comes up on your homework.

Some Remarks on Taylor Series

- In physics we never really want to work with full Taylor series - that is, the series with an infinite number of terms. Instead, we want to deal with only the first few terms in the series and use them as an approximation to the function.
- This is useful because polynomials are *nice* - we can easily take their derivatives or manipulate them algebraically, so replacing a complicated function by the first few terms in its Taylor series often greatly simplifies a problem.
- The Taylor series is most accurate close to the expansion point a - the further we go from a the more terms we need for the same amount of accuracy. *(Explain in context of plot below)*



- Just because we can write down the series doesn't mean the series itself will *converge*, that is, come closer and closer to some value. For example consider the geometric series

$$1 + x + x^2 + x^3 \dots$$

If $|x| < 1$ then the terms get smaller and smaller (since raising a number less than one to a power *decreases* it), so it seems like the series might converge (though of course just because the terms get smaller doesn't mean the series converges, e.g. the harmonic series). Indeed this is the case, as one can show that

$$1 + x + x^2 + x^3 \dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \ |x| < 1$$

Thus we can say that $\frac{1}{1-x}$ has a Taylor series given by the geometric series, that is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \dots$$

which is exactly what we'd get if we applied the above formulae to $\frac{1}{1-x}$ with $a = 0$. Note however, that this formula only holds for $|x| < 1$ - if this inequality doesn't hold then the Taylor series no longer converges and this expansion is no longer useful.

- Note that we can expand about *any* point a in our series - however, whether or not the Taylor series converges is dependent on what point we expand about. For instance, if we want to calculate the value of $\frac{1}{1-x}$ at $x = 2.5$ we could expand about $a = 2$ instead of 0 like we did above and the series would converge.
- As a follow-up, while many functions (e.g $e^x, \cos x, \sin x$) converge for all real numbers, there's often a better point a to expand about for certain situations. For example, if I want to approximate the value of $\sin(100.5)$ I *could* in principle do it by expanding $\sin(x)$ about $a = 0$ and taking an extremely large number of terms, but I'll get a much better approximation by expanding about $a = 100$.

Taylor Series in Solving Differential Equations

- You've probably actually already seen one example of using Taylor series to help solve physics problems - if we write the down the differential equation for the simple pendulum, we get

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

- This differential equation doesn't have a nice solution we can write in terms of functions we know. However, we know that $\sin\theta$ can be written in terms of its Taylor series (which converges for all x)

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

- For small x these values decrease very quickly, and so to a good approximation we can ignore all terms but the first, and write

$$\sin x \approx x$$

  in which case our differential equation becomes

$$\ddot{\theta} \approx -\frac{g}{\ell}\theta$$

  which should be recognizable as the equation that defines SHM.

Taylor Series in Solving Algebraic Equations

- We can also use Taylor series to solve algebraic equations, e.g. solving for the time a projectile spends in the air in the presence of air resistance. Let's work with a simpler example: Suppose we want to solve the equation

$$\cos(\lambda x) = x$$

- In general this equation is not solvable in terms of "nice" functions. However, if we assume that λ is "small enough" we can Taylor expand with respect to λ about 0 and keep only the first few terms in the expansion. That is, we rewrite our solution x as

$$x = \sum_{n=0}^{\infty} c_n \lambda^n = c_0 + c_1\lambda + c_2\lambda^2 + c_3\lambda^3 \dots$$

  where we need to determine the coefficients c_n. Our equation now becomes

$$\cos\left(\lambda \sum_{n=0}^{\infty} c_n \lambda^n\right) = \sum_{n=0}^{\infty} c_n \lambda^n$$

- From here we now want to expand $\sin x$ in its Taylor series, which gives

$$\sum_{n=0}^{\infty} (-1)^n \frac{\left(\lambda \sum_{i=0}^{\infty} c_i \lambda^i\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} c_n \lambda^n$$

  Notice that because we now have a double sum we had to introduce a new index - it would be incorrect to call both indices n. Calling in the index on the right n is fine since it isn't affected by what's on the left.

- This hardly looks like an improvement - in fact it looks like we've made the equation considerably more complicated. However, both sides involve power series expansions in terms of λ - and a useful Theorem on power series says that two power series are equal if and only if their coefficients are equal. Thus we can write a series of equations for the c_n by explicitly writing out terms on each side and then setting the coefficients of like powers of λ equal.

- Let's see how this work in practice - let's say we wish to go out to fourth order in λ. We explicitly write out the first few terms in the series

$$1-\frac{\left(\lambda\sum_{i=0}^{\infty}c_i\lambda^i\right)^2}{2!}+\frac{\left(\lambda\sum_{i=0}^{\infty}c_i\lambda^i\right)^4}{4!}\ldots=c_0+c_1\lambda+c_2\lambda^2+c_3\lambda^3+c_4\lambda^4\ldots$$
$$1-\lambda^2\frac{\left(c_0+c_1\lambda+c_2\lambda^2+c_3\lambda^3\ldots\right)^2}{2}+\lambda^4\frac{\left(c_0+c_1\lambda+c_2\lambda^2+c_3\lambda^3\ldots\right)^4}{24}\ldots=c_0+c_1\lambda+c_2\lambda^2+c_3\lambda^3+c_4\lambda^4\ldots$$

- In this form it's much easier to match the coefficients. First, we see that the constant term on the left is 1, and on the right it's c_0, so we immediately have

$$c_0=1$$

There is no term of order λ on the left, so

$$c_1=0$$

Now we need to worry about squaring out the infinite sum. The only way to get a term of order λ^2 is from the first term multiplied by itself, so we have

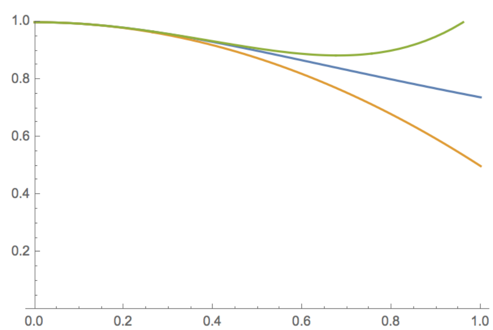
$$-\frac{c_0^2}{2}=c_2\Rightarrow c_2=-\frac{1}{2}$$

For the λ^3 there is again only one way to get a term of this order from the left - by multiplying $c_1\lambda$ by c_0. But $c_1=0$, so this term will drop out. Finally, the λ^4 term is a bit more complicated. There are now *two* ways to get this term from squaring the sum - either from terms of the form $c_0c_2\lambda^2$ or from $c_1^2\lambda^2$. For the same reason as above however, the latter will be 0. We also note there will be *two* of these terms. We now finally have to take into account the quartic term as well - thankfully we will only be concerned with the first term, $c_0\lambda^4$. Bringing this all together gives the equation

$$-\frac{2c_0c_2}{2}+\frac{c_0^4}{24}=c_4$$
$$c_4=\frac{13}{24}$$

Thus our perturbative solution is

$$\boxed{x\approx1-\frac{\lambda^2}{2}+\frac{13}{24}\lambda^4}$$



Numerical Methods

- In contrast to perturbative methods, we can also problems numerically when an exact solution is not possible. Let's go back to our pendulum example and discuss how to solve it numerically.
- If we want to solve the simple pendulum numerically we need to know exact values for the parameters in the problem, in this case we'd need the values of g and ℓ. If we know these numbers a program like Mathematica can solve the differential equation numerically.
- The advantage here is that this is generally faster - with Mathematica this can be done in about a minute if you know what you're doing - and we no longer need to make assumptions about the size of θ; this will work for *any* θ.

- The disadvantage however, is that we really only gain insight about the problem *for the specific values we chose*. We don't learn anything about how the problem behaves as we change g and ℓ unless we solve the problem many times for different values of the variables, which can quickly become too computationally expensive, especially for more complicated equations. For example our perturbative solution tells us that the motion is approximately simple harmonic for small θ, but the numerical solution only tells us the solution *looks* simple harmonic if we *choose* a small initial value of θ, and we can't even be sure that this isn't just a relic of the specific values we chose for g and ℓ without solving many times. Even then, we can't be sure that there isn't *some* combination of the parameters when the motion looks completely different (those of course this isn't the case, as the perturbative method tells us).

In Conclusion

- Each method has its pros and cons - think carefully about which will be more helpful given the problem you're trying to solve.