Physics 201 Lab 9: Torque and the Center of Mass

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Theoretical Discussion

For each of the linear kinematic variables; displacement $\vec{r}$, velocity $\vec{v}$ and acceleration $\vec{a}$; there is a corresponding angular kinematic variable; angular displacement $\vec{\theta}$, angular velocity $\vec{\omega}$, and angular acceleration $\vec{\alpha}$, respectively. Associated with these kinematic variables are dynamical variables—momentum and force for linear variables—angular momentum and torque for angular variables. The rotational analogue of the inertial mass m is the moment of inertia I[1]. The relationships are summarized in table I.

Linear variables	**analogous angular variable**
displacement $\vec{r}$	angular displacement $\vec{\theta}$
velocity $\vec{v} = \frac{d\vec{r}}{dt}$	angular velocity $\vec{\omega} = \frac{d\vec{\theta}}{dt}$
acceleration $\vec{a} = \frac{d\vec{v}}{dt}$	angular acceleration $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$
momentum $\vec{p} = m\frac{d\vec{r}}{dt} = m\vec{v}$	angular momentum $\vec{L} = \vec{r} \times \vec{p} = I\frac{d\vec{\theta}}{dt} = I\vec{\omega}$
force $\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$	torque $\vec{\tau} = \vec{r} \times \vec{F} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha}$

TABLE I: Linear variables and their angular analogues

According to the table, torque is the angular analogue of force: In other words, just as force acts to change the magnitude and/or direction of an object's linear velocity, torque acts to change the magnitude and/or direction of an object's angular velocity. The equation

$$\vec{\tau} \;=\; I\frac{d\vec{\omega}}{dt} = I\vec{\alpha} \tag{1}$$

describes the *effect* of a torque on the object's angular kinematic variables. It tells you what a torque does, but not where it comes from. A torque arises whenever a force acts upon a rigid body that is free to rotate about some axis. If the applied force is $\vec{F}$ and the displacement vector from the axis of rotation to the point where the force is applied is $\vec{r}$, then the torque is equal to

$$\vec{\tau} \;=\; \vec{r} \times \vec{F} \tag{2}$$

Using the definition of the vector cross product, the magnitude of the torque is

$$\tau \;=\; rF\sin\theta \tag{3}$$

where θ is the smallest angle between the vectors $\vec{F}$ and $\vec{r}$. The direction of the torque vector is given by the right-hand rule—place the fingers of your right hand along $\vec{r}$ and curl them into $\vec{F}$: your thumb will point in the direction of $\vec{\tau}$. The directional relationships between $\vec{F}$, $\vec{r}$, and $\vec{\tau}$ are shown in figure 1.

So long as the total net force on an object is zero, the velocity of its center of mass will not change. However, it is possible for an object to have zero net force acting on it, but to nevertheless have a non-zero torque

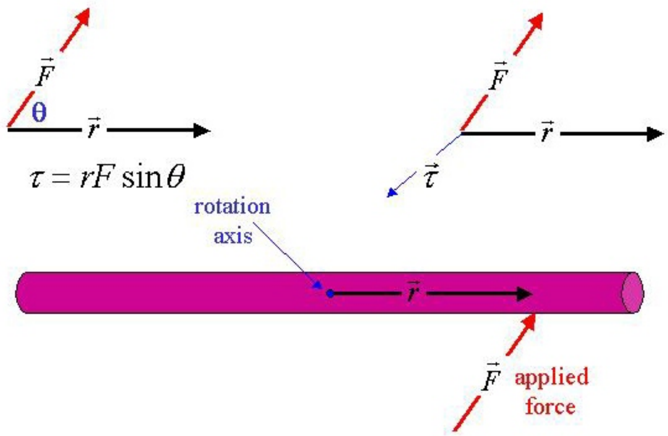

FIG. 1: Directional relationships between $\vec{F}$, $\vec{r}$, and $\vec{\tau}$

acting on it. Figure 2 shows one such possible scenario. The velocity of the center of mass of the object in figure 2, acted upon by two equal and opposite forces, will remain constant, but since the torque is non-zero, it will spin about its axis at an ever-increasing rate of rotation.

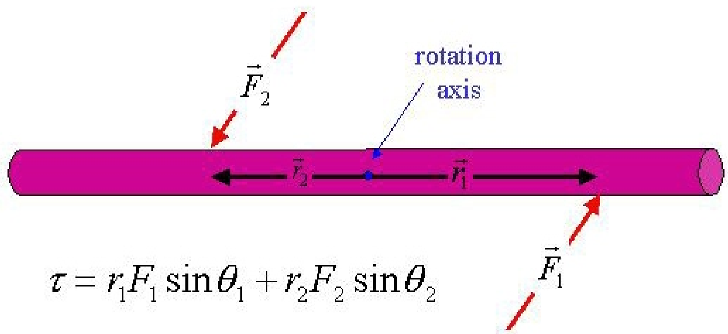

FIG. 2: Equal forces applied to opposite sides of a rotation axis

For an object to remain in *static equilibrium*, so that both the velocity of its center of mass and its angular velocity about any axis are constant, both the net force and the net torque on it must equal zero. The conditions for static equilibrium are:

$$\vec{F}_{\text{net}} = 0 \tag{4}$$
$$\vec{\tau}_{\text{net}} = 0 \tag{5}$$

Center of Mass

The center of mass coordinate of a system or an extended object is defined so that Newton's law of motion, in the form

$$\frac{d\vec{p}}{dt} = \vec{F}_{ext} \tag{6}$$

applies to the sytem or object as if it were a point particle located at the center of mass coordinate and the external force were applied at that point. If we define the center of mass coordinate as

$$\vec{r}_{c.m.} = \frac{\sum_j m_j \vec{r}_j}{\sum_j m_j} = \frac{\sum_j m_j \vec{r}_j}{M_{tot}} \tag{7}$$

where m_j is the mass of the j^{th} particle in the system or object, $\vec{r}_j$ is the position vector of the j^{th} particle, and M_{tot} is the system or object's total mass, we see that the center of mass so defined does indeed satisfy equation 6, since

$$\begin{aligned}
\frac{d\vec{p}}{dt} &= \frac{d}{dt}\left(M_{tot}\vec{v}_{c.m.}\right) \\
&= \frac{d}{dt}\left(M_{tot}\frac{d\vec{r}_{c.m.}}{dt}\right) \\
&= \frac{d}{dt}\left(\sum_j m_j \frac{d\vec{r}_j}{dt}\right) \\
&= \frac{d}{dt}\left(\sum_j \vec{p}_j\right) \\
&= \sum_j \vec{F}_j \\
&= \vec{F}_{ext}
\end{aligned}$$

Since the center of mass coordinate is defined so that the external force can be taken to act at that point, it is constant in time, and the geometric interpretation of Newton's Law for rotational motion (equation 8);

$$\vec{\tau}_{ext} = \frac{d\vec{L}_{c.m.}}{dt} = \frac{d}{dt}\left(\vec{r}_{c.m.} \times \vec{p}_{c.m.}\right) = \vec{r}_{c.m.} \times \frac{d\vec{p}_{c.m.}}{dt} = \vec{r}_{c.m.} \times \vec{F}_{ext} \tag{8}$$

is that the displacement vector $\vec{r}_{c.m.}$ points from the pivot point to the center of mass coordinate. Figure 3 depicts this geometric interpretation. Using the right-hand rule, we see that the torque in this case is negative (into the page), so that the celebrated swingee will rotate clockwise under the influence of gravity.
Since the magnitude of the vector cross product is given by

$$\left|\vec{A} \times \vec{B}\right| = |A||B|\sin\theta$$

where θ is the smallest angle between $\vec{A}$ and $\vec{B}$, then the magnitude of the torque is given by equation 9 below.

$$|\vec{\tau}_{ext}| = |\vec{r}_{c.m.}||\vec{F}_{ext}|\sin\theta \tag{9}$$

Equation 9 makes clear that the torque vanishes whenever the vector $\vec{r}_{c.m.}$ from the pivot point to the center of mass is colinear with the external force $\vec{F}_{ext}$. For an object suspended at rest under its own weight, this implies the following results, which we will make use of in this experiment:

- An object suspended at rest under its own weight is in static equilibrium. Therefore, both the net force and net torque on it are zero.

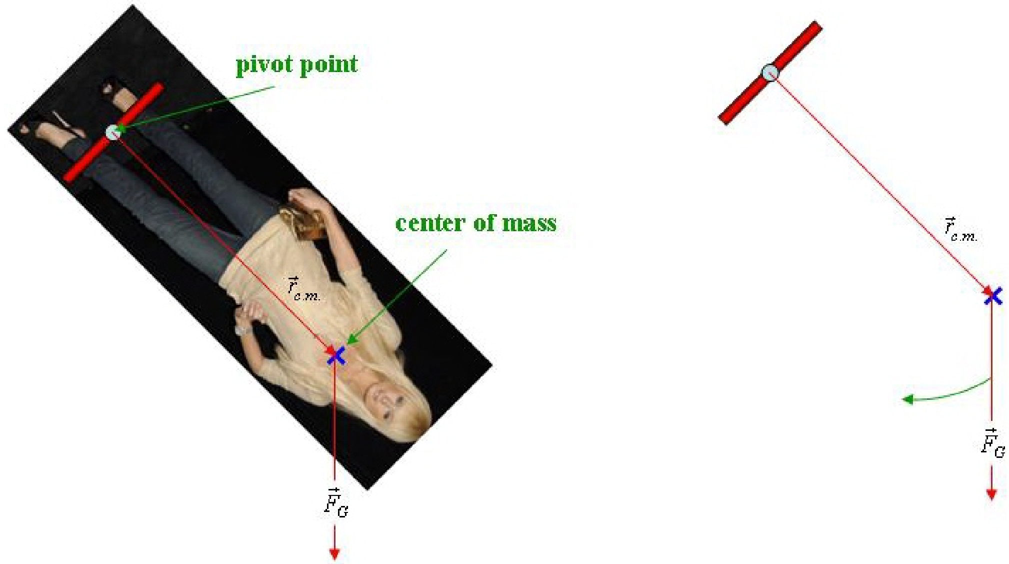

FIG. 3: Relative orientation of the vectors $\vec{r}_{c.m.}$ and $\vec{F}_{ext}$ for a celebrity suspended from a pivot.

- Since the torque will only vanish in the gravitational field if $\vec{r}_{c.m.}$ and $\vec{F}_G$ are parallel (or anti-parallel), the object will align itself so that the vector $\vec{r}_{c.m.}$ from the pivot point to the center of mass is *vertical.*

It follows that in suspending a massive object from a variety of different pivot points, all vertical lines originating from the respective pivot points will intersect at the center of mass. In today's lab you will use the conditions for static equilibrium to measure the mass of a meter stick that is balanced on a knife-edge fulcrum. You will also find the center of mass coordinate of an angled bar.

Procedure for Determining the Mass of a Meter Stick

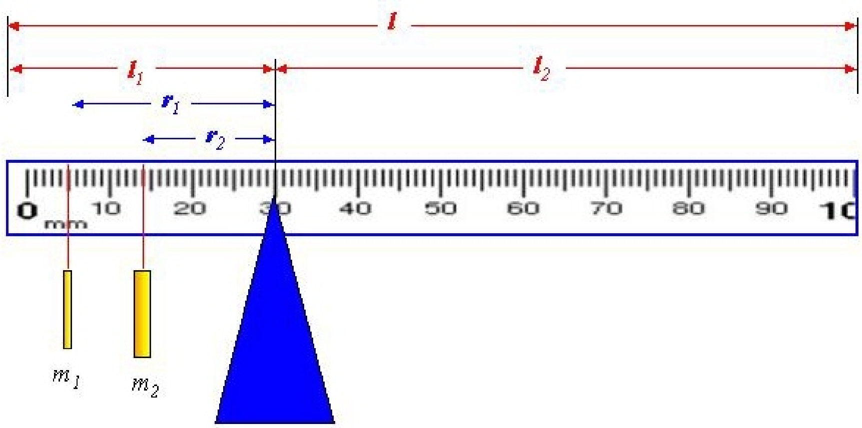

FIG. 4: Experimental setup for using torque to weigh a meter stick

The experimental setup is shown in figure 4. Suppose you can get your meter stick in balance as shown in the figure. There are three forces acting on the left-hand side of the stick and one force acting on the right-hand side. The mass of the section of ruler on the right hand side is $m_{rhs} = m\frac{l_2}{l}$, where $l = 100$ cm is the length of the ruler. The center of mass of the ruler material on the right-hand side is located a distance $\frac{l_2}{2}$ from the fulcrum, as illustrated in figure 5. Since the force acts at right angles to the displacement, the magnitude of the total torque acting on the right-hand side is

$$\tau_{\text{right}} = m_{rhs} g \frac{l_2}{2} = mg \frac{l_2^2}{2l} \tag{10}$$

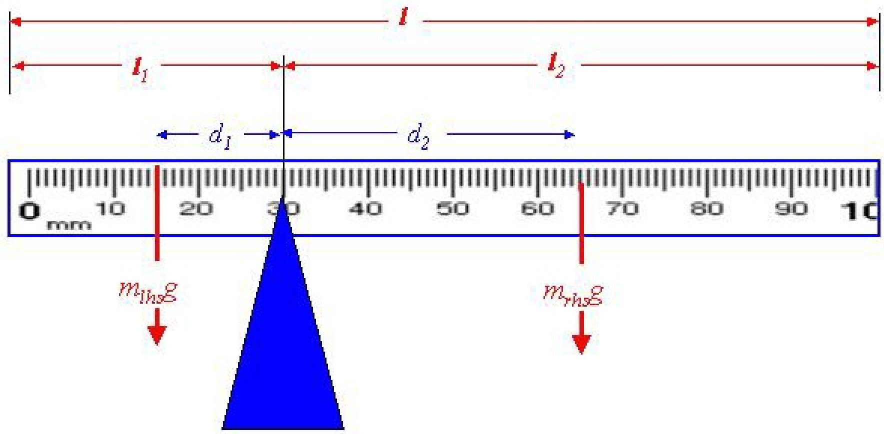

FIG. 5: Illustration of the balance of torques

On the left hand side, there are the gravitational forces due to m_1, m_2, and the mass of the ruler material m_{lhs} on the left-hand side of the stick. The magnitude of the forces arising from these three sources are, respectively,

$$\begin{aligned} F_1 &= m_1 g \\ F_2 &= m_2 g \\ F_3 &= m_{lhs} g = m \frac{l_1}{l} g \end{aligned}$$

The center-of mass of the ruler material on the left-hand side is located a distance $\frac{l_1}{2}$ from the fulcrum. Since all three forces act at right angles to the displacement, the total torque on the left-hand side is

$$\tau_{\text{left}} = m_1 g r_1 + m_2 g r_2 + mg \frac{l_1^2}{2l} \tag{11}$$

All of the downward forces acting on the ruler are countered by an equal and opposite upward reaction force (normal force) that acts at the point of the fulcrum. Since it acts at the fulcrum point, it exerts no torque, so that the equation for static equilibrium is

$$\tau_{\text{left}} = \tau_{\text{right}} \longrightarrow m_1 g r_1 + m_2 g r_2 + mg \frac{l_1^2}{2l} = mg \frac{l_2^2}{2l} \tag{12}$$

Cancelling the common factor of g, and re-arranging the equation, you get an equation for the mass of the ruler:

$$m = \frac{2l(m_1 r_1 + m_2 r_2)}{(l_2^2 - l_1^2)} \tag{13}$$

By varying the fulcrum point, and hence the values of l_1 and l_2, and adjusting the locations of m_1 and m_2 to achieve static equilibrium, you can obtain independent measurements of the mass of the ruler.

Procedure for Finding the Center of Mass of an Angled Bar

Figure 6 depicts the experimental set-up for this experiment. In brief, you will sequentially suspend the L-bracket from each of three pivot points. In each case, you will trace a vertical line along a plumb bob hanging from the pivot point. The point where all three lines intersect is the center of mass of the L-bracket. Note that the center of mass coordinate of a body need not lie within the body itself.

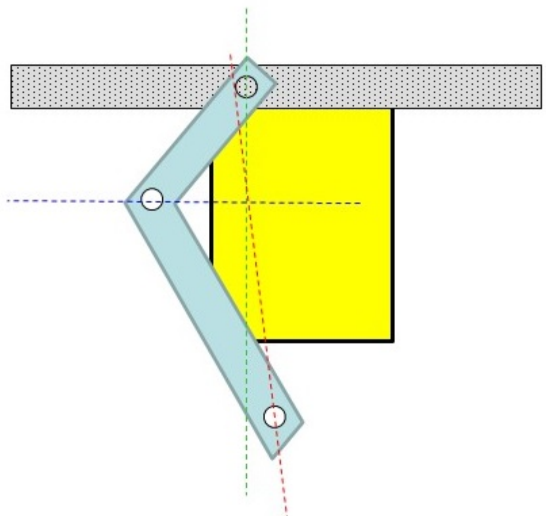

FIG. 6: Experimental scheme for measuring the center of mass.

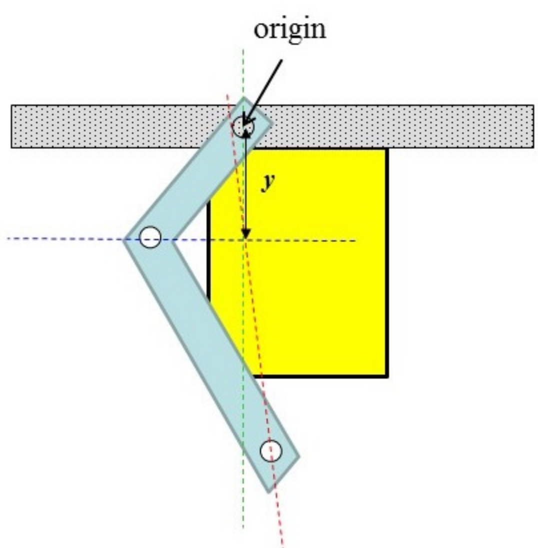

FIG. 7: Analysis method for reporting the center of mass.

Summary

- Weighing the meter stick

  1. Set the fulcrum location at approximately 30 cm.
  2. Bring the meter stick into balance (static equilibrium) by varying the positions of the two masses, m_1 and m_2. The inner mass m_2 (mass closest to the fulcrum) should be about 150 g. The outer mass m_1 should be about 10–20 g. You can use the position of m_1 to "fine-tune" the balance.
  3. Record the distances r_1, r_2, l_1, l_2, m_1 and m_2.
  4. Calculate the ruler's mass using the method of static equilibrium (equation 13) and label it m_{exp}.
  5. Weigh the ruler using the mass balance. Record this value as m_b.
  6. Calculate and report the fractional discrepancy δ between m_{exp} and m_b.

- Finding the center of mass of the angled bar

  1. Tape a sheet of paper into the inside corner of the L-bracket, as shown in figure 6.
  2. For each of the three possible pivot points:
     (a) Hang a plumb bob from the pivot point.
     (b) Mark two points along the plumb line, widely separated enough so that you can later draw an accurate line.
  3. Remove the L-bracket from the hanger and carefully draw in the three lines indicated by your points.
  4. The lines should intersect at a single point, or at worst, make a small triangle. In the former case, you have located the center of mass coordinate. In the later case, if all sides of the triangle are smaller than 1 cm, take the center of the triangle as the center of mass coordinate. Otherwise, repeat the measurements.
  5. Report your measured center-of-mass for the object, using the origin and method shown in figure 7.

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[1] The moment of inertia I is defined as $I = \int r^2 dm$, where the displacement r is from the axis of rotation to the location of the mass element dm.