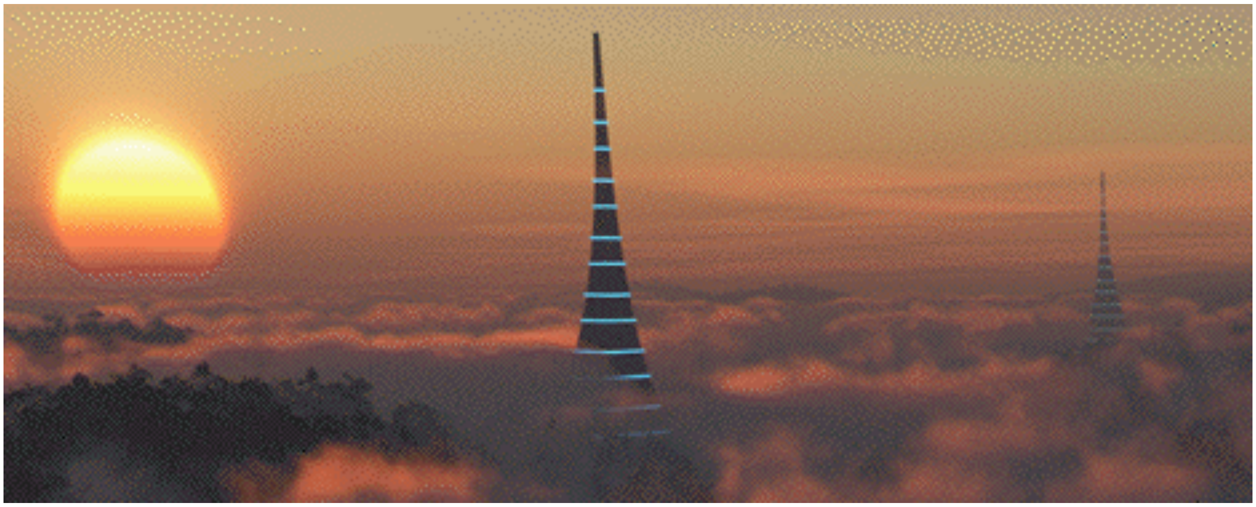

Physics 225

Relativity and Math Applications

Fall 2012

Unit 6

Conservation, Conversion, and Nuclear Power

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Unit 6: Conservation and Mass-Energy Conversion

Exercise 6.1: Relativistic Mechanics → Cleanup and Warmup

Last week, we modified Newtonian mechanics so that it works at speeds close to the speed of light. Following Einstein, we built the theory of **relativistic mechanics** from these 4 hypotheses:

① $\vec{F} = \dfrac{d\vec{p}}{dt}$ ② $\vec{p} = m_{\text{inertial}}\vec{v}$ ③ $W = \int \vec{F} \bullet d\vec{l} = \Delta E$ ④ $E = m_{\text{inertial}}c^2$

From these hypotheses, you yourself derived our new master relation (in the box below) between an object's total energy E, momentum p, and rest mass m_0. You also derived exactly how Einstein's m_{inertial} varies with speed. Third, we presented a definition of kinetic energy: it is simply the difference between an object's total energy and its rest energy.

$$m_{\text{inertial}} = \gamma m_0 \qquad \boxed{E = \sqrt{(pc)^2 + (m_0c^2)^2}} \qquad KE \equiv E - m_0c^2$$

Finally, you derived these useful combinations of the relations above:

$$\vec{p} = \gamma m_0 \vec{v} \qquad E = \gamma m_0 c^2 \qquad \beta = \frac{pc}{E} \qquad \gamma = \frac{E}{m_0c^2}$$

Today we'll add two key pieces to our theory: **energy & momentum conservation**. We will find that these lead to a profound consequence of relativity: **mass-energy conversion**. But first, let's spend a bit more time with our formulae and address some common points of confusion.

(a) As you work with these equations, you quickly realize that we can do some cleanup: <u>inertial mass can be removed from our theory entirely</u>. Replacing the "m" in $p = mv$ with a variable quantity "m_{inertial}" was an enormous leap of intuition, but now that our theory is built m_{inertial} plays no role → it is just a redundant variable representing the combination γm_0. To make this crystal clear, <u>**draw a big X**</u> through the equations above that involve m_{inertial}. Really! Find them, and X them out. *Do you see how no information is lost by removing this quantity?* Please be clear on this point; if you have any questions, ask!

So: **From now on, *m* means rest mass.**

You can still call it "m_0" if you want (I generally will in this unit); there's just no need to carry that extra subscript around any more as rest mass is the only mass you will ever see in any calculation or table of masses. In Special Relativity, the inertial mass γm_0 is essentially a historic concept that is now neatly absorbed into the formulae of relativistic mechanics.

(b) Momentum has direction → it's a vector. However, it often appears as just "p" without a vector sign, e.g. in some of the formulae above. Suppose you have a particle moving in the $-x$ direction. Should you assign a negative value to the symbol "p" in this case?

Answer: **NO!** The symbol "p" represents the **magnitude** of the vector $\vec{p}$ and *never* has a negative sign. So where does the direction information go? → in the **components** of $\vec{p}$. Magnitudes of vectors don't have signs, **only the components of vectors have signs**. In the previous example, you would assign the component p_x a negative value, but p is always positive.

(c) Newton's most famous formula is $F = ma$ … yet it is conspicuously absent from our summary of relativistic mechanics. Hmm. Most of Newton's other formulas are preserved, like $p = mv$. Honestly, it looks like all the relations *underlying* $F = ma$ are still there, yet our familiar friend is absent. What has happened to $F = ma$? Maybe it's fine, maybe it's so easy to derive from the other relations that we just didn't bother to write it down. What do you think? Try to derive $F = ma$ from $F = dp/dt$ and our new relation $p = \gamma m_0 v$ and see what happens.

(d) As we learned over the past week, the **photon** can be easily incorporated into our new mechanics as a particle of zero rest mass: that gives $E = \sqrt{(pc)^2 + (\cancel{m_0} c^2)^2} = pc$, which is exactly the energy-momentum relation for light that comes from Maxwell's equations. Wonderful! Now consider another of our relations: $p = \gamma m_0 v$. How does *that* apply for a photon??! With $m_0 = 0$, the formula seems to tell us that photons always have zero momentum! Is that so? Or is this formula somehow invalid for photons? Think a bit, and you'll figure it out.

(e) Our formulae for relativistic mechanics relate all the many properties of a particle. Here is the full list of every particle property that appears on the previous page: E, $\vec{p} = (p_x, p_y, p_z)$, KE, m_0, m_{inertial}, $\vec{v} = (v_x, v_y, v_z)$, β, γ. That's 12 symbols! It looks like a mess … but it's not really. ☺ The Key Question is: how many of those symbols are **independent**? i.e., what is the minimum number of particle properties we need to know in order to know *everything* about that particle?

The answer to that last question is … *drum roll* … **four**. A particle's kinematics can be completely described by exactly four quantities. A very standard set to choose is:

- total energy E
- all three components of momentum $\vec{p} = (p_x, p_y, p_z)$

If you know those four things about a particle, you can calculate everything else. *Any* set of four independent quantities will do. For example, if you know a particle's rest mass m_0 and velocity vector $\vec{v} = (v_x, v_y, v_z)$ you can also calculate everything about it. If direction is not known or not interesting, only **two** quantities are needed, e.g. m_0 and v.

(f) Particle physics detectors have many subsystems that measure different particle properties. A typical setup is to use a tracking system plus a magnetic field to measure the momentum vector $\vec{p}$ of each particle, while time-of-flight detectors determine each particle's speed $\beta = v / c$. Suppose your detector measures a track with these four properties:

$$p_x = 2 \text{ GeV/c}, \quad p_y = 1 \text{ GeV/c}, \quad p_z = 0, \quad \beta = 0.92$$

This is a complete set of information. Now you be the analysis software and do **particle identification** (**PID** for short): figure out what sort of particle this is. There are only five charged particles in nature that live long enough to leave observable tracks in a detector:

- electrons e^- and positrons e^+ : mass 0.51 MeV/c^2
- muons $\mu^{\pm}$: mass 106 MeV/c^2
- pions $\pi^{\pm}$: mass 140 MeV/c^2
- kaons $K^{\pm}$: mass 494 MeV/c^2
- protons p and antiprotons $\bar{p}$: mass 938 MeV/c^2

What sort of particle did you detect? (See footnote[1])

Exercise 6.2: Conservation of Energy and Momentum

It's time to add two more pieces to relativistic mechanics. Fortunately, they are old familiar friends: the laws of **conservation of energy** and **conservation of momentum**, which are fundamental to all of physics. What does "conservation" mean exactly? It means "remains constant for an isolated system". The total energy and momentum of an isolated system remain constant, so they are conserved quantities. To be precise, *four* things are conserved:

- total energy E is conserved
- *each component* of momentum is conserved: p_x, p_y, and p_z

[1] The mass you get will not *exactly* match any of the listed particle masses → this is a realistic, real-world problem and no detector has perfect accuracy. The number of significant digits provided for the momentum and speed measurements always indicates how precise those measurements were.

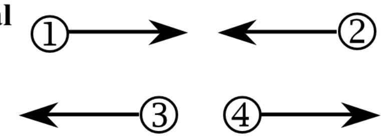
These conservation laws allow us to tackle **collision problems**: any situation where you have an isolated system consisting of one or more objects, then some interaction happens within the system, then the system continues in a modified state. Such situations include **elastic** collisions, **inelastic** collisions, and **decays**[2]. Relativistic collisions reveal something very interesting … but first, let's make sure you know how to set up conservation relations, particularly the *signs* …

(a) A "2-to-2" particle collision is shown below, with its conservation conditions written in different forms. Taking $+x$ to be to the right, put a <u>sign</u> (+ or –) in front of each p or E symbol:

- p vectors: $\vec{p}_1 \quad \vec{p}_2 \;=\; \vec{p}_3 \quad \vec{p}_4$
- p components: $p_{1x} \quad p_{2x} \;=\; p_{3x} \quad p_{4x}$
- p magnitudes: $p_1 \quad p_2 \;=\; p_3 \quad p_4$
- energy: $E_1 \quad E_2 \;=\; E_3 \quad E_4$

initial state:

final state:



(b) Two lumps of clay, each of rest mass m_0, collide head-on at speed 3/5 c (i.e. they are moving directly towards each other with equal and opposite velocities 3/5 c). They stick together, forming one big lump of clay. What is the rest mass M_0 of the composite lump?

Tactics for all collision problems: first make a sketch with labels *of course* ☺, then <u>write down the conservation relations</u> for E, p_x, p_y, and p_z. These are the very constraints that *nature* uses to decide what happens! Then work as usual to get <u>what you want</u> in terms of <u>what you know</u>.

Energy-to-Mass Conversion

If everything went correctly, you should find that the mass M_0 of the composite lump of clay is *larger* than the sum of the contributing rest masses m_0 (25% larger in this example). Wow, there is *more mass* around after the collision than before! This is **energy-to-mass conversion** at work → the kinetic energy of the original clay lumps was *converted into mass*.

[2] Jargon check on "elastic" and "inelastic": An **elastic** collision is one where you have the *same particles* in the initial and final states. A classic example is the scattering of billiard balls. In an **inelastic** collision, the nature of the system fundamentally changes: you may have more or fewer particles after the collection, or some of the particles may change state. An example would be a ground state atom which absorbs a photon and becomes an excited atom.

In Newtonian mechanics, this doesn't happen: mass is always conserved. But energy is conserved too. What happens to the kinetic energy of the initial state in Newtonian mechanics? In classical mechanics, we would say that this energy is converted into *heat*, a concept which represents the total *internal* energy of the tiny pieces making up the clay. Relativity doesn't dispute that. Rather, it says that *a particle's total energy is reflected in its inertial mass.* Quite literally, **a hot potato is heavier than a cold potato**!

(c) A charged pion at rest decays into a muon and a neutrino. Find the energy of the outgoing muon in terms of the rest mass m_π of the pion and the rest mass m_μ of the muon, taking the neutrino to have zero rest mass.[3]

(d) Plug in the actual rest-mass numbers from a couple of pages ago to get the energy of the muon in MeV. Now fill in this table with numbers to see what has happened:

	Initial State Pion π	Final State		
		Muon μ	Neutrino ν	**Total Final State**
Rest energy m_0c^2 (MeV)				
Kinetic energy *KE* (MeV)				
Energy *E* (MeV)				

[3] Only in the last few years have we discovered that the neutrino has non-zero mass, but the mass is extremely small.

Mass-to-Energy Conversion

This problem is the inverse of the previous one: we start with a massive object at rest (the pion) and we end up with two particles fleeing the scene at high speed. The kinetic energy of the final-state particles came from the mass of the initial-state pion. This is an example of **mass-to-energy conversion**. A more famous example is the Uranium-235 nucleus, which decays to a Thorium-231 nucleus and an alpha particle. The final-state products of this decay have kinetic energy, derived from the initial mass of Uranium-235, which is more massive than a Thorium-231 nucleus plus an alpha particle at rest. The final-state kinetic energy can be converted to heat (e.g. the decay products can be directed at a material, which heats up), and that heat can be further converted to mechanical energy. We have thus made a useful power generator! Uranium-235 is one of the key elements in modern nuclear reactors.

Potential Energy

The energy E in our kinematic formulae represents a particle's **total internal energy**, including its kinetic energy (due to its motion) and its internal energy not related to its motion (due to its rest mass, which for complex objects includes internal sources of energy like binding energy or heat). You may be wondering about **potential energy** … where is it? If your particle is an electron and is sitting near a charged plate, it has potential energy due to the force exerted on it by the plate's electric field. The potential energy of a particle in a force field of external origin is *not* included in $E = \gamma m_0 c^2$. Why? Because potential energy involves an outside force: it is *not internal* to the particle, but instead reflects the particle's environment.

Conserved Quantities: Newton vs Einstein

In **Newtonian mechanics**:

- the total (rest) mass m_0 of a system is conserved
- the total kinetic + potential energy, $KE+U$, of a system is also conserved (assuming, of course, that there are no other forms of energy like heat in the problem).

A glance at your table on the previous page reveals that relativistic mechanics does *not* conserve rest mass m_0! Instead, Einstein's $E = m_0 c^2 + KE$ is conserved. What about when potential energy is involved? The work energy theorem (hypothesis 3) on page 6.3 provides the answer: $W = \int \vec{F} \bullet d\vec{l} = \Delta E$, and since potential energy is defined by $\Delta U = -\int \vec{F} \bullet d\vec{l}$ we immediately obtain $\Delta E + \Delta U = 0$, i.e., that $E+U$ is conserved = never changes.

To summarize, in **relativistic mechanics**:

- the total rest mass m_0 of a system is ***not*** conserved
- the total internal + potential energy $E+U$ of a system is conserved

Einstein's E is the sum of rest energy $m_0 c^2$ and kinetic energy KE; it is this ***combination*** that, when added to potential energy, always remains the same for an isolated system.

Exercise 6.3: Now in 2D

So far in this unit, we've only considered problems involving motion along a single direction. That was all we needed to explore the new physics of energy-to-mass and mass-to-energy conversion. But now, on to more general collision problems involving some angles.

(a) To remind ourselves how this works, consider a simple non-relavistic problem. (You can use Newtonian or relativistic formulae, we've just kept the speeds of everything small enough that you'll get the same result to a high degree of accuracy.) A race car of mass 4,000 kg speeds along a straight track at 90 mph (which is 40 m/s) and crashes into a large crate left on the track. The crate is smashed into two pieces of debris, each of which goes flying away from the track with momentum 1.0×10^5 kg·m/s. One piece of debris flies off to the left and the other to the right, with both trajectories making an angle of 60° with the track. The race car continues along the track after the collision, but it has been slowed down. What is the momentum of the race car after the collision?

Self-check: the car's final momentum is in footnote [4]. If that's not what you got, scrutinize your work and see if you can find your error.

Now, on to a relativistic collision problem in 2D!

[4] The answer to (a) is 0.6×10^5 kg·m/s.

An electron beam of momentum 3 GeV/c is fired along the $+x$ direction at a stationary proton target. The (rest) masses of the electron and proton are 0.5 MeV/c^2 and 1 GeV/c^2 respectively. An elastic collision occurs. The electron beam is scattered at a sharp 90° angle, and heads off in the $+y$ direction. The proton target is deflected differently; after the collision, its trajectory makes an angle θ with the x axis. Your job is to calculate this angle θ.

(b) First, formulas only, no numbers: Obtain an expression that is *solvable* for the angle θ. Don't try to solve this expression (it's just a mess of uninstructive algebra), but make sure it contains *only* the angle θ and known quantities — that's what we mean by "solvable"!

(c) To obtain a numerical answer, notice how light the mass of the electron is compared to its momentum in this problem. This is a standard situation in nuclear and particle physics, we discussed in class, and it is very common to *approximate the electron as massless* (thus treating the electron like a photon). Taking this approximation, calculate the deflection angle θ of the scattered proton.[5]

[5] The answer is $\theta = 14°$. If you got it, congratulations! You are a master of collision problems ... and algebra. ☺

Summary of Relativistic Mechanics

The formulas for relativistic mechanics are derived from the following four postulates:

① $\vec{F} = \frac{d\vec{p}}{dt}$ ② $\vec{p} = m_{\text{inertial}}\vec{v}$ ③ $W = \int \vec{F} \bullet d\vec{l} = \Delta E$ ④ $E = m_{\text{inertial}}c^2$

I've greyed-out the relations involving the speed-dependent inertial mass because, as we discussed today, once we figured out that $m_{\text{inertial}} = \gamma m_0$, maintaining a separate symbol for that combination is unnecessary overhead. From the 4 hypotheses, you obtained these key relations:

$$m_{\text{inertial}} = \gamma m_0 \qquad \boxed{E = \sqrt{(pc)^2 + (m_0c^2)^2}} \qquad KE \equiv E - m_0c^2$$

Don't forget why the energy-momentum relation is in a box: (1) it is the one you are most likely to *avoid* because it is unfamiliar and has a nasty square root, and (2) it is generally the *most* useful of all the formulae! These other useful relations are readily derived from those above:

$$\vec{p} = \gamma m_0\vec{v} \qquad E = \gamma m_0c^2 \qquad \beta = \frac{pc}{E} \qquad \gamma = \frac{E}{m_0c^2}$$

The remaining elements of the theory we've developed so far are:

- The **photon** is incorporated into relativistic mechanics as a particle of <u>zero rest mass</u>.
- The **total energy *E*** of an isolated system is <u>conserved</u> (or $E+U$ if forces are present).
- **Each component p_i** of the total momentum $\vec{p}$ of an isolated system is <u>conserved</u>.

With that, we've successfully patched Newtonian mechanics to (1) apply to speeds close to c, and (2) incorporate the photon and its strange energy-momentum relation $E_\gamma = p_\gamma c$.
Finally, we discussed some important **calculational tips** that are especially valuable in conquering the jungle of relativistic mechanics formulae:

- <u>Pick the right formula</u> for the job → the one that has *only* the variables you care about
- <u>Avoid velocity</u>! → E, p, and m_0 are generally the best variables to work with.
- <u>Dimensionless terms are awesome</u> → "factor out the units" wherever you can.

For convenience, here are the formulae for boosting velocities between frames:

$$\boxed{u_x = \frac{u'_x + v}{1 + u'_x v / c^2} \qquad u_{y,z} = \frac{u'_{y,z}}{\gamma(1 + u'_x v / c^2)}}$$

One more week and we'll have the boost formulae for *all* of our kinematic quantities. ☺